Optimal transportation for multifractal random measures and applications.

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Abstract

In this paper, we study optimal transportation problems for multifractal random measures. Since these measures are much less regular than optimal transportation theory requires, we introduce a new notion of transportation which is intuitively some kind of multistep transportation. Applications are given for construction of multifractal random changes of times and to the existence of random metrics, the volume forms of which coincide with the multifractal random measures.

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1 Introduction

On the Borelian subsets of $\mathbb{R}^m$, consider a measure M formally defined by

$$M(A) = \int_A e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\, dx, \tag{1.1}$$

where $\gamma > 0$ is a parameter, $(X(x))_{x\in\mathbb{R}^m}$ is a Gaussian distribution with covariance function given by

$$K(x,y) = \mathrm{Cov}(\gamma X(x), \gamma X(y)) = \gamma^2 \ln_+ \frac{T}{|y-x|} + g(x,y), \tag{1.2}$$

and g is a continuous bounded function. Actually, whatever the function g is, the part that really matters in (1.2) is the logarithmic part. Such measures are called Gaussian multiplicative chaos associated to K and were first rigorously defined in [5].

The above situation can be generalized to the situation

$$M(A) = \int_A e^{\omega(x)}\, dx \tag{1.3}$$

where the process ω is a suitable Levy distribution: the resulting measures are called *log-infinitely divisible multifractal random measures*, MRM for short (see subsection 2.1 for a reminder of the construction). Such measures exhibit interesting properties like stationarity, isotropy, long-range dependence, fat tail distributions and, because of their log part in (1.2) (or a suitable generalization for Lévy distributions), possess a remarkable scaling property, the so-called **stochastic scale invariance**:

$$(M(\lambda A))_{A\subset B(0,T)} \overset{\text{law as } \lambda\to 0}{\simeq} (\lambda^m e^{\Omega_\lambda} M(A))_{A\subset B(0,T)}, \tag{1.4}$$

where Ω_λ is an infinitely divisible random variable, independent of $(M(A))_{A\subset B(0,T)}$. When M satisfies the above relation (1.4) with $=$ instead of $\simeq$, we will say that M satisfies the exact stochastic scale invariance property (or M is ESSI for short).

The purpose of this paper is to investigate (optimal) transportation problems associated to these measures. A transport map between two probability measures μ, ν is a map that pushes μ forward to ν. The transport map is said to be optimal if it realizes the infimum of a cost functional among all the possible transport maps. For usual cost functionals, existence and uniqueness of an optimal transport map are strongly connected to the regularity of the measures μ, ν. Concerning MRM, their regularity is much weaker than that required in optimal transportation theory. So we give new notions of transportation that can be applied to MRM. Though our result presents an intrinsic interest because we construct non trivial transport maps between measures that are much less regular than those usually involved in optimal transportation theory, this study is originally motivated by the construction of multifractal random changes of time and the construction of metric spaces the volume form of which is given by the MRM, as explained in subsection 3. The latter construction allows to construct random metric spaces exhibiting nice scaling properties (see subsection 3).

2 Background and main results

In this section, we first give the basic background in order to state rigorously the main results of the paper. Since the function g in (1.2) does not play a part in what follows, we focus on the case where the measure M satifies the exact scale invariance property.

2.1 Reminder of the construction of ESSI MRM

We present below the generalization of (1.1) to the situation where X is a Lévy distribution. For further details, the reader is referred to [9]. To characterize such a Lévy distribution, we consider the characteristic function of an infinitely divisible random variable Z, which can be written as $\mathbb{E}[e^{iqZ}] = e^{\varphi(q)}$ where (Lévy-Khintchine's formula)

$$\varphi(q) = ibq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}^*} (e^{iqx} - 1 - iq\sin(x))\,\nu(dx)$$

and $\nu(dx)$ is a so-called Lévy measure satisfying $\int_{\mathbb{R}^*} \min(1, x^2)\,\nu(dx) < +\infty$. We also introduce the Laplace exponent ψ of Z by $\psi(q) = \varphi(-iq)$ for each q such that both terms of the equality make sense, and we assume that $\psi(1) = 0$ (renormalization condition), $\psi(2) < +\infty$ and $\int_{[-1,1]} |x|\,\nu(dx) < +\infty$ (sufficient conditions for existence of MRM).

Now we define the process ω of (1.3). We remind that this process has to be stationary, isotropic and suitably scale invariant. Such properties come from the combination of several ingredients that we recall now. We introduce the unitary group G of $\mathbb{R}^m$, that is

$$G = \{M \in M_m(\mathbb{R}); MM^t = \mathrm{I}\},$$

and H its unique right translation invariant Haar measure with mass 1 defined on the Borel σ-algebra $\mathcal{B}(G)$. We also introduce the so-called space-scale half-space, that is

$$S = \{(t, y); t \in \mathbb{R}, y \in \mathbb{R}_+^*\},$$

with which we associate the measure (on the Borel σ-algebra $\mathcal{B}(S)$)

$$\theta(dt, dy) = y^{-2} dt\, dy.$$

Then we consider an independently scattered infinitely divisible random measure μ associated to $(\varphi, H \otimes \theta)$ and distributed on $G \times S$. It is worth recalling that such a measure satisfies the following properties:

- for every sequence of disjoint sets $(A_n)_n$ of $G \times S$, $(\mu(A_n))_n$ is a sequence of independent random variables and

$$\mu\Big(\bigcup_n A_n\Big) = \sum_n \mu(A_n), \quad \text{almost surely.}$$

- for any $H \otimes \theta$-measurable set A, $\mu(A)$ is an infinitely divisible random variable whose characteristic function is

$$\mathbb{E}(e^{iq\mu(A)}) = e^{\varphi(q) H \otimes \theta(A)}.$$

Given T, let us define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(l) = \begin{cases} l, & \text{if } l \leq T \\ T & \text{if } l \geq T \end{cases}$$

The cone-like subset $A_l(t)$ of S is defined by

$$A_l(t) = \{(s, y) \in S; y \geq l, -f(y)/2 \leq s - t \leq f(y)/2\}.$$

For any $x \in \mathbb{R}^m$ and $g \in G$, we denote by x_1^g the first coordinate of the vector gx. The cone product $C_l(x)$ is then defined as

$$C_l(x) = \{(g, t, y) \in G \times S; (t, y) \in A_l(x_1^g)\},$$

and the process ω_l $(0 < l < T)$ by $\omega_l(x) = \mu(C_l(x))$ for $x \in \mathbb{R}^m$.

The Radon measure M is then defined as the almost sure limit (in the sense of weak convergence of Radon measures) by

$$M(A) = \lim_{l \to 0^+} M_l(A) = \lim_{l \to 0^+} \int_A e^{\omega_l(x)}\, dx$$

for any Lebesgue measurable subset $A \subset \mathbb{R}^m$. The convergence is ensured by the fact that the family $(M_l(A))_{l>0}$ is a right-continuous positive martingale. The structure exponent of M is defined by

$$\forall q \geq 0, \quad \zeta(q) = dq - \psi(q)$$

for all q such that the right-hand side makes sense. The measure M is different from 0 if and only if there exists $\epsilon > 0$ such that $\zeta(1+\epsilon) > m$, (or equivalently $\psi'(1) < m$). In that case, we have:

Theorem 2.1. *The measure M is stationary, isotropic and satisfies the* **exact stochastic scale invariance property***: for any $\lambda \in]0,1]$,*

$$(M(\lambda A))_{A \subset B(0,T)} \overset{\text{law}}{=} (\lambda^m e^{\Omega_\lambda} M(A))_{A \subset B(0,T)},$$

where Ω_λ is an infinitely divisible random variable, independent of $(M(A))_{A \subset B(0,T)}$, the law of which is characterized by:

$$\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}.$$

Furthermore, the support of such measures is full in the sense that

Proposition 2.2. *If the measure M is non-degenerate, that is $\psi'(1) < m$, we have $\mathrm{Supp}(M) = \mathbb{R}^m$. Consequently, every Borelian subset of $\mathbb{R}^m$ with null M-measure has its complement dense in $\mathbb{R}^m$ for the Euclidian distance.*

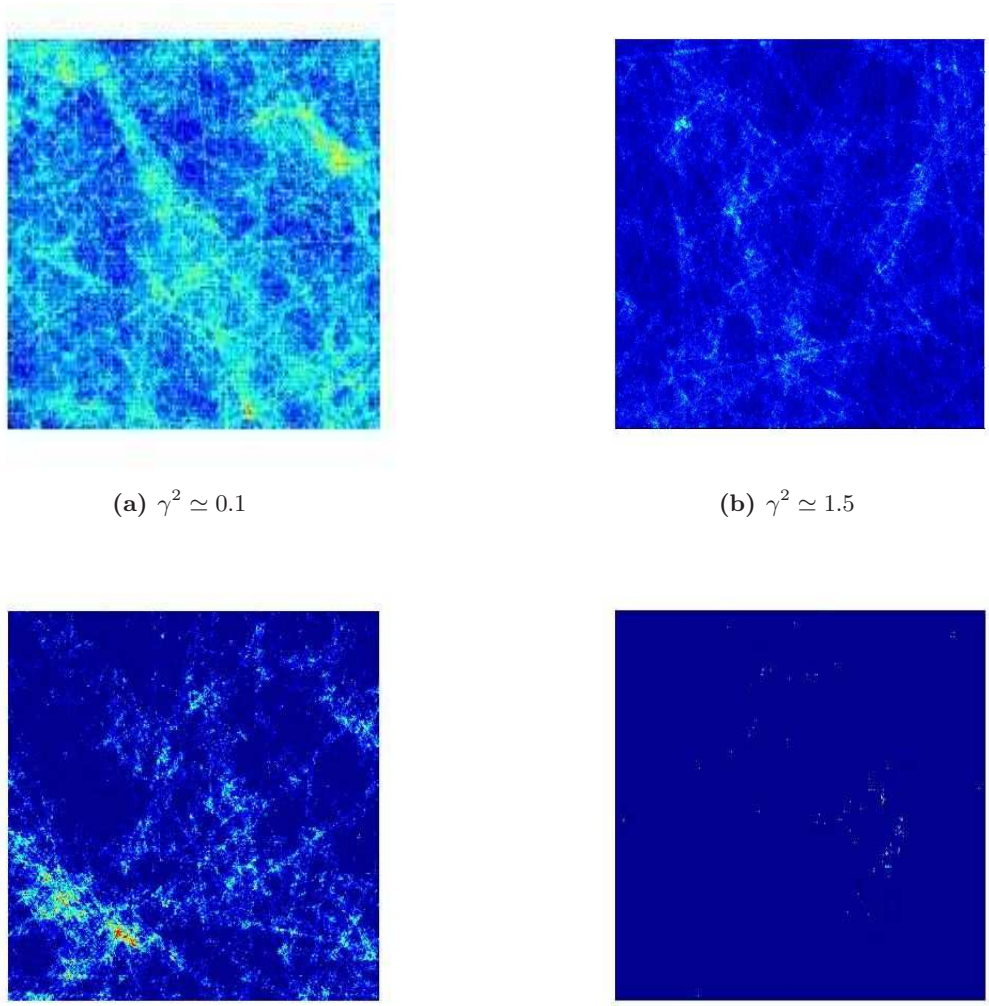
Proof. For a given ball $B(x,r) \subset \mathbb{R}^m$, the event $\{M(B(x,r)) > 0\}$ is measurable with respect to the asymptotic sigma algebra generated by $(\omega_l)_{l>0}$. By the 0-1 law, it has probability 0 or 1. Because of the uniform integrability of the martingale $(M_l(A))_{l>0}$ for each Borelian subset A with finite Lebesgue measure (denoted by $\lambda(A)$), we have $\mathbb{E}[M(B(x,r))] = \lambda(B(x,r)) > 0$. We deduce $\mathbb{P}(M(B(x,r)) > 0) = 1$. Hence, $\mathbb{P}$ a.s., $M(B(x,r)) > 0$ for all the balls $B(x,r)$ with rational centers and radii. □

2.2 Optimal transport for MRM

Though we only focus on MRMs as constructed in subsection 2.1, we stress that our results straightforwardly extend to more general MRMs as in [4, 5]. The reader may consult Appendix A for a brief reminder about optimal transport theory. In particular, we adopt the notations used in Appendix A. We denote by B_R the closed ball of $\mathbb{R}^m$ centered at 0 with radius R and by C_R its Lebesgue measure. We define λ_R as the renormalized Lebesgue measure on B_R with mass 1 and the renormalized probability measure on B_R

$$\overline{M}(dx) = \frac{1}{M(B_R)} M(dx).$$

Originally, our problem was to construct a nice (as much as possible) mapping that pushes the MRM M forward to the Lebesgue measure. Of course, there are several possibilities to achieve that construction. Among all the possibilities, we focus on optimal transportation theory because it features many interesting qualities: measurability with respect to the randomness, isotropy, regularity, etc... Basic results ensure that there exists an optimal transport map pushing $\overline{M}$ forward to the Lebesgue measure λ_R (for a quadratic cost function) provided that the measure $\overline{M}$ does not give mass to small sets. Though that condition is usually not satisfied by MRMs, this is true when the intermittency parameter $\psi'(1)$ is small:



(a) $\gamma^2 \simeq 0.1$

(b) $\gamma^2 \simeq 1.5$

(c) $\gamma^2 \simeq 2$

(d) $\gamma^2 \simeq 3.9$

Figure 1: Simulations of the density of a log-normal MRM with various intermittency parameters γ appearing in (1.1). The structure exponent matches $\xi(q) = (2 + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2$. In dimension 2, the measure is non-degenerate provided that $0 < \gamma^2 < 4$. The last two simulations are colored with a logarithmic intensity scale.

Theorem 2.3. *When $\psi'(1) < 1$, the measure $\overline{M}$ does not give mass to small sets. Hence there is a unique optimal transport map (for the Euclidian quadratic cost) that pushes the renormalized probability measure $\overline{M}$ forward to the renormalized Lebesgue measure.*

We can thus apply the classical transportation theory (see Theorem A.3). There exist two optimal transport maps

$$\chi : \mathrm{Supp}(\chi) \to \mathrm{Supp}(\Gamma) \quad \text{and} \quad \Gamma : \mathrm{Supp}(\Gamma) \to \mathrm{Supp}(\chi),$$

(the supports of which are Borelian and contained in B_R) that respectively push the Lebesgue measure λ_R forward to $\overline{M}$ and vice-versa, meaning

$$\chi_\# \lambda_R = \overline{M} \quad \text{and} \quad \Gamma_\# \overline{M} = \lambda_R. \tag{2.1}$$

They are both unique as gradients of convex functions satisfying 2.1. Moreover they are bijections, inverse from each other. They are respectively λ_R and $\overline{M}$-almost surely defined, meaning $\overline{M}\big(\mathrm{Supp}(\Gamma)\big) = 1$ and $\lambda_R\big(\mathrm{Supp}(\chi)\big) = 1$, so that both supports $\mathrm{Supp}(\Gamma)$, $\mathrm{Supp}(\chi)$ are dense in B_R for the Euclidian distance.

We equip B_R with the Riemannian metric

$$\forall x \in B_R, \forall u, v \in \mathbb{R}^m, g^e_x(u, v) = \frac{M(B_R)^2}{C_R^2}(u, v)$$

where $(\cdot, \cdot)$ denotes the usual inner product on $\mathbb{R}^m$. In that way, (B_R, g^e) is a Riemannian space in which the volume form matches $M(B_R) \times \lambda_R(dx)$ and the geodesic distance d_e is given by the Euclidian distance on B_R up to a random multiplicative constant:

$$d_e(x, y) = \frac{M(B_R)}{C_R}|x - y|.$$

On the support $\mathrm{Supp}(\Gamma)$ of Γ, we can define the distance d_Γ by

$$\forall x, y \in \mathrm{Supp}(\Gamma), \quad d_\Gamma(x, y) = d_e(\Gamma(x), \Gamma(y)).$$

Hence there is an isometry, namely Γ, between the metric-measure space $(\mathrm{Supp}(\Gamma), d_\Gamma, M)$ and the metric-measure space $(\mathrm{Supp}(\chi), d_e, M(B_R) \times \lambda_R)$. Since the closure of $\mathrm{Supp}(\chi)$ with respect to the Euclidian distance is equal to B_R, the completion of the metric space $(\mathrm{Supp}(\Gamma), d_\Gamma)$, denoted by (C, d_Γ), is isometric to (B_R, d_e). That isometry, which coincides with Γ on $\mathrm{Supp}(\Gamma)$ is still denoted by Γ and its inverse, which coincides with χ on $\mathrm{Supp}(\chi)$, is still denoted by χ. Obviously, the metric space (C, d_Γ) is compact.

Furthermore, since $\mathrm{Supp}(\Gamma)$ is a Borelian subset of B_R as well as a Borelian subset of C (for the respective topologies of B_R and C) and since $M(\mathrm{Supp}(\Gamma)) = M(B_R)$, the measure M can be extended to the whole of the Borelian subsets of C by prescribing:

$$\text{for any Borelian subset A of C}, \quad M(A) = M(A \cap \mathrm{Supp}(\Gamma)).$$

Via pullback, the space C inherits the structure of Riemannian manifold (smooth, complete, connected, m-dimensional manifold equipped with a smooth metric tensor). The (only) chart is given by $\Gamma : C \to B_R$. We summarize below what we have proved as well as the properties inherited from the pullback metric:

Theorem 2.4. *If $\psi'(1) < 1$, we can find a compact Riemannian manifold (C, g) and a Borelian subset B of B_R such that:*

1. *B is dense in B_R for the Euclidian distance and has full M-measure, namely $M(B_R \setminus B) = 0$,*
2. *C is the completion of B with respect to the geodesic distance on C,*
3. *the volume form on C coincides with the measure M on B,*
4. *in the system of local coordinates given by the chart Γ, the Riemannian metric tensor on C reads*

$$g = d(\omega)(dx_1^2 + \cdots + dx_m^2) \quad \text{with} \quad d(\omega) = \frac{M(B_R)^2}{C_R^2}.$$

The strongly intermittent case $1 \leq \psi'(1) < m$

Now we consider a log-infinitely divisible random measure M satisfying $1 \leq \psi'(1) < m$. The measure M gives mass to small sets so that we cannot use classical theorems of optimal transport theory if we consider the ambiant space B_R equipped with its Euclidian structure. It thus seems hopeless to solve the problem:

$$\text{(2.2)} \qquad \text{Find the mapping } \varphi \text{ realizing the infimum: } \inf_{\substack{\varphi: B_R \to B_R \\ \varphi_{\#}\overline{M} = \lambda_R}} \int_{B_R} |\varphi(x) - x|^2 \overline{M}(dx).$$

Our strategy is the following: can we find a pair (M', T'), where M' is a probability measure and T' is an optimal (for the Euclidian quadratic cost) transport map pushing M' forward to λ_R, such that the following Monge type optimization problem possesses a (unique) solution?

$$\text{(2.3)} \quad \text{Find the mapping } \varphi \text{ realizing the infimum: } \inf_{\substack{\varphi: B_R \to B_R \\ \varphi_{\#}\overline{M} = \lambda_R}} \int_{B_R} |\varphi(x) - T'(x)|^2 \overline{M}(dx).$$

It turns out that the above problem is a mathematical formulation of the following intuitive observation: if we cannot find an optimal transport pushing $\overline{M}$ forward to λ_R, can we find an intermediate measure M' and optimal transports T, T' respectively pushing $\overline{M}$ forward to M' and M' forward to λ_R? If the answer is positive, then the composition $T' \circ T$ pushes M forward to λ_R. Though the composition $T' \circ T$ is in general not optimal for the Euclidian quadratic cost, it is the composition of two gradients of convex functions, which is not bad in terms of regularity. So we have formalized some kind of two-step optimal transport. And more generally, if we cannot find a two-step optimal transport, is it possible to find a n-step optimal transport between $\overline{M}$ and λ_R, that is a composition of n gradients of convex functions?

The reader may have the following objection: from classical theorems, existence of a (unique) optimal transport map pushing $\overline{M}$ forward to M' or λ_R does not depend on the target measure (M' or λ_R) but only on $\overline{M}$ through the fact that $\overline{M}$ does or does not give mass to small sets. So, a priori, two-step optimal transports may be as difficult to exhibit as optimal transports. Our idea lies in the fact that Problem (2.3) can be reformulated

in a very simple way if we equip the ball B_R with an appropriate Riemannian structure. Indeed, if we change the unknown in (2.3), we get the following equivalent problem

(2.4)

Find the mapping ψ realizing the infimum: $\inf_{\substack{\psi:B_R\to B_R\\ \psi_{\#\overline{M}}=M'}} \int_{B_R} |T'(\psi(x))-T'(x)|^2\overline{M}(dx).$

It turns out that we can equip the ball B_R with a Riemannian structure, the distance of which matches $d(x,y)=|T'(x)-T'(y)|$ and the volume form of which matches M' (up to a multiplicative constant). Problem (2.4) thus reduces to a classical problem of optimal transportation theory on smooth Riemannian manifolds: for such a mapping to exist, it is mainly sufficient that $\overline{M}$ does not give mass to the small sets associated to the distance d. That is the main constraint when choosing the measure M': it must be the volume form associated to a Riemannian metric, the small sets of which are not charged by M. Of course, the argument generalizes to n-step optimal transports.

The main difficulty thus lies in choosing the number n of steps and the intermediate measures. The crucial point is the following. Given $n\geq 1$, M can be seen as the composition of n multiplicative chaos (see subsection 4.2): we can find n independent independently scattered log-infinitely divisible random measures $\mu^{(1)},\dots,\mu^{(n)}$ associated to $(\varphi/n,\theta)$ (see subsection 2.1). The corresponding processes ω_l associated to $\mu^{(1)},\dots,\mu^{(n)}$ are respectively denoted by $\omega_l^{(1)},\dots,\omega_l^{(n)}$. We define recursively for $k\leq n$:

$$M^{(0)}(dx)=dx \quad \text{and} \quad M^{(k)}(dx)=\lim_{l\to 0} e^{\omega_l^{(k)}(x)}M^{(k-1)}(dx),$$

where the limits have to be understood in the sense of weak convergence of Radon measures. Then both measures M and $M^{(n)}$ have the same law so that we assume, in what follows, that M and $M^{(n)}$ coincide. That procedure allows to see each measure $M^{(k)}$ as a chaos with respect to $M^{(k-1)}$ with a reduced intermittency parameter $\psi'(1)/n$. If we can equip the ball B_R with a Riemannian metric $g^{(k-1)}$ the volume form of which coincides with $M^{(k-1)}$, it turns out that $M^{(k)}$ does not give mass to the $g^{(k-1)}$-small sets provided that the intermittency parameter $\psi'(1)/n$ is small enough. So it suffices to choose n big enough. In consequence there is a unique optimal transport map (w.r.t. to the quadratic cost function associated to the metric $g^{(k-1)}$) that pushes the renormalized measure $\overline{M}^{(k)}$ forward to the renormalized measure $\overline{M}^{(k-1)}$. And so on for the different values of $k\leq n$. Thus we claim:

Theorem 2.5. *If $1\leq\psi'(1)<m$, we can find $n\geq 1$ and n gradients of convex functions $T^{(1)},\dots,T^{(n)}$ such that $\forall k=1,\dots,n$, the mapping $\varphi^{(k)}=T^{(1)}\circ\cdots\circ T^{(k)}$ pushes the measure $\overline{M}^{(k)}$ forward to the Lebesgue measure and minimizes the quantity*

$$\inf_{\substack{T:B_R\to B_R\\ T_{\#\overline{M}^{(k)}}=\lambda_R}} \int_{B_R} |T(x)-\varphi^{(k-1)}(x)|^2\overline{M}^{(k)}(dx).$$

As a corollary, we get

Theorem 2.6. *We can find a compact Riemannian manifold (C,g) and a Borelian subset B of B_R such that:*

1. *B is dense in B_R for the Euclidian distance and has full M-measure, namely $M(B_R \setminus B) = 0$,*

2. *C is the completion of B with respect to the geodesic distance on C.*

3. *the volume form on C coincides with the measure M on B*

4. *in the system of local coordinates given by the chart $\varphi^{(n)}$, the Riemannian metric tensor on C reads*

$$g = d(\omega)(dx_1^2 + \cdots + dx_m^2) \quad \text{with} \quad d(\omega) = \frac{M(B_R)^2}{C_R^2}.$$

Remark 2.7. *In Theorems 2.4 and 2.5, if $R < T$, by scale invariance, the random variable d has the same law as $e^{2\Omega_{\frac{R}{T}}} \frac{M(B_T)^2}{C_T^2}$ where $\Omega_{\frac{R}{T}}$ is an infinitely divisible random variable the law of which is characterized by $\mathbb{E}[e^{q\Omega_{\frac{R}{T}}}] = \left(\frac{R}{T}\right)^{-\psi(q)}$. So the radius R of the ball B_R influences the random metric through a log-infinitely divisible random variable, which turns out to be log-normal for log-normal MRM.*

We conclude the presentation of our results by the following remarks. First, the choice of the quadratic cost can be discussed since it does not, a priori, exhibit any intrinsic property. However it possesses the main advantage of being tractable. Second, in the strongly intermittent case, it is worth emphasizing that it is hopeless to maintain uniqueness of the transport map as soon as it is not characterized as the infimum of a Monge type optimization problem. Hence, there are plenty of transport maps pushing the renormalized measure M forward to the renormalized Lebesgue measure. But many of them are untractable and very singular (see for instance the measurable isomorphism [11, Introduction]). We justify our approach by the fact that our transport map enjoys, as much as possible, usual properties of an optimal transport map:

- in terms of uniqueness, the mappings $T^{(1)}, \ldots, T^{(n)}$ (and thus the mappings $\varphi^{(k)}$ for $k = 1, \ldots, n$) are entirely determined as soon as the intermediate measures $M^{(k)}$ (for $k = 1, \ldots, n$) are prescribed. This maintains some tractabality when we want to characterize our transport map. Of course, there is some flexibility in the choice of the measure $M^{(k)}$ and it would be interesting to know if there is an optimal choice for these intermediate measures. For instance, one can imagine choosing a sequence that minimizes a quantity like

$$\inf_{\overline{M}^{(1)},\ldots,\overline{M}^{(n-1)}} \sum_{k=1}^{n} \inf_{\substack{T:B_R \to B_R \\ T_{\#\overline{M}^{(k)}} = \lambda_R}} \int_{B_R} |T(x) - \varphi^{(k-1)}(x)|^2 \overline{M}^{(k)}(dx).$$

  To be honest, we fell short of establishing any global uniqueness criterion.

- in terms of regularity, our transport map is "as nice as possible" as a composition of gradients of convex functions. We have the feeling that this is a legitimate requirement when it is not possible to express it as the gradient of a single convex function.

3 Applications

3.1 Multifractal random changes of time

The main motivation of our paper is to construct multidimensional multifractal random changes of times via optimal transportation maps associated with MRMs. This idea originates from Mandelbrot in the 1-d case: if B is a 1-d Brownian motion and M is a 1-d MRM, we can define the process $t \mapsto B_{M[0,t]}$. This process has been widely used in financial modelling since it enjoys nice properties: it is a square-integrable martingale with long-range correlated stationary increments and nonlinear power law spectrum. Extending that construction to the multidimensional case is the purpose of what follows.

Let us fix $R > 0$ (with $T < R$) and a m-dimensional MRM as constructed in 2.1. Let Γ, χ be the maps pushing M forward to $\frac{M(B_R)}{C_R}\,dx$ and vice versa (sticking to the previous notations, we denote by B the support of Γ). We further consider an independent Gaussian white noise $W(dx)$ defined on $\mathbb{R}^m$. Given $x \in \mathbb{R}^m$, we denote by $C(x)$ the cube $[0, x_1] \times \cdots \times [0, x_m]$ (with the convention that, if $x_i < 0$, the interval $[0, x_i]$ stands for $[x_i, 0]$). Finally, we define the process

$$\forall x \in B_R, \quad B(x) = W\big(\Gamma(B \cap C(x))\big).$$

It is readily seen that the process B is isotropic and stochastically scale invariant:

$$\forall \lambda \leq 1, \quad \sqrt{M(B_R)}\big(B(\lambda x)\big)_{x \in B(0,T)} \overset{law}{=} \sqrt{C_R}\lambda^{m/2} e^{\frac{1}{2}\Omega_\lambda}\big(B(x)\big)_{x \in B(0,T)}, \tag{3.1}$$

where Ω_λ is an infinitely divisible random variable, independent of $\big(B(x)\big)_{x \in B(0,T)}$, the law of which is characterized by:

$$\mathbb{E}[e^{iq\Omega_\lambda}] = \lambda^{-\varphi(q)}.$$

This scale invariance property depends on the radius R. An interesting question is to pass to the infinite volume limit: by the ergodic theorem, $\frac{M(B_R)}{C_R}$ converges almost surely to 1 as $R \to \infty$. Therefore, equation (3.1) can be seen for R large as an approximate stochastic scale invariance equation. We leave as an open problem the study of Γ as $R \to \infty$ (existence, characterization of a limit). We stress that, in great generality, infinite volume transportation theory is a wide open area.

Of course, we can generalize this approach to construct multifractal random changes of time to many other multidimensional stochastic processes like Levy noise, fractional Brownian motion, and so on...

3.2 Random metrics associated with MRMs

A special case of (1.1) in a bounded domain of dimension 2 has recently received much attention. When X is the Gaussian Free Field (GFF), that is a Gaussian process with covariance function given by the Green function of the Laplacian, the measure M (Gaussian multiplicative chaos associated to the Green function) is called the Liouville quantum measure (see [3]). For several years, much effort has been made to understand the geometry of these measures. A recent important step was to prove the so-called KPZ formula [2, 3, 8].

Roughly speaking, the KPZ formula gives the correspondence between the Hausdorff dimension of a set as seen by the Lebesgue measure and the Hausdorff dimension of this set as seen by the measure M. More precisely, for a given compact set $E \subset B_R$ and a mesure ν, the s-dimensional Hausdorff measure of E wrt to ν is the quantity:

(3.2)

$$H^s(\nu, E) = \lim_{\delta \to 0} H^s_\delta(E, \nu), \qquad \text{where}$$
$$H^s_\delta(\nu, E) = \inf\{\sum_n \nu(B_n)^{s/2}; E \subset \bigcup B_n, B_n \text{ open Euclidian ball } 0 < M(B_n) \leq \delta\}.$$

Then we define the Hausdorff dimension $\dim^\nu_H(E)$ of the set E wrt to the measure ν:

$$\dim^\nu_H(E) = \inf\{s > 0; H^s(\nu, E) = 0\} = \sup\{s > 0; H^s(\nu, E) = +\infty\}. \tag{3.3}$$

When ν is given by the Lebesgue measure, the corresponding Hausdorff measures and dimensions will be called Euclidian, and denoted with the superscript e. The KPZ formula asserts that the Euclidian Hausdorff dimension $\dim^e_H(E)$ and the random Hausdorff dimension $\dim^M_H(E)$ are linked by the relation

KPZ formula. *Almost surely, we have*

$$\xi\Big(\frac{\dim^M_H(E)}{2}\Big) = \dim^e_H(E) \tag{3.4}$$

where ξ is the so-called structure exponent of M:

$$\xi(q) = \big(2 + \frac{\gamma^2}{2}\big)q - \frac{\gamma^2}{2}q^2.$$

Actually, formula (3.4) remains valid for any log infinitely divisible MRM as soon as M possesses moments of negative order (see [2, 8]).

A further step in the understanding of the KPZ formula is to construct random metric spaces exhibiting KPZ phenomena: we want the KPZ formula to remain true when defining the random Hausdorff dimension with the help of the constructed random metric instead of the measure M. Of special interest for specialists of quantum gravity is to make that random metric space closely related to random planar maps (see [3] and references therein).

Inspired by the aforementioned problem, we present below a toy construction of a flat Riemannian manifold the volume form of which coincides with M. We stick to the context of quantum measure but our construction remains true for more general MRMs (as in (2.1), see also [4, 1]). Theorem 2.6 allows to understand the measure M on B_R as the volume form of a flat Riemannian manifold (up to a set of null M-measure). Such a structure permits to define fundamental objects such as distance, arclength, geodesics associated to the measure M. The geodesics are easily described via the transport maps. By sticking to the notations of Theorem 2.5, we define $\Gamma = \varphi^{(n)}$ and $\chi = \Gamma^{-1}$. Since Γ is an isometry of metric spaces, a curve $\gamma : [0, 1] \to C$ is a geodesic on C if and only if $\Gamma(\gamma) \subset B_R$ is a geodesic for the Euclidian metric, that is a segment. The geodesic joining $x, y \in B \subset B_R$ (parameterized by constant speed) is thus given by

$$\gamma^{x,y} : t \in [0, 1] \to \chi\big(t\Gamma(x) + (1 - t)\Gamma(y)\big) \in C.$$

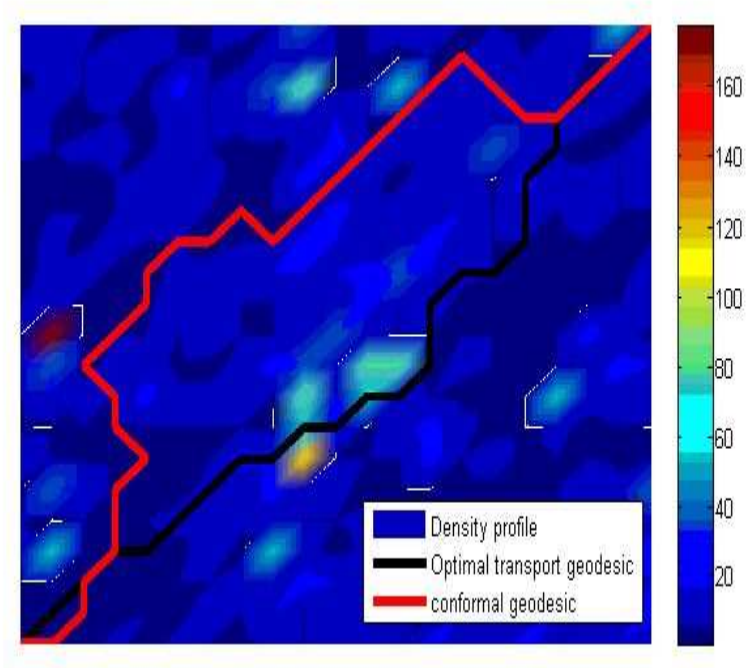


Figure 2: Simulations of conformal geodesics (weighted by the density of a properly regularized MRM along the path) and geodesics obtained via optimal transport for a log-normal MRM for $\gamma^2 \simeq 1.5$.

Because the general theory of optimal transport suffers a definite lack of strong estimates concerning the optimal transport maps, we cannot prove the KPZ formula connecting the Hausdorff dimensions defined in terms of distances (not measures) and we leave that point as an open question.

Furthermore we stress that our model is likely to have no relation with the expected random metric space that should appear in the context of Liouville quantum gravity: no branching properties of geodesics, not a conformal metric,...For instance, with a properly regularized version of the weights, we illustrate in Figure 2 that our geodesics are sensitive to the oriented directions of mass deformation of M whereas properly defined conformal geodesics are sensible to mass allocation only.

4 Proofs

We first prove the key estimate of the paper. We adapt a strategy first used in Kahane's original paper on multiplicative chaos [5].

4.1 Fundamental result

Let B be a Borelian subset of B_R and κ be a probability measure on B_R supported by B, meaning $\kappa(B) = 1$. We assume that the set B is equipped with a distance d and that the completion of B with respect to the distance d, denoted by (C, d), is compact. We

assume that the Borelian subsets of B with respect to the Euclidian topology coincide with the Borelian subsets of B w.r.t. the distance d so that we can extend the measure κ to the whole Borelian sigma-algebra of C by prescribing

$$\forall A \subset C \text{ Borelian}, \quad \kappa(A) = \kappa(A \cap B).$$

The classes $\mathbf{R}_\alpha, \mathbf{R}_\alpha^-$. For any $\alpha > 0$, we introduce the set R_α of Radon measures ν on C satisfying: for any $\epsilon > 0$, there are $\delta > 0$, $D > 0$ and a d-compact subset $K_\epsilon \subset C$ such that $\nu(C \setminus K_\epsilon) < \epsilon$ and the measure $\nu_\epsilon(dx) = \mathbb{1}_{K_\epsilon}(x)\nu(dx)$ satisfies

$$\forall U \, d\text{-open ball}, \quad \nu_\epsilon(U) \leq D \times \operatorname{Diam}_d(U)^{\alpha+\delta}, \tag{4.1}$$

where Diam_d denotes the diameter of the ball U with respect to the distance d. We further define the set of Radon measures $R_\alpha^- = \bigcap_{\beta<\alpha} R_\beta$.

We further give an energy condition for a Radon measure ν to be in the class R_α^-. For $\alpha > 0$, we define

$$C_\alpha(\nu) = \int_C \int_C \frac{1}{d(x,y)^\alpha} \nu(dx)\nu(dy). \tag{4.2}$$

It is plain to see that

$$\nu \text{ satisfies } C_\alpha(\nu) < +\infty \Rightarrow \nu \in R_\alpha^-.$$

Conversely a measure ν obeying (4.1) satisfies $C_\beta(\nu) < +\infty$ for each $\beta < \alpha + \delta$.

Notations: In what follows, we use the superscript e (i.e. R_α^e, R_α^{e-} and C_β^e) to mention that the distance d is equal to the Euclidian distance. □

Proposition 4.1. *Assume that the measure κ belongs to R_α^e. Let N be the Radon measure on B_R defined, $\mathbb{P}$-almost surely, as the following limit (in the sense of weak limit of Radon measures):*

$$N(dx) = \lim_{l \to 0} N_l(dx), \qquad \textit{where } N_l(dx) \overset{\text{def.}}{=} e^{\omega_l(x)}\kappa(dx).$$

If $\psi(2) < \alpha$ then the martingale $\big(N_l(B)\big)_l$ is uniformly integrable and, consequently, N is non-trivial and almost surely supported by B.

Proof. We remind the reader of the fact that the family $\big(N_l(B)\big)_l$ is a right-continuous positive martingale and thus converges almost surely.

Since κ belongs to R_α, for each $\epsilon > 0$, we can find $\delta > 0$, $D > 0$ and a compact subset K such that $\kappa(C \setminus K) < \epsilon$ and

$$\limsup_{r \to \infty} \frac{\ln \kappa(B_r^x)}{r} < -\alpha - \delta \quad \text{uniformly for } x \in K,$$

where B_r^x stands for the d-ball of radius e^{-r} and center x. This implies $C_\beta(\kappa_K) < +\infty$ for each $\alpha < \beta < \alpha + \delta$.

Then we compute

$$\mathbb{E}\Big[N_l(B \cap K)^2\Big] = \int_{B\cap K} \int_{B \cap K} e^{\psi(2)K_l(|y-x|)}\kappa(dx)\kappa(dy)$$

where K_l is given on $\mathbb{R}^m$ by

$$K_l(x) = \int_G \rho_l(gx)\, H(dg)$$

where $\rho_l(y)$ ($y \in \mathbb{R}$) is defined by $\ln(T/|y|)$ if $l \le |y| \le T$, $\rho_l(y) = \ln(T/l) + 1 - |y|/l$ if $|x| \le l$ and 0 otherwise. In particular, on B, K_l is not greater than $C + \ln \frac{T}{|x|}$ for some positive constant C. As a consequence, for some positive constant C' which may change along the inequalities, we have

$$\sup_l \mathbb{E}\Big[N_l(B \cap K)^2\Big] \le C' \int_{B\cap K} \int_{B\cap K} e^{\psi(2) \ln \frac{T}{|y-x|}} \kappa(dx)\kappa(dy) \le C' C_\beta(\kappa) < +\infty$$

since $\psi(2) < \beta$. The martingale $\big(\widetilde{\nu}_l(B \cap K)\big)_l$ is bounded in $L^2(\Omega)$ and is therefore uniformly integrable. It is plain to deduce that the martingale $\big(\widetilde{\nu}_l(B)\big)_l$ is uniformly integrable. □

Theorem 4.2. *1) Assume that the measure κ belongs to $R_\alpha \cap R^e_\varsigma$ for some $\varsigma > \psi(2)$. Then*

$$N \in R_{\alpha\frac{\varsigma-\psi(2)}{\varsigma}+\psi(2)-\psi'(1)}.$$

Consequently, we also have:

$$\kappa \in R^-_\alpha \cap R^{e-}_\varsigma \text{ for some } \varsigma > \psi(2) \;\Rightarrow\; N \in R^-_{\alpha\frac{\varsigma-\psi(2)}{\varsigma}+\psi(2)-\psi'(1)}.$$

2) In particular, we have in the Euclidian case: if $\kappa \in R^e_\alpha$ (resp. R^{e-}_α) and $\psi(2) < \alpha$ then $N \in R^e_{\alpha-\psi'(1)}$ (resp. $R^{e-}_{\alpha-\psi'(1)}$).

Proof. Since κ belongs to $R_\alpha \cap R^e_\varsigma$, for each $\epsilon > 0$, we can find $\delta > 0$ and a compact subset K' for the d-topology (d-compact for short) such that $\kappa(C \setminus K') < \epsilon/2$ and

$$\limsup_{r\to\infty} \frac{\ln \kappa(B^x_r)}{r} < -\alpha - \delta \quad \text{uniformly for } x \in K', \tag{4.3}$$

where B^x_r still stands for the d-ball of radius e^{-r}.

Since κ belongs to R^e_ς, for each $\epsilon > 0$, we can find a compact subset K'' for the Euclidian topology such that $\kappa(B_R \setminus K'') < \epsilon/4$ and $C_\beta(\mathbb{1}_{K''}(x)\kappa(dx)) < +\infty$ for each $\beta < \varsigma$. $K'' \cap B$ is a Borelian subset of B (for the Euclidian topology) and hence a Borelian subset of C. Since M is inner regular on C (recall that C is compact), we can find a d-compact set $\widetilde{K}$ of C such that $\widetilde{K} \subset K''$ and $M(K'' \setminus \widetilde{K}) < \epsilon/4$. Finally we set $K = K' \cap \widetilde{K}$, which is d-compact. The set K also satisfies: $M(C \setminus K) < \epsilon$, (4.3) is valid on K and $C^e_\beta(\mathbb{1}_{B\cap K}) < +\infty$ for any $\beta \le \varsigma$.

Even if it means multiplying κ by a constant, we assume $\kappa(K) = 1$. We consider on $\Omega \times K$ the probability measure $\mathbb{Q}$ defined by

$$\int_{\Omega\times K} f(\omega, x)\, d\mathbb{Q} = \mathbb{E}\Big[\int_K f(\omega, x) N(dx)\Big]$$

for all measurable nonnegative functions f.

Given $l' < l$, we define

$$\omega_{l',l}(x) = \omega_l(x) - \omega_{l'}(x).$$

For a sequence $l_1 < \cdots < l_n$, the random variables $\omega_{l_1,l_2}, \ldots, \omega_{l_{n-1},l_n}$ are $\mathbb{Q}$-independent and

$$\int e^{\lambda \omega_{l_i,l_j}}\, d\mathbb{Q} = \mathbb{E}\big[e^{(1+\lambda)\omega_{l_i,l_j}(x)}\big] = e^{\psi(1+\lambda)\ln(l_j/l_i)}.$$

The process $u \in \mathbb{R}_+ \mapsto \omega_{e^{-u}}$ is therefore an integrable Lévy process (we can consider a version that is right-continuous with left limits). From the strong law of large numbers, we have

$$\mathbb{Q} \text{ a.s.}, \quad \frac{\omega_{e^{-u}}}{u} \to \psi'(1) \quad \text{as } u \to \infty.$$

This implies that, $\mathbb{P}$ a.s.,

$$N \text{ a.s.}, \quad \frac{\omega_{e^{-u}}}{u} \to \psi'(1) \quad \text{as } u \to \infty.$$

Therefore, $\mathbb{P}$ a.s., for each $\epsilon > 0$ we can find a compact $K^1_\epsilon \subset K$ such that $N(K \setminus K^1_\epsilon) < \epsilon$ and $\frac{\omega_{e^{-u}}(x)}{u} \to \psi'(1)$ uniformly w.r.t. $x \in K^1_\epsilon$ as $u \to \infty$. Now we define

$$N_q(dy) = \lim_{l \to 0} e^{\omega_{l,e^{-q}}(y)}\, \kappa(dy) \quad \text{and} \quad P_q(x) = \int_{B_q x \cap K} N_q(dy).$$

We further define the function θ_q by

$$\theta_q(x,y) = \begin{cases} 1 & \text{if } d(x,y) \le e^{-q} \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have $P_q(x) = \int_K \theta_q(x,y) N_q(dy)$ and

$$\begin{aligned} \int P_q\, d\mathbb{Q} &= \mathbb{E} \int_K \int_K \theta_q(x,y) N_q(dy) N(dx) \\ &\le \int_{B\cap K} \int_{B \cap K} \theta_q(x,y) e^{\psi(2)\left(C + \ln(\frac{e^{-q}}{|x-y|})\right)}\, \kappa(dx)\kappa(dy) \\ &\le e^{\psi(2)C} \int_{B\cap K} \int_{B\cap K} \theta_q(x,y) e^{-q\psi(2)} \frac{1}{|y-x|^{\psi(2)}}\, \kappa(dx)\kappa(dy). \end{aligned}$$

By using the above relation, we obtain

$$\int \sum_{n \ge 1} e^{\beta n} P_n\, d\mathbb{Q} = \sum_{n \ge 1} \int_{B\cap K} \int_{B \cap K} e^{(\beta - \psi(2))n} \theta_n(y,x) \frac{1}{|y-x|^{\psi(2)}}\, \kappa(dx)\kappa(dy).$$

Note that (for some positive constant D)

$$\sum_{n \ge 1} e^{(\beta-\psi(2))n} \chi_n(y,x) = \sum_{1 \le n \le -\ln d(x,y)} e^{(\beta - \psi(2))n} \le D \frac{1}{d(x,y)^{\beta - \psi(2)}}$$

in such a way that we obtain

$$\int \sum_{n \ge 1} e^{\beta n} P_n\, d\mathbb{Q} \le D \int_{B\cap K} \int_{B \cap K} \frac{1}{|y-x|^{\psi(2)}} \frac{1}{d(x,y)^{\beta - \psi(2)}}\, \kappa(dx)\kappa(dy)$$

We want to prove that the latter integral is finite for a well chosen β. We fix

$$\psi(2) < \beta = (\alpha + \delta/2) \frac{\varsigma - \psi(2)}{\varsigma} + \psi(2).$$

We consider $p, q > 1$ satisfying the relation $\frac{1}{p} + \frac{1}{q} = 1$ and given by

$$p = \frac{\varsigma}{\psi(2)} \quad \text{and} \quad q = \frac{\varsigma}{\varsigma - \psi(2)}.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\Big[\int_{B\cap K}\int_{B\cap K} \frac{1}{|y-x|^{\psi(2)}} \frac{1}{d(x,y)^{\beta-\psi(2)}}\,\kappa(dx)\kappa(dy)\Big] \\
&\qquad \le \Big[\int_{B\cap K}\int_{B\cap K} \frac{1}{|y-x|^{p\psi(2)}}\,\kappa(dx)\kappa(dy)\Big]^{1/p}\Big[\int_{B\cap K}\int_{B\cap K} \frac{1}{d(x,y)^{q(\beta-\psi(2))}}\,\kappa(dx)\kappa(dy)\Big]^{1/q} \\
&\qquad \le \Big(C^e_\varsigma(\mathbb{1}_{K\cap B}(x)\kappa(dx))\Big)^{1/p}\Big(C_{\alpha+\delta/2}(\mathbb{1}_{K\cap B}(x)\kappa(dx))\Big)^{1/q},
\end{aligned}$$

in such a way that the above integrals are finite.

We deduce that, $\mathbb{Q}$ a.s., $e^{\beta n}P_n \to 0$ as $n \to \infty$. Therefore, $\mathbb{P}$ a.s., $e^{\beta n}P_n \to 0$ as $n \to \infty$ N-almost surely. So we can find a compact $K^2_\epsilon \subset C$ such that $N(C \setminus K^2_\epsilon) < \epsilon$ and

$$\limsup_{n\to\infty} \frac{\ln P_n(x)}{n} \le -\beta \quad \text{uniformly for } x \in K^2_\epsilon.$$

Finally we can set $\bar{K} = K^1_\epsilon \cap K^2_\epsilon$ and $N_{\bar{K}}(dx) = \mathbb{1}_{\bar{K}}(x)N(dx)$. We obtain

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\ln N_{\bar{K}}(B^x_n)}{n} &= \limsup_{n\to\infty} \frac{\ln \int_{\bar{K}\cap B^x_n} e^{\omega_{e^{-n}}(x)} N_n(dx)}{n} \\
&\le -\beta + \psi'(1)
\end{aligned}$$

uniformly w.r.t. $t \in \bar{K}$. We have proved $N \in R_{\alpha\frac{\varsigma-\psi(2)}{\varsigma}+\psi(2)-\psi'(1)}$ $\mathbb{P}$-almost surely. □

4.2 Composition of MRM

Now we prove that a non-degenerate log-infinitely divisible random measure M, i.e. satisfying $\psi(2) < +\infty$ and $\psi'(1) < d$, can be decomposed as an iterated multiplicative chaos (see also [13]).

Since $\psi(2) < +\infty$ we can find an integer n such that

$$m\psi(2) < n(m - \psi'(1)). \tag{4.4}$$

Then we can find n independent independently scattered log-infinitely divisible random measures $\mu^{(1)}, \dots, \mu^{(n)}$ associated to $(\varphi/n, \theta)$ (remind of the definition in subsection 2.1). We assume that the random measures $\mu^{(1)}, \dots, \mu^{(n)}$ are constructed on the probability spaces $(\Omega^{(1)}, \mathbb{P}^{(1)}), \dots, (\Omega^{(n)}, \mathbb{P}^{(n)})$. We define $\Omega = \Omega^{(1)} \times \cdots \times \Omega^{(n)}$ equipped with the product σ-algebra and the product probability measure $\mathbb{P} = \mathbb{P}^{(1)} \otimes \cdots \otimes \mathbb{P}^{(n)}$. The corresponding processes ω_l associated to $\mu^{(1)}, \dots, \mu^{(n)}$ are respectively denoted by $\omega^{(1)}_l, \dots, \omega^{(n)}_l$. Finally, we denote by $\mathbb{E}^{(i)}$ the conditional expectation given the variables $(\mu^{(k)})_{k\neq i}$.

We define recursively for $k \le n$:

$$M^{(0)}(dx) = dx \quad \text{and} \quad M^{(k)}(dx) = \lim_{l\to 0} e^{\omega^{(k)}_l(x)} M^{(k-1)}(dx),$$

where the limits have to be understood in the sense of weak convergence of Radon measures. Note that the choice of n makes valid the relation

$$\forall k \leq n-1, \quad \frac{m\psi(2)}{n} < m - \frac{k}{n}\psi'(1).$$

Hence, we can apply recursively Proposition 4.1 and Theorem 4.2 (with the distance d equal to the Euclidian distance and $\kappa \in R_m^{e-}$ is the Lebesgue measure) to prove that, for each $k \leq n$,

$$M^{(k)} \in R^{e-}_{m-\frac{k}{n}\psi'(1)} \quad \text{and} \quad \mathbb{E}^{(k)}[M^{(k)}(B_R)] = M^{(k-1)}(B_R) \quad \mathbb{P} \text{ a.s..}$$

Thus we have $M^{(n)} \in R^{e-}_{m-\psi'(1)}$ and $\mathbb{E}[M^{(n)}(B_R)] = \lambda(B_R)$ (the Lebesgue measure of B_R).

What we now want to prove is that the measure $M^{(n)}$ has the same law as the measure

$$M(dx) = \lim_{l\to 0} \; e^{\omega_l^{(1)}(x)+\cdots+\omega_l^{(n)}(x)}\, dx.$$

We consider on Ω the σ-algebra $\mathcal{G}_l$ generated by $\{\omega_r^{(1)}(x), \ldots, \omega_r^{(n)}(x); x \in \mathbb{R}^m, T > r > l\}$. The conditional expectation of $M^{(n)}(E)$ w.r.t. $\mathcal{G}_l$ is easily computed since, for each $k \leq n$, the martingale $(M_l^{(k)}(A))_l$ is $\mathbb{P}^{(k)}$-uniformly integrable. Indeed, we have:

$$\begin{aligned}
\mathbb{E}\big[M^{(n)}(A)|\mathcal{G}_l\big] &= \mathbb{E}\big[\mathbb{E}\big[M^{(n)}(A)|\mu^{(1)}, \ldots, \mu^{(n-1)}, (\omega_r^{(n)}(x))_{x\in\mathbb{R}^m, T>r>l}\big]|\mathcal{G}_l\big] \\
&= \mathbb{E}\big[\mathbb{E}^{(n)}\big[M^{(n)}(A)|(\omega_r^{(n)}(x))_{x\in\mathbb{R}^m, T>r>l}\big]|\mathcal{G}_l\big] \\
&= \mathbb{E}\big[\int_A e^{\omega_l^{(n)}(x)} M^{(n-1)}(dx)|\mathcal{G}_l\big] \\
&= \cdots \\
&= \int_A e^{\omega_l^{(n)}(x)+\cdots+\omega_l^{(1)}(x)}\, dx.
\end{aligned}$$

This latter quantity has the same law as $M_l(A)$. Since the martingale $\big(\mathbb{E}\big[M^{(n)}(A)|\mathcal{G}_l\big]\big)_l$ is uniformly integrable, we deduce that the family $(M_l(A))_l$ is uniformly integrable. Hence, both random variables $M(A)$ and $M^{(n)}(A)$ have the same law. In particular, $M \in R^e_{m-\psi'(1)}$.

Corollary 4.3. *If $\psi(2) < +\infty$ and $\psi'(1) < m$, then M belongs to $R^{e-}_{m-\psi'(1)}$.*

Remark 4.4. *The same composition argument shows that if a measure $\kappa \in R^e_\alpha$ and if M is defined as the limit*

$$M(dx) = \lim_{l\to 0} \; e^{\omega_l(x)}\kappa(dx)$$

and satisfies $\psi(2) < +\infty$ then $M \in R_{\alpha-\psi'(1)}$. Though we won't use it in this paper, this is an important result concerning the structure of the support of multifractal random measures.

4.3 Proof of Theorem 2.3

In the case where $\psi(2) < +\infty$ and $\psi'(1) < 1$, we will prove that the measure M does not give mass to small sets. Let $S \subset B_R$ be a small set, that is a set with Hausdorff dimension (w.r.t. the Euclidian distance) not larger than $m-1$.

From Corollary 4.3, M belongs to the class $R^{e-}_{m-\psi'(1)}$. Since $\psi'(1) < 1$, M then belongs to the class $R^{e}_{m-1+\beta}$ for some $\beta > 0$. We fix $\epsilon > 0$. So, $\mathbb{P}$ a.s., we can find a compact set $K \subset B_R$ and $\delta, D > 0$ such that $M(B_R \setminus K) \leq \epsilon$ and for all open balls $U \subset B_R$:

$$M(U \cap K) \leq D \mathrm{diam}_e(U)^{m-1+\beta+\delta}.$$

Since $m-1+\beta+\delta > \dim_H(S)$, we can find a covering of S by open balls $(U_i)_i$ such that

$$\sum_i \mathrm{diam}_e(U_i)^{m-1+\beta+\delta} < \epsilon.$$

Then we have

$$M(S) \leq M(S \setminus K) + M(S \cap K) \leq \epsilon + \sum_i M(U_i \cap K) \leq \epsilon + \sum_i \mathrm{diam}_e(U_i)^{1+\beta+\gamma} \leq 2\epsilon.$$

As we can make ϵ as small as we please, we deduce $M(S) = 0$. □

4.4 Proof of Theorems 2.5 and 2.6

We proceed recursively to prove the existence of an optimal transport between the measure $M^{(k-1)}$ and $M^{(k)}$ on some appropriate Riemannian manifold:

1) Step 1: We focus on the measure $M^{(1)}$, which has structure exponent $\xi^{(1)}(q) = mq - \frac{\psi(q)}{n}$. Relation (4.4) and the convexity of ψ imply the following inequalities

$$\frac{\psi'(1)}{n} \leq \frac{\psi(2)}{n} = \frac{1}{m}\frac{m\psi(2)}{n} < \frac{1}{m}(m - \psi'(1)) < 1.$$

Hence we can apply Theorem 2.3 to find two optimal transport maps $\chi^{(1)}, \Gamma^{(1)}$ that respectively push λ_R forward to $\overline{M}^{(1)}$ and vice versa. Furthermore, the quantity

$$\inf_{\substack{T: B_R \to B_R \\ T_\# \overline{M}^{(1)} = \lambda_R}} \int_{B_R} |T(x) - x|^2 \overline{M}^{(1)}(dx)$$

is achieved at $\Gamma^{(1)}$. We can also apply Theorem 2.4 to find a compact Riemannian manifold $(C^{(1)}, g^{(1)})$ and a Borelian subset $B^{(1)}$ of B_R such that:

-$B^{(1)}$ is dense in B_R for the Euclidian distance and has full $M^{(1)}$-measure, that is $M^{(1)}(B_R \setminus B^{(1)}) = 0$,

-$C^{(1)}$ is the completion of $B^{(1)}$ with respect to the geodesic distance on $C^{(1)}$.

-the volume form on $C^{(1)}$ coincides with the measure $M^{(1)}$ on $B^{(1)}$

-in a system of local coordinates, the Riemannian metric tensor on $C^{(1)}$ reads

$$g^{(1)} = \theta^{(1)}(\omega)(dx_1^2 + \cdots + dx_m^2) \quad \text{with} \quad \theta^{(1)}(\omega) = \frac{M^{(1)}(B_R)^2}{C_R^2}.$$

Furthermore, from Proposition 4.1 and Theorem 4.2, $M^{(1)} \in R^{e-}_{m-\psi'(1)/n}$. This ends up the first step of the induction.

2) Step 2: we assume that, for some $k < n$, we may find a compact Riemannian manifold $(C^{(k)}, g^{(k)})$ and a Borelian subset $B^{(k)}$ of B_R such that:

-$B^{(k)}$ is dense in B_R for the Euclidian distance and has full M-measure, that is $M^{(k)}(B_R \setminus B^{(k)}) = 0$,

-$C^{(k)}$ is the completion of $B^{(k)}$ with respect to the geodesic distance on $C^{(k)}$.

-the volume form on $C^{(k)}$ coincides with the measure $M^{(k)}$ on $B^{(k)}$

-in a system of local coordinates, the Riemannian metric tensor on $C^{(k)}$ reads

$$g^{(k)} = \theta^{(k)}(\omega)(dx_1^2 + \cdots + dx_m^2) \quad \text{with} \quad \theta^{(k)}(\omega) = \frac{M^{(k)}(B_R)^2}{C_R^2}.$$

We denote by $\varphi^{(k)} = \Gamma^{(1)} \circ \cdots \circ \Gamma^{(k)} : (C^{(k)}, g^{(k)}) \to (B_R, d_e)$ the isometry constructed recursively. From Proposition 4.1, the measure $M^{(k+1)}$ is supported by $B^{(k)}$ almost surely, that is $M^{(k+1)}(B_R \setminus B^{(k)}) = 0$ almost surely, so that $M^{(k+1)}$ extends to a measure on $C^{(k)}$ by prescribing:

$$\forall A \subset C^{(k)} \text{ Borelian}, \quad M^{(k+1)}(A) = M^{(k+1)}(A \cap B^{(k)}).$$

Furthermore, from subsection 4.2, we have $M^{(k)} \in R^{e-}_{m-\frac{k}{n}\psi'(1)}$. We now apply Theorem 4.2 where the distance d is equal to the geodesic distance on $C^{(k)}$, denoted by $d^{(k)}$ (the corresponding class R_α will be denoted by $R^{(k)}_\alpha$). Since $M^{(k)}$ is the volume form on $(C^{(k)}, g^{(k)})$, we have

$$M^{(k)}(U) = Dr^m, \quad \text{for any open ball U with radius r.}$$

Hence $M^{(k)} \in R^{(k)-}_m$. The assumptions of Theorem 4.2 are thus satisfied with $\alpha = m$ and $\varsigma = m - \frac{k}{n}\psi'(1)$ (and thus we have $\psi(2)/n < \varsigma$ because of (4.4)). It follows that $M^{(k+1)} \in R^{(k)-}_{m(1-\frac{\psi(2)}{mn-k\psi'(1)})+\frac{\psi(2)-\psi'(1)}{n}}$. Because of (4.4) again, we have

$$\begin{aligned} m(1 - \frac{\psi(2)}{mn - k\psi'(1)}) + \frac{\psi(2) - \psi'(1)}{n} &> m(1 - \frac{\psi(2)}{mn - k\psi'(1)}) \\ &> m - \frac{m\psi(2)}{n(m - \psi'(1))} \\ &> m - 1, \end{aligned}$$

so that we can show, as in the proof of Theorem 2.3, that $M^{(k+1)}$ does not charge the small sets of $(C^{(k)}, g^{(k)})$. Hence we can apply Theorem A.3 to find two optimal transport maps $\alpha^{(k+1)}, \beta^{(k+1)}$ that respectively push $\overline{M}^{(k)}$ forward to $\overline{M}^{(k+1)}$ and vice versa. Furthermore, $\beta^{(k+1)}$ can be rewritten as $(\varphi^{(k)})^{-1} \circ \Gamma^{(k+1)} \circ \varphi^{(k)}$ where $\Gamma^{(k+1)} : B_R \to B_R$ is the gradient of some convex function. The function $\varphi^{(k+1)} = \varphi^{(k)} \circ \beta^{(k+1)} = \Gamma^{(k+1)} \circ \cdots \circ \Gamma^{(1)}$ thus pushes $M^{(k+1)}$ forward to λ_R. Besides the quantity

$$\inf_{\substack{T: B_R \to B_R \\ T_\# \overline{M}^{(k+1)} = \overline{M}^{(k)}}} \int_{B_R} d^{(k)}(T(x), x)^2 \overline{M}^{(k+1)}(dx)$$

is achieved at $\beta^{(k+1)}$. Since $d^{(k)}(x,y) = d_e(\varphi^{(k)}(x), \varphi^{(k)}(y))$, we deduce that $\beta^{(k+1)}$ realizes the above infimum if and only if $\varphi^{(k+1)} = \varphi^{(k)} \circ \beta^{(k+1)}$ realizes the infimum

$$\inf_{\substack{T:B_R \to B_R \\ T_\# \overline{M}^{(k+1)} = \lambda_R}} \int_{B_R} |T(x) - \varphi^{(k)}(x)|^2 \overline{M}^{(k+1)}(dx).$$

Then we can apply the same machinery as in subsection 2.2 to construct the $k+1$-th Riemannian structure, which ends up the induction. □

4.5 Proof of relation (3.1)

Let $x_1, \dots, x_n \in B(0,T)$, $u_1, \dots, u_n \in \mathbb{R}$ and $\lambda \leq 1$. We have

$$\begin{aligned}
&\mathbb{E}\Big[e^{iu_1\sqrt{M(B_R)}B(\lambda x_1)+\dots+iu_n\sqrt{M(B_R)}B(\lambda x_n)}\Big] \\
&\quad = \mathbb{E}\Big[e^{iu_1\sqrt{M(B_R)}W\left(\Gamma(B\cap C(\lambda x_1))\right)+\dots+iu_n\sqrt{M(B_R)}W\left(\Gamma(B\cap C(\lambda x_n))\right)}\Big] \\
&\quad = \mathbb{E}\Big[e^{iu_1\sqrt{M(B_R)}\int_{\mathbb{R}^d}\left(\sum_{k=1}^n u_k \mathbb{1}_{\Gamma(B\cap C(\lambda x_k))}(x)\right)W(dx)}\Big].
\end{aligned}$$

By conditioning w.r.t. M, Γ that are independent from W, we deduce:

$$\begin{aligned}
&\mathbb{E}\Big[e^{iu_1\sqrt{M(B_R)}B(\lambda x_1)+\dots+iu_n\sqrt{M(B_R)}B(\lambda x_n)}\Big] \\
&\quad = \mathbb{E}\Big[e^{-\frac{1}{2}M(B_R)\int_{\mathbb{R}^d}\left(\sum_{k=1}^n u_k \mathbb{1}_{\Gamma(B\cap C(\lambda x_k))}(x)\right)^2 dx}\Big] \\
&\quad = \mathbb{E}\Big[e^{-\frac{1}{2}M(B_R)\int_{\mathbb{R}^d}\left(\sum_{k=1}^n u_k \mathbb{1}_{B\cap C(\lambda x_k)}(\chi(x))\right)^2 dx}\Big] \\
&\quad = \mathbb{E}\Big[e^{-\frac{C_R}{2}\int_{\mathbb{R}^d}\left(\sum_{k=1}^n u_k \mathbb{1}_{C(\lambda x_k)}(x)\right)^2 M(dx)}\Big].
\end{aligned}$$

Now we use the scale invariance property of the measure M (see Theorem 2.1):

$$\begin{aligned}
&\mathbb{E}\Big[e^{iu_1\sqrt{M(B_R)}B(\lambda x_1)+\dots+iu_n\sqrt{M(B_R)}B(\lambda x_n)}\Big] \\
&\quad = \mathbb{E}\Big[e^{-\frac{C_R}{2}\lambda^m e^{\Omega_\lambda}\int_{\mathbb{R}^d}\left(\sum_{k=1}^n u_k \mathbb{1}_{C(x_k)}(x)\right)^2 M(dx)}\Big].
\end{aligned}$$

By using the same computations we have the relation:

$$\begin{aligned}
&\mathbb{E}\Big[e^{iu_1\sqrt{C_R}\lambda^{m/2}e^{\frac{1}{2}\Omega_\lambda}B(x_1)+\dots+iu_n\sqrt{C_R}\lambda^{m/2}e^{\frac{1}{2}\Omega_\lambda}B(x_n)}\Big] \\
&\quad = \mathbb{E}\Big[e^{-\frac{C_R}{2}\lambda^m e^{\Omega_\lambda}\int_{\mathbb{R}^d}\left(\sum_{k=1}^n u_k \mathbb{1}_{C(x_k)}(x)\right)^2 M(dx)}\Big],
\end{aligned}$$

from which relation (3.1) follows. □

A Background about optimal transport theory

A.1 Monge problem

We remind the reader of the following classical results

Definition A.1. Push-forward of measures. *Let μ, ν be two measures respectively defined on the measured spaces E and F. We will say that a measurable mapping $\varphi : F \to E$ pushes the measure ν forward to μ if both measures μ and $\nu \circ \varphi^{-1}$ coincide. In that case, we write $\varphi_{\#}\nu = \mu$.*

Definition A.2. Small sets. *Given a metric space (X, d) with Hausdorff dimension n, a small set is a set with Hausdorff dimension not greater than $n - 1$.*

Given two probability measures μ and ν on B_R, a coupling of (μ, ν) is a probability measure π on $B_R \times B_R$ with marginals μ and ν. A coupling π is said to be deterministic if there is a measurable map $T : B_R \to B_R$ such that the map $x \in B_R \mapsto (x, T(x))$ pushes μ forward to π. In particular, for all ν-integrable function φ, one has

$$\int_{B_R} \varphi(y)\, d\nu(y) = \int_{B_R} \varphi(T(x))\, d\mu(x).$$

Such a map T is called a transport map between μ and ν.

The Monge-Kantorovich problem on the ball B_R can be formulated as follows. Given a cost function c defined on $B_R \times B_R$, one looks for a coupling π of (μ, ν) that realizes the infimum

$$C(\mu, \nu) = \inf \int_{B_R \times B_R} c(x, y) d\pi(x, y)$$

where the infimum runs over all the coupling π of (μ, ν). Such a coupling is called optimal transference plan. If the coupling π is deterministic, the corresponding transport map T is called optimal transport map. The optimal transport cost is then the value

$$\int_{B_R \times B_R} c(x, T(x)) d\mu(x).$$

The search of deterministic optimal transference plans is called the Monge problem.

A.2 Solution to the Monge problem

We have (see [11, Theorem 10.28] and [12, Theorem 2.12 iv] for the last statement)

Theorem A.3. *Let $\mathcal{X}$ be a Riemannian manifold isometric (as a smooth Riemannian manifold) to the closed ball B_R. We denote by $f : \mathcal{X} \to B_R$ the corresponding isometry of Riemannian structures. Let $c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be the cost function given by*

$$c(x, y) = d(x, y)^2$$

and μ, ν two probability measures on $\mathcal{X}$. Assume that the measure μ does not give mass to small sets. Then:

1) there is a unique (in law) optimal coupling π of (μ, ν) and it is deterministic.

2) there is a unique optimal transport map T (i.e. uniquely determined μ almost everywhere) solving the Monge problem. Furthermore, we can find a lower semi-continuous convex function ϕ defined on B_R such that

$$T(x) = f^{-1} \circ \nabla\phi \circ f(x)$$

for every $x \in f^{-1}\big(\{y \in \mathbb{R}; \phi \text{ is differentiable at } y\}\big)$.

3) $\mathrm{Supp}(\nu) = \overline{T(\mathrm{Supp}(\mu))}$.

4) Finally, if ν does not give mass to small sets either, then there is also a unique optimal transport map T' solving the Monge problem (of pushing ν forward to μ). We can also find a lower semi-continuous convex function ψ defined on B_R such that

$$T'(x) = f^{-1} \circ \nabla\phi \circ f(x)$$

for every $x \in f^{-1}\big(\{y \in \mathbb{R}; \psi \text{ is differentiable at } y\}\big)$. T and T' satisfy, for μ almost every $x \in \mathcal{X}$ and ν almost every $y \in \mathcal{X}$,

$$T' \circ T(x) = x, \quad T \circ T'(y) = y.$$

Proof. There is an easy way to deduce the above theorem from [11, Theorem 10.28]. Because of the isometry with the closed ball B_R, the above theorem is basically of Euclidian nature. Indeed, it is plain to see that π is an optimal coupling of μ, ν for the cost function $c(x, y) = d(x, y)^2$ on $\mathcal{X} \times \mathcal{X}$ if and only if $\Pi = \pi\#(f, f)$ is a coupling of the probability measures of $f\#\mu, f\#\nu$ on B_R. In the same way, $T : \mathcal{X} \to \mathcal{X}$ is an optimal transport map such that $T\#\mu = \nu$ if and only if $\theta = f \circ T \circ f^{-1}$ is an optimal transport map such that $\theta\#(f\#\mu) = f\#\nu$ for the Euclidian quadratic cost on B_R. So the proof of the above theorem boils down to applying [11, Theorem 10.28] in the Euclidian case with quadratic cost function. It is then plain to complete the proof. □

Remark A.4. *In case 4) is satisified, it is more convenient to restrict the support of T and T' respectively to $\{x \in \mathcal{X}; \phi$ is differentiable at $f(x)$ and $T' \circ T(x) = x\}$ and $\{x \in \mathcal{X}; \psi$ is differentiable at $f(x)$ and $T \circ T'(x) = x\}$. In that way, $T : \mathrm{Supp}(T) \to \mathrm{Supp}(T')$ and $T' : \mathrm{Supp}(T') \to \mathrm{Supp}(T)$ are both bijections.*

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