

A latent factor model for spatial data with informative missingness

Brian J. Reich^{a1} and Dipankar Bandyopadhyay^b

^a Department of Statistics,
North Carolina State University

^b Division of Biostatistics and Epidemiology,
Medical University of South Carolina

July 29, 2009

Abstract

A large amount of data is typically collected during a periodontal exam. Analyzing these data poses several challenges. Several types of measurements are taken at many locations throughout the mouth. These spatially-referenced data are a mix of binary and continuous responses, making joint modeling difficult. Also, most patients have missing teeth. Periodontal disease is a leading cause of tooth loss, so it is likely that the number and location of missing teeth informs about the patient's periodontal health. In this paper, we develop a multivariate spatial framework for these data which jointly models the binary and continuous responses as a function of a single latent spatial process representing general periodontal health. We also use the latent spatial process to model the location of missing teeth. We show using simulated and real data that exploiting spatial associations and jointly modeling the responses and locations of missing teeth mitigates the problems presented by these data.

Key words: Binary spatial data; Informative cluster size; Multivariate data; Periodontal data; Probit regression; Shared parameter model.

¹Corresponding author, email: reich@stat.ncsu.edu.

1 Introduction

Periodontal disease or periodontitis is an inflammatory disease affecting periodontium, the tissues that support and maintain teeth. Periodontitis causes progressive bone loss around the tooth which can lead to tooth loosening and eventually tooth loss. It has been estimated that about 50% of US adults over the age of 35 experience early stages of periodontal disease (Oliver, Brown and Loe, 1998), making periodontitis the primary cause of adult tooth loss. To measure periodontal status, dental hygienists often use a periodontal probe to measure several disease markers throughout the mouth. Three of the most popular markers are (a) clinical attachment loss (CAL), (b) periodontal pocket depth (PPD), and (c) bleeding on probing (BOP). PPD and CAL are continuous variables, usually rounded to the nearest millimeter. CAL is the distance down a tooth's root that is no longer attached to the surrounding bone by the periodontal ligament, and PPD is the distance from the gingival margin to the base of the pocket. BOP is a binary response and is indicative of whether a particular site bled with the application of a dental probe. During a full periodontal exam, all three markers are usually measured at six pre-specified sites (Darby and Walsh, 1995) for each tooth (excluding the third molars, i.e., the wisdom teeth). So for a patient with no missing teeth there are $S = 168$ measurements for each marker (Figure 1).

The motivating example is a clinical study conducted at the Medical University of South Carolina (MUSC) to determine the periodontal disease status for Type-2 diabetic Gullah-speaking African-Americans, originally presented in Fernandez et al. (2009). The objective of this analysis is to quantify the disease status of this population, and to study the associations between disease status and patient-level covariates such as age, BMI, gender, HbA1C, and smoking status.

Quantifying a patient's disease status from the extensive data collected during a periodontal exam is difficult. For example, it is common to summarize disease status using the whole-mouth

average CAL or the number of teeth with CAL above a certain threshold. Using the whole-mouth average CAL as the response in a regression with patient-level covariates is reasonable when the patients' residual distributions are identical. However, this assumption is often violated in practice, as different patients have different error variances, spatial covariances, and missing data patterns. In this paper we present a multivariate spatial model to jointly analyze periodontal data from multiple markers and multiple measurement locations to improve estimation of disease status, and hence develop a more powerful method for studying the association between patient-level covariates and periodontal disease.

We use spatial factor analysis (Wang and Wall, 2003; Hogan and Tchernis, 2004; Lopes et al., 2008) to model these multivariate spatial data. We postulate that the three markers are all related to a single latent spatial process (factor) measuring periodontal health. The latent periodontal health factor varies from site to site and is smoothed spatially using a conditionally autoregressive prior (Besag et al., 1991; Banerjee et al., 2004). The data collected for this study provide interesting challenges that require extensions of the spatial factor model. First, the data are a mix of continuous and binary responses. To jointly model these data types we develop a spatial probit model for binary responses, which has the advantage of being fully-conjugate and leads to rapid MCMC sampling and convergence. Also, we have data from multiple patients, and exploratory analysis suggests that the covariance of the latent spatial factor varies by patient. Therefore we develop a hierarchical model which allows the covariance to vary between patient, but pools information across patients to estimate the covariance parameters. We show in a simple case that in terms of estimating the effect of patient-level covariates, this model is equivalent to a weighted multiple regression, where the patient's scalar response is a linear combination of all data across location and marker, and the patient's weight decreases with the spatial correlation and variability.

Another challenging aspect of analyzing periodontal data is the considerable number of missing

teeth (around 20% for these data). The assumption that teeth are missing completely at random is not valid because periodontal disease is the leading cause of adult tooth loss, so patients with several missing teeth likely have poor periodontal health. For non-spatial data a common method to handle so-called “informative cluster size” is to include the number of observations as a covariate, or in the weights of a weighted regression (Hoffman et al., 2001; Williamson et al., 2003; Follman et al., 2003; Lu, 2005; Benhin et al., 2005; Panageas et al., 2007; Cong et al., 2007; Williamson et al., 2008). Dunson et al. (2003) take a different approach. They propose a joint model for clustered mixed (continuous and binary) data and the number of responses in each cluster, using a continuation ratio probit model for cluster size. Another approach is the shared parameter model (e.g., Wu and Carroll, 1998; Follman and Wu, 1995). The shared parameter model accounts for informative missingness by introducing random effects that are shared between the missing data process and the measurement process. Conditioned on the random effects, the missing data and measurement processes are assumed to be independent.

We propose a shared variable model to jointly model missing teeth with the other markers of periodontal disease. However, in our spatial setting both the number *and* location of missing teeth is informative. For example, a missing tooth in the front of the mouth surrounded by teeth with low CAL may not be informative; in contrast, a missing tooth in the back of the mouth (where periodontal disease is often the most advanced) surrounded by teeth with high CAL is indicative of poor periodontal health in that region of the mouth. Therefore, we model the number and spatial distribution of missing teeth using our latent spatial factor model. In this model, CAL, PPD, BOP, and the location of missing teeth are all modeled simultaneously in terms of a latent periodontal health factor; this approach uses all available information to estimate periodontal disease status.

The paper proceeds as follows. Section 2 presents our unified approach to modeling multivariate spatially-referenced periodontal data, as well as our model for informatively missing teeth. Section

3 offers some influence diagnostics to determine which patients and response types are the most informative about the patient-level covariates. Computing details are given in Section 4. Section 5’s simulation study shows that accounting for spatial association and informative observation location can lead to a substantial improvement in estimating the patient-level covariate effects. We analyze the periodontal data in Section 6. Section 7 concludes.

2 Latent spatial factor model for periodontal data

In this section we describe our approach for spatially-referenced mixed periodontal data with informative missingness. We begin in Section 2.1 by specifying a latent spatial factor model assuming no missing teeth. Section 2.2 introduces the spatial probit model for missing teeth and Section 2.3 specifies priors and discusses identifiability of the latent variable model.

2.1 Complete data model

We assume our multivariate spatial data has J types of responses (for the periodontal data the $J = 3$ responses are CAL, PPD, and BOP) at each spatial location for each patient. If the j^{th} type of response is continuous (CAL and PPD), let $y_{ij}(s)$ be the response at spatial location s for patient i , $s = 1, \dots, S$ and $i = 1, \dots, N$. Our data also has binary responses (BOP). If the j^{th} type of response is binary, let $y_{ij}^*(s)$ be the response at spatial location s for patient i . We model binary responses using probit regression, that is, $y_{ij}^*(s) = I(y_{ij}(s) > 0)$, where $I()$ is the binary indicator function and $y_{ij}(s)$ is a Gaussian latent variable.

All J responses are modeled as functions of the latent spatial disease status, $\mu_i(s)$, which represents the overall periodontal health of patient i at location s . Let

$$y_{ij}(s) = a_j + b_j \mu_i(s) + \varepsilon_{ij}(s), \tag{1}$$

where a_j is the intercept for response j , b_j relates the latent factor to response type j , and $\varepsilon_{ij}(s) \sim \text{N}(0, \sigma_{ij}^2)$ is error. As is customary for probit regression, we assume $\sigma_{ij}^2 = 1$ for binary responses for identification. Since all J responses depend on the common latent factor they are correlated with

$$\text{Cor}(y_{ij}(s), y_{il}(s)) = \frac{b_j b_l \text{Var}[\mu_i(s)]}{\sqrt{b_j^2 \text{Var}[\mu_i(s)] + \sigma_{ij}^2} \sqrt{b_l^2 \text{Var}[\mu_i(s)] + \sigma_{il}^2}}. \quad (2)$$

The slopes b_j and b_l determine the sign and magnitude of the correlation; if either b_j or b_l is zero then $y_{ij}(s)$ and $y_{il}(s)$ are uncorrelated, and if b_j and b_l share (do not share) the same sign then $y_{ij}(s)$ and $y_{il}(s)$ are positively (negatively) correlated.

The latent vector $\boldsymbol{\mu}_i = (\mu_i(1), \dots, \mu_i(S))'$ has a multivariate normal prior with conditionally autoregressive covariance (“CAR” , Besag et al., 1991). The mean of $\boldsymbol{\mu}_i$ is

$$E(\boldsymbol{\mu}_i) = W\boldsymbol{\alpha} + \Omega_i\boldsymbol{\beta},$$

where W is an $S \times q$ matrix of spatial covariates (e.g., tooth number) that do not vary across patient, X_i is the p -vector of patient-level covariates (e.g., age) that do not vary across space within patient, $\Omega_i = X_i' \otimes \mathbf{1}_S$, $\mathbf{1}_S$ is the S -vector of ones, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the corresponding regression parameters. The covariance of $\boldsymbol{\mu}_i$ is $\tau_i^2 Q(\rho_i)^{-1}$, where $Q(\rho_i) = M - \rho_i D$, $D_{ss'}$ is one if locations s and s' are adjacent and zero otherwise, M is diagonal with diagonal elements $M_{ss} = \sum_{s'} D_{ss'}$. In this spatial model, $\rho_i \in [0, 1]$ controls the degree of spatial association and $\tau_i^2 > 0$ controls the magnitude of variation. Let $r_i(s) = \mu_i(s) - E(\mu_i(s))$. A convenient interpretation of the CAR prior is that the conditional distribution of $r_i(s)$ given $r_i(s')$ for all $s' \neq s$ is normal with mean $\rho_i \bar{r}_i(s)$ and variance $\tau_i^2 / m(s)$, where $\bar{r}_i(s)$ is the average of $r_i(s)$ at location s 's $m(s)$ neighbors.

The degree of spatial variation is allowed to differ between patients by means of σ_{ij}^2 , τ_i^2 , and ρ_i .

To pool information across patients we use models

$$\begin{aligned}\sigma_{ij}^{-2}|c_j, d_j &\sim \text{Gamma}(c_j, d_j) \\ \tau_i^{-2}|e, f &\sim \text{Gamma}(e, f) \\ \rho_i|g, h &\sim \text{Beta}(g, h)\end{aligned}\tag{3}$$

where $\{c_j\}$, $\{d_j\}$, e , f , g , and h are hyperparameters.

2.2 Model for the location of missing teeth

For our data described in Section 1, roughly 20% of the teeth are missing. The locations of the missing teeth are not random, but rather related to the periodontal health in that region of the mouth. Therefore we propose a model for the location of missing teeth as a function of the underlying latent factor $\mu_i(s)$.

For our data either the six observations on a tooth for all J responses are all observed or all unobserved. That is, if a tooth is missing we have no data for the tooth, and if a tooth is not missing we have complete data. Let $y_{i0}^*(t)$ be an indicator of whether tooth $t = 1, \dots, T$ is missing for patient i . As with the binary data in Section 2 we model $y_{i0}^*(t)$ using probit regression. Let $y_{i0}^*(t) = I(y_{i0}(t) > 0)$, where $y_{i0}(t)$ is a latent continuous variable. Then

$$y_{i0}(t) = a_0 + b_0 Z_t' \boldsymbol{\mu}_i + \varepsilon_{i0}(t)\tag{4}$$

where Z_t is such that $Z_t' \boldsymbol{\mu}_i$ is the mean of $\boldsymbol{\mu}_i$ at the six observations on tooth t and $\varepsilon_{i0}(t) \stackrel{iid}{\sim} N(0, 1)$. a_0 and b_0 relate the latent process to the missing tooth indicator. Note that since $\mu_i(s)$ is included in both the model for presence of and value of the responses, both presence and value of the data

contribute to the posterior of $\mu_i(s)$, and thus the posterior of β . Also note that $b_{i0} = 0$ corresponds to independence between the latent factor and the location of missing teeth, in which case the location of missing teeth does not contribute to estimating β .

2.3 Identifiability and prior choice

Identifiability is a key issue in latent variable modeling. To see this we inspect the first two moments of the multivariate response at location s for patient i after integrating over the latent factor μ_i ,

$$\begin{aligned} \mathbb{E}(y_{ij}(s)) &= a_j + b_j [W(s)\alpha + X'_i\beta] \\ \text{Cov}(y_{ij}(s), y_{il}(s)) &= b_j b_l \tau_i^2 q(s) + I(j = l) \sigma_{ij}^2, \end{aligned} \tag{5}$$

where $W(s)$ is the row of W corresponding to location s and $q(s)$ is the (s, s) diagonal element of $Q(\rho_i)^{-1}$. Identifiability concerns arise in both moment expressions as multiplying all of the slopes b_j by scalar c and dividing α , β , and τ_i^2 by c gives identical moments. Although there are other ways to address this issue, we fix $b_1 \equiv 1$. This identifies both the regression coefficients α and β via the mean of the first response and the CAR variance τ_i^2 via the variance of the first response. In our analysis of periodontal data of Section 6 we take the first response with fixed slope to be clinical attachment loss, the most commonly used measure of periodontal disease. We also compare these results with other baseline assignments and discuss sensitivity to this assumption.

The regression coefficients $\{a_j\}$, $\{b_j\}$ ($j \neq 1$), α and β have independent $N(0, w^2)$ priors. The hyperparameters $\{c_j\}$, $\{d_j\}$, e , f , g , and h have independent $\text{Gamma}(u, v)$ priors. In the simulation study (Section 5) and data analysis (Section 6) we take $u = v = 0.1$ and $w = 10$ to give vague yet proper priors. We conduct a sensitivity analysis in Section 6 which shows that the results are not sensitive to these priors for this large periodontal data set.

3 Influence diagnostics

Our primary interest is in the patient-level parameters β . In this complicated hierarchical Bayesian model, we would like to identify the sources of data that are most informative about β . In this section we develop diagnostics to determine which patients, spatial locations, and response types are the most influential. We assume no missing teeth, that all responses are Gaussian, and that no covariates depend on space (W is null). In this case the regression coefficients only affect the overall average response for each patient, and a tempting simplification is to collapse data over space and use each patient's overall average as a scalar response. We show that even in this case different areas of the mouth are more than less informative, and that patients are weighted differently depending on their spatial covariance parameters. This motivates the hierarchical spatial model even in this simple case.

Integrating over latent effect μ_i , but conditioning on σ_{ij}^2 , τ_i^2 , ρ_i , a_j , and b_j , the posterior for β is Gaussian with

$$\begin{aligned} COV(\beta) &= \left(\sum_{i=1}^N w_i \mathbf{x}_i \mathbf{x}_i' \right)^{-1} \\ E(\beta) &= \left(\sum_{i=1}^N w_i \mathbf{x}_i \mathbf{x}_i' \right)^{-1} \sum_{i=1}^N w_i \mathbf{x}_i' z_i \end{aligned} \quad (6)$$

where

$$w_i = \tau_i^{-2} \mathbf{1}' \left[Q(\rho_i) - Q(\rho_i) (\delta_i I_S + Q(\rho_i))^{-1} Q(\rho_i) \right] \mathbf{1}, \quad (7)$$

$\delta_i = \tau_i^2 \sum_{j=1}^J b_j^2 / \sigma_{ij}^2$, and

$$z_i = \frac{1}{w_i} \mathbf{1}' Q(\rho_i) (\delta_i I_S + Q(\rho_i))^{-1} \sum_{j=1}^J b_j \sigma_{ij}^{-2} (y_{ij} - a_j).$$

The posterior in (6) is equivalent to a weighted linear regression where each patient contributes the scalar response z_i and is weighted according to w_i . Analyzing z_i and w_i shows which sites, patients, and outcomes contribute the most to β 's posterior.

First we consider z_i .

$$\begin{aligned} z_i &= \frac{1}{w_i} \mathbf{1}' \left[Q(\rho_i)(\delta_i I_s + Q(\rho_i))^{-1} \right] \sum_{j=1}^J \frac{b_j}{\sigma_{ij}^2} (y_{ij} - a_j) \\ &= \sum_{j=1}^J \frac{b_j}{\sigma_{ij}^2} \left[\sum_{s=1}^S k_i(s) (y_{ij}(s) - a_j) \right], \end{aligned} \quad (8)$$

where the vector $k_i = \mathbf{1}' [Q(\rho_i)(\delta_i I_s + Q(\rho_i))^{-1}] / w_i$. Therefore z_i is a linear combination of all the observations for patient i , with $k(s)$ controlling the relative weight of observations at location s and b_j/σ_{ij}^2 controlling the relative weight of response type j . Figure 2a plots $k(s)$ (scaled to sum to S) for four combinations of ρ_i and δ_i . Observations in the gaps between teeth have the highest weight; these sites have the most neighbors and thus the smallest prior variance. Observations in the back of the mouth and on the sides of teeth get less weight.

The patient weights w_i are plotted as a function of ρ_i , δ_i , and τ_i in Figure 2b. The weight decreases with ρ_i and τ_i^2 , and increases with δ_i (inversely related to error variances σ_{ij}^2). That is, patients with little spatial association and small variances τ_i^2 and σ_{ij}^2 (and thus large δ_i) have the most influence on β 's posterior. To search for overly-influential patients we compute the weights by evaluating (7) using posterior means $\hat{\rho}_i$, $\hat{\tau}_i^2$, and $\hat{\sigma}_{i1}^2$. However, the marginal posterior for β is not available in closed-form for Section 6's data with informative missing teeth and binary responses. Therefore we use only the CAL error variance $\hat{\sigma}_{i1}^2$, i.e., $\delta_i = \hat{\tau}_i^2 / \hat{\sigma}_{i1}^2$ ($b_1 = 1$, Section 2.3), as an approximation. This approximation is not meant to be definitive, but rather a useful heuristic device.

4 MCMC sampling algorithm

MCMC sampling is carried out using the free software *R* (<http://www.r-project.org/>), although it would also be straightforward to implement the model using WinBUGS (<http://www.mrc-bsu.cam.ac.uk/bugs/>). We draw 20,000 MCMC samples and discard the first 5,000 as burn-in. Convergence is monitored using trace plots of the deviance as well as several representative parameters.

The patient-specific parameters are conditionally-conjugate except for the CAR spatial association parameters ρ_i , which are updated using Metropolis-Hastings sampling with a Beta($50\rho_i^*$, $50(1 - \rho_i^*)$) candidate distribution, where ρ_i^* is the value at the previous iteration. The remaining parameters are updated using Gibbs sampling with full conditionals given below. The latent continuous variables corresponding to the probit model for binary responses, $y_{ij}(s)$, are updated from their truncated full conditionals $y_{ij}(s) \sim N(a_j + b_j\mu_i(s), 1)$, restricted to $(-\infty, 0)$ if $y_{ij}^*(s) = 0$ and $(0, \infty)$ if $y_{ij}^*(s) = 1$. The vector of latent effects for patient i , $\boldsymbol{\mu}_i$, is multivariate normal with

$$\begin{aligned} V(\boldsymbol{\mu}_i|\text{rest})^{-1} &= Z'Zb_0^2 + Q(\rho_i)/\tau_i^2 + \left(\sum_{j=1}^J b_j^2/\sigma_{ij}^2 \right) I_n \\ E(\boldsymbol{\mu}_i|\text{rest}) &= V(\boldsymbol{\mu}_i|\text{rest}) \left(b_0Z'(y_{i0} - a_0) + Q(\rho_i)(W\boldsymbol{\alpha} + \Omega_i\boldsymbol{\beta})/\tau_i^2 + \sum_{j=1}^J b_j(y_{ij} - a_j)/\sigma_{ij}^2 \right), \end{aligned}$$

where $Z = (Z_1, \dots, Z_T)$, $y_{ij} = (y_{ij}(1), \dots, y_{ij}(S))$, and $y_{i0} = (y_{i0}(1), \dots, y_{i0}(T))$. The measurement error variances for the continuous responses have full conditional

$$\begin{aligned} \sigma_{ij}^2|\text{rest} &\sim \text{InvGamma}(S/2 + c_j, \sum_{s=1}^S (y_{ij}(s) - a_j - b_j\mu_i(s))^2 / 2 + d_j) \\ \tau_j^2|\text{rest} &\sim \text{InvGamma}(S/2 + e, r_i'Q(\rho_i)r_i/2 + f), \end{aligned} \tag{9}$$

where $r_i = \boldsymbol{\mu}_i - W\boldsymbol{\alpha} - \Omega_i\boldsymbol{\beta}$.

The intercept/slope pairs (a_j, b_j) have bivariate normal full conditionals with mean

$$\begin{aligned} V((a_j, b_j)'|\text{rest})^{-1} &= w^{-2}I_2 + \sum_{i=1}^N \Delta_i' \Delta_i / \sigma_{ij}^2 \\ E((a_j, b_j)'|\text{rest}) &= V((a_j, b_j)'|\text{rest}) \left(\sum_{i=1}^N \Delta_i' y_{ij} / \sigma_{ij}^2 \right), \end{aligned}$$

where $\Delta_i = (\mathbf{1}, \boldsymbol{\mu}_i)$. The regression coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have multivariate normal full conditionals with

$$\begin{aligned} V(\boldsymbol{\alpha}|\text{rest})^{-1} &= w^{-2}I_{p_s} + \sum_{i=1}^N W' Q(\rho_i) W / \tau_i^2 \\ E(\boldsymbol{\alpha}|\text{rest}) &= V(\boldsymbol{\beta}_s|\text{rest}) X_s' \sum_{i=1}^N Q(\rho_i) (\boldsymbol{\mu}_i - \Omega_i' \boldsymbol{\beta}) / \tau_i^2, \end{aligned}$$

and

$$\begin{aligned} V(\boldsymbol{\beta}|\text{rest})^{-1} &= w^{-2}I_p + \sum_{i=1}^N \Omega_i' Q(\rho_i) \Omega_i / \tau_i^2 \\ E(\boldsymbol{\beta}|\text{rest}) &= V(\boldsymbol{\beta}|\text{rest}) \sum_{i=1}^N \Omega_i' Q(\rho_i) (\boldsymbol{\mu}_i - W \boldsymbol{\alpha}) / \tau_i^2. \end{aligned}$$

The remaining parameters $\{c_j\}$, $\{d_j\}$, e , f , and g are updated using Metropolis sampling with Gaussian candidate distributions tuned to give acceptance ratios near 0.40.

5 Simulation study

In this section we conduct a simulation study to demonstrate the effects of spatial correlation and informative missingness on the analysis of patient-level fixed effects. For computational purposes we assume only one quadrant (i.e., half jaw) for each patient leaving $S = 42$, that there are no spatial covariates W , and the same CAR spatial association parameter for each patient, i.e., $\rho_i = \rho$.

We also assume there is only a single continuous response. Data are generated from the model

$$P(y_i(s) = \text{observed}) = 1 - \Phi(a_0 + b_0\mu_i(s)) \quad (10)$$

$$y_i(s)|y_i(s) \text{ observed} \sim N(a_1 + b_1\mu_i(s), \sigma_i^2),$$

where $\boldsymbol{\mu}_i \sim N([x_i'\boldsymbol{\beta}]\mathbf{1}_S, \tau_i^2 Q^{-1}(\rho))$. Each simulated data set contains data generated from this model for $N = 50$ patients. The $p = 6$ patient-level covariates \mathbf{x}_i are generated independently from the standard normal distribution and the regression coefficients are $\boldsymbol{\beta} = (0, 0, 0, 1, 2, 3)/20$. Finally, $a_1 = b_1 = 1$ and $a_0 = -1$.

$M = 100$ data sets are generated from each of six designs specified by varying the true value of the covariance parameters σ_i^2 , τ_i^2 , and ρ and the missing data mechanism b_0 :

- **Design 1:** $\rho = 0.0$, $b_0 = 0$, and $\sigma_i^2 = \tau_i^2 = 1$
- **Design 2:** $\rho = 0.9$, $b_0 = 0$, and $\sigma_i^2 = \tau_i^2 = 1$
- **Design 3:** $\rho = 0.9$, $b_0 = 0$, and $\sigma_i^2 = \tau_i^2 = 1.5 \cdot \mathbf{I}(i \text{ is odd}) + 0.5$
- **Design 4:** $\rho = 0.9$, $b_0 = 1$, and $\sigma_i^2 = \tau_i^2 = 1$
- **Design 5:** $\rho = 0.9$, $b_0 = 1$, and $\sigma_i^2 = \tau_i^2 = 1.5 \cdot \mathbf{I}(i \text{ is odd}) + 0.5$
- **Design 6:** $\rho = 0.5$, $b_0 = 1$, and $\sigma_i^2 = \tau_i^2 = 1.5 \cdot \mathbf{I}(i \text{ is odd}) + 0.5$

Observations within patients are independent under the first design and spatially correlated under all other designs. The variances are the same for all patients under Design 2 and vary across patients for Design 3. Designs 4 and 5 are similar to Designs 2 and 3 except that the locations of missing observations are informative with $b_0 = 1$. Design 6 is the same as Design 5 except with moderate spatial association $\rho = 0.5$.

We analyze each simulated data set using five models:

- **Model 1:** Linear regression, $\bar{y}_i = \sum_{s \in S_i} y_i(s) / |S_i| \sim N(x_i'\boldsymbol{\beta}, \sigma^2)$

- **Model 2:** Section 2's spatial model without informative missingness or patient-specific variances, i.e., $b_0 = 0$, $\sigma_i^2 = \sigma^2$, and $\tau_i^2 = \tau^2$
- **Model 3:** Section 2's spatial model with patient-specific variances but without informative missingness, i.e., $b_0 = 0$
- **Model 4:** Section 2's spatial model with informative missingness but without patient-specific variances, i.e., $\sigma_i^2 = \sigma^2$ and $\tau_i^2 = \tau^2$
- **Model 5:** Section 2's full spatial model

where S_i in Model 1 is the set of locations of observed data for patient i . Model 1 ignores spatial associations and missing teeth, and simply uses each patient's average observed response in a multiple regression. Models 2-5 explicitly model all observations individually and account for spatial associations between nearby observations.

The results are presented in Table 1. For each model and each design, we calculate the proportion of the 95% posterior intervals for b_0 and the regression coefficients that exclude zero. We also compute the mean squared error and relative bias, $\text{MSE} = \frac{1}{pM} \sum_{m=1}^M \sum_{j=1}^p \left(\hat{\beta}_j^{(m)} - \beta_j \right)^2$ and $\text{RelBias}_j = \frac{1}{M} \sum_{m=1}^M \left(\hat{\beta}_j^{(m)} - \beta_j \right) / \beta_j$, where $\hat{\beta}_j^{(m)}$ is the posterior mean of β_j for the m^{th} simulated data set and β_j is the true value. Relative bias is only presented for the largest coefficient, β_6 .

Data for the first design are generated without spatial association or informative missingness. In this case all five models give nearly identical results, demonstrating that the spatial models are able to approximate the simple regression model if appropriate. The five models are also nearly identical for Design 2 where the data are generated with spatial correlation and the same variances for each patient. In this case the patient means \bar{y}_i are Gaussian with mean $a_1 + x_i' \beta$ and the same variances, satisfying the usual regression assumptions.

The linear regression model does not perform well for Design 3's spatial model with patient-dependent variances. In this case the patient means \bar{y}_i have different variances, violating the usual regression assumptions. The spatial models that allow for patient-dependent variances (Models

3 and 5) give dramatic improvements in both power and mean squared error compared to the homoskedastic models.

The locations of missing observations are informative for Designs 4, 5, and 6. For these two designs the models (Models 1-3) that do not account for informative missingness are biased for β_6 . The models that allow for informative location consistently identify b_0 as non-zero (power 1.0 in all three designs), which alleviates the bias for the non-null predictors and improves power. Design 5 has both informative missingness and patient-dependent variances, common traits of periodontal data. In this case our full model is more than three times more powerful for β_6 (0.26 to 0.94) and has roughly one fourth the mean squared error (0.193 to 0.780) of the usual non-spatial regression approach.

6 Analysis of periodontal data

The motivating data were collected from a clinical study (Fernandes et al., 2009) conducted by the Center for Oral Health Research (COHR) at the Medical University of South Carolina (MUSC). The relationship between periodontal disease and diabetes level has been previously studied in the dental literature (Faria-Almeida et al., 2006; Taylor and Borgnakke, 2008). The objective of this study was to explore the relationship between periodontal disease and diabetes level (determined by the popular marker HbA1c, or “glycosylated hemoglobin”) in the Type-2 diabetic adult (13 years or older) Gullah-speaking African-American population residing in the coastal sea-islands of South Carolina. Since this is part of an ongoing study, we selected 199 patients with complete covariate information and with at least 50% responses available.

For each patient CAL, PPD, and BOP are measured at six locations on each non-missing tooth, as shown in Figure 1. Additionally several patient-level covariates were obtained, including

age (in years), gender (1=Female, 0=Male), body mass index or BMI (in kg/m²), smoking status (1=a smoker, 0=never), and HbA1c (1=High, 0=controlled). We also include spatial covariates for the site in the gap between teeth (1=in the gap, 0=on the side of a tooth), jaw (1=maxilla, 0=mandible), and six tooth number indicators with the first tooth (front of the mouth, Figure 1) serving as the reference tooth. All covariates are standardized to have mean zero and variance one. The spatial adjacency structure is shown in Figure 1; we consider neighboring sites on the same tooth as well as neighboring sites on the consecutive teeth to be adjacent.

We begin by fitting several models with the same variances for all patients, i.e., $\sigma_{ij}^2 = \sigma_j^2$, $\tau_i^2 = \tau^2$, and $\rho_i^2 = \rho^2$. We fit four models by assuming spatial association ($\rho \sim \text{Unif}[0,1]$) and independence ($\rho = 0$), and assuming missing teeth are informative ($b_0 \neq 0$) and not informative ($b_0 = 0$). Table 2 gives posterior 95% intervals for several parameters. The slopes b_j for pocket depth and bleeding on probing (as described in Section 2.3, slope for attachment loss is fixed at one) are significantly positive for all models, suggesting strong positive associations between the three responses. Several covariates are significant for all models, including patient effects gender, smoking status and HbA1c status, indicators of a site in the gap between teeth and a site on the upper jaw, and several tooth number indicators with sites in the back of the mouth having higher mean responses.

The slope relating the latent spatial process with the probability of a missing tooth, b_0 , is also significantly positive. This matches the intuition that patients with poor periodontal health generally have more missing teeth. Figure 3 plots the data and fitted values for a typical patient to illustrate the effects of accounting for informative missing teeth. This plot compares the spatial models with b_0 set to zero (solid lines) and b_0 not set to zero (dashed lines). The posterior means (Figures 3a - 3c) and credible sets (3d) are nearly identical for observations on non-missing teeth. However, for missing teeth the fitted values for all three responses are larger (worse periodontal

health) when accounting for informative observation location.

Accounting for informative observation location also affects the patient effect for age. The 95% interval ignoring spatial association and informative observations location is (-0.002, 0.022), compared to (0.036, 0.115) for the full model. The measures of periodontal disease are cumulative, so it seems reasonable that age should be an important predictor. Our data show a relationship between age and the number of missing teeth; patients that are younger than 54 (the mean age) have an average of 135.8 (sd=20.1) observations and patients that are older than 54 have an average of 124.7 (sd=22.2) observations. By accounting for this relationship we identify age as a significant predictor of periodontal health.

Section 5's simulation study shows that the fixed effects can also be affected if patients have different spatial covariances. To explore this possibility for our periodontal data, we apply Section 2's model with variances σ_{ij}^2 and τ_i^2 varying across patients. Figures 4a and 4b summarize the posteriors of the variance parameters. Here we see considerable variation across patients; Figure 4b shows that the posterior 95% intervals for τ_i are non-overlapping for patients with small and large τ_i .

Section 3's w_i diagnostic in (7) indicates which patients are the most influential on the regression coefficients. The w_i (computed using only the CAL error variance) have median 28.0 and vary greatly across patients with 95% interval (5.8, 61.7). Figures 4c and 4d plot CAL for the patients with the smallest and largest w_i . The responses for the patient with smallest w_i vary considerably from site-to-site within the mouth, with attachment loss ranging from 0 to 11 mm. Information about the patient-level covariates accumulate via the mean of the latent parameters μ_i ; due to spatial variability the mean is quite uncertain for this patient and thus this patient provides little information about β . In contrast, the patient with largest w_i is stable from site-to-site, providing reliable information the mean of μ_i and thus about β .

Table 3 gives the 95% posterior intervals for several parameters from the model with patient-dependent variances. Comparing the spatial models with informative missingness, the results for the patient-level covariates are fairly similar for the models with and without patient-dependent variances (i.e., the final columns of 2 and 3). However, we note that the width of the credible intervals are smaller for the model with patient-dependent variances.

Finally, we conducted a sensitivity analysis to determine the effect of modeling assumptions for the full spatial model with patient-dependent variances and informative observation location. We modified the analysis by changing the reference group with slope b_j fixed to one from CAL to PPD and BOP, changing the hyperparameters $u = v = 0.1$ to $u = v = 0.0001$, and changing the hyperparameter $w = 10$ to $w = 1000$. The posterior 95% intervals are given in Table 4 for the patient effects, scaled by b_1 for comparison across reference group. The modification with the largest effect is changing the reference group from CAL to BOP. The patient level effects are generally closer to zero using BOP as the reference group. Despite this change in scale, the signs of the coefficients and the subset of coefficients with intervals that exclude zero remains the same as the original analysis.

Also, we consider modifying the adjacency structure shown by the gray lines in Figure 1 (“spatial grid 1”) in two ways: first by not considering sites on the opposite side of a tooth to be neighbors (“spatial grid 2”) to give independent AR(1) models to the sites on the buccal and lingual sides of each jaw, and second by considering all pairs of observations on the same tooth to be neighbors (“spatial grid 3”). To determine how well each of these spatial grids fit our data, we use the deviance information criteria (DIC) of Spiegelhalter et al. (2002). To compare spatial structures using DIC we analyze only a single continuous response, CAL, and do not consider informative missing teeth. DIC prefers grid 1 (DIC = 66128) over grids 2 (DIC = 68168) and 3 (DIC = 68562). Table 4 gives the posterior of the subject-level effects for the full data analysis using the three spatial grids; the

results are not sensitive to the choice of spatial structure.

7 Discussion

In this paper we develop a latent factor model for multivariate spatial periodontal data with a mix of binary and continuous responses. Our model allows for a different spatial covariance for each patient and for informative missing teeth. We show using simulated and real data that accounting for these factors leads to a substantial improvement for estimating covariate effects compared to standard regression techniques.

We have assumed throughout that the patient’s periodontal health can be captured by a single latent factor. It would be straightforward, conceptually if not computationally, to include more latent factors. However this leads to the problem of selecting the appropriate number of latent factors, interpreting the roles of the different latent factors, and understanding the effects of the covariates on the different latent factors. For these data with three strongly-correlated responses we prefer the single factor model for computational simplicity and interpretability. If multiple factors are allowed, the number of factors could be chosen using the deviance information criteria. Another approach would be to allow the number of factors to be unknown. Lopes, Salazar, and Gamerman (2008) and Salazar, Gamerman, and Lopes (2009) use reversible jump MCMC to account for uncertainty in the number of latent factors. Extending this approach to our setting may be complicated by the large number of subjects, since the proposal density would have to propose spatial models that simultaneously fit well for all 199 subjects. Another possibility would be to extend the parameter expansion method of Ghosh and Dunson (2008) to the spatial setting.

We have also assumed that the latent spatial process is Gaussian. For non-spatial data several authors have proposed methods that avoid assuming the shared random effects are Gaussian (Lin

et al., 2000; Song et al., 2002; Beunckens et al., 2008; Tsonaka et al., 2009). These approaches could be extended to the periodontal setting by replacing the Gaussian spatial model with a non-Gaussian spatial model (e.g., Gelfand et al., 2005; Griffin and Steel, 2006; Reich and Fuentes, 2007).

An area of future work is to apply this method to longitudinal periodontal data. Periodontal data is often collected repeatedly for a single patient over time to monitor disease progression. Reich and Hodges (2008) propose a spatiotemporal model for attachment loss. It should be possible to extend this model to accommodate mixed multivariate responses and informative missing teeth.

Acknowledgements

This research was supported in part by NIH/NCRR Grant P20 RR017696-06 and NIH Grant 1R01LM009153. The authors thank the Center for Oral Health Research (COHR) at the Medical University of South Carolina for providing the data and context for this work, in particular Drs. S. London, J. Fernandes, C. Salinas, W. Zhao, Ms. L. Summerlin, Ms. P. Hudson. We also wish to acknowledge several helpful discussions with Dr. James Hodges of the University of Minnesota.

References

- Banerjee S, Carlin BP, Gelfand AE (2004). *Hierarchical Modeling and Analysis for Spatial Data*, 1st edition, Chapman and Hall/CRC, USA.
- Benhin E, Rao JNK, Scott AJ (2005). Mean estimating equation approach to analysing cluster-correlated data with nonignorable cluster sizes. *Biometrika*, **92**, 435–450.
- Besag J, York JC, Mollié A (1991). Bayesian image restoration, with two applications in spatial statistics (with discussion). *Annals of the Institute of Statistical Mathematics*, **43**, 1–59.
- Beunckens C, Molenberghs G, Verkeke G, Mallinchrout C (2008). A latent-class mixture model for incomplete longitudinal Gaussian data. *Biometrics*, **64**, 96–105.
- Cong XJ, Yin G, Shen Y (2007). Marginal Analysis of Correlated Failure Time Data with Informative Cluster Sizes. *Biometrics*, **63**, 663–672.
- Darby ML, Walsh MM (1995). *Dental Hygiene: Theory and Practice*, 1st edition, W.B.Saunders, USA.

- Dunson DB, Chen Z, Harry J (2003). A Bayesian approach for joint modeling of cluster size and subunit-specific outcomes. *Biometrics*, **59**, 521–530.
- Faria-Almeida R, Navarro A, Bascones A (2006). Clinical and metabolic changes after conventional treatment of Type-2 diabetic patients with chronic periodontitis. *Journal of Periodontology*, **77**, 591–598.
- Fernandez JK, Wiegand RE, Salinas CF, Grossi SG, Sanders JJ, Lopes-Virella MF, Slate EH (2009). Periodontal disease status in Gullah African Americans with Type 2 diabetics living in South Carolina. *Journal of Periodontology*, **80**, 1062–1068.
- Follman D, Wu MC (1995). An approximate generalized linear model with random effects and informative missing data. *Biometrics*, **51**, 151–168.
- Follman D, Proschan M, Leifer E (2003). Multiple outputation: Inference for complex clustered data by averaging analyses from independent data. *Biometrics*, **59**, 420–429.
- Gelfand AE, Kottas A, MacEachern SN (2005). Bayesian Nonparametric Spatial Modeling with Dirichlet Process Mixing. *Journal of the American Statistical Association*, **100**: 1021–1035.
- Ghosh J, Dunson DB (2008). Bayesian model selection in factor analytic models. In *Random Effect and Latent Variable Model Selection*, ed. D.B. Dunson. John Wiley & Sons.
- Griffin JE, Steel MFJ (2006). Order-based dependent Dirichlet processes. *Journal of the American Statistical Association*, **101**:179–194.
- Hoffman EB, Sen PK, Weinberg CR (2001). Within-cluster resampling. *Biometrika*, **88**, 1121–1134.
- Hogan JW, Tchernis R (2004). Bayesian factor analysis for spatially correlated data, with application to summarizing area-level material deprivation from census data. *Journal of the American Statistical Association*, **99**, 314–324.
- Lin H, McCulloch CE, Turnbull BW, Slate E, Clark L (2000). A latent class mixed model for analysing biomarker trajectories with irregularly scheduled observations. *Statistics in Medicine*, **19**, 1303–1318.
- Lopes HF, Salazar E, Gamerman D (2008). Spatial dynamic factor analysis. *Bayesian Analysis*, **3**, 759–792.
- Lu W (2005). Marginal regression of multivariate event times based on linear transformation models. *Lifetime Data Analysis*, **11**, 389–404.
- Oliver RC, Brown LJ, Loe H (1998). Periodontal diseases in the United States population. *Journal of Periodontology*, **69**, 269–278.
- Panageas KS, Schrag D, Localio AR, Venkatraman ES, Begg CB (2007). Properties of analysis methods that account for clustering in volume-outcome studies when the primary predictor is cluster size. *Statistics in Medicine*, **26**, 2017–2035.
- Pihlstrom B (1992). Issues in the evaluation of clinical trials of periodontitis: A clinical perspective, *Journal of Periodontal Research*, **27**, 433–441.
- Reich BJ, Fuentes M (2007). A multivariate semiparametric Bayesian spatial modeling framework for hurricane surface wind fields. *Annals of Applied Statistics*, **1**: 249–264.
- Reich BJ, Hodges JS (2008). Modeling longitudinal spatial periodontal data: A spatially-adaptive model with tools for specifying priors and checking fit. *Biometrics*, **64**, 790–799.

- Salazar E, Gamerman D, Lopes HF (2009). Generalized spatial dynamic factor analysis, Technical report, University of Chicago.
- Song X, Davidian M, Tsiatis AA (2002). A semiparametric likelihood approach to joint modeling of longitudinal and time to event data. *Biometrics*, **58**, 742–753.
- Spiegelhalter DJ, Best NG, Carlin BP, van der Linde A (2002). Bayesian measures of model complexity and fit (with discussion). *Journal of the Royal Statistical Society B*, **64**, 583–639.
- Taylor GW, Borgnakke WS (2008). Periodontal disease: associations with diabetes, glycemic control and complications. *Oral Diseases*, **14**, 191–203.
- Tsonaka R, Verbeke G, Lesaffre E (2009). A semi-parametric shared parameter model to handle nonmonotone nonignorable missingness. *Biometrics*, **65**, 81–87.
- Wang F, Wall M (2003). Generalized common spatial factor model, *Biostatistics*. **4**, 569–582.
- Williamson JM, Datta S, Satten GA (2003). Marginal analysis of clustered data when the cluster size is informative. *Biometrics*, **59**, 36–42.
- Williamson JM, Kim HY, Manatunga A, Addiss DG (2008). Modeling survival data with informative cluster size. *Statistics in Medicine*, **27**, 543–555.
- Wu MC, Carroll, R (1988). Estimation and comparison of changes in the presence of informative right censoring by modeling the censoring process. *Biometrics*, **44**, 175–188.

Figure 1: Observed CAL for a typical patient. The shaded boxes represent teeth, the circles represent measurement sites, and the gray lines represent neighbor pairs connecting adjacent sites on the same tooth and sites that share a gap between teeth. “Maxillary” and “Mandibular” refer to upper and lower jaws respectively. The small numbers beside each tooth are the “tooth numbers”. The maxilla’s second tooth on the left is missing; third molars (“wisdom teeth”) are excluded.

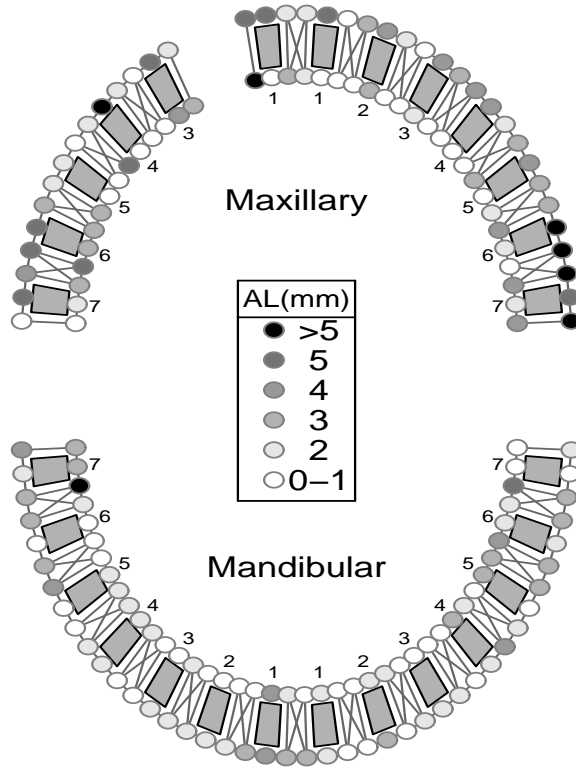


Table 1: Simulation study results. Columns labels “ b_0 ” – “ β_6 ” give the proportion of 95% intervals that exclude zero. The Monte Carlo standard errors (not shown) are between 0.007 and 0.045 for $100 \cdot \text{MSE}$ and between 0.003 and 0.006 for the bias.

Design	Model	b_0	β_1	β_2	β_3	β_4	β_5	β_6	$100 \cdot \text{MSE}$	RelBias_6
1	1	–	0.06	0.05	0.03	0.29	0.85	1.00	0.103	0.022
	2	–	0.05	0.07	0.04	0.34	0.86	1.00	0.103	0.021
	3	–	0.05	0.06	0.04	0.35	0.86	1.00	0.104	0.022
	4	0.10	0.05	0.05	0.03	0.35	0.85	1.00	0.104	-0.001
	5	0.12	0.06	0.05	0.04	0.35	0.83	1.00	0.106	0.006
2	1	–	0.03	0.04	0.01	0.13	0.52	0.80	0.289	0.042
	2	–	0.03	0.05	0.02	0.15	0.56	0.80	0.287	0.034
	3	–	0.03	0.03	0.03	0.17	0.48	0.79	0.298	0.045
	4	0.06	0.03	0.06	0.01	0.16	0.52	0.83	0.285	0.037
	5	0.06	0.03	0.05	0.01	0.16	0.46	0.81	0.297	0.043
3	1	–	0.03	0.07	0.04	0.12	0.31	0.51	0.657	0.077
	2	–	0.04	0.08	0.08	0.13	0.31	0.52	0.655	0.072
	3	–	0.09	0.09	0.07	0.36	0.69	0.95	0.181	0.027
	4	0.08	0.03	0.08	0.09	0.12	0.31	0.56	0.653	0.077
	5	0.08	0.09	0.12	0.08	0.39	0.68	0.96	0.178	0.034
4	1	–	0.04	0.08	0.05	0.14	0.43	0.70	0.266	-0.150
	2	–	0.06	0.05	0.07	0.16	0.43	0.76	0.267	-0.146
	3	–	0.04	0.06	0.06	0.18	0.42	0.72	0.265	-0.141
	4	1.00	0.04	0.10	0.04	0.18	0.58	0.89	0.278	0.048
	5	1.00	0.02	0.06	0.05	0.19	0.58	0.86	0.267	0.026
5	1	–	0.05	0.04	0.08	0.07	0.19	0.26	0.780	-0.229
	2	–	0.11	0.07	0.12	0.15	0.26	0.34	0.725	-0.217
	3	–	0.12	0.11	0.18	0.31	0.67	0.89	0.221	-0.075
	4	1.00	0.06	0.10	0.09	0.16	0.46	0.71	0.693	0.187
	5	1.00	0.06	0.09	0.08	0.29	0.66	0.94	0.193	0.023
6	1	–	0.06	0.04	0.07	0.10	0.28	0.47	0.409	-0.200
	2	–	0.16	0.11	0.12	0.18	0.43	0.66	0.383	-0.191
	3	–	0.07	0.09	0.11	0.45	0.89	1.00	0.086	-0.062
	4	1.00	0.09	0.07	0.11	0.30	0.76	0.97	0.279	0.130
	5	1.00	0.06	0.07	0.08	0.61	0.96	1.00	0.070	0.024

Table 2: Posterior 95% intervals for models assuming variances σ_{ij}^2 and τ_i^2 are constant across patients. “Spatial” models take $\rho \neq 0$ and models with informative missing teeth (“Info missing”) have $b_0 \neq 0$.

Spatial Info missing	No No	No Yes	Yes No	Yes Yes
Age	(-0.002, 0.022)	(0.009, 0.033)	(0.000, 0.079)	(0.036, 0.115)
Female	(-0.129, -0.104)	(-0.128, -0.103)	(-0.181, -0.103)	(-0.173, -0.096)
BMI	(-0.016, 0.007)	(-0.014, 0.011)	(-0.048, 0.030)	(-0.033, 0.046)
Smoker	(0.028, 0.051)	(0.028, 0.051)	(0.014, 0.091)	(0.010, 0.088)
Hba1c	(0.114, 0.139)	(0.118, 0.143)	(0.123, 0.199)	(0.128, 0.207)
a_0 - missing	–	(-1.390, -1.239)	–	(-1.349, -1.172)
a_1 - CAL	(1.008, 1.052)	(1.021, 1.087)	(0.993, 1.112)	(1.002, 1.139)
a_2 - PPD	(1.015, 1.055)	(1.034, 1.101)	(1.092, 1.214)	(1.104, 1.139)
a_3 - BOP	(-0.369, -0.323)	(-0.359, -0.312)	(-0.399, -0.327)	(-0.394, -0.309)
b_0 - missing	–	(0.432, 0.513)	–	(0.434, 0.544)
b_2 - PPD	(1.144, 1.178)	(1.131, 1.160)	(1.021, 1.047)	(1.014, 1.043)
b_3 - BOP	(0.473, 0.510)	(0.475, 0.510)	(0.510, 0.547)	(0.434, 0.544)
ρ	–	–	(0.972, 0.978)	(0.972, 0.978)
τ	(1.454, 1.499)	(1.464, 1.508)	(0.832, 0.870)	(0.838, 0.874)
σ_1 - CAL	(0.942, 0.961)	(0.972, 0.978)	(0.881, 0.900)	(0.935, 0.953)
σ_2 - PPD	(0.106, 0.182)	(0.177, 0.219)	(0.454, 0.486)	(0.464, 0.493)
Tooth 2	(-0.027, 0.038)	(-0.038, 0.034)	(-0.063, 0.027)	(-0.072, 0.022)
Tooth 3	(0.037, 0.101)	(0.022, 0.091)	(-0.007, 0.106)	(-0.025, 0.100)
Tooth 4	(0.174, 0.241)	(0.179, 0.253)	(0.106, 0.242)	(0.138, 0.275)
Tooth 5	(0.237, 0.306)	(0.263, 0.339)	(0.214, 0.369)	(0.296, 0.451)
Tooth 6	(0.751, 0.825)	(0.849, 0.932)	(0.597, 0.773)	(0.853, 1.037)
Tooth 7	(0.866, 0.954)	(0.955, 1.048)	(0.597, 0.773)	(0.986, 1.151)
Gap	(0.953, 1.001)	(0.938, 0.994)	(0.992, 1.030)	(0.986, 1.022)
Maxilla	(-0.289, -0.246)	(-0.293, -0.247)	(-0.376, -0.235)	(-0.363, -0.211)

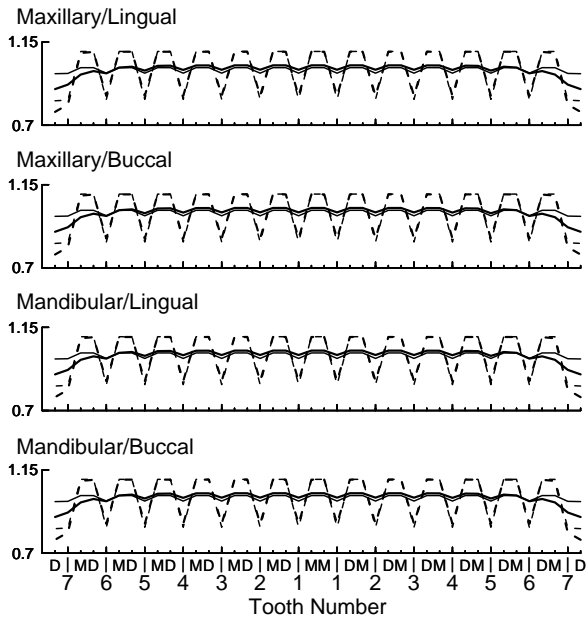
Table 3: Posterior 95% intervals for models assuming variances σ_{ij}^2 and τ_i^2 vary across patients. “Spatial” models take $\rho \neq 0$ and models with informative missing teeth (“Info missing”) have $b_0 \neq 0$.

Spatial Info missing	No No	No Yes	Yes No	Yes Yes
Age	(-0.019, 0.007)	(-0.013, 0.013)	(-0.002, 0.057)	(0.023, 0.086)
Female	(-0.114, -0.086)	(-0.115, -0.087)	(-0.159, -0.096)	(-0.168, -0.104)
BMI	(-0.008, 0.019)	(-0.006, 0.020)	(-0.015, 0.040)	(-0.009, 0.050)
Smoker	(0.038, 0.062)	(0.037, 0.061)	(0.021, 0.078)	(0.019, 0.075)
Hba1c	(0.089, 0.114)	(0.090, 0.118)	(0.095, 0.155)	(0.106, 0.171)
a_0 - missing	–	(-1.154, -1.004)	–	(-1.201, -1.040)
a_1 - CAL	(0.859, 0.921)	(0.851, 0.930)	(0.855, 0.942)	(0.892, 0.989)
a_2 - PPD	(0.899, 0.960)	(0.889, 0.966)	(0.920, 1.012)	(0.958, 1.058)
a_3 - BOP	(-0.425, -0.373)	(-0.424, -0.370)	(-0.482, -0.414)	(-0.464, -0.394)
b_0 - missing	–	(0.265, 0.378)	–	(0.294, 0.410)
b_2 - PPD	(1.002, 1.014)	(1.001, 1.014)	(1.017, 1.036)	(1.013, 1.031)
b_3 - BOP	(0.462, 0.499)	(0.463, 0.498)	(0.518, 0.559)	(0.521, 0.560)
ρ	–	–	(0.954, 0.962)	(0.958, 0.965)
Tooth 2	(-0.051, 0.010)	(-0.050, 0.019)	(-0.054, 0.023)	(-0.063, 0.016)
Tooth 3	(0.022, 0.081)	(0.021, 0.086)	(0.007, 0.105)	(-0.009, 0.095)
Tooth 4	(0.183, 0.246)	(0.190, 0.258)	(0.138, 0.251)	(0.141, 0.254)
Tooth 5	(0.307, 0.374)	(0.318, 0.391)	(0.267, 0.385)	(0.291, 0.412)
Tooth 6	(0.776, 0.850)	(0.825, 0.909)	(0.634, 0.763)	(0.753, 0.899)
Tooth 7	(0.854, 0.940)	(0.902, 0.991)	(0.630, 0.769)	(0.782, 0.938)
Gap	(0.887, 0.929)	(0.886, 0.932)	(0.894, 0.926)	(0.890, 0.922)
Maxilla	(-0.280, -0.241)	(-0.278, -0.238)	(-0.305, -0.208)	(-0.301, -0.201)

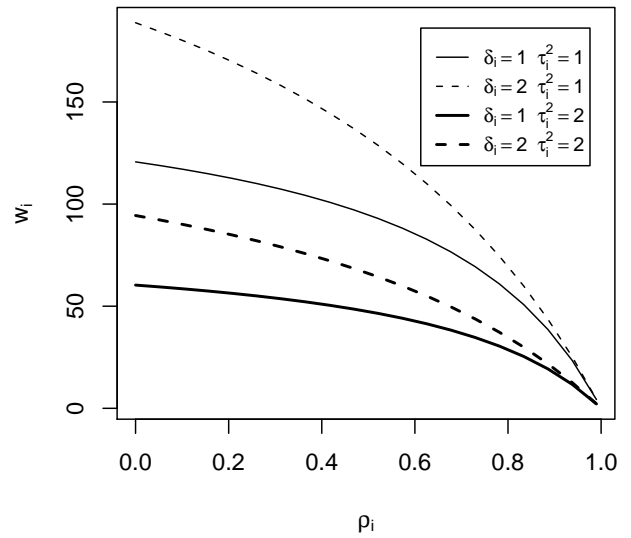
Table 4: Posterior 95% intervals for the patient effects for various modeling/prior choices for the full model with spatial correlation, patient dependent variance, and informative observation location. “Ref group” refers to the response that has slope b_j fixed to one, “ u, v ” and “ w ” are the hyperparameters for the covariance parameters and regression coefficients, respectively, as described in Section 2.3. To facilitate comparison across reference groups the intervals for β/b_1 are presented.

Ref group	CAL	PPD	BOP	CAL
u, v	0.1	0.1	0.1	0.001
w	10	10	10	10
Spatial grid	1	1	1	1
Age	(0.023, 0.086)	(0.025, 0.088)	(0.007, 0.079)	(0.039, 0.110)
Female	(-0.168, -0.104)	(-0.175, -0.108)	(-0.052, -0.031)	(-0.169, -0.097)
BMI	(-0.009, 0.050)	(-0.012, 0.052)	(-0.003, 0.015)	(-0.015, 0.054)
Smoker	(0.019, 0.075)	(0.020, 0.077)	(0.005, 0.023)	(0.009, 0.075)
Hba1c	(0.106, 0.171)	(0.110, 0.174)	(0.033, 0.054)	(0.122, 0.190)
Ref group	CAL	CAL	CAL	
u, v	0.1	0.1	0.1	
w	1000	10	10	
Spatial grid	1	2	3	
Age	(0.039, 0.109)	(0.023, 0.068)	(0.038, 0.105)	
Female	(-0.174, -0.096)	(-0.150, -0.103)	(-0.167, -0.097)	
BMI	(-0.017, 0.050)	(-0.009, 0.035)	(-0.011, 0.054)	
Smoker	(0.009, 0.076)	(0.024, 0.067)	(0.010, 0.074)	
Hba1c	(0.119, 0.191)	(0.117, 0.164)	(0.111, 0.176)	

Figure 2: Panel (a) plots the spatial weights $k(s)$ for various δ and ρ . “Maxillary” and “Mandibular” refer to upper and lower jaws respectively, while “buccal” and “lingual” refer to the cheek and the tongue sides of the teeth, respectively. The thin lines have $\rho = 0.1$, the wide lines have $\rho = 0.99$; the solid lines have $\delta = 0.2$, the dashed lines have $\delta = 5$. Panel (b) plots the patient weights w_i for various δ_i , τ_i , and ρ_i .

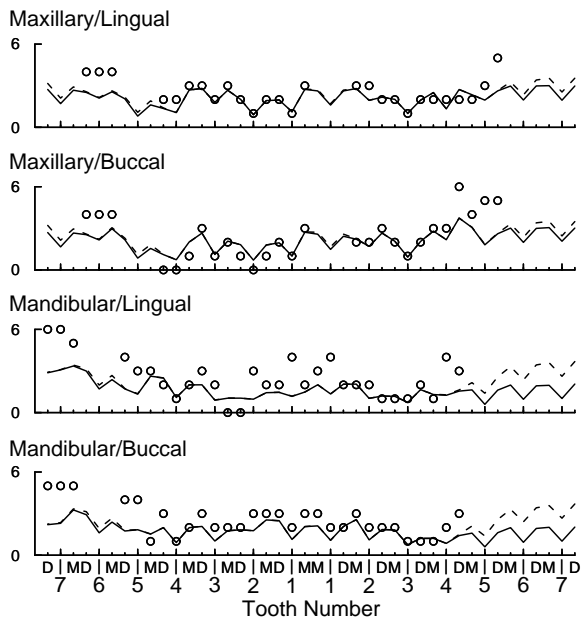


(a) Spatial weights $k(s)$

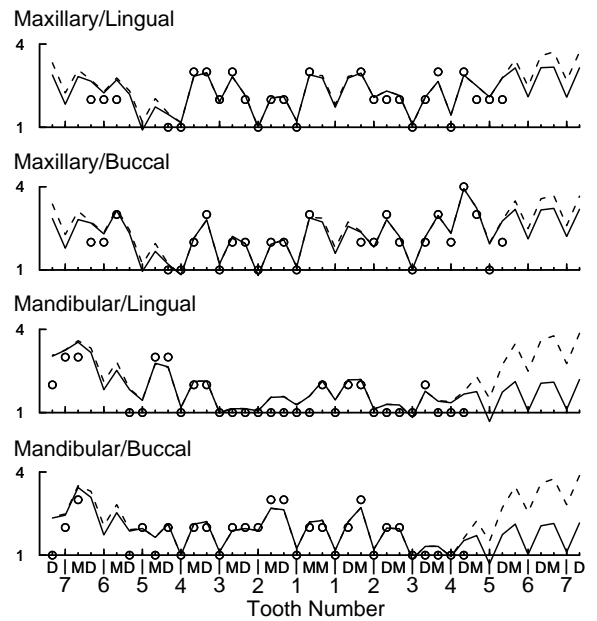


(b) Patient weights w_i

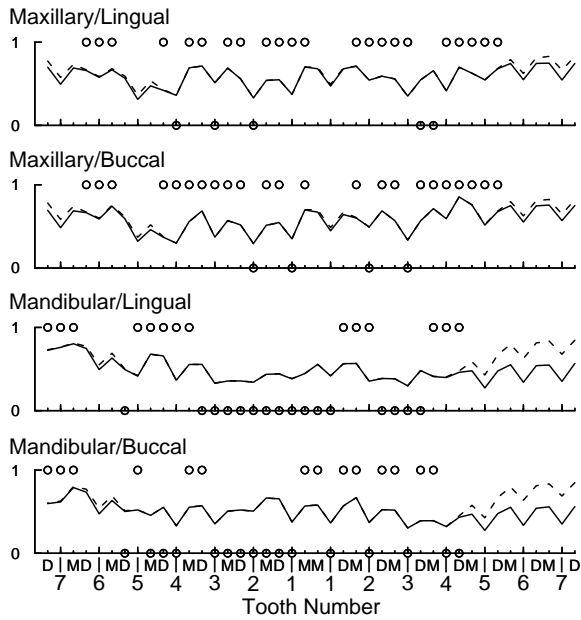
Figure 3: Panels (a)-(c) plot the data (dots) and posterior mean of the expected response (lines) for a typical patient. Panel (d) plots the posterior mean (bold) and 95% interval (thin) for the latent spatial process $\mu(s)$. All plots include results for both the model with (dashed line) and without (solid line) informative observation location. “Maxillary” and “Mandibular” refer to upper and lower jaws respectively, while “buccal” and “lingual” refer to the cheek and the tongue sides of the teeth, respectively.



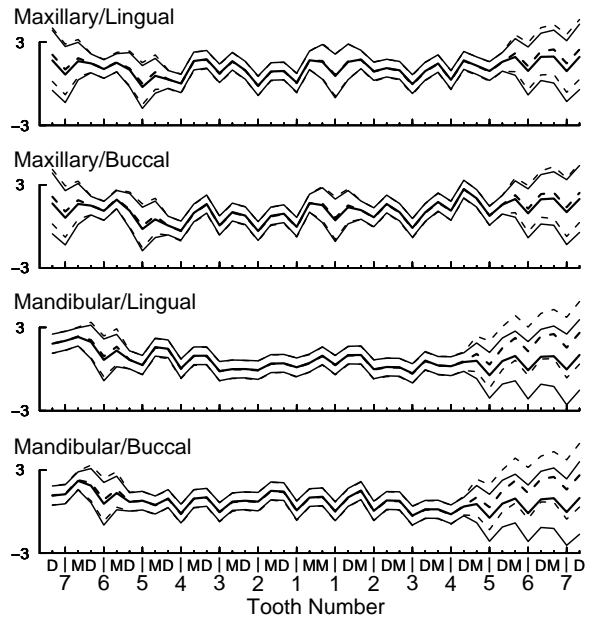
(a) Attachment loss (mm)



(b) Pocket depth (mm)

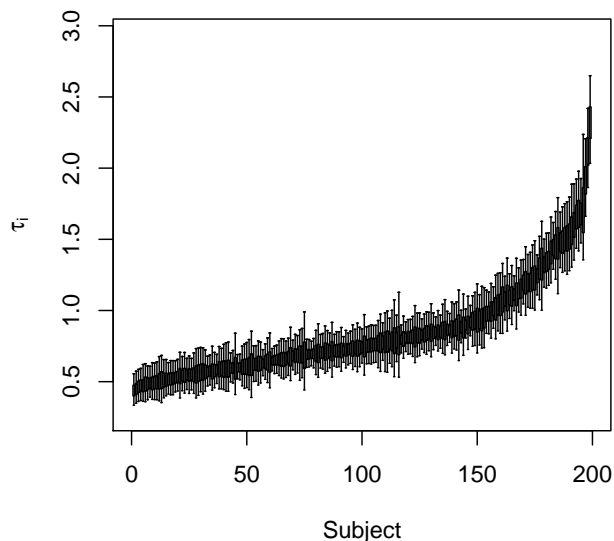
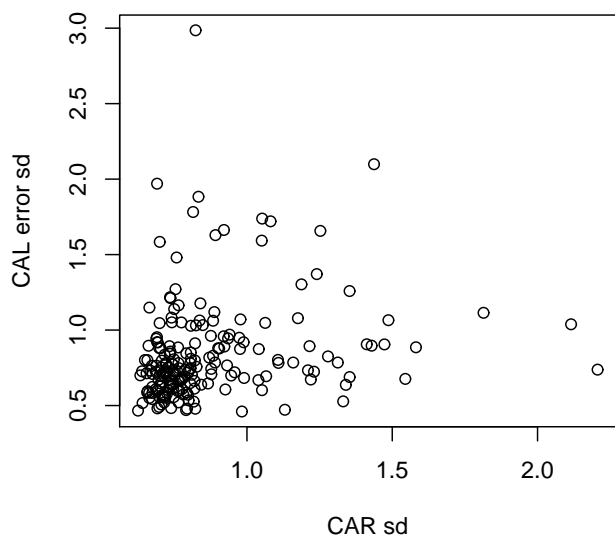


(c) Bleeding on probing



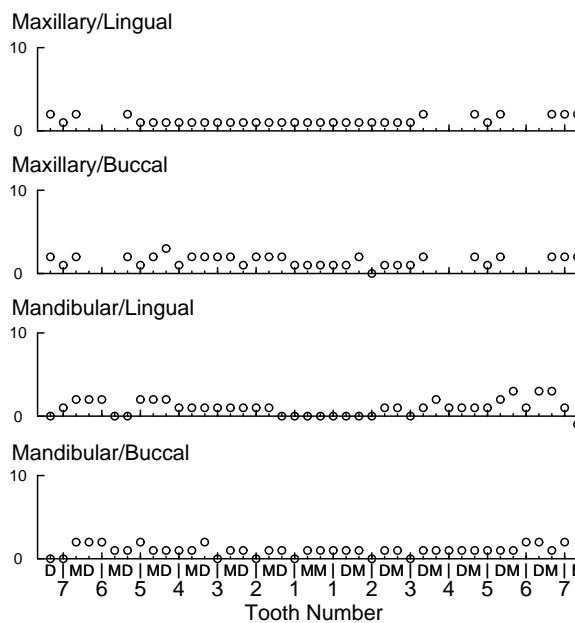
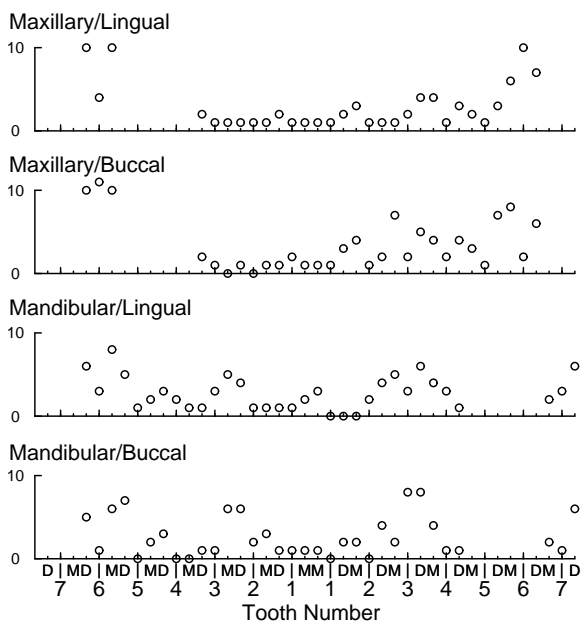
(d) Latent process $\mu(s)$

Figure 4: Panel (a) gives the posterior medians of the patient-specific standard deviations (τ_i, σ_{ij}) for attachment loss, Panel (b) plots the posterior of the CAR standard deviations τ_i (the horizontal lines in the i^{th} column are the posterior 0.025, 0.25, 0.5, 0.75, and 0.975 quantiles for τ_i). Panels (c) and (d) plot the attachment loss for the patients with smallest and largest weights w_i , respectively. “Maxillary” and “Mandibular” refer to upper and lower jaws respectively, while “buccal” and “lingual” refer to the cheek and the tongue sides of the teeth, respectively.



(a) Plot of posterior mean (τ_i, σ_{ij})

(b) Posterior of τ_i



(c) CAL (mm) for the patient with $w_i = 3.0$

(d) CAL (mm) for the patient with $w_i = 74.1$