

# How Do Heterogeneities in Operating Environments Affect Field Failure Predictions and Test Planning?

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June 9, 2013

## Abstract

The main objective of accelerated life tests (ALTs) is to predict fraction failings of products in the field. However, there are often discrepancies between the predicted fraction failing from the lab testing data and that from the field failure data, due to the yet unobserved heterogeneities in usage and operating conditions. Most previous research on ALT planning and data analysis ignores the discrepancies, resulting in inferior test plans and biased predictions. In this paper, we model the heterogeneous environments together with their effects on the product failures as a frailty term to link the lab failure time distribution and field failure time distribution of a product. We show that in the presence of the heterogeneous operating conditions, the hazard rate function of the field failure time distribution exhibits a range of shapes. Statistical inference procedure for the frailty models is developed when both the ALT data and the field failure data are available. Based on the frailty models, optimal ALT plans aimed at predicting the field failure time distribution are obtained. The developed methods are demonstrated through a real life example.

**Keywords:** Accelerated life test data, frailty model, field failure data, heterogeneous operating conditions, optimal plan.

# 1 Introduction

## 1.1 Motivation

Most commercial products are sold with warranties. Before a new product is launched to the market, it is extremely important to accurately estimate the proportion of field returns within a given warranty period in order to determine the monetary reserves for covering future warranty claims. The failure information can be obtained through pre-launch accelerated life tests (ALTs) in a timely fashion. In an ALT, a number of samples are tested under harsh conditions, e.g., a combination of high voltage, temperature, pressure, use rate, etc., which yields information on product reliability within a reasonable time frame. Failure time data from the test are collected, analyzed and extrapolated to estimate lifetime characteristics of interest at *nominal* use conditions based on some stress-life models. There is a bulk of literature on ALT data analysis and optimal design of ALT experiments. See [Pascual \(2006\)](#), [Ma and Meeker \(2008\)](#), [Guo and Liao \(2012\)](#) and [Liu \(2012\)](#), among others. The use conditions are implicitly assumed to be homogeneous (same for all customers) in most ALT research, including the above references.

After the product is sold to customers with a warranty, units that fail within the warranty period are returned to the manufacturer for repair or replacement, which are known as warranty claims. These warranty claim data reflect failure behaviors of the product under *actual* use conditions. Analysis of these warranty return data are useful because it validates the results from ALT data analysis, and can be used to improve the accuracy of parameter estimation from the ALT. See [Blischke et al. \(2011\)](#) for an overview of this topic.

However, large discrepancies between the results of ALT data analysis and field failure data analysis are often found. Analysis of field failure data tends to suggest higher variability in the product's failure times compared with the result based on ALT data analysis. Conceivably, this is because products in the field are usually exposed to heterogeneous usage and operating conditions. A motivating example is as follows.

[Meeker et al. \(2009\)](#) described an application involving an appliance, which is called Appliance B. Appliance B contains a turbine device which has two major failure modes: crack failure modes and wear failure modes. Engineering knowledge suggests that it is

reasonable to assume that these two failure modes are independent. For illustration, we only consider the wear failure mode, accounting for around 80% of the total field failures. Appliance B was sold with a two-year warranty. Before its entry into the market, an ALT was conducted to obtain reliability information of the product, in which 10 units were subject to a wear test. Field failure data were also available during the subsequent warranty tracking study of 4708 units with 93 wear failures. More details can be found in [Meeker et al. \(2009\)](#).

According to the analysis in Section 5, the Weibull distribution provides a good fit to the failure data from ALT, but it does not provide an adequate fit to the field data. As we will argue, the discrepancy is largely due to the varying operating conditions in the field. When varying operating conditions are taken into account, theory suggests the use of other distributions for the field data, such as the Burr-XII distribution, which do fit well.

## 1.2 Heterogeneous Operating Conditions

The operating conditions are dynamic in a number of ways. First off, products are used in different geographical areas because of customer locations. Therefore, the operating environments (e.g., temperature, humidity, etc.) are heterogeneous for units across the product population. Secondly, different users have different usage behaviors. In a two-dimensional warranty analysis, it is commonly assumed that the use rate of a customer is constant and it varies across the customer population ([Lawless et al. 2009](#); [Ye et al. 2013](#)). [Yang \(2010\)](#) also observed that the field stress level may vary over the product population. Moreover, the usage profile can be time dependent. As an example, [Nelson \(2001\)](#) reported a problem where the stress profile, e.g., pressure and temperature, over time for a seal in brake cylinders is stochastic. The presence of variable operating conditions significantly influences failures of the product. As suggested from consumer reports in February 1991 ([Padmanabhan 1995](#)), the percentage of washer-dryer machines that ended up with a warranty claim went up from 14% among those who reported an average of one to four laundry loads per week to 25% among those who reported an average in excess of eight loads per week. Furthermore, this pattern was observed across brands consistently.

In the presence of heterogeneous operating conditions of the product population, direct prediction of the proportion of warranty returns from ALT data analysis can be highly

biased. In principle, the failure time distribution of the in-lab testing units can be linked to that of the field population by taking into account information about these dynamics in environments. The information includes the types of significant dynamic environmental factors, the distributions for these factors as well as the acceleration relationships that relate each factor to the failure process. Among these environmental factors, information about the use rate may be the easiest to collect. For example, [Meeker et al. \(2009\)](#) and [Yang \(2010\)](#) focused on modeling the effects of usage rates. Both studies assumed a constant usage rate for an individual unit and a lognormal distribution for usage rates across the product population. However, the field failure time distribution in [Yang \(2010\)](#) does not have closed form expressions, which makes analysis of field return data and verification of model assumptions (e.g., the lognormal assumption of the usage rate distribution) very difficult, and which greatly complicates the ALT planning for a new vintage of the product under similar environments. Even if the distribution of the usage rate is available, say, from a customer survey, the models in these two studies still ignore other influential factors such as heterogeneous customer locations. In fact, it is almost impossible to directly collect information (i.e., distributions for each environmental factor and their respective effects on the failure process) about all heterogeneous environmental factors other than the usage rate.

### 1.3 Objectives and Overview

This paper is an endeavor to answer the question how heterogeneities in operating environments affect predictions of field failures and planning of ALTs. We treat the *unobservable operating factors* as well as *their effects* on the product failure process as a “frailty”, through which the lab failure time distribution of a product can be linked to the field failure time distribution. The “frailty” is an unobservable random variable used to account for heterogeneities caused by unobservable covariates. In its simplest form, the frailty is an unobserved random proportionality factor that modifies the baseline failure rate function of an individual, which is similar to the multiplicative effect of a covariate on the failure rate in the Cox’s proportional hazard model. In biostatistics, lifetime models with frailties have attracted much attention, e.g., see [Hanagal \(2011\)](#) for a book length treatment on this area. In reliability engineering, the frailty is often called a random effect and also receives some

applications, e.g., see [Stefanescu and Turnbull \(2006\)](#), [Lawless and Crowder \(2010\)](#) and [Ye and Chen \(2013\)](#), among others. However, one challenge of using frailty is that the resulting marginal distribution is often mathematically intractable.

This paper develops tractable frailty models that relate ALT failures to warranty failures. We show that in the presence of the frailty, the hazard rate of a field unit exhibits various shapes. An appropriate distribution for the frailty can be determined through joint modeling of both ALT data and warranty return data. Detailed procedures to analyze the data and to collate the frailty distribution are developed. The results enable the prediction of field failures for a future product through analysis of ALT data. We also derive optimal designs of ALT experiments for a new vintage and show how the heterogeneities affect the optimal ALT design.

The remainder of the paper is organized as follows. [Section 2](#) introduces the gamma frailty model for linking lab test data and field failure data and investigates possible shapes of the field failure rate. In [Section 3](#), a procedure for statistical inference of the frailty model is developed. We also extensively discuss the model validation through hypothesis testing. Optimal ALT plans under the frailty model are obtained in [Section 4](#). [Section 5](#) applies the frailty model to the Appliance B example. [Section 6](#) concludes the paper.

## 2 Linking Lab Failures and Field Failures

Under the stable lab testing conditions, we assume the lifetime  $X$  of the product follows a Weibull distribution, which is one of the most commonly used lifetime distributions. However, existence of the heterogeneous operating conditions influences lifetime of a field unit. The basic idea is to introduce into the hazard rate an additional random parameter  $Z$  that accounts for the heterogeneities. The frailty  $Z$  links the distribution of  $X$  to that of the field failure time  $T$ . In this section, the frailty model is developed and the hazard rate of  $T$  is investigated.

## 2.1 Failures in Lab Testing

As suggested by the extreme value theory, the Weibull distribution is an appropriate lifetime model when the failure is caused by the weakest flaw/link in a unit. It has been widely used for modeling lifetime of products and components. The failure time  $X$  of a lab testing unit is assumed to follow a Weibull distribution with the respective cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_X(x) = 1 - \exp \left[ - \left( \frac{x}{\alpha} \right)^\beta \right], \quad x > 0,$$

and

$$f_X(x) = \frac{\beta}{\alpha} \left( \frac{x}{\alpha} \right)^{\beta-1} \exp \left[ - \left( \frac{x}{\alpha} \right)^\beta \right], \quad x > 0, \quad (1)$$

where  $\alpha > 0$  is the scale parameter and  $\beta > 0$  is the shape parameter. The hazard rate function of  $X$  is given by

$$h_X(x) = \frac{\beta}{\alpha} \left( \frac{x}{\alpha} \right)^{\beta-1}. \quad (2)$$

It is well-known that the hazard rate function is monotone increasing when  $\beta > 1$ , and monotone decreasing when  $0 < \beta < 1$ .

## 2.2 Field Failures: A Gamma Frailty Model

When the product is sold to customers, the operating conditions are heterogeneous and unobservable. The unobservable effects are described by a frailty  $Z$ . The frailty  $Z$  is constant for a unit and varies across the product population. Conditional on  $Z$ , the lifetime of a field unit follows the Weibull distribution with a hazard rate function given by

$$h_T(t; Z) = Zh_X(t) = Z \times \frac{\beta}{\alpha} \left( \frac{t}{\alpha} \right)^{\beta-1}. \quad (3)$$

Because the baseline distribution is Weibull, this frailty model is similar to assuming a random scale parameter  $\alpha$  (Meeker and Escobar 1998, pp 457). Previously, Meeker et al. (2009) and Yang (2010) adopted such method to accommodate information on the heterogeneities. However, the reason we do not use a random scale parameter is that it is difficult, if not impossible, to find a distribution for  $\alpha$  such that the resulting field failure time distribution has a closed form.

The distribution of  $Z$  depends on the heterogeneities of the field environments as well as the effects of the random environments on the product. For example, when the heterogeneities are caused by the random use rate  $U$ , previous research suggests that the effect of  $U$  on product failures can be empirically described by a power law relation  $Z = aU^b$ ,  $a, b > 0$  are parameters, while the use rate distribution tends to be unimodal and positively skewed. This leads to a unimodal and positively skewed distribution for  $aU^b$ . Therefore, distributions like the gamma (Majeske 2007; Lawless et al. 2009), lognormal (Lawless et al. 1995; Meeker et al. 2009) and inverse Gaussian distributions are appropriate for  $Z$ . Occasionally, the uniform distribution is also recommended (Iskandar et al. 2005). The frailty  $Z$  includes the random usage rate, and thus it is reasonable to assume that it is also unimodal and positively skewed. To specify a distribution family for the frailty  $Z$ , it is of advantage that the resulting field failure distribution is tractable. This is because when the distribution of  $Z$  has a closed form, we can easily collate the validity of the frailty distribution through data analysis. We find that the families of gamma, inverse Gaussian and uniform distributions for the frailty result in tractable distributions for  $T$ . In the motivating example described in Section 1.1, the frailty is found to be well described by the gamma distribution. Therefore, this paper focuses on the gamma frailty model. *Development of the inverse Gaussian frailty model and the uniform frailty model is put in the supplemental material.* In fact, as suggested by Singpurwalla (2006), the gamma distribution is highly flexible to reflect pdfs of most shapes and thus the gamma frailty model is applicable to similar problems other than the Appliance B example.

In this section, we consider the gamma distribution with a threshold parameter in order to demonstrate the fact that the hazard rate function of  $T$  exhibits various shapes. The three-parameter gamma distribution with a threshold parameter  $\gamma$  has a pdf given by

$$\varphi(z) = \frac{\mu^k(z - \gamma)^{k-1}}{\Gamma(k)} \exp[-\mu(z - \gamma)], \quad z > \gamma. \quad (4)$$

When the frailty follows a distribution specified by (4), it can be shown by marginalizing  $Z$

out of (3) that the cdf and pdf of  $T$  are respectively given by

$$\begin{aligned} F_T(t) &= 1 - [(t/\alpha)^\beta/\mu + 1]^{-k} \exp[-\gamma(t/\alpha)^\beta], \\ f_T(t) &= \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} \left[\frac{(t/\alpha)^\beta}{\mu} + 1\right]^{-k} \left\{ \gamma + k \left[\left(\frac{t}{\alpha}\right)^\beta + \mu\right]^{-1} \right\} \exp\left[-\gamma\left(\frac{t}{\alpha}\right)^\beta\right]. \end{aligned} \quad (5)$$

It is interesting to note that when  $\gamma = 0$ , model (5) reduces to the Burr-XII distribution. The Burr-XII distribution has been used in reliability analysis by a few researchers, e.g., see Zimmer et al. (1998); Shao (2004); Soliman (2005) and Wang and Cheng (2010), to name a few. However, the Burr-XII distribution is much less popular than the lognormal distribution. Nevertheless, this distribution has several advantages over the lognormal distribution. Similar to the lognormal distribution, the Burr-XII distribution also has a unimodal hazard rate. But compared with the lognormal distribution, the Burr-XII distribution is more flexible in analysis of survival data. For example, parameters of the Burr-XII distribution can be determined through a simple probability plotting procedure (Zimmer et al. 1998). In addition, it has greater mathematical tractability when dealing with censored data which are very common in lifetime data analysis. The contribution of a right-censored observation to the likelihood is equal to the value of the survival function at the time of censoring, which can be evaluated explicitly for the Burr-XII distribution, but not for the log-normal distribution.

When  $\gamma = 0$ , the mean and variance of the frailty variable  $Z$  are  $k/\mu$  and  $k/\mu^2$ , respectively. If we fix  $k/\mu$  at a constant and let  $\mu \rightarrow \infty$ , then the distribution of  $Z$  will degenerate to a single point, and the Burr-XII distribution will also degenerate to a Weibull distribution. This is legitimate because under such circumstance, there is no variation in the frailty. The log-logistic distribution, a common distribution used in lifetime data analysis, is also a special case of model (5), when  $\gamma = 0$  and  $k = 1$ .

### 2.3 Hazard Rate for Units in the Field

In reliability assessment, the shape of the hazard rate reflects the early failure and aging behavior of the product. Therefore, it is important to know the shape with a view to scheduling preventive maintenance and detecting possible early failure modes. The hazard

rate function of  $Z$  can be readily obtained by dividing the pdf by the survival function, i.e.,  $1 - F_T(t)$ , which gives

$$h_T(t) = \frac{\gamma\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} + \frac{k\beta t^{\beta-1}}{t^\beta + \mu\alpha^\beta}. \quad (6)$$

The hazard rate of this distribution exhibits various shapes, as can be checked through the first order derivative of (6) with respect to  $t$ . By and large, the hazard rate could have four possible shapes, as summarized below.

**Case 1.**  $\beta \leq 1$ .

The hazard rate  $h_T(t)$  is decreasing in  $t$ . Specifically, when  $\beta < 1$ ,  $h_T(t)$  decreases from  $\infty$  to 0. When  $\beta = 1$ ,  $h_T(t)$  decreases from  $\gamma + k/\mu$  to  $\gamma$ . This is because a mixture of distributions with decreasing hazard rates has a non-increasing hazard rate.

**Case 2.**  $\gamma > 0, \beta > 1, \beta^2 - \beta < \frac{k}{4\gamma\mu}$ .

The hazard rate  $h_T(t)$  exhibits an N-shape.

**Case 3.**  $\gamma > 0, \beta > 1, \beta^2 - \beta > \frac{k}{4\gamma\mu}$ .

The hazard rate  $h_T(t)$  is increasing.

**Case 4.**  $\gamma = 0$  and  $\beta > 1$ .

The hazard rate  $h_T(t)$  has an upside-down bathtub shape.

Some typical curves of the hazard rate are depicted in Figure 1. It is interesting to see that when  $\beta > 1$ , the hazard rate under lab conditions is increasing, but the hazard rate of a field unit can be either increasing, unimodal, or N-shape. When the hazard rate of  $T$  is unimodal or N-shape, the initial hazard rate can be very high, as can be seen from the dash dotted lines in Figure 1. In practice, when a manufacturer observes a high hazard rate at the early stage, he may suspect that it is the infant mortality caused by defects. The analysis in this section reveals that early failures can also be caused by units operated under harsh environments (i.e., large realizations of  $Z$ ). These units are more likely to fail, and hence, more “frail” than other field units.

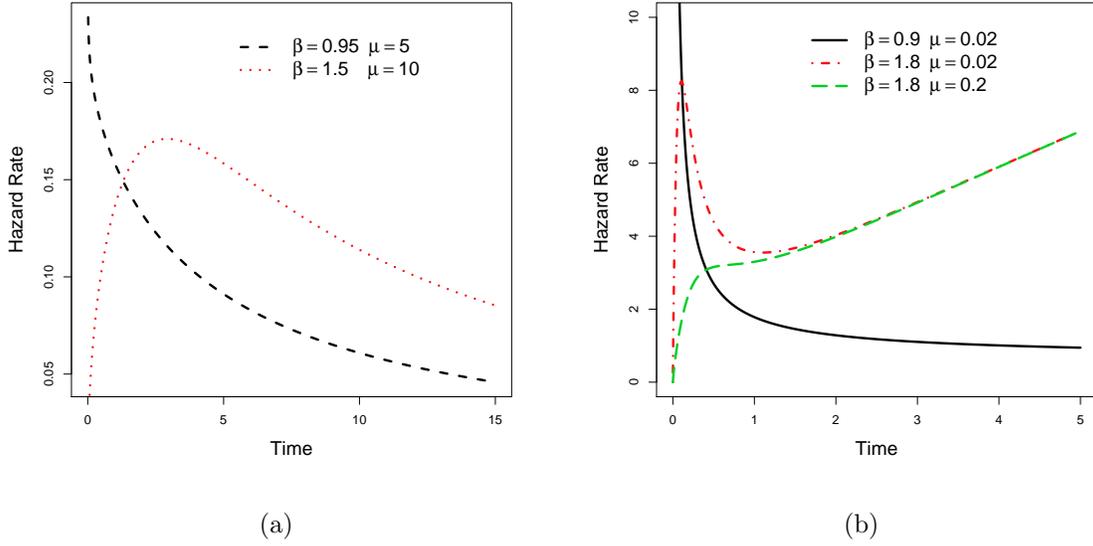


Figure 1: Illustrations of some shapes of the hazard function in (6): (a)  $\gamma = 0, \alpha = 1, k = 1$ ; and (b)  $\gamma = 1, \alpha = 1, k = 1$ .

### 3 Statistical Inference

Information about the distribution of the frailty can be obtained through a joint analysis of lab data and field data. In the previous section, we adopt the three-parameter gamma distribution with a threshold parameter  $\gamma$  for the frailty  $Z$  to set forth the fact that the field hazard rate can have various shapes in the presence of heterogeneous operating conditions. In reality, the frailty  $Z$  often ranges from zero to infinity. Thus, this section focuses on the case when the frailty follows a regular two-parameter gamma distribution (i.e.,  $\gamma = 0$ ), under which the field failure time  $T$  follows the Burr-XII distribution.

Suppose that  $n$  units are tested in the lab and  $x_i$  is the observed failure time or censoring time for the  $i$ -th unit. Further let  $\delta_i$  be the censoring indicator, where  $\delta_i = 0$  when the unit is right censored and 1 otherwise. Therefore, for the  $i$ -th lab unit, we observe  $(x_i, \delta_i)$ . Similarly, suppose we observe the failure times of  $N$  field units  $(t_j, \tilde{\delta}_j)$ ,  $j = 1, 2, \dots, N$ , where the field-data censoring indicator  $\tilde{\delta}_j = 0$  when the  $j$ -th unit is right censored and 1 if it fails and is returned as a warranty claim.

### 3.1 Estimation and Hypothesis Tests

Given the lab testing data and the warranty return data for the same product, we develop a procedure to analyze the data by capitalizing on the model in Section 2. In this procedure, we need to first collate the Weibull distribution (1) for the ALT data, and then check if the warranty return data conform to the Burr-XII distribution with cdf

$$G(t) = 1 - [(t/\lambda)^\beta + 1]^{-k}, \quad t > 0. \quad (7)$$

It is noted from (1) and (7) that when the gamma frailty model holds, the Weibull shape parameter in (1) should be equal to  $\beta$  in the Burr-XII distribution (7), and  $\lambda = \alpha\mu^{1/\beta}$ . The equality of  $\beta$  provides us a means to collate the correctness of the gamma frailty model. Details of the procedure are as follows.

**Step 1.** Fit the lab test data using the Weibull model with cdf given by (1). To underscore the fact that the shape parameter  $\beta$  is estimated from the lab data, we replace it with  $\beta_L$  in the following presentation. The maximum likelihood (ML) estimate of  $(\alpha, \beta_L)$ , denoted as  $(\hat{\alpha}, \hat{\beta}_L)$ , is obtained by maximizing the log-likelihood function (up to a constant)

$$l_L(\alpha, \beta_L | \text{Lab Data}) = \sum_{i=1}^n [\delta_i (\ln \beta_L + \beta_L \ln x_i - \beta \ln \alpha_L) - (x_i/\alpha)^{\beta_L}]. \quad (8)$$

Assess goodness-of-fit of the Weibull model. If the Weibull distribution provides a good fit to the lab data, then proceed to Step 2.

**Step 2.** Fit the field return data with the Burr-XII distribution. Here,  $\beta$  in (7) is replaced with  $\beta_W$  to stress the fact that this parameter is estimated from field data. The ML estimate of  $(\lambda, \beta_W, k)$ , denoted as  $(\hat{\lambda}, \hat{\beta}_W, \hat{k})$ , is obtained by maximizing the log-likelihood function (up to a constant)

$$l_W(\lambda, \beta_W, k | \text{Field Data}) = \sum_{j=1}^N \tilde{\delta}_j \{ \ln(k\beta_W) + \beta_W \ln(t_j/\lambda) - \ln[(t_j/\lambda)^{\beta_W} + 1] \} \quad (9) \\ - \sum_{j=1}^N k \ln[(t_j/\lambda)^{\beta_W} + 1].$$

Assess the goodness of fit of the Burr-XII distribution. If it provides a good fit, proceed to Step 3.

**Step 3.** Test the hypothesis  $H_0: k = 1$  versus the alternative hypothesis  $k \neq 1$ . If we accept the null hypothesis, the frailty follows an exponential distribution and the field failure time follows a log-logistic distribution, so we can fit the field data with the log-logistic distribution. If the hypothesis is rejected, stick to the Burr-XII distribution.

**Step 4.** Test the hypothesis  $H_0: \beta_L = \beta_W$  versus the alternative hypothesis  $\beta_L \neq \beta_W$ . If the null hypothesis is accepted, then there are statistical evidences that the frailty follows a gamma/exponential distribution, and then we can proceed to Step 5.

**Step 5.** Estimate the parameters in the gamma frailty model (5) by combining the lab test data and field return data. The ensemble log-likelihood function is

$$l(\alpha, \beta, \lambda, k | \text{All Data}) = l_L(\alpha, \beta | \text{Lab Data}) + l_W(\lambda, \beta, k | \text{Field Data}). \quad (10)$$

To test the hypothesis in Step 3, we can use either the likelihood ratio test or the score test. These two tests are expected to be accurate as the size of field return data is often large. However, these two tests may not be accurate enough when testing the hypothesis in Step 4, insofar as the lab test data are often limited. When both the ALT data and the field return data are complete or Type II censored, the following theorem shows that  $\hat{\beta}_L/\hat{\beta}_W$  is a “pivotal statistic” – that is, its distribution is independent of the unknown parameters  $\alpha$ ,  $\lambda$  and  $\beta$ . This ratio and its distribution will therefore be helpful in testing the hypothesis that  $\beta_L = \beta_W$  in Step 4.

**Theorem 1** *Suppose the lab failure times follow a Weibull distribution given by (1), while the field failure times conform to a Burr-XII distribution given in (7). Consider the hypothesis  $H_0: \beta_L = \beta_W \equiv \beta$  versus the alternative hypothesis  $\beta_L \neq \beta_W$  and assume the parameter  $k$  in (7) is known. When both the lab test data and field failure data are complete or Type II censored,  $\hat{\beta}_L/\hat{\beta}_W$  is a pivotal statistic independent of  $(\alpha, \lambda, \beta)$ .*

Proof of this theorem is in the appendix. The proof is based on the fact that  $\hat{\beta}_L/\beta$  and  $\hat{\beta}_W/\beta$  are pivotal statistics under the Type II censored (or complete) lab data and field data,

respectively. The constant  $k$  assumption is meaningful for the log-logistic distribution where  $k = 1$ . When  $k \neq 1$  and  $k$  is estimated from field data, we can treat  $\hat{k}$  as the true value of  $k$ . This approximation should work well because the field data are often abundant and thus the estimation error of  $k$  is small. Theorem 1 is not restricted by the problem of limited ALT data, and hence it is expected to perform better for testing the hypothesis of  $\beta_L = \beta_W$  compared with the likelihood ratio test. In a real-life application, we would recommend conducting both tests. When the results of both tests tally, there is sufficient evidence to accept or reject the hypothesis. When the results differ, we shall stick to the test based on Theorem 1. The distribution of  $\hat{\beta}_L/\hat{\beta}_W$  can be obtained through simulation as follows.

**Algorithm 1:**

1. Generate  $n$  samples from Weibull(1, 1) and  $N$  samples from BXII(1, 1,  $k$ ). For Type II censoring, the number of events will be the same as the number of events in the datasets. For Type I censoring, the expected number of events will be the same as the number of events in the datasets.
2. Estimate  $\hat{\beta}_L^*$  and  $\hat{\beta}_W^*$  from these two datasets separately.
3. Repeat the above two steps  $B$  times to get  $\hat{\beta}_L^{*i}/\hat{\beta}_W^{*i}$ ,  $i = 1, 2, \dots, B$ .
4. Use the  $B$  samples to estimate the empirical cdf and sample quantiles of  $\hat{\beta}_L/\hat{\beta}_W$ .

In Algorithm 1, one can use  $\hat{k}$  as the value of  $k$  in the simulation. The performance of this substitution will be evaluated through simulation. During the lab test, both Type I and Type II censoring are common. For warranty return data, Type I censoring or progressive Type I censoring are more common due to staggered entries and warranty limits. Under this scenario,  $\hat{\beta}_L/\hat{\beta}_W$  is an approximate pivotal.

**Theorem 2** *Suppose the lab test data follow a Weibull distribution given by (1), while the field failure data conform to a Burr-XII distribution given in (7). Consider the hypothesis  $H_0: \beta_L = \beta_W \equiv \beta$  versus the alternative hypothesis  $\beta_L \neq \beta_W$  and assume the parameter  $k$  in (7) is known. When the lab test data and/or field failure data are Type-I censored, then  $\hat{\beta}_L/\hat{\beta}_W$  is an approximate pivotal statistic.*

Under Type I censoring, the distribution of  $\hat{\beta}_W/\beta_W$  depends on the unknown fraction failing at the censoring time (e.g., [Jeng and Meeker 2001](#)). Thus it is an approximate pivotal. The approximation improves as the sample size increases. Because the sample size of field return data is often large, the performance of the approximation of  $\hat{\beta}_W/\beta_W$  is typically satisfactory. On the other hand, according to the Type of the lab test data, we have the following two discussions.

- When the lab test data is Type II censoring, then  $\hat{\beta}_L/\beta$  is an exact pivotal. Thus  $\hat{\beta}_L/\hat{\beta}_W = (\hat{\beta}_L/\beta)/(\hat{\beta}_W/\beta)$  is an approximate pivotal because  $\hat{\beta}_W/\beta_W$  is an approximate pivotal.
- When the lab test data is Type I censoring, then  $\hat{\beta}_L/\beta$  is an approximate pivotal. Thus  $\hat{\beta}_L/\hat{\beta}_W = (\hat{\beta}_L/\beta)/(\hat{\beta}_W/\beta)$  is also an approximate pivotal.

Algorithm 1 can still be used to do the test and the performance will be evaluated by simulations in the next subsection.

### 3.2 Simulation Study

In this section, we conduct simulation studies to show the performance of the statistics proposed in [Theorems 1 and 2](#). In particular, we consider three scenarios:

- Scenario I: Type II censoring for lab data and Type II censoring for field data.
- Scenario II: Type II censoring for lab data and Type I censoring for field data.
- Scenario III: Type I censoring for lab data and Type I censoring for field data.

We assume that the ALT uses 10 testing units whose lifetime follows a Weibull distribution. For Scenarios I and II of the simulation, the test is run until 8 of the units fail (i.e., Type II censoring). For Scenario III, the expected number of failures is 8 out of 10 testing units in the ALT (the censoring time is 733 in the simulation). For the field data,  $N$  units of the same product are sold to customers and the environmental frailty follows Gamma  $(k, \mu)$ . For the Type II censoring setting (Scenario I), we stop the follow-up when  $0.1N$  failures have been observed. For the Type I setting (Scenarios II and III), the failure times are censored

at  $\tau$ . The censoring time  $\tau$  is so chosen that the expected proportion of field failures is 10%. In the simulation, we use  $\alpha = 534$ ,  $k = 1$ ,  $\mu = 19$ , and  $\tau = 878$ . We examine  $N = 2000, 5000$  and  $\beta = 1.5, 2.0$ .

Under each combination of  $(\beta, N)$ , we replicate the simulation 2,000 times. In each replication, we compute the likelihood ratio statistic and the statistic in Theorem 1. The hypothesis is rejected or accepted according to the  $\tilde{\alpha}$  level. The estimated Type I error is obtained as the proportion of incorrect rejections. To obtain the distribution of the pivotal, we use  $B = 5,000$  in each run. In the simulation, we use normal approximation to simulate  $\hat{\beta}_W^{*i}$  and use the distribution of  $\hat{\beta}_W^{*i}/\hat{\beta}_W$  to approximate the distribution of  $\hat{\beta}_W/\beta_W$ . In particular,  $\hat{\beta}_W^{*i}$  is simulated from  $\mathcal{N}(\hat{\beta}_W, \sigma_{\hat{\beta}_W}^2)$  where  $\sigma_{\hat{\beta}_W}^2$  is the large sample approximate variance estimate of  $\hat{\beta}_W$ .

Table 1 shows the estimated Type I errors of the test procedure in Theorem 1 and the likelihood ratio test procedure, under three different scenarios. The nominal Type I errors that are considered in the simulation are  $\tilde{\alpha} = 0.1, 0.05$  and  $0.01$ . Under all scenarios, the estimated Type I errors of the testing procedure in Theorem 1 are closer to the nominal ones compared with the likelihood ratio statistic. In addition, the magnitude of  $N$  tends to have little effect on the Type I errors of the likelihood ratio statistic. This is best explained by our conjecture that the bias/error of the likelihood ratio statistic is attributed to the small lab testing samples. Overall, we can see that the performance of the approximate pivotal is satisfactory.

## 4 Optimal Accelerated Life Tests

Over the course of product evaluation and customer feedback, the manufacturer will generate a number of design changes and come up with a new vintage. ALTs can again be used to evaluate reliability of this new vintage by making use of the frailty information obtained from joint analysis of lab data and field data of previous generations. The ALTs need to be conducted within stringent cost and time constraints, and the testing samples need to be used efficiently. In addition, the heterogeneous field conditions should be taken into account when estimating life characteristics of interest. It is expected that the operating conditions

Table 1: Estimated Type I error of the test procedure in Theorem 1 and the likelihood ratio test procedure, under three different scenarios. The nominal Type I errors are  $\tilde{\alpha} = 0.1, 0.05, 0.01$ .

Scenarios	$\beta$	$N$	$\tilde{\alpha} = 0.1$		$\tilde{\alpha} = 0.05$		$\tilde{\alpha} = 0.01$	
			Thm	LR	Thm	LR	Thm	LR
Scenario I	1.5	2000	0.108	0.142	0.057	0.072	0.010	0.016
	1.5	5000	0.100	0.149	0.058	0.078	0.008	0.024
	2.0	2000	0.098	0.134	0.047	0.078	0.012	0.014
	2.0	5000	0.101	0.138	0.052	0.071	0.012	0.020
Scenario II	1.5	2000	0.108	0.128	0.056	0.077	0.014	0.016
	1.5	5000	0.092	0.136	0.048	0.080	0.010	0.018
	2.0	2000	0.110	0.135	0.056	0.075	0.014	0.018
	2.0	5000	0.110	0.146	0.058	0.082	0.014	0.022
Scenario III	1.5	2000	0.101	0.117	0.044	0.062	0.004	0.012
	1.5	5000	0.099	0.126	0.050	0.062	0.004	0.014
	2.0	2000	0.100	0.116	0.048	0.060	0.008	0.012
	2.0	5000	0.094	0.118	0.040	0.060	0.008	0.012

and the effects of the environments on the new generation be approximately the same. This implies that the new vintage will have the same frailty  $Z$  with the old generation. Based on this fact, optimal ALT plans can be developed.

For the new generation of interest, suppose its lifetime  $X$  under the stable lab testing conditions follows a Weibull distribution specified by (1). Let  $S_0$  be the *nominal* design stress (say, the same as the old generation), and  $S_H$  be the highest allowable test stress that has been pre-specified. For convenience, the stress is re-parameterized as  $\xi = (S - S_H)/(S_0 - S_H)$ . It is noted that under the nominal stress  $S_0$ ,  $\xi = 1$ . When  $X$  follows a Weibull distribution,  $Y = \ln X$  conforms to a smallest extreme value distribution with the location parameter  $\eta = \ln \alpha$  and the scale parameter  $\sigma = 1/\beta$ . Following the convention of ALT design for the Weibull distribution (e.g., [Meeker and Escobar 1998](#), Chapter 17), we work with the extreme value distribution whose Fisher information matrix has a closed form, and assume that the scale parameter  $\sigma$  is a constant independent of the stress while the location parameter  $\eta$  depends on the stress through a linear stress-life model

$$\eta(\xi) = v_0 + v_1\xi.$$

Usually, the optimal test plans use only two test stresses with the higher stress being the highest allowable stress  $S_H$ . Therefore, an ALT plan is specified by the lower stress level  $\xi_L$  and the proportion of units  $\pi$  for this stress. The combination  $(\xi_L, \pi)$  is called a test plan. The purpose of the ALT design is to find out the optimal test plan  $(\xi_L^*, \pi^*)$  in order to optimize a certain index of interest. When we are interested in life characteristics under the *nominal* conditions (i.e., characteristics based on  $X$ ) and ignore the heterogeneous field operating conditions, optimal constant-stress ALTs for the extreme value distribution have been well studied, e.g., see [Nelson and Meeker \(1978\)](#) for the optimal Type I censoring plan and [Escobar and Meeker \(1986\)](#) for the Type II censoring case. In the presence of the heterogeneities, however, the criteria of ALT planning should be based on field failure times  $T$ , and thus the existing plans are no longer optimal. Optimal plans that take the frailty  $Z$  into account will be developed in this section. Denote  $\mathcal{I}(\xi_L, \pi)$  as the Fisher information matrix for  $(v_0, v_1, \sigma)$  under the test plan  $(\xi_L, \pi)$ . The matrix  $\mathcal{I}(\xi_L, \pi)$  has been derived by [Nelson and Meeker \(1978\)](#) under the Type I censoring scheme, and by [Escobar and Meeker](#)

(1986) with Type II censoring. Therefore, use will be directly made of these existing results.

#### 4.1 Minimization of the Asymptotic Variance of the $p$ -Quantile

Consider the common criterion of ALT planning that minimizes the asymptotic variance of the ML estimator  $\hat{t}_p$  of the  $p$  quantile of field failure time  $T$ . Based on (5) with  $\gamma = 0$ , the  $p$ -quantile of  $T$  is given by

$$t_p = \alpha [\mu(1-p)^{-1/k} - \mu]^{1/\beta} = \exp(v_0 + v_1) [\mu(1-p)^{-1/k} - \mu]^\sigma. \quad (11)$$

The asymptotic variance of the ML estimator  $\hat{t}_p$  is  $AV(\hat{t}_p) = (\nabla t_p)' \mathcal{I}(\xi_L, \pi) \nabla t_p$ , where  $\nabla t_p$  is the first derivative of  $t_p$  with respect to  $(v_0, v_1, \sigma)$ . The expression of  $\nabla t_p$  is quite involved. Alternatively, it is not difficult to show that minimization of  $AV(\hat{t}_p)$  amounts to minimization of the asymptotic variance of  $\ln \hat{t}_p$ , which is equivalent to minimizing the asymptotic variance of  $\hat{t}_p/t_p$ . The asymptotic variance of  $\ln \hat{t}_p$  is  $AV(\ln \hat{t}_p) = (\nabla \ln t_p)' \mathcal{I}(\xi_L, \pi) \nabla \ln t_p$ , where  $\nabla \ln t_p$  is the gradient of  $\ln t_p$  with respect to  $(v_0, v_1, \sigma)$  as

$$\begin{aligned} \nabla \ln t_p(1) &= \frac{\partial \ln t_p}{\partial v_0} = 1, \\ \nabla \ln t_p(2) &= \frac{\partial \ln t_p}{\partial v_1} = 1, \\ \nabla \ln t_p(3) &= \frac{\partial \ln t_p}{\partial \sigma} = \ln [\mu(1-p)^{-1/k} - \mu]. \end{aligned} \quad (12)$$

Optimal test plans can be obtained by minimizing  $AV(\ln \hat{t}_p)$  under some constraints, e.g., time constraint, budget constraint or sample size constraint.

#### 4.2 Minimization of the Asymptotic Variance of the Failure Probability

The  $p$ -quantile criterion considered above is often used to determine a suitable warranty period for a new product (Ye et al. 2011). For a product with a given warranty period  $\tau$ , what the manufacturer is most concerned with is the proportion of field returns within  $\tau$ . Therefore, another rational planning criterion is to minimize the asymptotic variance of  $\hat{p}_\tau$ , the ML estimate of the probability of warranty failures. This probability is given by

$$p_\tau = 1 - [(\tau/\alpha)^\beta/\mu + 1]^{-k} = 1 - \left[ \frac{[\tau \exp(-v_0 - v_1)]^{1/\sigma}}{\mu} + 1 \right]^{-k}. \quad (13)$$

The first derivative of  $p$  with respect to  $(v_0, v_1, \sigma)$  can be obtained as

$$\begin{aligned}\nabla p_{\tau}(1) &= \frac{\partial p}{\partial v_0} = -\frac{k\Omega^{1/\sigma}}{\mu\sigma} \left(\frac{\Omega^{1/\sigma}}{\mu} + 1\right)^{-k-1}, \\ \nabla p_{\tau}(2) &= \frac{\partial p}{\partial v_1} = -\frac{k\Omega^{1/\sigma}}{\mu\sigma} \left(\frac{\Omega^{1/\sigma}}{\mu} + 1\right)^{-k-1}, \\ \nabla p_{\tau}(3) &= \frac{\partial p}{\partial \sigma} = -\frac{k\Omega^{1/\sigma}}{\mu\sigma^2} \left(\frac{\Omega^{1/\sigma}}{\mu} + 1\right)^{-k-1} \ln \Omega,\end{aligned}\tag{14}$$

where  $\Omega = \tau \exp[-(v_0 + v_1)]$ . Based on the delta method, the asymptotic variance is  $AV(\hat{p}_{\tau}) = (\nabla p_{\tau})' \mathcal{I}(\xi_L, \pi) \nabla p_{\tau}$ . Optimal test plans can be determined by minimizing this asymptotic variance subject to possible constraints on available resources.

## 5 Illustrative Example

### 5.1 Weibull Fit to Lab Test Data

10 units of Appliance B were subject to a lab test. The experiment ended at  $t = 687$  units of time, upon which 8 failures were observed and 2 were censored. In order to demonstrate Theorem 1, we assume that the experiment ended when the 8th failure is observed. After this modification, the data are Type II censored. The observed failure times of the 8 failed samples are presented in Table 2.

Table 2: Ordered failure time data observed from the ALT test.

99	141	163	300	350	523	602	687
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We use the Weibull model to fit the ALT data, and the ML estimates (standard errors) of the two parameters are  $\hat{\alpha} = 529.4$  (121.0) and  $\hat{\beta}_L = 1.55$  (0.470), respectively. In order to visualize the goodness-of-fit, we also fit the data using the Kaplan-Meier method. The estimated cdfs by means of the Weibull model and the Kaplan-Meier method are depicted in Figure 2. As can be seen from this figure, the estimated Weibull cdf passes through the empirical cdf and falls well within the 95% simultaneous confidence band (SCB). Therefore, the Weibull model is considered as an appropriate model for the product under nominal conditions.

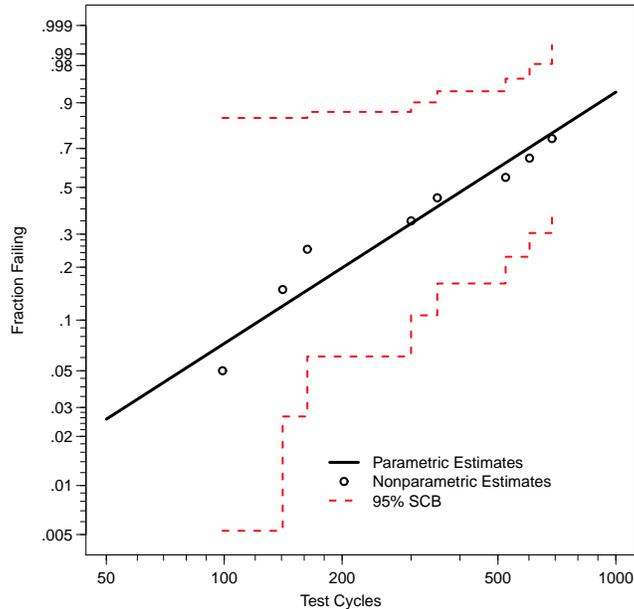


Figure 2: Weibull probability plot showing the Weibull fit to the lab data and the 95% nonparametric SCB.

## 5.2 Burr-XII Fit to the Field Failure Data

We first use the Weibull distribution to fit the field return data. The maximum log-likelihood value is  $-977.2$ . The estimated Weibull cdf as well as the empirical cdf using the Kaplan-Meier method is shown in Figure 3. As can be seen from this figure, the Weibull distribution cannot capture the curvature of the nonparametric estimates in the lower tail. We suspect that the inconsistency between failures in the lab and in the field is caused by heterogeneous operating conditions. Therefore, the gamma frailty model is invoked to solve the problem.

We apply (5) to fit the data and use the likelihood ratio statistic to test the threshold parameter  $\gamma = 0$ . The test reveals no evidence to reject the hypothesis. Therefore, we set  $\gamma = 0$  in the following analysis. We apply the Burr-XII distribution to fit the data, and the maximum log-likelihood value is  $-973.8$ . The estimated values of the parameters are  $\hat{\lambda} = 298.6$  (83.9),  $\hat{\beta}_W = 2.66$  (0.452) and  $\hat{k} = 0.0223$  (0.0109), respectively. We use the Akaike information criterion (AIC) to compare the Burr-XII model and the Weibull model for the field data. The AIC is specified by  $AIC = -2l + 2m$ , where  $l$  is the maximum log-likelihood

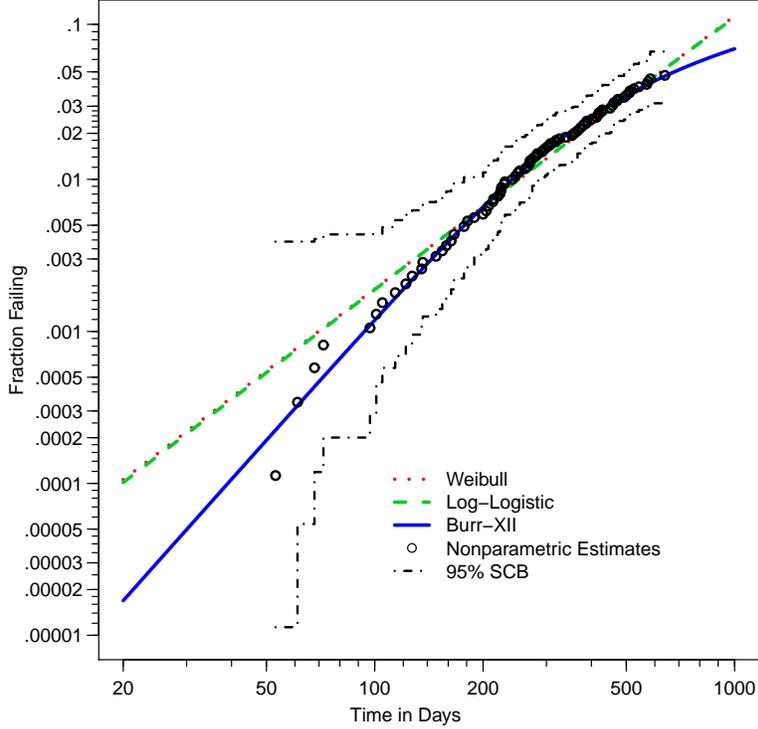


Figure 3: Weibull probability plot showing the ML estimates of the Weibull, log-logistic, and Burr-XII fits to the field data and the 95% nonparametric SCB.

value of a model and  $m$  is the number of parameters in the model. The respective AIC values for the Weibull and the Burr-XII distributions are 1958.4 and 1953.5. The Burr-XII distribution has a smaller AIC value, indicating a better fit. As can be seen from Figure 3, the Burr-XII distribution captures the curvature of the nonparametric estimates in the lower tail very well, indicating a better fit than the Weibull distribution.

We also fit the data by using the log-logistic distribution, leading to a maximum likelihood value of  $-977.0$ . This value is very similar to that of the Weibull model. Overall, the analysis suggests that the Burr-XII distribution is more appropriate for the field data than the Weibull model.

### 5.3 The Gamma Frailty Model

As can be seen from the above analysis,  $\hat{\beta}_L$  is quite close to  $\hat{\beta}_W$ . We apply the statistic developed in Theorem 1 to quantitatively check the correctness of the gamma frailty model by testing  $H_0 : \beta_L = \beta_W$ . It is easy to see that  $\hat{\beta}_L/\hat{\beta}_W = 0.585$ . By making use of Algorithm 1, the  $p$ -value is 0.217. We then apply the likelihood ratio test. The likelihood ratio statistic is 1.356 with a  $p$ -value of 0.244. Both tests suggest that there is no reason to reject this hypothesis. Therefore, we can believe that the discrepancies between the lab test data and the field data can be explained by the frailty model, and the gamma frailty model is appropriate for the problem.

At the last step, we estimate the parameters in (5) by combining both the ALT data and the field failure data. The ensemble of the likelihood function consists of the Weibull likelihood contributed from the lab data and the Burr-XII likelihood contributed from the field data. Maximization of this function yields the ML estimates of the four parameters (standard errors) as  $\hat{\alpha} = 545.15$  (84.7),  $\hat{\beta} = 2.28$  (0.32),  $\hat{\lambda} = 385.05$  (136.5) and  $\hat{k} = 0.0341$  (0.019). Using the invariance property of the MLE, the estimated scale parameter for the gamma frailty is  $\hat{\mu} = (\hat{\lambda}/\hat{\alpha})^{\hat{\beta}} = 0.452$  with a standard error 0.23. The estimated cdfs for the lab failure time distribution and the field failure time distribution can be updated based on these parameter estimates, as shown in Figure 4.

### 5.4 Optimal ALT Plans

In order to improve product reliability and cater to market changes, the manufacturer may make a number of changes to the product and come up with a new generation. The new generation, if sold to the market, would be operated under the same environments as the old ones and the environments will have the same effect on the product failures. Therefore, we assume the frailty  $Z$  follows the same gamma distribution  $\text{Gamma}(k, \mu)$  with  $\mu = 0.452$  and  $k = 0.0341$ . Suppose that the manufacturer is interested in knowing the 5% quantile of the field lifetime of the new generation, and a maximum test time of 50 is allowed for the ALT. During the test, all units are run simultaneously. Assume that the lifetime of the new vintage follows a Weibull distribution under the nominal use condition, and the planning values of

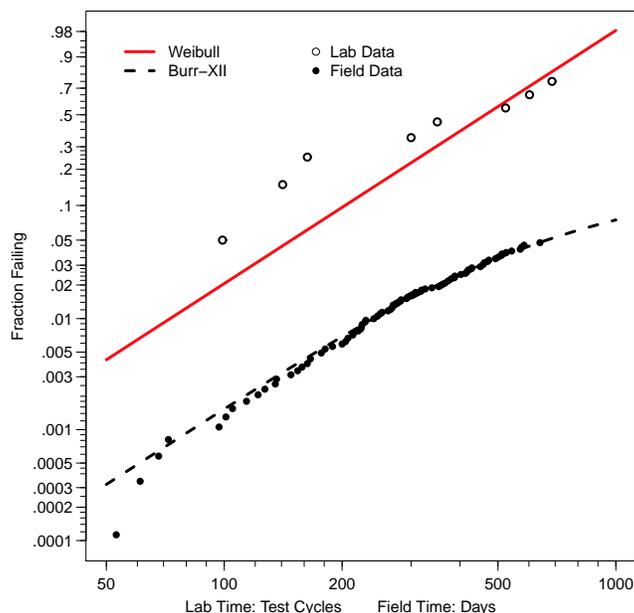


Figure 4: Weibull probability plot showing the ML estimates of Weibull fit to lab data and Burr-XII fit to the field data with a common  $\beta$ .

the ALT are  $v_0 = 3$ ,  $v_1 = 3.4$  and  $\beta = 2.28$ . Based on the above settings, optimal test plans can be obtained by numerically optimizing the asymptotic variance given in Section 4.1. For example, if the objective is to minimize the asymptotic standard deviation, i.e., square root of the variance, of  $\ln \hat{t}_p$ , then the optimal test plan is  $(\epsilon^*, \pi^*) = (0.338, 0.649)$  and the associated minimal standard deviation is 3.23. Figure 5(a) shows the contour of the asymptotic standard deviation with respect to  $\epsilon$  and  $\pi$ . This test plan also minimizes the asymptotic standard deviation of  $\hat{t}_p$ , as can be seen from Figure 5(b). If we ignore the heterogeneous field environments, the optimal test plan will be  $(\epsilon, \pi) = (0.419, 0.766)$ , which is quite different from  $(\epsilon^*, \pi^*)$ .

## 6 Conclusions

This study has explained the discrepancies between in-lab failures and field failures through the frailty model. The frailty term of each field unit represents the unobserved operating

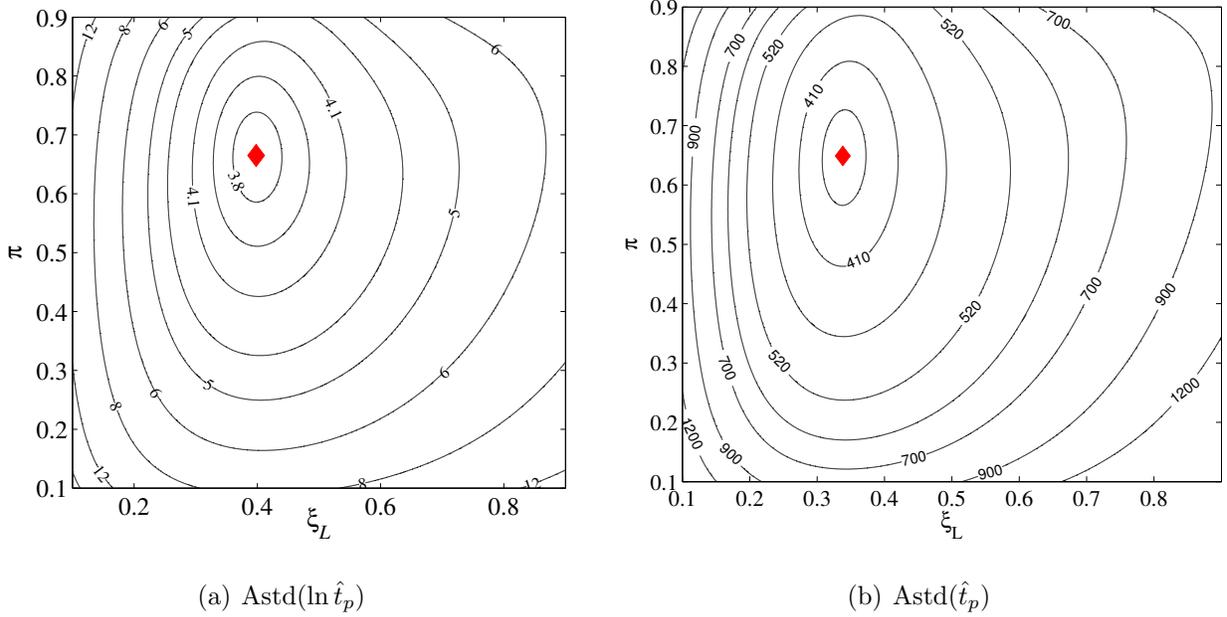


Figure 5: Contours of the asymptotic standard deviation for the two-stress optimum ALT plan.

conditions and their complicated effects on the product failures. In the presence of heterogeneous operating conditions, we showed that the field failure rate can exhibit a variety of shapes, and some units may fail very early due to severe working conditions rather than defects. ALTs should take these heterogeneities into account. Previous research assumed homogeneous operating conditions, which will inevitably underestimate the variation of the field failures and in turn underestimates the proportion of field returns. In addition, test plans derived under the homogeneity assumption may be quite different from the true optimum due to ignorance of the heterogeneity. We overcame these deficiencies and derived the optimal plans by considering the frailty. A procedure was developed to obtain the frailty information and to collate the validity of the gamma distribution for the frailty. Instead of using the likelihood ratio statistic to test the equality of  $\beta_L$  and  $\beta_W$ , we suggested the use of the statistic  $\hat{\beta}_L/\hat{\beta}_W$ . This statistic is pivotal under complete or Type II censored data. Under Type I censoring, this statistic is approximately a pivotal quantity and its good performance is demonstrated through a simulation study. In the supplement, we further developed the inverse Gaussian frailty models and the uniform frailty models. These two models yield tractable field failure distributions, and supplement the class of frailty models for linking lab

failures and field failures. We also proposed an ensemble inference procedure in consideration of all the gamma, inverse Gaussian, and uniform frailty models in the supplement.

## Acknowledgments

We would like to thank the editor, the associate editor, and one referee for their helpful comments on this paper. The work by Ye was partially supported by the Sate Key Laboratory of Industrial Control Technology with project code ICT1313. Hong and Xie were supported by the National Science Foundation under Grant CMMI-1068933 to Virginia Tech and the 2011 DuPont Young Professor Grant.

## Appendix

### Shapes of the Hazard Rate Function of the Gamma Frailty Model

Taking the first derivative of (6) with respect to  $t$  yields

$$h'(t) = \left[ (2\beta\gamma\mu - 2\gamma\mu - k)t^\beta + \mu\alpha^\beta(\mu\gamma + k)(\beta - 1) + \frac{(\beta - 1)\gamma t^{2\beta}}{\alpha^\beta} \right] \frac{\beta t^{\beta-2}}{(t^\beta + \mu\alpha^\beta)^2}. \quad (\text{A.1})$$

The second term on the right-hand side of (A.1) is always larger than 0. So we can focus on the first term

$$r(x) = (\beta - 1)\gamma\alpha^{-\beta}x^2 + (2\beta\gamma\mu - 2\gamma\mu - k)x + \mu\alpha^\beta(\mu\gamma + k)(\beta - 1). \quad (\text{A.2})$$

**Case 1.** When  $\beta < 1$ , it is easy to see that  $r(x) < 0$ , and hence,  $h'(t) < 0$ .

When  $\beta > 1$ , the minimum of  $r(x)$  is achieved at the point

$$\left( \frac{-2\gamma\mu(\beta - 1) + k}{2(\beta - 1)\gamma\alpha^{-\beta}}, \frac{4\gamma\mu\beta(\beta - 1) - k}{4(\beta - 1)\gamma\alpha^{-\beta}k^{-1}} \right).$$

**Case 2.,** When  $\gamma > 0, \beta > 1$  and  $4\gamma\mu\beta(\beta - 1) - k < 0$ , we see from  $\beta > 1$  that

$$4\gamma\mu\beta(\beta - 1) - 2k\beta < 0, \quad \text{so} \quad 2\gamma\mu(\beta - 1) - k < 0.$$

This means that

$$\frac{-2\gamma\mu(\beta - 1) + k}{2(\beta - 1)\gamma\alpha^{-\beta}} > 0 \quad \text{and} \quad \frac{4\gamma\mu\beta(\beta - 1) - k}{4(\beta - 1)\gamma\alpha^{-\beta}k^{-1}} < 0.$$

By noting that  $r(0) = \mu\alpha^\beta(\mu\gamma+k)(\beta-1) > 0$ , we can know that when  $x \geq 0$ ,  $r(x)$  is positive initially, followed by a negative period, and then become positive again. From (10), we can know  $h'(t)$  also has this positive-negative-positive sign change. Therefore,  $h(t)$  increases initially, followed by a decreasing period, and then increases again, i.e.,  $h(t)$  has an N-shape.

**Case 3.** When  $\gamma > 0, \beta > 1$  and  $4\gamma\mu\beta(\beta-1) - k > 0$ ,  $r(x)$  is always greater than 0, and so is  $h'(t)$ . Therefore,  $h(t)$  is increasing over  $[0, \infty)$ .

**Case 4.** When  $\gamma = 0$  and  $\beta > 1$ ,  $r(x)$  reduces to  $r(x) = -kx + (\beta-1)k\mu\alpha^\beta$ . This linear function is monotone decreasing with  $r(0) > 0$  and  $r(\infty) < 0$ . Therefore,  $h(t)$  is increasing at the outset and decreasing afterwards. Hence,  $h(t)$  has an upside-down bathtub shape.

## Proof of Theorem 1

Before proceeding to the proof of Theorem 1, two lemmas are first presented.

**Lemma 1** *For the Burr-XII distribution given by (7), conditional on  $k$ ,  $\hat{\beta}_W/\beta_W$  is a pivotal statistic under Type II or complete data.*

Proof: Let  $\mathbf{x} = (x_1, \dots, x_n)$  be an ordered random sample of size  $n$  from  $\text{BXII}(1, 1, k)$ . An ordered random sample  $\mathbf{t} = (t_1, \dots, t_n)$  conforming to  $\text{BXII}(\beta_W, \lambda, k)$  can be obtained by taking  $t_i = \lambda x_i^{1/\beta_W}$ . Suppose the sample was censored after the  $r$ -th observation. The ML estimates of the parameters in the Burr-XII distribution can be obtained by deriving the score functions, equating them to zero, and solving for the solution. Denote the ML estimates based on  $\mathbf{x}$  and  $\mathbf{t}$  as  $(\hat{\beta}_{W_0}, \hat{\lambda}_0)$  and  $(\hat{\beta}_W, \hat{\lambda})$ , respectively. Now, we proceed to investigate the relationship between  $(\hat{\beta}_{W_0}, \hat{\lambda}_0)$  and  $(\hat{\beta}_W, \hat{\lambda})$ . The log-likelihood function based on  $\mathbf{t}$ , up to a constant, can be written as

$$L(\beta_W, \lambda) = r \ln \beta_W + \sum_{i=1}^r \ln(t_i/\lambda)^{\beta_W} - (k+1) \sum_{i=1}^r \ln[(t_i/\lambda)^{\beta_W} + 1] \\ - k(n-r) \ln[(t_r/\lambda)^{\beta_W} + 1].$$

Therefore, the ML estimator  $(\hat{\beta}_W, \hat{\lambda})$  satisfies the following equation.

$$(k+1) \sum_{i=1}^r \frac{(t_i/\hat{\lambda})^{\hat{\beta}_W} \ln(t_i/\hat{\lambda})^{\hat{\beta}_W}}{(t_i/\hat{\lambda})^{\hat{\beta}_W} + 1} - r - \sum_{i=1}^r \ln(t_i/\hat{\lambda})^{\hat{\beta}_W} + k(n-r) \frac{(t_r/\hat{\lambda})^{\hat{\beta}_W} \ln(t_r/\hat{\lambda})^{\hat{\beta}_W}}{(t_r/\hat{\lambda})^{\hat{\beta}_W} + 1} = 0 \\ (k+1) \sum_{i=1}^r \frac{(t_i/\hat{\lambda})^{\hat{\beta}_W}}{(t_i/\hat{\lambda})^{\hat{\beta}_W} + 1} + k(n-r) \frac{(t_r/\hat{\lambda})^{\hat{\beta}_W}}{(t_r/\hat{\lambda})^{\hat{\beta}_W} + 1} - r = 0$$

If we substitute  $t_i = \lambda x_i^{1/\beta_W}$  into the above two equations, we can obtain

$$(k+1) \sum_{i=1}^r \frac{\Lambda_i \ln \Lambda_i}{\Lambda_i + 1} - r - \sum_{i=1}^r \ln \Lambda_i + k(n-r) \frac{\Lambda_r \ln \Lambda_r}{\Lambda_r + 1} = 0, \quad (\text{A.3})$$

$$(k+1) \sum_{i=1}^r \frac{\Lambda_i}{\Lambda_i + 1} + k(n-r) \frac{\Lambda_r}{\Lambda_r + 1} - r = 0, \quad (\text{A.4})$$

where  $\Lambda_i = \left( x_i / (\hat{\lambda} / \lambda)^{\beta_W} \right)^{\hat{\beta}_W / \beta_W}$ . The left-hand sides of the above two equations are the score functions based on the sample  $\mathbf{x}$ . Therefore, it is readily seen that  $\hat{\lambda}_0 = (\hat{\lambda} / \lambda)^{\beta_W}$  and  $\hat{\beta}_{W_0} = \hat{\beta}_W / \beta_W$ . This means that the distribution of  $\hat{\beta}_W / \beta_W$  is the same as  $\hat{\beta}_{W_0}$ , which does not depend on  $\beta_W$  and  $\lambda$ .

**Lemma 2** For the Weibull distribution specified by (1),  $\hat{\beta}_L / \beta_L$  is a pivotal statistic under Type II or complete data.

Proof: See [Thoman et al. \(1969\)](#).

**Proof of Theorem 1.** By using Lemmas 1 and 2 above, we can see that  $\frac{\hat{\beta}_W / \beta_W}{\hat{\beta}_L / \beta_L}$  is a pivotal statistic. Under the null hypothesis, this pivotal statistic is exactly  $\hat{\beta}_L / \hat{\beta}_W$ , which completes the proof.

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