Characterizations of processes with stationary and independent increments under G-expectation

Yongsheng Song *
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing, China;
yssong@amss.ac.cn

Abstract

Our purpose is to investigate properties for processes with stationary and independent increments under G-expectation. As applications, we prove the martingale characterization of G-Brownian motion and present a pathwise decomposition theorem for generalized G-Brownian motion.

Key words: stationary increments; independent increments; martingale characterization; decomposition theorem; G-Brownian motion; G-expectation

MSC-classification: 60G10, 60G17, 60G48, 60G51

1 Introduction

Recently, motivated by the modelling of dynamic risk measures, Shige Peng ([P07a], [P07b], [P08a]) introduced the notion of a G-expectation space. It is a generalization of probability spaces (with their associated linear expectation) to spaces endowed with a nonlinear expectation. As the counterpart of Wiener space in the linear case, the notion of G-Brownian motion was introduced under the nonlinear G-expectation.

*Y. Song was supported by NCMIS; Youth Grant of National Science Foundation (No.11101406/A011002); the National Basic Research Program of China (973 Program) (No.2007CB814902); Key Lab of Random Complex Structures and Data Science, Chinese Academy of Sciences (Grant No. 2008DP173182).

Recall that if $\{A_t\}$ is a continuous process over a probability space $(\Omega, \mathcal{F}, P)$ with stationary, independent increments and finite variation, then there exists some constant c such that $A_t = ct$. However, it's not the case in the G-expectation space $(\Omega_T, L_G^1(\Omega_T), \hat{E})$. A counterexample is $\{\langle B\rangle_t\}$, the quadratic variation process for the coordinate process $\{B_t\}$, which is a G-Brownian motion. We know that $\{\langle B\rangle_t\}$ is a continuous, increasing process with stationary and independent increments, but it is not deterministic.

The process $\{\langle B\rangle_t\}$ is very important in the theory of G-expectation, which shows, in many aspects, the difference between probability spaces and G-expectation spaces. For example, we know that for a probability space continuous local martingales with finite variation are trivial processes. However, [P07b] proved that in a G-expectation space all processes in form of $\int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$, $\eta \in M_G^1(0,T)$(see Section 2 for the definitions of the function $G(\cdot)$ and the space $M_G^1(0,T)$), are nontrivial G-martingales with finite variation (In fact, they are even nonincreasing) and continuous paths. [P07b] also conjectured that any G-martingale with finite variation should have such representation. Up to now, some properties of the process $\{\langle B\rangle_t\}$ remain unknown. For example, we know that, if $G(x) = \frac{1}{2}\sup_{\underline{\sigma}\le\sigma\le\overline{\sigma}} \sigma^2 x$ generates the G-expectation, we have $\underline{\sigma}^2(t-s) \le \langle B\rangle_t - \langle B\rangle_s \le \overline{\sigma}(t-s)$ for all $s < t$, but we don't know whether $\{\frac{d}{ds}\langle B\rangle_s\}$ belongs to $M_G^1(0,T)$. This is a very important property since $\{\frac{d}{ds}\langle B\rangle_s\} \in M_G^1(0,T)$ would imply that the representation mentioned above of G-martingales with finite variation is not unique.

For the case of a probability space, a continuous local martingale $\{M_t\}$ is a standard Brownian motion if and only if the quadratic variation process $\langle M\rangle_t = t$. However, it's not the case for G-Brownian motion since its quadratic variation process is only an increasing process with stationary and independent increments. How can we give a characterization for G-Brownian motion?

In this article, we shall prove that if $A_t = \int_0^t h_s ds$ (respectively $A_t = \int_0^t h_s d\langle B\rangle_s$) is a process with stationary, independent increments and $h \in M_G^1(0,T)$ (respectively $h \in M_G^{\beta,+}(0,T)$, for some $\beta > 1$), then there exists some constant c such that $h \equiv c$. As applications, we prove the following conclusions (Question 1 and 3 are put forward by Prof. Shige Peng in private communications):

1. $\{\frac{d}{ds}\langle B\rangle_s\} \notin M_G^1(0,T)$.

2.(Martingale characterization)

A symmetric G-martingale $\{M_t\}$ is a G-Brownian motion if and only if its quadratic variation process $\{\langle M\rangle_t\}$ has stationary and independent increments;

A symmetric G-martingale $\{M_t\}$ is a G-Brownian motion if and only if its quadratic variation process $\langle M\rangle_t = c\langle B\rangle_t$ for some $c \ge 0$.

The sufficiency of the second assertion is trivial, but not the necessity.

3. *Let $\{X_t\}$ be a generalized G-Brownian motion with zero mean, then we have the following decomposition:*

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a (symmetric) G-Brownian motion, and $\{L_t\}$ is a nonpositive, nonincreasing G-martingale with stationary and independent increments.

This article is organized as follows: In section 2 we recall some basic notions and results of G-expectation and the related space of random variables. In section 3 we characterize processes with stationary and independent increments. In section 4, as application, we prove the martingale characterization of G-Brownian motion and present a decomposition theorem for generalized G-Brownian motion. In section 5 we present some properties for G-martingales with finite variation.

2 Preliminary

We recall some basic notions and results of G-expectation and the related space of random variables. More details of this section can be found in [P07a, P07b, P08a, P08b, P10].

Definition 2.1 Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ satisfying the following properties: For all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space. □

Definition 2.2 Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ is the space of real continuous functions defined on R^n such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \text{for all } x, y \in R^n,$$

where k and C depend only on φ. □

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$, under $\hat{E}(\cdot)$, denoted by $Y \perp\!\!\!\perp X$, if for every test function $\varphi \in C_{b,Lip}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$. □

Definition 2.4 (G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if for every $a, b \in R_+$ we have

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : S_d \to R,$$

where S_d denotes the collection of $d \times d$ symmetric matrices. □

The function $G(\cdot) : S_d \to R$ is a monotonic, sublinear mapping on S_d and $G(A) = \frac{1}{2}\hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr(\gamma A). \tag{2.0.1}$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta Tr(A - B)$ for any $A \geq B$, we call the G-normal distribution nondegenerate. This is the case we consider throughout this article.

Definition 2.5 i) Let $\Omega_T = C_0([0, T]; R^d)$ be endowed with the supremum norm and $\{B_t\}$ be the coordinate process. Set $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n}) | n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{l,Lip}(R^{d\times n})\}$. G-expectation is a sublinear expectation defined by

$$\hat{E}[X] = \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_{i+1} is independent of $(\xi_1, \cdots, \xi_i)$ for every $i = 1, \cdots, m - 1$. $(\Omega_T, \mathcal{H}_T^0, \hat{E})$ is called a G-expectation space.

ii) Let us define the conditional G-expectation $\hat{E}_t$ of $\xi \in \mathcal{H}_T^0$ knowing $\mathcal{H}_t^0$, for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq m$, and we put

$$\begin{aligned}&\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]\\ &= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),\end{aligned}$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

□

Define $\|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then for all $t \in [0, T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$ with respect to the norm $\|\cdot\|_{1,G}$ and therefore can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under the norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) | n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{b,Lip}(R^{d\times n})\}$, where $C_{b,Lip}(R^{d\times n})$ denotes the set of bounded Lipschitz functions on $R^{d\times n}$. [DHP11] proved that the completions of $C_b(\Omega_T)$, $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same; we denote them by $L_G^p(\Omega_T)$.

Definition 2.6 i) We say that $\{X_t\}$ on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ is a process with independent increments if for any $0 < t < T$ and $s_0 \leq \cdots \leq s_m \leq t \leq t_0 \leq \cdots \leq t_n \leq T$,

$$(X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}) \perp\!\!\!\perp (X_{s_1} - X_{s_0}, \cdots, X_{s_m} - X_{s_{m-1}}).$$

ii) We say that $\{X_t\}$ on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ with $X_t \in L_G^1(\Omega_t)$ for every $t \in [0, T]$ is a process with independent increments **w.r.t. the filtration** if for any $0 < s < T$ and $s_0 \leq \cdots \leq s_m \leq s \leq t_0 \leq \cdots \leq t_n \leq T$,

$$(X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}) \perp\!\!\!\perp (B_{s_1} - B_{s_0}, \cdots, B_{s_m} - B_{s_{m-1}}).$$

□

Remark 2.7 i) Let $\xi \in L_G^1(\Omega_T)$. If there exists $s \in [0, T]$ such that for any $s_0 \leq \cdots \leq s_m \leq s$, $\xi \perp\!\!\!\perp (B_{s_1} - B_{s_0}, \cdots, B_{s_m} - B_{s_{m-1}})$, then we have $\hat{E}_s(\xi) = \hat{E}(\xi)$. In fact, there is no loss of generality, we assume $\hat{E}(\xi) = 1$ and $C \geq \xi \geq \varepsilon$ for some $C, \varepsilon > 0$. Set $\eta = \hat{E}_s(\xi)$. For any $n \in N$, we have

$$\hat{E}(\eta^{n+1}) = \hat{E}(\eta^n \xi).$$

Since $\xi \perp\!\!\!\perp \eta^n$, we have

$$\hat{E}(\eta^{n+1}) = \hat{E}(\eta^n) = \cdots = \hat{E}(\eta) = 1.$$

By this, we have

$$\eta \leq 1, \quad q.s..$$

On the other hand, we have

$$\hat{E}[(\eta - 1)^2] = \hat{E}[\eta(\eta - 2)] + 1 = \hat{E}[\eta(\xi - 2)] + 1.$$

Since $\xi - 2 \perp\!\!\!\perp \eta$, we have

$$\hat{E}[(1 - \eta)^2] = \hat{E}(1 - \eta).$$

By Theorem 2.12 below, there exists $P \in \mathcal{P}$ such that

$$E_P[(1-\eta)^2] = \hat{E}[(1-\eta)^2].$$

Noting that

$$E_P(1-\eta) \leq \hat{E}(1-\eta) = \hat{E}[(1-\eta)^2] = E_P[(1-\eta)^2] \leq E_P(1-\eta),$$

we have

$$E_P[(1-\eta)^2] = E_P(1-\eta).$$

By this, we have

$$\eta^2 = \eta, \ P - a.s..$$

Since $\eta \geq \varepsilon$, we have $\eta = 1$, P-a.s. So we have

$$\hat{E}[(1-\eta)^2] = E_P[(1-\eta)^2] = 0.$$

ii) Let $\{X_t\}$ on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ be a process with stationary and independent increments and let $c = \hat{E}(X_T)/T$. If $\hat{E}(X_t) \to 0$ as $t \downarrow 0$, then for any $0 \leq s < t \leq T$, we have $\hat{E}(X_t - X_s) = c(t-s)$.□

Definition 2.8 Let $\{X_t\}$ be a d-dimensional process defined on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ such that

(i) $X_0 = 0$;

(ii) $\{X_t\}$ is a process with stationary and independent increments w.r.t. the filtration;

(iii) $\lim_{t\to 0} \hat{E}[|X_t|^3]t^{-1} = 0$.

Then $\{X_t\}$ is called a generalized G-Brownian motion.

If in addition $\hat{E}(X_t) = \hat{E}(-X_t) = 0$ for all $t \in [0,T]$, $\{X_t\}$ is called a (symmetric) G-Brownian motion. □

Remark 2.9 i) Clearly, the coordinate process $\{B_t\}$ is a (symmetric) G-Brownian motion and its quadratic variation process $\{\langle B\rangle_t\}$ is a process with stationary and independent increments (w.r.t. the filtration).

ii) [P07b] gave a characterization for the generalized G-Brownian motion: Let $\{X_t\}$ be a generalized G-Brownian motion. Then

$$X_{t+s} - X_t \sim \sqrt{s}\xi + s\eta \text{ for } t, s \geq 0, \tag{2.0.2}$$

where (ξ, η) is G-distributed(see, e.g., [P08b] for the definition of G-distributed random vectors). In fact, this characterization presented a decomposition of generalized G-Brownian motion in the sense of distribution. In this article, we shall give a pathwise decomposition for the generalized G-Brownian motion. □

Let $H_G^0(0,T)$ be the collection of processes of the following form: for a given partition $\{t_0,\cdots,t_N\}=\pi_T$ of $[0,T]$, $N\geq 1$,

$$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)1_{]t_j,t_{j+1}]}(t),$$

where $\xi_i\in L_{ip}(\Omega_{t_i})$, $i=0,1,2,\cdots,N-1$. For every $\eta\in H_G^0(0,T)$, let $\|\eta\|_{H_G^p}=\{\hat{E}(\int_0^T|\eta_s|^2ds)^{p/2}\}^{1/p}$, $\|\eta\|_{M_G^p}=\{\hat{E}(\int_0^T|\eta_s|^pds)\}^{1/p}$ and denote by $H_G^p(0,T)$, $M_G^p(0,T)$ the completions of $H_G^0(0,T)$ under the norms $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Definition 2.10 For every $\eta\in H_G^0(0,T)$ with the form

$$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)1_{]t_j,t_{j+1}]}(t),$$

we define

$$I(\eta)=\int_0^T\eta(s)dB_s:=\sum_{j=0}^{N-1}\xi_j(B_{t_{j+1}}-B_{t_j}).$$

By B-D-G inequality(see Proposition 4.3 in [Song11a] for this inequality under G-expectation), the mapping $I:H_G^0(0,T)\to L_G^p(\Omega_T)$ is continuous under $\|\cdot\|_{H_G^p}$ and thus can be continuously extended to $H_G^p(0,T)$. □

Definition 2.11 i) A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G-martingale if $\hat{E}_s(M_t)=M_s$ for any $s\leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both G-martingales, we call $\{M_t\}$ a symmetric G-martingale.

ii) A random variable $\xi\in L_G^1(\Omega_T)$ is called symmetric if $\hat{E}(\xi)+\hat{E}(-\xi)=0$. □

A G-martingale $\{M_t\}$ is symmetric if and only if M_T is symmetric.

Theorem 2.12([DHP11]) There exists a tight subset $\mathcal{P}\subset\mathcal{M}_1(\Omega_T)$ such that

$$\hat{E}(\xi)=\max_{P\in\mathcal{P}}E_P(\xi)\ \text{ for all }\xi\in\mathcal{H}_T^0.$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Remark 2.13 i) Let $(\Omega^0,\mathcal{F}^0,P^0)$ be a probability space and $\{W_t\}$ be a d-dimensional Brownian motion under P^0. Let $F^0=\{\mathcal{F}_t^0\}$ be the augmented filtration generated by W. [DHP11] proved that

$$\mathcal{P}_M:=\{P_h|P_h=P^0\circ X^{-1},X_t=\int_0^t h_sdW_s,h\in L_{F^0}^2([0,T];\Gamma^{1/2})\}$$

is a set that represents $\hat{E}$, where $\Gamma^{1/2}:=\{\gamma^{1/2}|\gamma\in\Gamma\}$ and Γ is the set in the representation of $G(\cdot)$ in the formula (2.0.1).

ii) For the 1-dimensional case, i.e., $\Omega_T = C_0([0,T],R)$,

$L^2_{F^0} := L^2_{F^0}([0,T];\Gamma^{1/2}) = \{h|\ h$ is adapted w.r.t. F^0 and $\underline{\sigma} \le h_s \le \overline{\sigma}\}$,

where $\overline{\sigma}^2 = \hat{E}(B_1^2)$ and $\underline{\sigma}^2 = -\hat{E}(-B_1^2)$.

$G(a) = 1/2\hat{E}[aB_1^2] = 1/2[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-]$ for $a \in R$.

iii) Set $c(A) = \sup_{P\in\mathcal{P}_M} P(A)$, for $A \in \mathcal{B}(\Omega_T)$. We say $A \in \mathcal{B}(\Omega_T)$ is a polar set if $c(A) = 0$. If an event happens except on a polar set, we say the event happens q.s..

3 Characterization of processes with stationary and independent increments

In what follows, we only consider the G-expectation space $(\Omega_T, L^1_G(\Omega_T), \hat{E})$ with $\Omega_T = C_0([0,T],R)$ and $\overline{\sigma}^2 = \hat{E}(B_1^2) > -\hat{E}(-B_1^2) = \underline{\sigma}^2 > 0$.

Lemma 3.1 For $\zeta \in M^1_G(0,T)$ and $\varepsilon > 0$, let

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon}\int_{(t-\varepsilon)^+}^t \zeta_s ds$$

and

$$\zeta_t^{\varepsilon,0} = \sum_{k=1}^{k_\varepsilon - 1} \frac{1}{\varepsilon}\int_{(k-1)\varepsilon}^{k\varepsilon} \zeta_s ds 1_{]k\varepsilon,(k+1)\varepsilon]}(t),$$

where $t \in [0,T]$, $k_\varepsilon\varepsilon \le T < (k_\varepsilon + 1)\varepsilon$. Then as $\varepsilon \to 0$

$$\|\zeta^\varepsilon - \zeta\|_{M^1_G(0,T)} \to 0 \ \text{ and } \ \|\zeta^{\varepsilon,0} - \zeta\|_{M^1_G(0,T)} \to 0.$$

Proof. The proofs of the two cases are similar. Here we only prove the second case. Our proof starts with the observation that for any $\zeta, \zeta' \in M^1_G(0,T)$

$$\|\zeta^{\varepsilon,0} - \zeta'^{\varepsilon,0}\|_{M^1_G(0,T)} \le \|\zeta - \zeta'\|_{M^1_G(0,T)}. \tag{3.0.3}$$

By the definition of the space $M^1_G(0,T)$, we know that for every $\zeta \in M^1_G(0,T)$, there exists a sequence of processes $\{\zeta^n\}$ with

$$\zeta_t^n = \Sigma_{k=0}^{m_n-1} \xi^n_{t_k^n} 1_{]t_k^n, t_{k+1}^n]}(t)$$

and $\xi^n_{t_k^n} \in L_{ip}(\Omega_{t_k^n})$ such that

$$\|\zeta - \zeta^n\|_{M^1_G(0,T)} \to 0 \text{ as } n \to \infty. \tag{3.0.4}$$

It is easily seen that for every n

$$\|\zeta^{n;\varepsilon,0} - \zeta^n\|_{M_G^1(0,T)} \to 0 \text{ as } \varepsilon \to 0. \tag{3.0.5}$$

Thus we get

$$\begin{aligned}
& \|\zeta^{\varepsilon,0} - \zeta\|_{M_G^1(0,T)} \\
\leq\ & \|\zeta^{\varepsilon,0} - \zeta^{n;\varepsilon,0}\|_{M_G^1(0,T)} + \|\zeta^n - \zeta^{n;\varepsilon,0}\|_{M_G^1(0,T)} + \|\zeta^n - \zeta\|_{M_G^1(0,T)} \\
\leq\ & 2\|\zeta^n - \zeta\|_{M_G^1(0,T)} + \|\zeta^n - \zeta^{n;\varepsilon,0}\|_{M_G^1(0,T)}.
\end{aligned}$$

The second inequality follows from (3.0.3). Combining (3.0.4) and (3.0.5), first letting $\varepsilon \to 0$, then letting $n \to \infty$, we have

$$\|\zeta^{\varepsilon,0} - \zeta\|_{M_G^1(0,T)} \to 0 \text{ as } \varepsilon \to 0.$$

□

Theorem 3.2 Let $A_t = \int_0^t h_s ds$ with $h \in M_G^1(0,T)$ be a process with stationary and independent increments (w.r.t. the filtration). Then we have $h \equiv c$ for some constant c.

Proof. Let $\overline{c} := \hat{E}(A_T)/T \geq -\hat{E}(-A_T)/T =: \underline{c}$. For $n \in N$, set $\varepsilon = T/(2n)$, and define $h^{T/(2n),0}$ as in Lemma 3.1. Then we have

$$\begin{aligned}
& \|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
=\ & \hat{E}[\sum_{k=0}^{2n-1} \int_{kT/(2n)}^{(k+1)T/(2n)} |h_s - h_s^{T/(2n),0}| ds] \\
\geq\ & \hat{E}[\sum_{k=1}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} (h_s - h_s^{T/(2n),0}) ds] \\
=\ & \hat{E}[\sum_{k=1}^{n-1} (\int_{2kT/(2n)}^{(2k+1)T/(2n)} h_s ds - \int_{(2k-1)T/(2n)}^{2kT/(2n)} h_s ds)] \\
=\ & \hat{E}\sum_{k=1}^{n-1} [(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)T/2n})].
\end{aligned}$$

Consequently, from the condition of independence of the increments and their stationarity, we have

$$\begin{aligned}
& \|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
\geq\ & \sum_{k=1}^{n-1} \hat{E}[(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)T/2n})] \\
=\ & \sum_{k=1}^{n-1} (\overline{c} - \underline{c}) T/(2n) \\
=\ & (\overline{c} - \underline{c})(n-1)T/(2n).
\end{aligned}$$

So by Lemma 3.1, letting $n \to \infty$, we have $\overline{c} = \underline{c}$. Furthermore, we note that $M_t := A_t - \overline{c}t$ is a G-martingale. In fact, for $t > s$, we see

$$\begin{aligned} & \hat{E}_s(M_t) \\ = \; & \hat{E}_s(M_t - M_s) + M_s \\ = \; & \hat{E}(M_t - M_s) + M_s \\ = \; & M_s. \end{aligned}$$

The second equality is due to the independence of increments of M w.r.t. the filtration.

So $\{M_t\}$ is a symmetric G-martingale with finite variation, from which we conclude that $M_t \equiv 0$, hence that $A_t = \overline{c}t$.□

Corollary 3.3 Assume $\overline{\sigma} > \underline{\sigma} > 0$. Then we have that $\{\frac{d}{ds}\langle B\rangle_s\} \notin M_G^1(0,T)$.

Proof. The proof is straightforward from Theorem 3.2. □

Corollary 3.4 There is no symmetric G-martingale $\{M_t\}$ which is a standard Brownian motion under G-expectation(i.e. $\langle M\rangle_t = t$).

Proof. Let $\{M_t\}$ be a symmetric G-martingale. If $\{M_t\}$ is also a standard Brownian motion, by Theorem 4.8 in [Song11a] or Corollary 5.2 in [Song11b], there exists $\{h_s\} \in M_G^2(0,T)$ such that

$$M_t = \int_0^t h_s dB_s$$

and

$$\int_0^t h_s^2 d\langle B\rangle_s = t.$$

Thus we have $\frac{d}{ds}\langle B\rangle_s = h_s^{-2} \in M_G^1(0,T)$, which contradicts the conclusion of Corollary 3.3. □

Proposition 3.5 Let $A_t = \int_0^t h_s ds$ with $h \in M_G^1(0,T)$ be a process with independent increments. Then A_t is symmetric for every $t \in [0,T]$.

Proof. By arguments similar to those in the proof of Theorem 3.2, we have

$$\begin{aligned} & \|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\ \geq \; & \hat{E}\sum_{k=0}^{n-1}[(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)^+T/2n})] \\ = \; & \sum_{k=0}^{n-1}\{\hat{E}(A_{(2k+1)T/2n} - A_{2kT/2n}) + \hat{E}[-(A_{2kT/2n} - A_{(2k-1)^+T/2n})]\}. \end{aligned}$$

The right side of the first inequality is only the sum of the odd terms. Summing up the even terms only, we have

$$\begin{aligned}
&\|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
\geq\ & \sum_{k=0}^{n-1}\{\hat{E}(A_{(2k+2)T/2n} - A_{(2k+1)T/2n}) + \hat{E}[-(A_{(2k+1)T/2n} - A_{2kT/2n})]\}.
\end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned}
&2\|h - h^{T/(2n),0}\|_{M_G^1(0,T)} \\
\geq\ & \sum_{k=0}^{2n-1}\{\hat{E}(A_{(k+1)T/2n} - A_{(kT/2n)}) + \hat{E}[-(A_{(k+1)T/2n} - A_{kT/2n})]\} \\
\geq\ & \hat{E}\sum_{k=0}^{2n-1}(A_{(k+1)T/2n} - A_{(kT/2n)}) + \hat{E}\sum_{k=0}^{2n-1}[-(A_{(k+1)T/2n} - A_{kT/2n})] \\
=\ & \hat{E}(A_T) + \hat{E}(-A_T).
\end{aligned}$$

Thus by Lemma 3.1, letting $n \to \infty$, we have $\hat{E}(A_T) + \hat{E}(-A_T) = 0$, which means that A_T is symmetric. □

For $n \in N$, define $\delta_n(s)$ in the following way:

$$\delta_n(s) = \sum_{i=0}^{n-1}(-1)^i 1_{]\frac{iT}{n},\frac{(i+1)T}{n}]}(s), \text{ for all } s \in [0,T].$$

In [Song12] we proved that $\lim_{n\to\infty}\hat{E}(\int_0^T \delta_n(s)h_s ds) = 0$ for $h \in M_G^1(0,T)$.

Let $\mathcal{F}_t = \sigma\{B_s | s \leq t\}$ and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$.

In the following, we shall use some notations introduced in Remark 2.13.

For every $P \in \mathcal{P}_M$ and $t \in [0,T]$, set $\mathcal{A}_{t,P} := \{Q \in \mathcal{P}_M |\ Q_{|\mathcal{F}_t} = P_{|\mathcal{F}_t}\}$. Proposition 3.4 in [STZ11] gave the following result: For $t \in [0,T]$, assume $\xi \in L_G^1(\Omega_T)$ and $\eta \in L_G^1(\Omega_t)$. Then $\eta = \hat{E}_t(\xi)$ if and only if for every $P \in \mathcal{P}_M$

$$\eta = \operatorname{ess\,sup}^P_{Q\in\mathcal{A}_{t,P}} E_Q(\xi|\mathcal{F}_t),\ P\text{-}a.s.,$$

where $\operatorname{ess\,sup}^P$ denotes the essential supremum under P.

Theorem 3.6 Let $A_t = \int_0^t h_s d\langle B\rangle_s$ be a process with stationary, independent increments (w.r.t. the filtration) and $h \in M_G^{1,+}(0,T)$. If $A_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$, we have $A_t = c\langle B\rangle_t$ for some constant $c \geq 0$.

Proof. For the readability, we divide the proof into several steps:

Step 1. Set $K_t := \int_0^t h_s ds$. We claim that K_T is symmetric.

Step 1.1. Let $\overline{\mu} = \hat{E}(A_T)/T$ and $\underline{\mu} = -\hat{E}(-A_T)/T$. First, we shall prove that $\frac{\overline{\mu}}{\overline{\sigma}^2} = \frac{\underline{\mu}}{\underline{\sigma}^2}$.

Actually, for any $0 \le s < t \le T$, we have

$$\hat{E}_s(\int_s^t h_r dr) = \hat{E}_s(\int_s^t \theta_r^{-1} dA_r) \ge \frac{1}{\overline{\sigma}^2}\hat{E}_s(\int_s^t dA_r) = \frac{\overline{\mu}}{\overline{\sigma}^2}(t-s) \quad q.s.,$$

where the inequality holds due to $\theta_s := \frac{d\langle B\rangle_s}{ds} \le \overline{\sigma}^2$, q.s.. Noting that $\underline{\mu}t - A_t$ is nonincreasing by Lemma 4.3 in Section 4 since it is a G-martingale with finite variation, we have, for every $\eta \in L^2_{F^0}$, P_η-a.s.,

$$\begin{aligned}
&\hat{E}_s(\int_s^t h_r dr) \\
= \;& \text{ess sup}^{P_\eta}_{Q\in\mathcal{A}_{t,P_\eta}} E_Q(\int_s^t h_r dr|\mathcal{F}_s) \\
= \;& \text{ess sup}^{P_\eta}_{Q\in\mathcal{A}_{t,P_\eta}} E_Q(\int_s^t \theta_r^{-1} dA_r|\mathcal{F}_s) \\
\ge \;& \underline{\mu}\,\text{ess sup}^{P_\eta}_{Q\in\mathcal{A}_{t,P_\eta}} E_Q(\int_s^t \theta_r^{-1} dr|\mathcal{F}_s) \\
= \;& \frac{\underline{\mu}}{\underline{\sigma}^2}(t-s).
\end{aligned}$$

So $\hat{E}_s(\int_s^t h_r dr) \ge \max\{\frac{\overline{\mu}}{\overline{\sigma}^2}, \frac{\underline{\mu}}{\underline{\sigma}^2}\}(t-s) =: \overline{\lambda}(t-s),\ q.s.$.

On the other hand,

$$\hat{E}_s(-\int_s^t h_r dr) = \hat{E}_s(\int_s^t -\theta_r^{-1} dA_r) \ge \frac{1}{\underline{\sigma}^2}\hat{E}_s(-\int_s^t dA_r) = -\frac{\underline{\mu}}{\underline{\sigma}^2}(t-s),\ q.s.$$

and for every $\eta \in L^2_{F^0}$, P_η-a.s.,

$$\begin{aligned}
&\hat{E}_s(-\int_s^t h_r dr) \\
= \;& \text{ess sup}^{P_\eta}_{Q\in\mathcal{A}_{t,P_\eta}} E_Q(-\int_s^t h_r dr|\mathcal{F}_s) \\
= \;& \text{ess sup}^{P_\eta}_{Q\in\mathcal{A}_{t,P_\eta}} E_Q(-\int_s^t \theta_r^{-1} dA_r|\mathcal{F}_s) \\
\ge \;& \overline{\mu}\,\text{ess sup}^{P_\eta}_{Q\in\mathcal{A}_{t,P_\eta}} E_Q(-\int_s^t \theta_r^{-1} dr|\mathcal{F}_s) \\
= \;& -\frac{\overline{\mu}}{\overline{\sigma}^2}(t-s)
\end{aligned}$$

since $A_t - \overline{\mu}t$ is nonincreasing. So

$$\hat{E}_s(-\int_s^t h_r dr) \geq -\min\{\frac{\overline{\mu}}{\overline{\sigma}^2}, \frac{\underline{\mu}}{\underline{\sigma}^2}\}(t-s) =: -\underline{\lambda}(t-s),\ q.s..$$

Noting that

$$\begin{aligned}
&\hat{E}(\int_0^T \delta_{2n}(s)h_s ds) \\
= \ & \hat{E}[\int_0^{\frac{(2n-1)T}{2n}} \delta_{2n}(s)h_s ds + \hat{E}_{\frac{(2n-1)T}{2n}}(-\int_{\frac{(2n-1)T}{2n}}^T h_s ds)] \\
\geq \ & (-\underline{\lambda})\frac{T}{2n} + \hat{E}[\int_0^{\frac{(2n-2)T}{2n}} \delta_{2n}(s)h_s ds + \hat{E}_{\frac{(2n-2)T}{2n}}(\int_{\frac{(2n-2)T}{2n}}^{\frac{(2n-1)T}{2n}} h_s ds)] \\
\geq \ & \frac{\overline{\lambda}-\underline{\lambda}}{2n}T + \hat{E}[\int_0^{\frac{(2n-2)T}{2n}} \delta_{2n}(s)h_s ds],
\end{aligned}$$

we have

$$\hat{E}(\int_0^T \delta_{2n}(s)h_s ds) \geq \frac{\overline{\lambda}-\underline{\lambda}}{2}T.$$

So

$$0 = \lim_{n\to\infty} \hat{E}(\int_0^T \delta_{2n}(s)h_s ds) \geq \frac{\overline{\lambda}-\underline{\lambda}}{2}T,$$

and $\frac{\overline{\mu}}{\overline{\sigma}^2} = \frac{\underline{\mu}}{\underline{\sigma}^2} =: \lambda$.

Step 1.2. For every $\eta \in L^2_{F^0}$, $E_{P_\eta}(K_T) = \lambda T$, which implies that K_T is symmetric.

Step 1.2.1. We now introduce some notations: For $0 \leq s < t \leq T$ and $\eta \in L^2_{F^0}$, set $\overline{\eta} = \overline{\sigma}$, $\underline{\eta} = \underline{\sigma}$, $\eta^* = \sqrt{\frac{\overline{\sigma}^2+\underline{\sigma}^2}{2}}$ on $]s,t]$ and $\overline{\eta} = \underline{\eta} = \eta^* = \eta$ on $]s,t]^c$. For $n \in N$, set $\eta^n_r = \sum_{i=0}^{n-1}(\underline{\sigma}1_{]t_{2i},t_{2i+1}]}(r) + \overline{\sigma}1_{]t_{2i+1},t_{2i+2}]}(r))$ on $]s,t]$ and $\eta^n = \eta$ on $]s,t]^c$, where $t_j = s + \frac{j}{2n}(t-s)$, $j = 0, \cdots, 2n$.

Step 1.2.2. $E_{P_{\eta^n}}(\int_s^t (h_r - \lambda)dr|\mathcal{F}_s) \to 0$, P_η-a.s., as $n \to \infty$.

Actually, we have, P_η-a.s.,

$$\overline{\mu}(t-s) = \hat{E}_s(\int_s^t h_r d\langle B\rangle_r) \geq E_{P_{\overline{\eta}}}(\int_s^t h_r d\langle B\rangle_r|\mathcal{F}_s) = \overline{\sigma}^2 E_{P_{\overline{\eta}}}(\int_s^t h_r dr|\mathcal{F}_s).$$

So

$$E_{P_{\overline{\eta}}}(\int_s^t h_r dr|\mathcal{F}_s) \leq \lambda(t-s),\ P_\eta - a.s.. \tag{3.0.6}$$

By similar arguments we have that

$$E_{P_{\underline{\eta}}}(\int_s^t h_r dr|\mathcal{F}_s) \geq \lambda(t-s),\ P_\eta - a.s.. \tag{3.0.7}$$

Let's compute the following conditional expectations:

$$\begin{aligned}
& E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)\delta_{2n}(r)dr|\mathcal{F}_s) \\
= \ & E^{\mathcal{F}_s}_{P_{\eta^n}}[\sum_{i=0}^{n-1}\{E^{\mathcal{F}_{t_{2i}}}_{P_{\eta^n}}\int_{t_{2i}}^{t_{2i+1}}(h_r-\lambda)dr + E^{\mathcal{F}_{t_{2i+1}}}_{P_{\eta^n}}\int_{t_{2i+1}}^{t_{2i+2}}(\lambda-h_r)dr\}] \\
= \ & : E^{\mathcal{F}_s}_{P_{\eta^n}}[\sum_{i=0}^{n-1}(A_i+B_i)],
\end{aligned}$$

where $\delta_{2n}(r) = \sum_{i=0}^{n-1}(1_{]t_{2i},t_{2i+1}]}(r) - 1_{]t_{2i+1},t_{2i+2}]}(r))$, $t_j = s + \frac{j}{2n}(t-s)$, $j = 0,\cdots,2n$;

$$E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)dr|\mathcal{F}_s) = E^{\mathcal{F}_s}_{P_{\eta^n}}[\sum_{i=0}^{n-1}(A_i-B_i)].$$

By (3.0.6) and (3.0.7)(noting that η and s,t are all arbitrary), we conclude that $A_i, B_i \geq 0$, P_{η_n}-a.s.. So

$$|E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)dr|\mathcal{F}_s)| \leq E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)\delta_{2n}(r)dr|\mathcal{F}_s),\ P_\eta - a.s..$$

Noting that

$$E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)\delta_{2n}(r)dr|\mathcal{F}_s) \leq \hat{E}_s[\int_s^t (h_r-\lambda)\delta_{2n}(r)dr],\ P_\eta - a.s.$$

and

$$\hat{E}_s[\int_s^t (h_r-\lambda)\delta_{2n}(r)dr] \to 0\ q.s.,\ as\ n\to\infty,$$

we have $E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)dr|\mathcal{F}_s) \to 0$, P_η-a.s., as $n\to\infty$.

Step 1.2.3. For any $\xi \in L^1_G(\Omega_t)$, $E_{P_{\eta^n}}(\xi|\mathcal{F}_s) \to E_{P_{\eta^*}}(\xi|\mathcal{F}_s)$, P_η-a.s., as $n\to\infty$.

In fact, for $\xi = \varphi(B_{s_1}-B_{s_0},\cdots,B_{s_m}-B_{s_{m-1}}) \in L_{ip}(\Omega_t)$, the conclusion is obvious. For general $\xi \in L^1_G(\Omega_t)$, there exists a sequence $\{\xi^m\} \subset L_{ip}(\Omega_t)$ such that $\hat{E}[|\xi^m-\xi|] = \hat{E}[\hat{E}_s(|\xi^m-\xi|)] \to 0$. So we can assume $\hat{E}_s(|\xi^m-\xi|) \to 0$ q.s..

Then, P_η-a.s., we have

$$\begin{aligned}
& |E_{P_{\eta^n}}(\xi|\mathcal{F}_s) - E_{P_{\eta^*}}(\xi|\mathcal{F}_s)| \\
\leq\ & |E_{P_{\eta^n}}(\xi|\mathcal{F}_s) - E_{P_{\eta^n}}(\xi^m|\mathcal{F}_s)| + |E_{P_{\eta^n}}(\xi^m|\mathcal{F}_s) - E_{P_{\eta^*}}(\xi^m|\mathcal{F}_s)| \\
& +|E_{P_{\eta^*}}(\xi^m|\mathcal{F}_s) - E_{P_{\eta^*}}(\xi|\mathcal{F}_s)| \\
\leq\ & 2\hat{E}_s(|\xi^m - \xi|) + |E_{P_{\eta^n}}(\xi^m|\mathcal{F}_s) - E_{P_{\eta^*}}(\xi^m|\mathcal{F}_s)|.
\end{aligned}$$

First letting $n \to \infty$, then letting $m \to \infty$, we have $E_{P_{\eta^n}}(\xi|\mathcal{F}_s) \to E_{P_{\eta^*}}(\xi|\mathcal{F}_s)$, P_η-a.s.. So combining Step 1.2.2 and Step 1.2.3, we have

$$E_{P_{\eta^*}}(\int_s^t h_r dr|\mathcal{F}_s) = \lambda(t-s)\ P_\eta - a.s.. \tag{3.0.8}$$

Step 1.2.4. For $0 \leq s < t \leq T$, $\eta \in L^2_{F^0}$ $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, set $\eta^\sigma = \sigma$ on $]s,t]$ and $\eta^\sigma = \eta$ on $]s,t]^c$. We have

$$E_{P_{\eta^\sigma}}(\int_s^t h_r dr|\mathcal{F}_s) = \lambda(t-s)\ P_\eta - a.s..$$

In fact, Step 1.2.2-Step 1.2.3 proved the following fact: If (3.0.6), (3.0.7) hold for some $\sigma, \sigma' \in [\underline{\sigma}, \overline{\sigma}]$, then (3.0.8) holds for $\sqrt{\frac{\sigma^2+\sigma'^2}{2}}$. So by repeating the Step 1.2.2-Step 1.2.3, we get the desired result.

Step 1.2.5. For any simple process $\eta \in L^2_{F^0}$, $E_{P_\eta}(K_T) = \lambda T$.

Let $\eta_r = \sum_{i=0}^{m-1} \eta_{t_i} 1_{]t_i,t_{i+1}]}(r) \in L^2_{F^0}$ with $\eta_{t_i} = \sum_{j=1}^{n_i} a^i_j 1_{A^i_j}$ an $\mathcal{F}^0_{t_i}$ measurable simple function, where $\{t_0, \cdots, t_m\}$ is a given partition of $[0,T]$. Set $X_t = \int_0^t \eta_r dW_r$. Let $F^X = \{\mathcal{F}^X_t\}$ be the filtration generated by X.

Fix $0 \leq i < m$. Set $\eta^{j,\varepsilon}_s = \eta_s 1_{[0,t_i+\varepsilon]}(s) + a^i_j 1_{]t_i+\varepsilon,T]}(s)$ and $X^{j,\varepsilon}_t = \int_0^t \eta^{j,\varepsilon}_s dW_s$ for $\varepsilon > 0$ small enough. Let $F^{X^{j,\varepsilon}} = \{\mathcal{F}^{X^{j,\varepsilon}}_t\}$ be the filtration generated by $X^{j,\varepsilon}$. Then

$$E_{P_\eta}(\int_{t_i+\varepsilon}^{t_{i+1}} h_r dr) = E_{P^0}(\int_{t_i+\varepsilon}^{t_{i+1}} h_r \circ X dr) = E_{P^0}[E_{P^0}(\int_{t_i+\varepsilon}^{t_{i+1}} h_r \circ X dr|\mathcal{F}^X_{t_i+\varepsilon})].$$

Since $A^i_j \in \mathcal{F}^X_{t_i+\varepsilon} = \mathcal{F}^{X^{j,\varepsilon}}_{t_i+\varepsilon}$ and $X_t = \sum_{j=0}^{n_i} X^{j,\varepsilon}_t 1_{A^i_j}$ on $[0, t_{i+1}]$, we have

$$\begin{aligned}
& E_{P^0}(\int_{t_i+\varepsilon}^{t_{i+1}} h_r \circ X dr|\mathcal{F}^X_{t_i+\varepsilon}) \\
=\ & \sum_{j=1}^{n_i} E_{P^0}(1_{A^i_j} \int_{t_i}^{t_{i+1}} h_r \circ X^{j,\varepsilon} dr|\mathcal{F}^X_{t_i+\varepsilon}) \\
=\ & \sum_{j=1}^{n_i} 1_{A^i_j} E_{P^0}(\int_{t_i}^{t_{i+1}} h_r \circ X^{j,\varepsilon} dr|\mathcal{F}^{X^{j,\varepsilon}}_{t_i+\varepsilon}).
\end{aligned}$$

Noting that

$$E_{P^0}(\int_{t_i+\varepsilon}^{t_{i+1}} h_r \circ X^{j,\varepsilon} dr | \mathcal{F}_{t_i+\varepsilon}^{X^{j,\varepsilon}}) = E_{P_{\eta^{j,\varepsilon}}}(\int_{t_i+\varepsilon}^{t_{i+1}} h_r dr | \mathcal{F}_{t_i+\varepsilon}) \circ X^{j,\varepsilon} = \lambda(t_{i+1} - t_i - \varepsilon)\ P^0 - a.s.,$$

by Step 1.2.4, we have $E_{P_\eta}(\int_{t_i}^{t_{i+1}} h_r dr) = \lambda(t_{i+1} - t_i)$ and $E_{P_\eta}(K_T) = \lambda T$.

Step 2. $h \equiv \lambda$.

Let $M_t = \int_0^t h_r d\langle B\rangle_s - \int_0^t 2G(h_s)ds$ and $N_t = \int_0^t h_s d\langle B\rangle_s - \overline{\mu}t$. As is mentioned in the introduction, [P07b] proved that $\{M_t\}$ is a G-martingale. Since $\{\int_0^t h_s d\langle B\rangle_s\}$ is a process with stationary and independent increments w.r.t. the filtration, We know that $\{N_t\}$ is also a G-martingale. Let $L_t = \hat{E}_t(\overline{\mu}T - \overline{\sigma}^2 K_T)$. Then $\{L_t\}$ is a symmetric G-martingale since K_T is symmetric. By the symmetry of $\{L_t\}$ we have

$$M_t = \hat{E}_t(M_T) = \hat{E}_t(L_T + N_T) = L_t + N_t.$$

By the uniqueness of the G-martingale decomposition, we get $L \equiv 0$ and $h \equiv \lambda$. □

Remark 3.7 Clearly, $h \in M_G^\beta(0,T)$ for some $\beta > 1$ implies $A_T = \int_0^T h_s d\langle B\rangle \in L_G^\beta(\Omega_T)$.

4 Characterization of the G-Brownian motion

A version of the martingale characterization for the G-Brownian motion was given in [XZ09], where only symmetric G-martingales with Markovian property were considered. Here we shall present a martingale characterization in a quite different form, which is a natural but nontrivial generalization of the classical case in a probability space.

Theorem 4.1(Martingale characterization of the G-Brownian motion)

Let $\{M_t\}$ be a symmetric G-martingale with $M_T \in L_G^\alpha(\Omega_T)$ for some $\alpha > 2$ and $\{\langle M\rangle_t\}$ a process with stationary and independent increments (w.r.t. the filtration). Then $\{M_t\}$ is a G-Brownian motion;

Let $\{M_t\}$ be a G-Brownian motion on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$. Then there exists a positive constant c such that $\langle M\rangle_t = c\langle B\rangle_t$.

Proof. By Corollary 5.2 in [Song11b], there exists $h \in M_G^2(0,T)$ such that $M_t = \int_0^t h_s dB_s$. So $\langle M\rangle_t = \int_0^t h_s^2 d\langle B\rangle_s$. By the assumption, we know that $\langle M\rangle_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. By Theorem 3.6, there exists some constant $c \geq 0$ such that $h^2 \equiv c$. Thus by Theorem 2.12 and Remark 2.13, $\{M_t\}$ is a G-Brownian motion with M_t distributed as $N(0, [c\underline{\sigma}^2 t, c\overline{\sigma}^2 t])$.

On the other hand, if $\{M_t\}$ is a G-Brownian motion on $(\Omega_T, L_G^1(\Omega_T)$, then $\{M_t\}$ is a symmetric G-martingale. By the above arguments, we have $\langle M\rangle_t = c\langle B\rangle_t$ for some positive constant c. □

Let

$$\mathcal{H} = \{a|\ a(t) = \Sigma_{k=0}^{n-1} a_{t_k} 1_{]t_k,t_{k+1}]}(t), n \in N, 0 = t_0 < t_1 < \cdots < t_n = T\}$$

and $H = \{a \in \mathcal{H} | \lambda[a = 0] = 0\}$, where λ is the Lebesgue measure.

Lemma 4.2 Let $\{L_t\}$ be a process with absolutely continuous paths. Assume that there exist real numbers $\underline{c} \le \overline{c}$ such that $\underline{c}(t-s) \le L_t - L_s \le \overline{c}(t-s)$ for any $s < t$. Let $C(a) = \overline{c}a^+ - \underline{c}a^-$ for any $a \in R$. If

$$\hat{E}(\int_0^T a(s)dL_s) = \int_0^T C(a(s))ds, \text{ for all } a \in \mathcal{H},$$

we have that $\{L_t\}$ is a process with stationary and independent increments such that $\underline{c}t = -\hat{E}(-L_t) \le \hat{E}(L_t) = \overline{c}t$, i.e., its distribution is determined by $\underline{c}, \overline{c}$.

Proof. It suffices to prove the lemma for the case $\underline{c} < \overline{c}$. For any $a \in H$, let

$$\theta_s^a = \overline{c}1_{[a(s)\ge 0]} + \underline{c}1_{[a(s)<0]}.$$

By assumption,

$$\hat{E}(\int_0^T a(s)dL_s) = \int_0^T a(s)\theta_s^a ds.$$

On the other hand, by Theorem 2.12, there exists some weakly compact subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that

$$\hat{E}(\xi) = \max_{P\in\mathcal{P}} E_P(\xi), \text{ for all } \xi \in L_G^1(\Omega_T),$$

which means that there exists $P_a \in \mathcal{P}$ such that

$$E_{P_a}(\int_0^T a(s)dL_s) = \int_0^T a(s)\theta_s^a ds.$$

By the assumption for $\{L_t\}$, we have $P_a\{L_t = \int_0^t \theta_s^a ds, \text{ for all } t \in [0,T]\} = 1$. From this we have

$$\hat{E}[\varphi(L_{t_1} - L_{t_0}, \cdots, L_{t_n} - L_{t_{n-1}})] \ge \varphi(\int_{t_0}^{t_1} \theta_s^a ds, \cdots, \int_{t_{n-1}}^{t_n} \theta_s^a ds)$$

for any $\varphi \in C_b(R^n)$ and $n \in N$. Consequently,

$$\begin{aligned} & \hat{E}[\varphi(L_{t_1} - L_{t_0}, \cdots, L_{t_n} - L_{t_{n-1}})] \\ \ge\ & \sup_{a\in H} \varphi(\int_{t_0}^{t_1} \theta_s^a ds, \cdots, \int_{t_{n-1}}^{t_n} \theta_s^a ds) \\ =\ & \sup_{c_1,\cdots,c_n\in[\underline{c},\overline{c}]} \varphi(c_1(t_1 - t_0), \cdots, c_n(t_n - t_{n-1})). \end{aligned}$$

The converse inequality is obvious. Thus $\{L_t\}$ is a process with stationary and independent increments such that $\underline{c}t = -\hat{E}(-L_t) \leq \hat{E}(L_t) = \overline{c}t$. □

Lemma 4.3 Let $\{L_t\}$ be a G-martingale with finite variation and $L_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. Then $\{L_t\}$ is nonincreasing. Particularly, $L_t \leq L_0 = \hat{E}(L_T)$.

Proof. By Theorem 4.5 in [Song11a], we know $\{L_t\}$ has the following decomposition

$$L_t = \hat{E}(L_T) + M_t + K_t,$$

where $\{M_t\}$ is a symmetric G-martingale and $\{K_t\}$ is a nonpositive, nonincreasing G-martingale. Since both $\{L_t\}$ and $\{K_t\}$ are processes with finite variation, we get $M_t \equiv 0$. Therefore, we have $L_t = \hat{E}(L_T) + K_t \leq \hat{E}(L_T) = L_0$. □

Theorem 4.4 Let $\{X_t\}$ be a generalized G-Brownian motion with zero mean. Then we have the following decomposition:

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric G-Brownian motion, and $\{L_t\}$ is a nonpositive, nonincreasing G-martingale with stationary and independent increments.

Proof. Clearly $\{X_t\}$ is a G-martingale. By Theorem 4.5 in [Song11a], we have the following decomposition

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric G-martingale, and $\{L_t\}$ is a nonpositive, nonincreasing G-martingale. Noting that $X_t \in L_G^3(\Omega_T)$ from the definition of generalized G-Brownian motion, we know that $M_t, L_t \in L_G^\beta(\Omega_T)$ for any $1 \leq \beta < 3$ by Theorem 4.5 in [Song11a].

In the sequel, we first prove that $\{L_t\}$ is a process with stationary and independent increments. Noting that $\hat{E}(-L_t) = \hat{E}(-X_t) = ct$ for some positive constant c since $\{X_t\}$ is a process with stationary and independent increments, we claim that $-L_t - ct$ is a G-martingale. To prove this, it suffices to show that for any $t > s$, $\hat{E}_s[-(L_t - L_s)] = c(t-s)$. In fact, since $\{M_t\}$ is a symmetric G-martingale, we have

$$\hat{E}_s[-(L_t - L_s)] = \hat{E}_s[-(X_t - M_t - X_s + M_s)] = \hat{E}_s[-(X_t - X_s)].$$

Noting that $\{X_t\}$ is a process with independent increments(w.r.t. the filtration),

$$\hat{E}_s[-(X_t - X_s)] = \hat{E}[-(X_t - X_s)] = c(t-s).$$

Combining this with Lemma 4.3, we have $-(L_t - L_s) - c(t-s) \leq 0$ for any $s < t$. On the other hand, for any $a \in \mathcal{H}$, noting that $\{M_t\}$ is a symmetric G-martingale, we have

$$\hat{E}[\int_0^T a(s)dL_s] = \hat{E}[\int_0^T a(s)dX_s] = \hat{E}[\Sigma_{k=0}^{n-1} a_{t_k}(X_{t_{k+1}} - X_{t_k})].$$

Since $\{X_t\}$ is a process with stationary, independent increments, we have

$$\begin{aligned}
& \hat{E}[\int_0^T a(s)dL_s] \\
= \ & \Sigma_{k=0}^{n-1}\hat{E}[a_{t_k}(X_{t_{k+1}} - X_{t_k})] \\
= \ & \Sigma_{k=0}^{n-1} c a_{t_k}^{-}(t_{k+1} - t_k) \\
= \ & \int_0^T c a^{-}(s)ds = \int_0^T C(a(s))ds,
\end{aligned}$$

where $C(a(s))$ is defined as in Lemma 4.2 with $\overline{c} = 0, \underline{c} = -c$. By Lemma 4.2, $\{L_t\}$ is a process with stationary and independent increments.

Now we are in a position to show that $\{M_t\}$ is a (symmetric) G-Brownian motion. To this end, by Theorem 4.1, it suffices to prove that $\{\langle M\rangle_t\}$ is a process with stationary and independent increments (w.r.t. the filtration). For $n \in N$, let

$$X_t^n = \sum_{k=0}^{2^n-1} X_{\frac{kT}{2^n}} 1_{]\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]}(t)$$

and

$$\Omega_t^n(X) = \sum_{k=0}^{2^n-1} (X_{\frac{(k+1)t}{2^n}} - X_{\frac{kt}{2^n}})^2.$$

Observing that $\Omega_t^n(X) = X_t^2 - 2\int_0^t X_s^n dX_s$, we have

$$\begin{aligned}
& |\Omega_t^n(X) - \Omega_t^{m+n}(X)| \\
\leq \ & 2(|\int_0^t (X_s^n - X_s^{m+n})dM_s| + |\int_0^t (X_s^n - X_s^{m+n})dL_s|) \\
= \ & 2(|I| + |II|).
\end{aligned}$$

for any $n, m \in N$. It's easy to check that

$$\hat{E}(|II|) \leq c\int_0^t \hat{E}(|X_s^n - X_s^{m+n}|)ds \to 0 \text{ as } m, n \to \infty.$$

Noting that

$$\begin{aligned}
I \ &= \ \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^m-1} (X_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}} - X_{\frac{it}{2^n}})(M_{\frac{it}{2^n}+\frac{(j+1)t}{2^{n+m}}} - M_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}}) \\
&= \ \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^m-1} I_i^j,
\end{aligned}$$

we get

$$\hat{E}(I^2) \leq \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} \hat{E}[(I_i^j)^2].$$

Let's estimate the expectation $\hat{E}[(I_i^j)^2]$:

$$\begin{aligned}
& \hat{E}[(I_i^j)^2] \\
= \; & \hat{E}[(X_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}} - X_{\frac{it}{2^n}})^2 (M_{\frac{it}{2^n}+\frac{(j+1)t}{2^{n+m}}} - M_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}})^2] \\
\leq \; & 2\hat{E}[(X_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}} - X_{\frac{it}{2^n}})^2 \{(X_{\frac{it}{2^n}+\frac{(j+1)t}{2^{n+m}}} - X_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}})^2 + (L_{\frac{it}{2^n}+\frac{(j+1)t}{2^{n+m}}} - L_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}})^2\}]
\end{aligned}$$

Noting that $-c(t-s) \leq L_t - L_s \leq 0$, we have

$$\hat{E}[(I_i^j)^2] \leq \hat{E}[(X_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}} - X_{\frac{it}{2^n}})^2 \{(X_{\frac{it}{2^n}+\frac{(j+1)t}{2^{n+m}}} - X_{\frac{it}{2^n}+\frac{jt}{2^{n+m}}})^2 + c^2 \frac{t^2}{2^{2(n+m)}}\}].$$

By (2.0.2), $\hat{E}[(X_t - X_s)^2] \leq C_1|t-s|$ for some constant C_1. From the condition of independent increments of X, we have $\hat{E}[(I_i^j)^2] \leq C\frac{j}{2^{2(n+m)}}$ for some constant C, hence that $\hat{E}(I^2) \to 0$, and finally that $\hat{E}(|\Omega_t^n(X) - \Omega_t^{m+n}(X)|) \to 0$ as $m, n \to \infty$. Then

$$\langle X \rangle_t := \lim_{L_G^1(\Omega_T), n\to\infty} \Omega_t^n$$

is a process with stationary and independent increments (w.r.t. the filtration). Noting that $\langle M \rangle_t = \langle X \rangle_t$, $\langle M \rangle_t$ is also a process with stationary and independent increments (w.r.t. the filtration). □

5 G-martingales with finite variation

Proposition 5.1 Let $\eta \in M_G^1(0,T)$ with $|\eta| \equiv c$ for some constant c. Then

$$K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds \tag{5.0.9}$$

is a process with stationary and independent increments. Moreover, for fixed c, all processes in the above form have the same distribution.

Proof. Since $-c(\overline{\sigma}^2 - \underline{\sigma}^2)(t-s) \leq K_t - K_s \leq 0$ for any $s < t$, by Lemma 4.2, it suffices to prove that for any $a \in \mathcal{H}$

$$\hat{E}(\int_0^T a_s dK_s) = \int_0^T C(a_s)ds,$$

where $C(a_s)$ is defined as in Lemma 4.2 with $\overline{c} = 0, \underline{c} = -c(\overline{\sigma}^2 - \underline{\sigma}^2)$. In fact, noting that

$$\int_0^T a_s dK_s \leq \int_0^T 2G(a_s\eta_s)ds - \int_0^T 2a_s G(\eta_s)ds = \int_0^T C(a_s)ds,$$

we have

$$\hat{E}(\int_0^T a_s dK_s) \leq \int_0^T C(a_s)ds.$$

On the other hand, we have

$$\hat{E}(\int_0^T a_s dK_s) \geq -\hat{E}\{-[\int_0^T 2G(a_s\eta_s)ds - \int_0^T 2a_s G(\eta_s)ds]\} = \int_0^T C(a_s)ds.$$

So $\{K_t\}$ is a process with stationary and independent increments and its distribution is determined by c. □

Just like the conjecture by Shige Peng for the representation of G-martingales with finite variation, we guess that any G-martingale with stationary, independent increments and finite variation should have the form of (5.0.9). At the end we present a characterization for G-martingales with finite variation.

Proposition 5.2 Let $\{M_t\}$ be a G-martingale with $M_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. Then $\{M_t\}$ is a G-martingale with finite variation if and only if $\{f(M_t)\}$ is a G-martingale for any nondecreasing $f \in C_{b,Lip}(R)$.

Proof. Necessity. Assume $\{M_t\}$ is a G-martingale with finite variation. By Lemma 4.3, we know that $\{M_t\}$ is nonincreasing. By Theorem 5.4 in [Song11b], there exists a sequence $\{\eta_t^n\} \subset H_G^0(0,T)$ such that

$$\hat{E}[\sup_{t\in[0,T]} |M_t - L_t(\eta^n)|^\beta] \to 0$$

as n goes to infinity, where $L_t(\eta^n) = \int_0^t \eta_s^n d\langle B\rangle_s - \int_0^t 2G(\eta_s^n)ds$. It suffices to prove that for any $\eta \in H_G^0(0,T)$ and nondecreasing $f \in C_b^2(R)$, $f(L_t(\eta))$ is a G-martingale. In fact,

$$\begin{aligned} f(L_t(\eta)) &= f(L_0) + \int_0^t f'(L_s(\eta))dL_s(\eta) \\ &= f(L_0) + \int_0^t f'(L_s(\eta))\eta_s d\langle B\rangle_s - \int_0^t 2f'(L_s(\eta))G(\eta_s)ds. \end{aligned}$$

Since $f'(L_s(\eta)) \geq 0$ and $f'(L_s(\eta))\eta_s \in M_G^1(0,T)$, we conclude that

$$f(L_t(\eta)) = f(L_0) + L_t(f'(L(\eta))\eta)$$

is a G-martingale.

Sufficiency. Assume $\{f(M_t)\}$ is a G-martingale for any nondecreasing $f \in C_{b,Lip}(R)$. Let $X_t := \arctan M_t$. Then $\{X_t\}$ is a bounded G-martingale and $\{f(X_t)\}$ is a G-martingale for any nondecreasing $f \in C_{b,Lip}(R)$. By Theorem 4.5 in [Song11a], we know $\{X_t\}$ has the following decomposition

$$X_t = \hat{E}(X_T) + N_t + K_t,$$

where $\{N_t\}$ is a symmetric G-martingale and $\{K_t\}$ is a nonpositive, nonincreasing G-martingale. Then by Itô's formula

$$e^{\alpha X_t} = e^{\alpha X_0} + \alpha \int_0^t e^{\alpha X_s} dX_s + \frac{\alpha^2}{2} \int_0^t e^{\alpha X_s} d\langle N \rangle_s.$$

For any $\alpha > 0$, by assumption, $e^{\alpha X_t}$ is a G-martingale. So $L_t := \int_0^t e^{\alpha X_s} dK_s + \frac{\alpha}{2} \int_0^t e^{\alpha X_s} d\langle N \rangle_s$ is a G-martingale with finite variation. By Lemma 4.3, L_t is nonincreasing, by which we conclude that $K_t + \frac{\alpha}{2}\langle N \rangle_t$ is nonincreasing. So

$$\frac{\alpha}{2}\hat{E}(\langle N \rangle_T) \leq \hat{E}(-K_T) \text{ for all } \alpha > 0.$$

By this, we conclude that $\hat{E}(\langle N \rangle_T) = 0$ and $N_t \equiv 0$. Then $X_t = \hat{E}(X_T) + K_t$ is nonincreasing, and consequently, M_t is nonincreasing. □

Particularly, Proposition 5.2 provides a method to convert G-martingales with finite variation into bounded G-martingales with finite variation.

Acknowledgement

I am very grateful to the conscientious anonymous referee for his very serious and very valuable comments.

References

[DHP11] Denis, L., Hu, M. and Peng S. *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes*. Potential Anal 34:139-161, 2011.

[HP09] Hu, M. and Peng, S. *On representation theorem of G-expectations and paths of G-Brownian motion*. Acta Math. Appl. Sin. Engl. Ser. 25, no. 3, 539-546, 2009.

[P07a] Peng, S. *G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type*, Stochastic analysis and applications, 541-567, Abel Symp., 2, Springer, Berlin, 2007.

[P07b] Peng, S. *G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty*, arXiv:0711.2834v1 [math.PR] 19 Nov 2007.

[P08a] Peng, S. *Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation*, Stochastic Processes and their Applications, 118(12), 2223-2253, 2008.

[P08b] Peng, S. *A New Central Limit Theorem under Sublinear Expectations*, arXiv:0803.2656v1 [math.PR] 18 Mar 2008.

[P09] Peng, S. *Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations*, Science in China Series A: Mathematics, 52, No.7, 1391-1411, 2009.

[P10] Peng, S. *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, arXiv:1002.4546v1 [math.PR] 24 Feb 2010.

[STZ11] Soner, M., Touzi, N., Zhang, J. *Martingale Representation Theorem under G-expectation*, Stochastic Processes and their Applications 121, 265-287, 2011.

[Song11a] Song, Y. *Some properties on G-evaluation and its applica- tions to G-martingale decomposition,* Science China Mathematics, Volume 54, Number 2, 287-300, 2011.

[Song11b] Song, Y. *Properties of hitting times for G-martingales and their applicaions,* Stochastic Processes and their Applications 121, 1770-1784, 2011.

[Song12] Song, Y. *Uniqueness of the representation for G-martingales with finite variation,* Electron. J. Probab. 17 , no. 24, 1-15, 2012.

[XZ09] Xu J, Zhang B. *Martingale characterization of G-Brownian motion.* Stochastic Processes Appl., 119(1): 232-248, 2009.