INVARIANCE PRINCIPLE FOR MOTT VARIABLE RANGE HOPPING AND OTHER WALKS ON POINT PROCESSES

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ABSTRACT. We consider a random walk on a homogeneous Poisson point process with energy marks. The jump rates decay exponentially in the α-power of the jump length and depend on the energy marks via a Boltzmann–like factor. The case $\alpha = 1$ corresponds to the phonon-induced Mott variable range hopping in disordered solids in the regime of strong Anderson localization. We prove that for almost every realization of the marked process, the diffusively rescaled random walk, with an arbitrary start point, converges to a Brownian motion whose diffusion matrix is positive definite and independent of the environment. Finally, we extend the above result to other point processes including diluted lattices.

Key words: random walk in random environment, Poisson point process, percolation, stochastic domination, invariance principle, corrector.

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1. Introduction and results

Random walks on random point processes such as Mott variable range hopping have been proposed in the physics literature as effective models for the analysis of the conductivity of disordered systems; see e.g. [AHL, SE]. They provide natural models of reversible random walks in random environments, which generalize in several ways the well known random conductance lattice model. Recently, several aspects of random walks on random point processes have been analyzed with mathematical rigor: diffusivity [FSS, CF2, FM]; isoperimetry and mixing times [CF1]; and transience vs. recurrence [CFG].

1.1. **The model.** Let ξ denote the realization of a simple point process on $\mathbb{R}^d$, $d \geqslant 1$, and identify ξ with the countable collection of its points. For example, one can take ξ to be a homogeneous Poisson point process, or a Bernoulli process on $\mathbb{Z}^d$. To each point x of ξ we associate an *energy mark* E_x, such that the family of energy marks is independent from the point process and is given by i.i.d. random variables taking values in the interval $[-1, 1]$. We write $\mathbb{P}$ for the law of the resulting marked simple point process $\omega = \big(\xi, \{E_x\}_{x\in\xi}\big)$, which plays the role of the random environment. Then, we consider the discrete–time random walk $(X_n,\ n \geqslant 0)$ on ξ jumping, at each time step, from a point x to a point y with probability

$$p(x, y) = \frac{r(x, y)e^{-u(E_x,E_y)}}{w(x)}\,, \quad w(x) = \sum_{z\in\xi} r(x, z)e^{-u(E_x,E_z)}\,, \tag{1.1}$$

where the functions u and r satisfy the following properties for some constants $c, \alpha > 0$:

(i) $u : [-1, 1]^2 \to \mathbb{R}_+$ is a bounded nonnegative symmetric function:

$$0 \leqslant u(E_x, E_y) = u(E_y, E_x) \leqslant c\,, \tag{1.2}$$

(ii) r is symmetric and translation invariant, i.e. $r(x,y) = r(y,x) = r(y-x)$, and

$$c^{-1} \exp\left(-c\,|x|^{\alpha}\right) \leqslant r(x) = r(-x) \leqslant c \exp\left(-c^{-1}\,|x|^{\alpha}\right), \qquad x \in \mathbb{R}^d\,, \tag{1.3}$$

Here and below $|\cdot|$ denotes euclidean distance. For this model to be well defined it suffices to assume that $w(x) < \infty$ for all $x \in \xi$ and almost all realizations of the environment (see Lemma B.3). Below, we write $X_t := X_{\lfloor t \rfloor}$, $t \geqslant 0$, and consider the associated distribution on the space $\mathcal{D} = D([0,\infty), \mathbb{R}^d)$ of right-continuous paths with left limits, equipped with the Skorohod topology.

Similarly, consider the continuous–time version of the above random walk, with state space ξ and infinitesimal generator

$$\mathbb{L}f(x) = \sum_{y \in \xi} r(x,y)\, e^{-u(E_x,E_y)} \big(f(y) - f(x)\big)\,, \qquad x \in \xi\,, \tag{1.4}$$

for bounded functions $f : \xi \to \mathbb{R}$. With some abuse of notation, the resulting random process on $\mathcal{D}$ is again denoted by $(X_t : t \geqslant 0)$. To avoid confusion we shall refer to the two processes as the DTRW (discrete-time random walk) and the CTRW (continuous-time random walk). In words, the CTRW behaves as follows: having arrived at a point $x \in \xi$, it waits an exponential time with parameter $w(x)$, after which it jumps to a point $y \in \xi$ with probability $p(x,y)$. In Lemma B.3 we give some sufficient conditions ensuring that the CTRW is well defined, i.e. no explosion takes place.

An important special case of the model introduced above is *Mott variable range hopping*, obtained by choosing

$$r(v) = e^{-|v|}\,, \quad u(E_x,E_y) = \beta\ (|E_x| + |E_y| + |E_y - E_x|) \tag{1.5}$$

where β is a positive constant proportional to the inverse temperature. Here the underlying point process is often taken as the homogeneous Poisson process or the diluted lattice $\mathbb{Z}^d$, the common law ν of the energy marks is assumed to be of the form $\nu(dE) = c|E|^{\gamma} dE$ on $[-1,1]$ for some constants $c > 0$ and $\gamma \geqslant 0$, and the relevant issue is the asymptotic behavior as $\beta \to \infty$. Mott variable range hopping is a mean field dynamics describing low temperature phonon–assisted electron transport in disordered solids, in which the Fermi level (which is 0 above) lies in a region of strong Anderson localization. The points of ξ correspond to the impurities of the disordered solid and the electron Hamiltonian has exponentially localized quantum eigenstates with localization centers $x \in \xi$ and corresponding energy E_x. The rate of transitions between the localized eigenstates can be calculated from first principles by means of the Fermi golden rule [MA, SE]. Due to localization, one can approximate the above quantum system by an exclusion process, where the hard–core interaction comes from the Pauli blocking induced by the Fermi statistics. If, however, the blocking is treated in a mean field approximation, one obtains a family of independent random walks with rates described by (1.5) in the limit $\beta \to \infty$ [MA, AHL]. Mott's law represents a fundamental principle describing the decay of the DC conductivity at low temperature [Mo, SE]. In view of Einstein's relation [S], this law can be restated in terms of the diffusivity of Mott variable range hopping.

1.2. **Invariance principle.** When we need to emphasize the dependence on the environment ω and the starting point x_0, we write $X_t(x_0,\omega)$ for the two processes defined above and $P_{x_0,\omega}$ for the associated laws on $\mathcal{D}$. Asymptotic diffusive behavior of both DTRW and CTRW is studied via the rescaled process

$$X^{(\varepsilon)}(t) := \varepsilon\, X_{t/\varepsilon^2}\,, \tag{1.6}$$

and the associated laws $P^{(\varepsilon)}_{x_0,\omega}$ on $\mathcal{D}$.

Definition 1.1. *We say that the* strong invariance principle *(SIP) holds if there exists a positive definite $d\times d$ matrix D such that $\mathbb{P}$ almost surely, for every $x_0 \in \xi$, $P^{(\varepsilon)}_{x_0,\omega}$ converges weakly to d-dimensional Browninan motion with diffusion matrix D. We say that the* weak invariance principle *(WIP) holds if the above convergence takes place in $\mathbb{P}$-probability.*

The terms *quenched* and *annealed* are sometimes used to replace *strong* and *weak*, respectively, in the above definition. Diffusive behavior of the CTRW has been rigorously investigated in [FSS]. Under suitable assumptions on the law of the point process the authors prove the WIP. Moreover, [FSS] proves lower bounds on the diffusion coefficient D in agreement with Mott's law for the special case (1.5), as $\beta \to \infty$. The corresponding upper bound is proven in [FM]. In [CF2], the authors consider the case $d = 1$, where they obtain the SIP, and analyze the large β behavior of various generalizations of the model (1.5) at the edge of subdiffusivity.

The aim of the present work is to prove the strong invariance principle in dimension $d \geqslant 2$. To state our main result we need to introduce some more notation. We write $\xi(A)$ for the number of points of ξ in a bounded Borel set $A \subset \mathbb{R}^d$. Let $\mathbb{E}$ denote the expectation associated to the law $\mathbb{P}$ of the environment ω. Set $\rho_k = \mathbb{E}\left(\xi([0,1)^d)^k\right)$, so that ρ_1 is the density and ρ_2 stands for the second moment of the point process. If ξ is a stationary simple point process with finite density, then we can consider the associated Palm distribution. If ξ is a homogeneous Poisson point process (from now on, PPP), then its Palm distribution is simply the law of the point process obtained from ξ by adding a point at the origin. In general, if $\mathbb{P}$ is the law of $\omega = (\xi, \{E_x\})$, then we let $\mathbb{P}_0$ denote the associated Palm distribution (see Section 2 for the definition) and we write $\mathbb{E}_0$ for the expectation with respect to $\mathbb{P}_0$. As explained in Lemma B.3 in the appendix, if $\rho_2 < \infty$, then $\mathbb{P}$–a.s. the law $P_{x_0,\omega}$ on $\mathcal{D}$ is well defined for both DTRW and CTRW, for all $x_0 \in \xi$. Moreover, under the same assumption, the law $P_{0,\omega}$ on $\mathcal{D}$ with starting point 0 is well defined $\mathbb{P}_0$–a.s.

Our main result applies to several examples of point processes. These include homogeneous PPP, as well as Bernoulli fields on a lattice, referred to as the *diluted lattice* case below. In Section 2.3 we describe conditions on the point process, under which all our arguments apply. Below we restrict to $d \geqslant 2$ since the one dimensional case is already treated in [CF1].

Theorem 1.2. *Let $d \geqslant 2$, $\alpha > 0$, and fix an arbitrary law ν on $[-1, 1]$. Let ξ be the realization of a homogeneous PPP, or a diluted lattice, or else any stationary simple point process with $\rho_2 < \infty$, and satisfying the assumptions listed in Section 2.3. Then, the following holds for both the DTRW and the CTRW: $\mathbb{P}_0$ almost surely, as $\varepsilon \to 0$, $P^{(\varepsilon)}_{0,\omega}$ converges weakly to d-dimensional Brownian motion with positive diffusion matrix D_{DTRW} and D_{CTRW} respectively. Moreover, the diffusion coefficients are related by*

$$D_{CTRW} = \mathbb{E}_0 w(0)\, D_{DTRW}\,. \tag{1.7}$$

The desired result is then a consequence of Theorem 1.2 together with standard properties of the Palm distribution:

Corollary 1.3. *Under the assumptions of Theorem 1.2, the strong invariance principle holds for both DTRW and CTRW, with the same diffusion matrices appearing in Theorem 1.2.*

As a consequence of the above result, for the model (1.5) the quenched diffusion matrix D_{CTRW} satisfies stretched exponential estimates as $\beta \to \infty$, in agreement with Mott's law. This follows from the bounds of [FSS] and [FM] on the annealed diffusion matrix and the fact that the quenched and annealed diffusion matrices must coincide.

1.3. **Background and discussion.** To illustrate the kind of difficulties encountered in the proof of Theorem 1.2, let us briefly recall the standard approach (see [Koz, KV, DFGW] and references therein) for the invariance principle in the case of reversible random walks in random environment. The main idea is to consider the environment as seen from the moving particle, and to use this new Markov process to define a displacement field $\chi(x) = \chi(\omega, x)$ that compensates the local drift felt by the random walk X_t in such a way that the process $M_t := X_t + \chi(X_t)$ defines a martingale. The displacement field χ is usually referred to as the *corrector*. A strong invariance principle for the martingale M_t can be obtained in a rather standard way, so that what remains is to show that the corrector's contribution is negligible. In particular, one needs that for every $t > 0$:

$$\lim_{\varepsilon \to 0} \varepsilon \, \chi(X_{t/\varepsilon^2}, \omega) = 0 \,, \qquad \text{in } P_{x_0,\omega}\text{- probability} \,. \tag{1.8}$$

Roughly speaking, the L^2-theory developed in [KV, DFGW] allows to obtain the statement (1.8) in probability with respect to the random environment. This approach can then be used to prove the WIP, as detailed in [FSS]. Moreover, this approach provides an expression for the limiting diffusion matrix in terms of a variational principle. However, to have the strong invariance principle, (1.8) must hold almost surely with respect to the environment. This turns out to be related to a highly non trivial ergodic property of the field χ.

The same difficulty appears in analogous investigations for the random conductance model in $\mathbb{Z}^d$. In this model, one has i.i.d. non-negative weights $r(x, y)$ on the nearest neighbor edges $\{x, y\}$ of $\mathbb{Z}^d$, so that the random walk with generator (1.4) becomes a reversible nearest neighbor lattice walk. When the weights $r(x, y)$ are uniformly positive and bounded, the SIP for this model has been proved in [SS]; see also [Koz, Bo, BoDe]. In the case of super-critical Bernoulli weights, [SS] proved the SIP for $d \geqslant 4$. Later, [MP, BB] proved the SIP for all $d \geqslant 2$. These results were recently extended in [M, BP] to the general case of bounded but possibly vanishing weights, under the only assumption that positive weights percolate. More recent developments include the case of unbounded weights [BD]. All these works prove the SIP using the approach outlined above, although the techniques used may differ. Following [SS, BB, MP], an important ingredient for the proof of estimate (1.8) is represented by suitable heat kernel and isoperimetric estimates. However, it is known that such estimates cannot hold if the system lacks ellipticity, i.e. if arbitrarily small weights are allowed; see [FoMa, BBHK]. An important idea of [M, BP] to overcome this problem was then to consider the random walk embedded in an elliptic cluster and to control the corrector for this restricted process.

Our random walk on random point process has several features in common with the random conductance model. The lack of ellipticity corresponds to the existence in the point process of regions of isolated points, where the walk can be trapped. For instance, it was shown in [CF1] that the existence of these traps is responsible for the loss of diffusive isoperimetric and Poincaré inequalities, as soon as the power α in (1.3) is larger than the dimension d.

On the other hand, there are some important differences with respect to previous work on the random conductance model: the long-range nature of the jumps, the existence of

overcrowded regions (i.e. regions with atypically high density of points) and the intrinsically non-deterministic nature of the state space, that is the lack of a natural lattice structure for the point process. As we will see, these are the source of new technical difficulties.

As in [M, BP], we are forced to work with a suitable cluster of good points. In our setting this good set has to be defined in such a way that: (i) good points x must have uniformly bounded weights $w(x)$, (ii) given two good points x, y there must exist a path from x to y visiting only good points with uniformly bounded jump lengths. The requirement (ii) alone could be achieved by a simple local construction as in [M, BP]. On the other hand, due to long jumps, non negligible contributions to the weights $w(x)$ may come from arbitrarily far overcrowded regions. Therefore, requirement (i) forces a nonlocal construction of the family of good points, making harder any quantitative control on its geometry. For the homogeneous PPP, this problem is solved by showing that a suitable discretized $\{0,1\}$–random field with infinite–range spatial correlations stochastically dominates a supercritical Bernoulli field on $\mathbb{Z}^d$; see Section 2.

In addition, the ability of the walk to take long jumps has led us to the development of an extended version of the analysis of "holes" in the cluster of good points needed in [M, BP]. In particular, a suitable *enlargement of holes* is required in Section 4.1 to gain some control on the number of jumps and the distance traveled by the walk between successive visits to the cluster of good points.

A convenient way to deal with the lack of a lattice structure, and to obtain statements valid for every starting point $x_0 \in \xi$, is to work with the Palm distribution of the point process, as in the statement of Theorem 1.2. On the other hand, the method developed in [BB] to establish sublinearity of the corrector is intrinsically based on a lattice structure and, at a first analysis, the Palm distribution and the lattice strategy of [BB] seem to collide. To overcome this conceptual obstacle, in several steps of the proof, we have introduced intermediate "bridge" distributions (cf. Sections 7.2, 7.3, and 7.6). These distributions are probability measures on the space of the environments, having both a lattice structure and an absolutely continuous Radon–Nykodim derivative with respect to the Palm distribution (or other related distributions that appear along the proof). An alternative option would be to follow [MP, M] rather than [BB, BP] to establish (1.8). This approach is more naturally adapted to the continuum setting. However, it is more demanding in terms of heat kernel and tightness estimates and more extra work would be needed to establish the bounds used there.

It is worthy of note that similar problems are encountered when analyzing random walks on Voronoĭ tessellations or random walks on the infinite cluster of the supercritical continuous percolation for Poisson processes. Some of the methods developed here are likely to find applications in the analysis of these other models.

1.4. **Outline of the paper.** As we mentioned, the proof of Theorem 1.2 is entirely based on a suitable control of the corrector field. Since the energy marks play a very minor role in such an estimate, for the sake of simplicity we set $u(E_x, E_y) = 0$, throughout most of the paper, and we identify the environment ω with the point process ξ. The extension to non trivial energy marks will be discussed only in Section 8. Another simplification which causes very little loss of generality is obtained by setting $r(x) = e^{-|x|^\alpha}$, for some $\alpha > 0$, throughout the rest of the paper.

In Section 2 we take a close look at the random environment, state our main assumptions and define the cluster of good points. In particular, in Section 2.4 we verify that the

homogeneous PPP satisfies the main assumptions. The corrector field is introduced in Section 3. The main sublinearity estimate for the corrector is stated in Section 3.3, cf. Theorem 3.6. There, this estimate is shown to imply Theorem 1.2 and Corollary 1.3. The rest of the paper is then devoted to the proof of the sublinearity estimate. Section 4 introduces the restricted random walk, i.e. the random walk X_n embedded in the cluster of good points. In particular, we state two crucial estimates: the heat kernel bound and the expected distance bound. This section also contains the analysis of "holes" in the cluster of good points. The heat kernel bound is proved in Section 5, while the expected distance bound is proved in Section 6. Section 7 is entirely devoted to the proof of the sublinearity estimate. Finally, Section 8 deals with the slight modifications needed in the presence of energy marks. The appendix collects several technical results used in the main text.

2. The random environment

Since we have set $u(\cdot,\cdot) = 0$, we may disregard the energy marks, and the random environment coincides with the state space ξ of the random walk, i.e. the point process.

2.1. **Stationary, ergodic simple point process and Palm measure.** We denote by $\mathcal{N}$ the family of locally finite subsets ξ of $\mathbb{R}^d$ endowed with the σ–algebra generated by the sets $\{\xi(A_1) = n_1, \dots, \xi(A_k) = n_k\}$, $A_1, \dots, A_k$ being disjoint bounded Borel subsets of $\mathbb{R}^d$, $n_1, \dots, n_k$ varying in $\mathbb{N} = \{0, 1, \dots\}$ and $\xi(A) := |\xi \cap A|$. Elements $\xi \in \mathcal{N}$ are usually identified with the counting measure on ξ. Moreover, given $\xi \in \mathcal{N}$ and $x \in \mathbb{R}^d$, we denote by $\tau_x\xi$ the translated set $\xi - x$. A simple point process is a measurable map from a probability space to the measurable space $\mathcal{N}$.

Fix a simple point process on $\mathbb{R}^d$ with law $\mathbb{P}$, ergodic and stationary w.r.t. the group of space translations, having finite density $\rho = \rho_1 = \mathbb{E}(\xi([0,1)^d))$. Due to stationarity ρ can also be expressed as $\mathbb{E}(\xi(A))/\ell(A)$ for any bounded Borel subset $A \subset \mathbb{R}^d$ having positive Lebesgue measure $\ell(A)$. We denote by $\mathbb{P}_0$ the Palm distribution associated to $\mathbb{P}$. Considering the measurable subset $\mathcal{N}_0 = \{\xi \in \mathcal{N} : 0 \in \xi\}$, $\mathbb{P}_0$ is a probability law on $\mathcal{N}_0$ coinciding, roughly speaking, with "$\mathbb{P}(\cdot|0 \in \xi)$" (cf. Theorem 12.3.V in [DV]). A key relation between $\mathbb{P}$ and $\mathbb{P}_0$ is given by the Campbell identity [DV]: for any nonnegative measurable function f on $\mathbb{R}^d \times \mathcal{N}_0$

$$\int_{\mathbb{R}^d} dx \int_{\mathcal{N}_0} \mathbb{P}_0(d\xi) f(x,\xi) = \frac{1}{\rho} \int_{\mathcal{N}} \mathbb{P}(d\xi) \int_{\mathbb{R}^d} \xi(dx) f(x, \tau_x\xi)\,. \tag{2.1}$$

2.2. **Black and white boxes.** For any $K > 0$ we write $B_K = [0,K)^d$ for the cube of side K in $\mathbb{R}^d$. Boxes $B(z) := B_K + Kz$, $z \in \mathbb{Z}^d$, are generically called K–boxes. We also use the notation $B_z = B(z)$, for the K-box at $z \in \mathbb{Z}^d$. A K–box $B(z)$ is called *occupied* if $\xi \cap B(z) \neq \emptyset$. We encode this information in the field $\sigma = (\sigma_z : z \in \mathbb{Z}^d)$ defined on $\mathcal{N}$ by

$$\sigma_z(\xi) = \begin{cases} 1 & \text{if } B(z) \text{ is occupied}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.2}$$

Let us now introduce another parameter $T_0 > 0$. A K–box $B(z)$ is called *overcrowded* if the number of points of ξ in $B(z)$, $n_z := \xi(B(z))$, satisfies $n_z \geqslant T_0$. We define

$$R_z(\xi) = \begin{cases} (\log n_z)^{2/\alpha} & \text{if } B(z) \text{ is overcrowded}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.3}$$

Next, we define $\mathcal{G} = \cup_{z\in\mathbb{Z}^d} Q(z, R_z)$, where $Q(z,r) = \{z' \in \mathbb{Z}^d : |z - z'|_\infty < r\}$. Note that $Q(z, 0) = \emptyset$. Of course, $\mathcal{G}$ contains all points z such that $B(z)$ is overcrowded. The interest in the set $\mathcal{G}$ comes from the following simple estimate.

Lemma 2.1. *There exists a positive constant $T = T(\alpha, K, T_0)$ such that $w(x) \leqslant T$, for all $x \in \xi \cap B(z)$ with $z \in \mathbb{Z}^d \setminus \mathcal{G}$.*

Proof. Note that if $x \in B(z)$ and $y \in B(v)$ then $|x-y|_\infty \geqslant K|z-v|_\infty - 2K$. Therefore we can find positive constants c_1, c_2 (depending on α, K, T_0) such that, for any $x \in B(z) \cap \xi$ we have

$$w(x) \leqslant c_1 \sum_{v\in\mathbb{Z}^d} n_v\, e^{-c_2|z-v|_\infty^\alpha}\,. \tag{2.4}$$

Since $z \in \mathbb{Z}^d \setminus \mathcal{G}$, it must be that all points $v \in \mathbb{Z}^d$ satisfy $|z-v|_\infty \geqslant (\log n_v)^{2/\alpha}$. Therefore $n_v \leqslant \exp\{|z-v|_\infty^{\alpha/2}\}$ and using this in (2.4) one has $w(x) \leqslant c_3$ for some new constant $c_3 = c_3(\alpha, K, T_0)$. □

We call a point $z \in \mathbb{Z}^d$ *black* if z belongs to $\mathcal{G}$ or if the box $B(z)$ is unoccupied. If z is not black, we call it *white*. From Lemma 2.1, if z is white then $w(x) \leqslant T$, for every $x \in \xi \cap B(z)$, for some constant T. Finally, we introduce the field $\vartheta = (\vartheta_z : z \in \mathbb{Z}^d)$ defined on $\mathcal{N}$ as

$$\vartheta_z(\xi) = \begin{cases} 0 & \text{if } z \text{ is black}\,, \\ 1 & \text{if } z \text{ is white}\,. \end{cases} \tag{2.5}$$

The random fields $\sigma(\xi)$ and $\vartheta(\xi)$, where ξ is sampled with law $\mathbb{P}$, are often denoted simply σ, ϑ. We shall write σ^K, ϑ^{K,T_0}, when the dependence on the parameters K, T_0 needs to be emphasized. Clearly, these random fields are stationary w.r.t. $\mathbb{Z}^d$–translations due to the stationarity of $\mathbb{P}$.

2.3. **Main assumptions on the point process.** Given a stationary, ergodic point process with finite density ρ and law $\mathbb{P}$, we shall make the following assumptions:

(H1) For each $p \in (0,1)$ there exist $K, T_0 > 0$ such that the random field of white points ϑ^{K,T_0} stochastically dominates the independent Bernoulli process $Z(p)$ on $\mathbb{Z}^d$ with parameter p.

(H2) For each $K > 0$ and for each vector $e \in \mathbb{Z}^d$ with $|e|_1 = 1$, consider the product probability space $\Theta := \mathcal{N} \times ([0,K)^d \cup \{\partial\})^{\mathbb{Z}}$ whose elements $\big(\xi, (a_i : i \in \mathbb{Z})\big)$ are sampled as follows: choose ξ with law $\mathbb{P}$, and then choose independently for each index i a point $b_i \in \xi \cap B(ie)$ with uniform probability and set $a_i := b_i - iKe \in [0,K)^d$. If $\xi \cap B(ie) = \emptyset$, set $a_i = \partial$. We assume that the resulting law $P^{(K,e)}$ on $\mathcal{N} \times ([0,K)^d \cup \{\partial\})^{\mathbb{Z}}$ is ergodic w.r.t. the transformation

$$\tau : \big(\xi, (a_i : i \in \mathbb{Z})\big) \to \big(\tau_{Ke}\xi, (a_{i+1} : i \in \mathbb{Z})\big)\,. \tag{2.6}$$

2.3.1. *Remarks.* Since $\vartheta_z = 1$ implies $\sigma_z = 1$, it is clear that assumption (H1) implies the following statement, which we shall refer to as Property (A):

(A) For all $p \in (0,1)$, there exists $K > 0$ such that the random field σ^K stochastically dominates the independent Bernoulli process $Z(p)$ on $\mathbb{Z}^d$ with parameter p.

Also, it is not hard to check that if $p(K)$ is the largest p such that σ^K stochastically dominates $Z(p)$, then $p(mK) \geqslant (1-(1-p(K))^{m^d})$, and therefore $p(mK) \to 1$ as $m \to \infty$.

Observe that the law $P^{(K,e)}$ is invariant w.r.t. the transformation τ (due to the stationarity of $\mathbb{P}$). Assumption (H2) means that, for each $K > 0$ and for each vector $e \in \mathbb{Z}^d$ with

$|e|_1 = 1$, any measurable subset $\mathcal{A} \subset \Theta$ such that $\tau\mathcal{A} = \mathcal{A}$ must have $P^{(K,e)}$–probability 0 or 1. We point out that assumption (H2) alone implies that $\mathbb{P}$ is ergodic.

When working with the energy marks, assumption (H2) will be slightly modified as discussed in Section 8.

2.4. **The homogeneous PPP satisfies (H1)–(H2).** The homogeneous PPP with density ρ is plainly an ergodic, stationary simple point process with finite moments of any order. In order to prove assumption (H2), we fix $\mathcal{A} \subset \Theta$ such that $\tau\mathcal{A} = \mathcal{A}$ and set $P = P^{(K,e)}$. If $\mathcal{A}$ depends only on ξ restricted to $[-\ell K, \ell K]$ and on $\{a_i : |i| \leqslant \ell - 1\}$ for some integer ℓ, then $\mathcal{A}$ and $\tau^{m\ell}\mathcal{A}$ are independent for m large and therefore $P(\mathcal{A}) = P(\mathcal{A} \cap \tau^{m\ell}\mathcal{A}) = P(\mathcal{A})P(\tau^{m\ell}\mathcal{A}) = P(\mathcal{A})^2$. This implies that $P(\mathcal{A}) \in \{0, 1\}$. The general case can be treated by a standard approximation argument. The rest of this section is concerned with the proof of assumption (H1), which we reformulate as follows.

Theorem 2.2. *For every $p \in (0, 1)$ and $\rho > 0$, there exist constants K, T_0, depending on p and ρ, such that, for the homogeneous PPP with density ρ, the random field $\vartheta = \vartheta^{K,T_0}$ defined in* (2.5) *stochastically dominates the Bernoulli field $Z(p)$ with parameter p.*

2.4.1. *Preliminary estimates.* Before we start the proof of Theorem 2.2 we shall establish a few preliminary facts.

Lemma 2.3. *A Poisson variable N with mean λ satisfies*

$$\mathbb{P}(N > t) \leqslant \exp\{-t(\log t - \log\lambda) + t - \lambda\} \leqslant \exp\{-t\}\,, \qquad \forall t \geqslant e^2\lambda.$$

Proof. Take $s = \log(t/\lambda)$ in the following expression, valid for all $s \geqslant 0$:

$$\mathbb{P}(N > t) = \mathbb{P}(e^{sN} > e^{st}) \leqslant e^{-st}\mathbb{E}(e^{sN}) = \exp\{-st + \lambda e^s - \lambda\}\,. \qquad \square$$

Next, recall the definition (2.3) of the set $\mathcal{G}$. The random variables n_z are i.i.d. Poisson variables with mean ρK^d, and using Lemma 2.3, the variables R_z satisfy

$$\mathbb{P}(R_z > r) = \mathbb{P}\big(n_z \geqslant \exp\big(r^{\frac{\alpha}{2}}\big)\,,\, n_z \geqslant T_0\big) \leqslant \exp\big(-\exp\big(r^{\frac{\alpha}{2}}\big)\big) \tag{2.7}$$

whenever $\exp\big(r^{\frac{\alpha}{2}}\big) \geqslant e^2\rho K^d$.

Set $\gamma_m = \exp\big(-\exp(m^{\frac{\alpha}{4}})\big)$, $m \in \mathbb{N}$, and consider the Bernoulli random field $Z(\gamma_m)$ on $\mathbb{Z}^d$ with parameter γ_m. Next, let $\{Z(\gamma_m), m \in \mathbb{N}\}$ denote an independent sequence of the random fields $Z(\gamma_m)$ on some probability space with law P and set

$$\widetilde{R}_z := \sup\{m \geqslant m_0 \,:\, Z(\gamma_m)_z = 1\}\,,$$

with the convention that the supremum of the emptyset is 0. Here and below m_0 is a constant related to T_0 by

$$T_0 = \exp\Big(m_0^{\alpha/2}\Big)\,. \tag{2.8}$$

Note that the random variables $\widetilde{R}_z$, $z \in \mathbb{Z}^d$, are independent. Moreover, $\widetilde{R}_z$ is finite P–a.s. since

$$E\Big(\sum_{m=m_0}^{\infty} Z(\gamma_m)_z\Big) = \sum_{m=m_0}^{\infty} \gamma_m < \infty\,.$$

Lemma 2.4. *For all $\rho, K > 0$, there exists a constant T_0 such that, for all $z \in \mathbb{Z}^d$:*

$$P(\widetilde{R}_z < t) \leqslant \mathbb{P}(R_z < t)\,, \qquad \forall t > 0\,. \tag{2.9}$$

Proof. For every $t > 0$,

$$P(\widetilde{R}_z < t) = \prod_{k \geqslant \lceil t \rceil \vee m_0} (1 - \gamma_k) \leqslant \exp\Big\{ - \sum_{k \geqslant \lceil t \rceil \vee m_0} \gamma_k \Big\} .$$

If $t \leqslant m_0$, then $P(\widetilde{R}_z < t) = P(\widetilde{R}_z = 0) = \prod_{k=m_0}^{\infty}(1-\gamma_k)$ which can be bounded by $e^{-\gamma_{m_0}}$. Now, take m_0, T_0 as in (2.8) and assume $T_0 \geqslant e^2 \rho K^d$. In particular, $R_z < m_0$ is equivalent to $R_z = 0$, cf. (2.3), and (2.7) holds for all $r \geqslant m_0$. Therefore, for all $t \leqslant m_0$ one has

$$\mathbb{P}(R_z < t) = \mathbb{P}(R_z = 0) = 1 - \mathbb{P}(R_z \geqslant m_0) \geqslant 1 - \exp\big(-\exp\big(m_0^{\alpha/2}\big)\big).$$

Using that $e^{-2x} \leqslant 1 - x$ for $x \in [0, \frac{1}{2}]$, and that $\exp\big(-\exp\big(m_0^{\alpha/2}\big)\big) \leqslant \gamma_{m_0}/2$ for m_0 sufficiently large, we conclude that $\mathbb{P}(R_z < t) \geqslant 1 - \gamma_{m_0}/2 \geqslant e^{-\gamma_{m_0}}$ for all $t \leqslant m_0$. This concludes the proof of (2.9) for $0 < t \leqslant m_0$.

Suppose now that $m_0 \leqslant m - 1 < t \leqslant m$. Then $P(\widetilde{R}_z < t) = \prod_{k=m}^{\infty}(1-\gamma_k) \leqslant e^{-\gamma_m}$. On the other hand, reasoning as above we see that $\mathbb{P}(R_z \geqslant m-1) \leqslant \frac{1}{2}\gamma_m$ and therefore

$$\mathbb{P}(R_z < t) \geqslant \mathbb{P}(R_z \leqslant m-1) = 1 - \mathbb{P}(R_z \geqslant m-1) \geqslant 1 - \frac{1}{2}\gamma_m \geqslant e^{-\gamma_m} .$$

This ends the proof of the lemma. □

Lemma 2.4 implies that the random field $R = (R_z : z \in \mathbb{Z}^d)$ is stochastically dominated by the random field $\widetilde{R} = (\widetilde{R}_z : z \in \mathbb{Z}^d)$. Taking a coupling between R and $\widetilde{R}$ on an enlarged probability space such that $R_z \leqslant \widetilde{R}_z$ for all $z \in \mathbb{Z}^d$ a.s. we get that

$$\mathcal{G} = \cup_{a \in \mathbb{Z}^d} Q(a, R_a) \subset \widetilde{\mathcal{G}} := \cup_{a \in \mathbb{Z}^d} Q(a, \widetilde{R}_a) . \tag{2.10}$$

The random set $\widetilde{\mathcal{G}}$ can be described by the random field $Y = (Y_z : z \in \mathbb{Z}^d)$ (in the sense that $z \in \widetilde{\mathcal{G}}$ if and only if $Y_z = 1$) where

$$Y_z := \max\{Y_z^{(m)} : m \geqslant m_0\} ,$$

and, for each m,

$$Y_z^{(m)} := \begin{cases} 1 & \text{if } \exists a \in \mathbb{Z}^d : Z(\gamma_m)_a = 1, z \in Q(a,m) , \\ 0 & \text{otherwise} . \end{cases}$$

Lemma 2.5. *For every $K, \rho > 0$, there exist a constant T_0 such that for each $m \geqslant m_0$, the random field $Y^{(m)}$ is stochastically dominated by the Bernoulli random field $Z(q_m)$ with parameter $q_m := 2^{-m+1}$.*

Proof. We apply a result of [LSS] on stochastic domination, in the form which appears in Theorem (7.65) in [G]. Namely, set $\ell = 2(m+1)+1$, so that $Y^{(m)}$ is a ℓ–dependent field taking values in $\{0,1\}$. Note that

$$P(Y_0^{(m)} = 1) \leqslant \ell^d \gamma_m . \tag{2.11}$$

Suppose that there exist two parameters $u, v > 0$ such that

$$(1-u)(1-v)^{\ell^d} \geqslant \ell^d \gamma_m ,$$
$$(1-u)u^{\ell^d} \geqslant \ell^d \gamma_m .$$

Then, by Eqs. (7.114)–(7.115) in [G], we know that $Y^{(m)}$ is stochastically dominated by an independent Bernoulli random field with parameter $1 - uv$. Therefore, we have to prove

that u and v can be taken in such a way that $1 - uv \leqslant 2^{-m+1}$. This can be achieved by the choice $u = 1 - \gamma_m \ell^d 2^{m\ell^d}$ and $v = 1 - 2^{-m}$. Indeed, $(1-u)(1-v)^{\ell^d} = \ell^d \gamma_m$ and $(1-u)u^{\ell^d} \geqslant \gamma_m \ell^d$ if m is large enough (using that $1 - x \leqslant e^{-x}$). Moreover, by definition $1 - uv \leqslant \gamma_m \ell^d 2^{m\ell^d} + 2^{-m} \leqslant 2^{-m+1}$ if m is large enough. This concludes the proof. □

Since all random fields $Y^{(m)}$ are independent, thanks to the above lemma we can build, on a suitable probability space, the random fields $(Y^{(m)}, Z(q_m))$, $m \geqslant m_0$, such that they are all independent and

$$Y^{(m)}_z \leqslant Z(q_m)_z \qquad \forall z \in \mathbb{Z}^d\,, \qquad \text{a.s.}$$

In particular we have that $Y = \max\{Y^{(m)} : m \geqslant m_0\}$ is stochastically dominated by the random field $Z := \max\{Z(q_m) : m \geqslant m_0\}$. Note that Z is a Bernoulli random field, with parameter

$$q = \mathbb{P}(Z_0 = 1) \leqslant \sum_{m=m_0}^{\infty} q_m = \sum_{m=m_0}^{\infty} 2^{-m+1} = 2^{-m_0+2}\,. \tag{2.12}$$

2.4.2. *Proof of Theorem 2.2.* The results discussed above can be summarized as follows.

Proposition 2.6. *For every $K, \rho > 0$, and $\varepsilon > 0$, there exists T_0 such that, for the homogeneous PPP with intensity ρ, the random set $\mathcal{G} = \mathcal{G}(K, T_0)$ is stochastically dominated by the Bernoulli field $Z(\varepsilon)$ with parameter ε.*

Proof. From Eq. (2.10), Lemma 2.5 and Eq. (2.12), it suffices to take T_0 (and therefore, by (2.8), m_0) so large that $q \leqslant \varepsilon$. □

We can now conclude the proof of Theorem 2.2. Let us fix $p \in (0,1)$ and $\varepsilon = 1 - \sqrt{p}$. Then fix $K = K(\rho)$ such that $\mathbb{P}(\xi(B(0)) = 0) \leqslant \varepsilon/2$. Also, let $t_0 > 0$ be so large that $\mathbb{P}(\xi(B(0)) = 0 \,|\, \xi(B(0)) < t) \leqslant \varepsilon$ for all $t \geqslant t_0$.

Next, choose $T_0 = T_0(K, \rho)$ so large that $\mathcal{G}$ is stochastically dominated by $Z(\varepsilon)$ as in Proposition 2.6. Let $\omega_z = 1$ if $z \in \mathbb{Z}^d \setminus \mathcal{G}$ and $\omega_z = 0$ otherwise. For fixed z, let $A_z \subset \{0,1\}^{\mathbb{Z}^d}$ be an arbitrary measurable set such that $A_z \subset \{\omega_z = 1\}$. If $T_0 \geqslant t_0$ then, by independence of the Poisson field, one has

$$\mathbb{P}\big(\xi(B(z)) > 0 \,|\, \omega \in A_z\big) = \mathbb{P}\big(\xi(B(z)) > 0 \,|\, \xi(B(z)) < T_0\big) \geqslant 1 - \varepsilon\,.$$

Since this bound is uniform over all possible values of $\omega_{z'}$, $z' \neq z$, it follows that the set of white boxes ϑ, i.e. $\vartheta_z = \omega_z \sigma_z$, $z \in \mathbb{Z}^d$, stochastically dominates the Bernoulli field with parameter $(1-\varepsilon)^2 = p$. This ends the proof.

3. The corrector field χ

Let μ be the measure on $\mathcal{N}_0 \times \mathbb{R}^d$ such that the scalar product in $L^2(\mathcal{N}_0 \times \mathbb{R}^d, \mu)$ is given by

$$(u, v)_\mu = \mathbb{E}_0\Big[\sum_{x \in \xi} r(x)\, u(\xi, x)\, v(\xi, x)\Big]\,. \tag{3.1}$$

Since $\mathbb{P}$ has a finite second moment, by Lemma B.1 in the appendix

$$(1,1)_\mu = \mathbb{E}_0\Big[\sum_{x \in \xi} r(x)\Big] = \mathbb{E}_0[w(0)] < \infty\,.$$

3.1. **Potential vs. solenoidal forms.** We call $u \in L^2(\mu)$ a *square integrable form*. In what follows we shall study this space in some detail. In general, we will call a *form* any measurable function $u\colon \mathcal{N}_0 \times \mathbb{R}^d \to \mathbb{R}$.

Given $\psi\colon \mathcal{N}_0 \to \mathbb{R}$ we define the *gradient form* $\nabla\psi$ as

$$\nabla\psi(\xi, x) := \psi(\tau_x\xi) - \psi(\xi)\,. \tag{3.2}$$

Hence, $\nabla\psi \in L^2(\mu)$ whenever ψ is bounded (written $\psi \in B(\mathcal{N}_0)$).

The space $\mathcal{H}_\nabla \subset L^2(\mu)$ of *potential forms* is defined as the closure of the subspace given by the gradient forms $\nabla\psi$ with $\psi \in B(\mathcal{N}_0)$. Its orthogonal complement $\mathcal{H}_\nabla^\perp$ is the space of *solenoidal forms*.

A form $u\colon \mathcal{N}_0 \times \mathbb{R}^d \to \mathbb{R}$ is called *curl–free* if for any $\xi \in \mathcal{N}_0$, $n \geqslant 1$ and any family of n points $x_0, x_1, \dots, x_n \in \xi$ with $x_0 = x_n$, we have

$$\sum_{j=0}^{n-1} u(\tau_{x_j}\xi, x_{j+1} - x_j) = 0\,. \tag{3.3}$$

A square integrable form $u \in L^2(\mu)$ is called curl–free if this holds for $\mathbb{P}_0$–a.e. ξ.

Lemma 3.1. *Each potential form $u \in \mathcal{H}_\nabla$ is curl–free.*

Proof. This is trivial to check for $u = \nabla\psi$, $\psi \in B(\mathcal{N}_0)$. In the general case, let ψ_n be a sequence in $B(\mathcal{N}_0)$ such that $\nabla\psi_n$ converges to u in $L^2(\mu)$. By taking a subsequence we can assume that the convergence holds also μ–a.s. Since each $\nabla\psi_n$ satisfies (3.3) $\mathbb{P}_0$–a.s. by taking the limit in (3.3) we conclude that the same identity holds for u. □

A form u is called *shift–covariant* if

$$u(\xi, x) = u(\xi, y) + u(\tau_y\xi, x - y)\,, \qquad \forall\, x, y \in \xi\,. \tag{3.4}$$

If u is a square integrable form, we call it shift–covariant if the above property holds for $\mathbb{P}_0$–a.a. ξ.

Lemma 3.2. *Each curl–free form is shift–covariant.*

Proof. Let $u : \mathcal{N}_0 \times \mathbb{R}^d \to \mathbb{R}$ be a curl–free form. Taking in (3.3) $n = 3$, $x_0 = x_3 = 0$ (recall that $\xi \in \mathcal{N}_0$), $x_1 = y$ and $x_2 = x$, we get that

$$u(\xi, y) + u(\tau_y\xi, x - y) + u(\tau_x\xi, -x) = 0\,. \tag{3.5}$$

On the other hand, taking in (3.3) $n = 2$, $x_0 = x_2 = 0$ and $x_1 = x$, we obtain

$$u(\xi, x) + u(\tau_x\xi, -x) = 0 \tag{3.6}$$

for any ξ. From (3.5) and (3.6) one obtains (3.4). □

Given $u \in L^2(\mu)$ we define the *divergence* as $\operatorname{div} u\,(\xi) = \sum_{x\in\xi} r(x)u(\xi, x)$. Since $\mathbb{E}_0|\operatorname{div} u| \leqslant (u,u)_\mu^{1/2}(1,1)_\mu^{1/2}$, we have that $\operatorname{div} u \in L^1(\mu)$. By these definitions, we have a key relation between the gradient and divergence.

Lemma 3.3. *For each $\psi \in B(\mathcal{N}_0)$ and each curl–free $u \in L^2(\mu)$*

$$(u, \nabla\psi)_\mu = -2\mathbb{E}_0\left[\psi\, \operatorname{div} u\right]\,. \tag{3.7}$$

In particular, a form $u \in L^2(\mu)$ is solenoidal (that is, $u \in \mathcal{H}_\nabla^\perp$) if and only if $\operatorname{div} u = 0$, $\mathbb{P}_0$*–a.s.*

Proof. We only need to prove (3.7), since the last statement is then obvious. Due to (3.6) (which holds for all $x \in \xi$, $\mathbb{P}_0$–a.s.) we can rewrite the l.h.s. of (3.7) as

$$(u, \nabla\psi)_\mu = -\mathbb{E}_0\Big[\sum_{x\in\xi} r(x)u(\tau_x\xi, -x)\psi(\tau_x\xi)\Big] - \mathbb{E}_0\Big[\sum_{x\in\xi} r(x)u(\xi, x)\psi(\xi)\Big]\,. \tag{3.8}$$

We define the function f on $\mathcal{N}_0\times\mathbb{R}^d$ as $f(\xi, x) = r(x)u(\xi, x)\psi(\xi)$. Then it holds $f(\tau_x\xi, -x) = r(x)u(\tau_x\xi, -x)\psi(\tau_x\xi)$. In addition,

$$\mathbb{E}_0\Big[\sum_{x\in\xi} f(\xi, x)\Big] \leqslant \|\psi\|_\infty \mathbb{E}_0[w(0)]^{1/2}(u, u)_\mu^{1/2} < \infty\,.$$

This allows us to apply Lemma B.1 (i) in the Appendix to the function f and to conclude that

$$\mathbb{E}_0\Big[\sum_{x\in\xi} r(x)u(\xi, x)\psi(\xi)\Big] = \mathbb{E}_0\Big[\sum_{x\in\xi} r(x)u(\tau_x\xi, -x)\psi(\tau_x\xi)\Big] \tag{3.9}$$

(by Lemma B.1 (i) we know that the integrand in the r.h.s. belongs to $L^1(\mathbb{P}_0)$). The above identity allows then to rewrite the r.h.s. of (3.8) as

$$-2\mathbb{E}_0\Big[\sum_{x\in\xi} r(x)u(\xi, x)\psi(\xi)\Big] = -2\mathbb{E}_0[\psi\,\mathrm{div}\,u]\,. \qquad \square$$

Lemma 3.4. *Let $u \in \mathcal{H}_\nabla^\perp$. Then for $\mathbb{P}_0$–a.a. ξ*

$$\sum_{y\in\xi} r(y)|u(\xi, y)| < \infty\,, \qquad \sum_{y\in\xi} r(y)u(\xi, y) = 0\,. \tag{3.10}$$

In particular, for $\mathbb{P}$–a.a. ξ and for all $x \in \xi$

$$\sum_{y\in\xi} r(y - x)|u(\tau_x\xi, y - x)| < \infty\,, \qquad \sum_{y\in\xi} r(y - x)u(\tau_x\xi, y - x) = 0\,. \tag{3.11}$$

Proof. The second statement (3.11) follows from the first one (3.10) by Lemma B.2 in the appendix. In order to prove (3.10), we first observe that

$$\int \mathbb{P}_0(d\xi) \sum_{y\in\xi} r(y)|u(y)| \leqslant \mathbb{E}_0\big[w(0)\big]^{\frac12}\Big\{\int \mathbb{P}_0(d\xi) \sum_{y\in\xi} r(y)|u(\xi, y)|^2\Big\}^{\frac12} = C(u, u)_\mu^{\frac12}\,,$$

and the last member is finite. This implies the upper bound in (3.10). The identity in (3.10) is equivalent to $\mathrm{div}\,u = 0$ $\mathbb{P}_0$–a.s. , which follows from the previous lemma. $\square$

3.2. **Corrector field.** We can now define the corrector field χ following the construction of [MP]. Consider the form $u_i\colon \mathcal{N}_0 \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, d$, defined by $u_i(\xi, x) = x_i$ (the i–th coordinate of $x \in \mathbb{R}^d$). Note that, since $\mathbb{P}$ has finite second moment, Lemma B.1 assures us that $u_i \in L^2(\mu)$. Let $\pi\colon L^2(\mu) \to \mathcal{H}_\nabla$ be the orthogonal projection on potential forms and define

$$\chi_i := \pi(-u_i)\,, \quad i = 1, \dots, d\,.$$

Setting $\Phi_i := x_i + \chi_i \in \mathcal{H}_\nabla^\perp$, from Lemma 3.4 we see that Φ_i is *harmonic*, i.e. for $\mathbb{P}_0$-a.a. ξ, $\Phi_i \in L^1(P_{0,\xi})$ and $E_{0,\xi}\Phi_i = 0$, for all $i = 1, \dots, d$. The vector form $\chi = (\chi_1, \dots, \chi_d)$ is the so called *corrector field*.

Up to now χ has been defined as element of $L^2(\mu)^d$, hence as a pointwise function it is defined modulo a set of zero μ–measure. It is convenient to work with a special representative of χ, which is everywhere defined on $\mathcal{N}_0 \times \mathbb{R}^d$ and has good properties:

Lemma 3.5. *There exists a representative $\bar\chi : \mathcal{N}_0 \times \mathbb{R}^d \to \mathbb{R}^d$ of the corrector $\chi \in L^2(\mu)^d$ such that*

$$\bar\chi(\xi, x) = \bar\chi(\xi, y) + \bar\chi(\tau_y\xi, x - y)\,, \qquad \forall \xi \in \mathcal{N}_0,\ \forall x, y \in \xi\,. \tag{3.12}$$

In particular, $\bar\chi(\xi, 0) = 0$ for all $\xi \in \mathcal{N}_0$.

Proof. The conclusion of the Lemma follows from (3.12) by taking $x = y = 0$. Let us therefore concentrate on (3.12). Due to Lemma 3.1, χ_i is a curl–free square integrable form. We fix a representative $\hat\chi_i$ of χ_i as pointwise function on $\mathcal{N}_0 \times \mathbb{R}^d$ and call $\mathcal{B}_i \subset \mathcal{N}_0$ the set of ξ satisfying (3.3) w.r.t. the form $\hat\chi_i$, for any family of n points $x_0, x_1, \dots, x_n$ in ξ. By definition, it must be $\mathbb{P}_0(\mathcal{B}_i) = 1$.

We claim that if $\xi \notin \mathcal{B}_i$ then $\tau_x\xi \notin \mathcal{B}_i$ for all $x \in \xi$. Suppose for the sake of contradiction that $\tau_x\xi \in \mathcal{B}_i$ and fix a family of n points $x_0, x_1, \dots, x_n$ in ξ. Then the points $y_0, y_1, \dots, y_n$ defined as $y_k = x_k - x$ lie in $\tau_x\xi$. Because $\tau_x\xi \in \mathcal{B}_i$, we conclude that

$$0 = \sum_{j=0}^{n-1} \hat\chi_i(\tau_{y_j}(\tau_x\xi), y_{j+1} - y_y) = \sum_{j=0}^{n-1} \hat\chi_i(\tau_{x_j}\xi, x_{j+1} - x_j)\,,$$

thus implying that $\xi \in \mathcal{B}_i$, which is a contradiction. This concludes the proof of our claim.

At this point we define $\mathcal{B} = \cap_{i=1}^d \mathcal{B}_i$ and

$$\bar\chi_i(\xi, x) := \begin{cases} \hat\chi_i(\xi, x) & \text{if } \xi \in \mathcal{B}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let us check (3.12). If $\xi \in \mathcal{B}$ and $x \in \xi$, then also $\tau_y\xi$ must belong to $\mathcal{B}$ (if it was not in $\mathcal{B}_i$ for some i, since $-y \in \tau_y\xi$ we would conclude that $\xi = \tau_{-y}(\tau_y\xi)$ does not belong to $\mathcal{B}_i \supset \mathcal{B}$). In particular, the identity (3.12) can be rewritten as

$$\hat\chi_i(\xi, x) = \hat\chi_i(\xi, y) + \hat\chi_i(\tau_y\xi, x - y)\,, \qquad \forall i = 1, \dots, d\,,$$

which is trivially true by definition of $\mathcal{B}$ and $\mathcal{B}_i$. Take now $\xi \notin \mathcal{B}$ and $x, y \in \xi$. By definition of $\bar\chi$ we get $\bar\chi(\xi, x) = \bar\chi(\xi, y) = 0$. Since for some i it must be $\xi \notin \mathcal{B}_i$, we know that also $\tau_y\xi$ does not belong to $\mathcal{B}_i \supset \mathcal{B}$. As consequence, it must be $\bar\chi(\tau_y\xi, x - y) = 0$ and the identity in (3.12) reduces to $0 = 0 + 0$. □

From now on, when working with the corrector field χ we will always refer to the pointwise function $\bar\chi : \mathcal{N}_0 \times \mathbb{R}^d \to \mathbb{R}^d$ of the above lemma.

3.3. **Sublinearity and the proof of Theorem 1.2 and Corollary 1.3.** The core of the proof of Theorem 1.2 lies in the following result:

Theorem 3.6. *Under the assumptions of Theorem 1.2: for $\mathbb{P}_0$–a.a. ξ*

$$\lim_{n\to\infty} \frac{1}{n} \max_{\substack{x \in \xi\,: \\ |x|_\infty \leqslant n}} \big|\chi(\xi, x)\big| = 0\,. \tag{3.13}$$

The proof of Theorem 3.6 is completed in Section 7. Here, we show how Theorem 1.2 follows from Theorem 3.6. The argument is standard; see [SS, BB, BP, BD] for very similar arguments. We only sketch the main steps.

3.3.1. *Proof of Theorem 1.2.* Let us start with the discrete parameter case. Set $\Phi(\xi, x) := x + \chi(\xi, x)$, so that $M_n = \Phi(\xi, X_n)$, $n \in \mathbb{N}$, is a martingale (see Lemma 3.4). Also, define $M_n^v := v \cdot M_n$, for $v \in \mathbb{R}^d$, and, for every $K \geqslant 0$:

$$F_K(\xi) = E_{0,\xi}\left(|M_1^v|^2\,;\ |M_1^v| \geqslant K\right)$$

Let $\widetilde{\mathbb{P}}_0$ denote the probability on $\mathcal{N}_0$ with density $\frac{w(0)}{\mathbb{E}_0 w(0)}$ with respect to $\mathbb{P}_0$, and let $\widetilde{\mathbb{E}}_0$ denote the associated expectation. Recall that the Markov chain on environments, $n \mapsto \tau_{X_n}\xi$, is ergodic with reversible invariant distribution given by $\widetilde{\mathbb{P}}_0$; see [FSS, Proposition 2]. Therefore, by the Ergodic theorem, for every $K \geqslant 0$:

$$\frac{1}{n}\sum_{k=0}^{n} F_K \circ \tau_{X_k}\xi \to \widetilde{\mathbb{E}}_0 F_K\,, \tag{3.14}$$

$P_{0,\xi}$–a.s., for $\mathbb{P}_0$–a.a. ξ, as $n \to \infty$. As in [BB, Section 6.1] and [BP, Section 5], using the above convergence together with the monotonicity of $K \mapsto F_K$, allows us to verify the assumptions of the Lindeberg-Feller Martingale Functional CLT (as in, e.g., [Du, Theorem 7.7.3]). It follows that, for every $v \in \mathbb{R}^d$, $\mathbb{P}_0$–a.s. $t \mapsto \varepsilon\, M^v_{\lfloor t/\varepsilon^2 \rfloor}$ converges weakly, as $\varepsilon \to 0$, to one-dimensional Brownian motion with diffusion coefficient

$$\langle v, D_{DTRW} v\rangle = \widetilde{\mathbb{E}}_0 E_{0,\xi}\left(|M_1^v|^2\right)\,. \tag{3.15}$$

In particular, with the notation (3.1), it holds $[D_{DTRW}]_{i,j} = (\Phi_i, \Phi_j)_\mu/(1,1)_\mu$, where $\Phi_i = x_i + \chi_i$, $i, j = 1, \dots, d$. That D_{DTRW} is positive definite follows from the fact that if (3.15) is zero for some coordinate axis $v = e_j$, $j = 1, \dots, d$, then $x_j = -\chi_j(\xi, x)$, $\mathbb{P}_0$–a.s. for every $x \in \xi$, and this is not compatible with Theorem 3.6.

To conclude the proof we argue as in [BB, Section 6.2]. Namely, from Theorem 3.6 we have that $\mathbb{P}_0$–a.s. for $\delta \in (0, 1/2)$, there exists $\kappa(\xi) < \infty$ such that for all $x \in \xi$ it holds $|\chi(\xi, x)| \leqslant \kappa(\xi) + \delta\,|x|$. Writing $x = \Phi(\xi, x) - \chi(\xi, x)$, one has

$$\max_{k \leqslant n} |\chi(\xi, X_k)| \leqslant 2\kappa(\xi) + 2\delta \max_{k \leqslant n} |M_k|\,,$$

which implies, by the arbitrariness of δ, that $\max_{k \leqslant n} |\chi(\xi, X_n)| = o(\sqrt{n})$ in $P_{0,\xi}$ probability, for $\mathbb{P}_0$–a.a. ξ. This ends the proof of Theorem 1.2 for the DTRW. It is worth noting that a separate tightness argument is not needed for this proof, since one uses the strong sublinearity given by Theorem 3.6.

To treat the CTRW, observe that it consists of a time change of the DTRW. Indeed, the CTRW waits at site x an exponential time of parameter $w(x)$ and then jumps to $y \in \xi$ with probability $p(x, y)$ (it could be $y = x$). With this notion of "jump", if $n^*(t)$ denotes the number of jumps of the CTRW up to time t, then the CTRW at time t coincides with the DTRW at $n^*(t)$. Therefore, arguing as in [DFGW, Theorem 4.5], it is sufficient to show that the limit

$$\lim_{t\to\infty} n^*(t)/t = \mathbb{E}_0 w(0)\,, \tag{3.16}$$

holds $P_{0,\xi}$–a.s., for $\mathbb{P}_0$–a.a. ξ. Let σ_i, $i = 0, 1, 2, \dots$ denote an independent family of i.i.d. exponentials of parameter 1 and write $T_i := \sigma_i / w(X_i)$ for the waiting time after the i-th jump of the discrete-time chain. Then, setting $R_0 = 0$, and $R_n = T_{n-1} + \cdots + T_0$, we have that $n^*(t) = n$ if and only if $R_{n-1} \leqslant t < R_n$. Observing that $w(x) = w(x, \xi) = w(0, \tau_x\xi)$ for every $x \in \xi$, and invoking the ergodicity of the environment process $n \mapsto \tau_{X_n}\xi$ as in (3.14) above, we see that $\mathbb{P}_0$–a.s. and $P_{0,\xi}$–a.s.

$$\lim_{n\to\infty} R_n/n = \widetilde{\mathbb{E}}_0\big(1/w(0)\big) = 1/\mathbb{E}_0 w(0)\,.$$

This implies (3.16). Moreover, this also shows that $[D_{CTRW}]_{i,j} = (\Phi_i, \Phi_j)_\mu$, and the relation (1.7) must hold. □

3.3.2. *Proof of Corollary 1.3.* This corollary is a direct consequence of Theorem 1.2 and Lemma B.2 in the appendix, taking in Lemma B.2 $\mathcal{A}_0$ as the set of configurations $\xi \in \mathcal{N}_0$ such that both the DTRW and the CTRW starting at the origin converge under diffusive rescaling to the Brownian motions described in Thereom 1.2.

4. Restricted random walk

Recall the definition of occupied boxes and white boxes, as in Section 2. As consequence of (H1), see also property (A), once p is taken large enough, $\mathbb{P}$–a.s. the random sets $\{x \in \mathbb{Z}^d : \sigma_x = 1\}$ and $\{x \in \mathbb{Z}^d : \vartheta_x = 1\}$ have a unique infinite connected component, the infinite cluster; see e.g. [G]. Here points x, y are thought of as connected if there exist points $x_0, x_1, \dots, x_n$ in the above random sets such that $x_0 = x, x_n = y$ and $|x_i - x_{i+1}|_1 = 1$ for all $i = 0, \dots, n-1$. We call C_∞ the infinite clusters in $\{x \in \mathbb{Z}^d : \sigma_x = 1\}$, and C^*_∞ the infinite clusters in $\{x \in \mathbb{Z}^d : \vartheta_x = 1\}$. By taking p large, from the domination assumptions (H1) we also know that there exists $c < \infty$ such that $\mathbb{P}$–a.s. the holes of C_∞ and C^*_∞ intersecting the box $[-n, n]^d$ have diameter at most $c \log n$ [BP, Prop. 2.3] (in particular, all holes have finite cardinality). Finally, we define

$$\mathfrak{C}_\infty = \cup_{x \in C_\infty} B(x)\,, \qquad \mathfrak{C}^*_\infty = \cup_{x \in C^*_\infty} B(x)\,.$$

The dependence on the parameters K, T_0 is understood. The set $\mathfrak{C}^*_\infty$ is often referred to as the cluster of white boxes, while $\mathfrak{C}_\infty$ is called the cluster of occupied boxes. Clearly, $\mathfrak{C}_\infty \supset \mathfrak{C}^*_\infty$. The points $\xi \cap \mathfrak{C}^*_\infty$ are sometimes referred to as the good points.

Given a starting point in $\xi \cap \mathfrak{C}^*_\infty$, the random walk Y_n is the discrete–time random walk made by the consecutive visits of X_n to the set $\xi \cap \mathfrak{C}^*_\infty$: setting

$$T_1 := \min\{n \geqslant 1 : X_n \in \xi \cap \mathfrak{C}^*_\infty\}\,, \tag{4.1}$$

the transition probability from x to y of Y is given by

$$\omega_{x,y}(\xi) := P_{x,\xi}(X_{T_1} = y)\,. \tag{4.2}$$

Thus $Y_n = X_{T_n}$, where T_n is the time of the n-th visit to $\xi \cap \mathfrak{C}^*_\infty$. The continuous–time random walk $\widetilde{Y}_t$ is defined as $Y_{N(t)}$, where $N(t)$ is the Poisson process with intensity 1. Equivalently, $\widetilde{Y}_t$ is the continuous–time Markov chain on $\xi \cap \mathfrak{C}^*_\infty$ whose infinitesimal generator is given by

$$\mathbb{L}_* f(x) = \sum_{y \in \xi \cap \mathfrak{C}^*_\infty} \omega_{x,y}(\xi)\big(f(y) - f(x)\big)\,. \tag{4.3}$$

In order to simplify the notation, we simply write Y_t instead of $\widetilde{Y}_t$ when no confusion can be generated. It is simple to check that $(w(x) : x \in \xi \cap \mathfrak{C}^*_\infty)$ is a reversible measure both for Y_n and for Y_t.

Following [BP], a crucial step towards the proof of Theorem 3.6 consists in establishing the following bounds on the distance and heat kernel of the restricted walk.

Proposition 4.1. *For a suitable deterministic sequence $b_n = o(n^2)$ and for $\mathbb{P}$–a.a. ξ:*

$$\limsup_{n\to\infty} \max_{\substack{x\in\xi\cap\mathfrak{C}^*_\infty:\\ |x|_\infty\leqslant n}} \sup_{y\in\xi\cap\mathfrak{C}^*_\infty} \sup_{t\geqslant n} t^{d/2} P_{x,\xi}(Y_t = y) < \infty\,, \tag{4.4}$$

$$\limsup_{n\to\infty} \max_{\substack{x\in\xi\cap\mathfrak{C}^*_\infty:\\ |x|_\infty\leqslant n}} \sup_{t\geqslant b_n} \frac{E_{x,\xi}|Y_t - x|}{\sqrt{t}} < \infty\,. \tag{4.5}$$

Before going to the proof of Proposition 4.1, which is given in Section 5 and Section 6, we start by developing some tools that will be repeatedly used in the sequel.

4.1. **Enlargement of holes.** Connected components in the complement of C^*_∞ and in the complement of $\mathfrak{C}^*_\infty$ are called *discrete holes* and *holes*, respectively. A generic discrete hole C is thus a finite set, while a subset $C' \subset \mathbb{R}^d$ is a hole if and only if it can be written as $C' = B(z_1) \cup \cdots \cup B(z_m)$, where $\{z_1, \dots, z_m\}$ is a discrete hole. For the moment we restrict our analysis to discrete holes.

Given $z \in \mathbb{Z}^d$, we use $G(z)$ to denote the unique discrete hole C such that $z \in C$. For a vertex $z \in C^*_\infty$, we set $G(z) = \emptyset$. Let $d_1(z, z') = |z - z'|_1$, $z, z' \in \mathbb{Z}^d$, denote the ℓ_1 distance, i.e. the graph distance in $\mathbb{Z}^d$. Also, we use $d_2(z, z') = |z - z'|$ for the usual ℓ_2 distance. Given an arbitrary $D \subset \mathbb{Z}^d$ and $i = 1, 2$, we write $d_i(z, D)$ for the point-to-set ℓ_i distance and $\mathrm{diam}_i(D) = \sup_{z,z'\in D} d_i(z, z')$ for the ℓ_i diameter of D. We write $|D|$ for the cardinality of D.

The *enlargement* of a discrete hole C is given by the set

$$\widetilde{C} = \{z \in \mathbb{Z}^d : d_2(z, C) \leqslant \mathrm{diam}_2(C)\}\,. \tag{4.6}$$

Note that an enlarged discrete hole $\widetilde{C}$ will in general contain some vertices $z \in C^*_\infty$, and a vertex $z \in C^*_\infty$ can be covered by several enlarged holes. When $z \notin C^*_\infty$, we use the notation $\widetilde{G}(z) := \widetilde{G(z)}$ for the enlargement of $G(z)$. When $z \in C^*_\infty$, we set $\widetilde{G}(z) = \emptyset$.

Two vertices $z, z' \in \mathbb{Z}^d$ are said to be *related* if they both belong to some enlarged discrete hole $\widetilde{C}$. This notion induces in an obvious way an equivalence relation between vertices: two vertices z, z' are *equivalent* (written $z \sim z'$) if and only if there exist vertices $z_0, \dots, z_n$ such that $z_0 = z$, $z_n = z'$, and z_i, z_{i+1} are related for all $i = 0, \dots, n-1$. Consider now the graph obtained from $\mathbb{Z}^d$ by identifying all equivalent vertices. Call $\bar{d}(z, z')$ the associated graph distance (each vertex is at distance 0 from any member of its equivalence class). Note that according to this definition, two distinct vertices $z, z' \in C^*_\infty$ may well have distance 0 (if there exists a nearest neighbor path γ connecting z, z' such that γ is fully contained in the union of all enlarged holes).

Clearly, $\bar{d}(z, z') \leqslant d_1(z, z')$ for any pair of vertices. Our assumptions allow us to compare the two distances in the opposite direction as well.

Proposition 4.2. *For all $a > 0$, there exist K, T_0 such that for all $z \in \mathbb{Z}^d$:*

$$\mathbb{P}\Big(\bar{d}(0, z) \leqslant \frac{1}{2}\, d_1(0, z)\Big) \;\leqslant\; e^{-a\, d_1(0,z)}\,. \tag{4.7}$$

Proof. Due to assumption (H1) we can find K, T_0 such that the field of white points ϑ dominates a supercritical Bernoulli field $Z(p)$ with parameter p. Therefore, the probability appearing in (4.7) is smaller than $\mathbb{P}_p\big(\bar{d}(0, z) \leqslant \frac{1}{2}\, d_1(0, z)\big)$, where $\mathbb{P}_p$ is the law of the Bernoulli random field $Z(p)$ (and $\bar{d}(0, z)$ is accordingly defined as a function of $Z(p)$ and its unique infinite cluster instead of ϑ and C^∞_*, respectively).

Let us first observe that if $\bar{d}(0,z) = 0$ then there exist discrete holes $C_1, \dots, C_m$ such that the union of the enlarged discrete holes $\widetilde{C}_1, \dots, \widetilde{C}_m$ contains a nearest neighbor path from 0 to z. In particular, there must exists a nearest neighbor path $(z_0 = 0, \dots, z_n = z)$ in $\mathbb{Z}^d$ of length $n \geqslant d_1(0,z)$ such that

$$n \leqslant \sum_{i=0}^{n} \mathrm{diam}_1(\tilde{G}(z_i)) 1_{\{z_i \notin G(z_j)\,,\,\forall j<i\}}\,.$$

More generally, by pasting together different paths as in the example above, one obtains that the event $\bar{d}(0,z) \leqslant \frac{1}{2}\, d_1(0,z)$ is contained in the event: there exist $n \geqslant d_1(0,z)$ and a nearest neighbor path $\gamma_n = (z_0 = 0, \dots, z_n = z)$ in $\mathbb{Z}^d$ such that

$$\frac{n}{2} \leqslant X(\gamma_n) := \sum_{i=0}^{n} \mathrm{diam}_1(\tilde{G}(z_i)) 1_{\{z_i \notin G(z_j)\,,\,\forall j<i\}}\,.$$

Therefore, using the exponential Chebyshev estimate and a union bound, for any $\lambda > 0$:

$$\mathbb{P}_p\Big(\bar{d}(0,z) \leqslant \frac{1}{2}\, d_1(0,z)\Big) \leqslant \sum_{n \geqslant d_1(0,z)} \sum_{\gamma_n} e^{-\lambda n/2}\, \mathbb{E}_p[e^{\lambda X(\gamma_n)}]\,,$$

where $\mathbb{E}_p$ denotes expectation w.r.t. $\mathbb{P}_p$. We claim that for every $\lambda > 0$ there exists $p \in (0,1)$ such that

$$\mathbb{E}_p[e^{\lambda X(\gamma_n)}] \leqslant 2^n\,, \tag{4.8}$$

for all nearest neighbor paths γ_n of length n. Once (4.8) is proved, estimating by $(2d)^n$ the total number of paths γ_n connecting 0 and z, one obtains that for all $a > 0$, there exist suitable constants $\lambda > 0$ and $p \in (0,1)$ such that

$$\mathbb{P}_p\Big(\bar{d}(0,z) \leqslant \frac{1}{2}\, d_1(0,z)\Big) \leqslant \sum_{n \geqslant d_1(0,z)} (4d)^n e^{-\lambda n/2} \leqslant e^{-a\, d_1(0,z)}\,,$$

and the proposition follows.

We turn to the proof of (4.8). Let the path $\gamma_n = (z_0 = 0, \dots, z_n = z)$ be fixed. Let $\mathcal{F}_i$ denote the σ–algebra generated by the random variables $G(z_0), \dots, G(z_i)$. To prove (4.8), it is sufficient to establish the uniform estimate: For any $\lambda > 0$ there exists $p \in (0,1)$ such that

$$\mathbb{E}_p\Big[\exp\big\{\lambda\ \mathrm{diam}_1(\widetilde{G}(z_i)) 1_{\{z_i \notin G(z_j)\,,\forall j<i\}}\big\}\,\Big|\,\mathcal{F}_{i-1}\Big] \leqslant 2\,. \tag{4.9}$$

Note that the definition (4.6) implies that $\mathrm{diam}_1\, \widetilde{G}(0) \leqslant c\, \mathrm{diam}_1\, G(0)$, for some finite constant $c = c(d)$. Therefore, it suffices to show that for any $\lambda > 0$ there exists $p \in (0,1)$ such that (4.9) holds with $\widetilde{G}(z_i)$ replaced by $G(z_i)$. At this point the conclusion follows from a standard Peierls argument, as in [BP, Lemma 3.1; proof of Eq. (3.12)]. □

We extend the definition of $\bar{d}$ to all points in the process ξ using the corresponding K-boxes. Namely, for any $x \in \xi$, let $z(x)$ denote the unique point of $\mathbb{Z}^d$ such that $x \in B(z(x))$. Then, we set

$$\bar{d}(x,y) := \bar{d}(z(x), z(y))\,, \qquad x, y \in \xi\,. \tag{4.10}$$

The next estimate is a useful corollary of Proposition 4.2.

Corollary 4.3. *Take K, T_0 satisfying (4.7) for some $a > 0$. Then, $\mathbb{P}$–a.s. there exists $\kappa = \kappa(\xi, K) < \infty$ such that for all $x, y \in \xi$:*

$$|x - y| \leqslant \kappa\, \big(1 + \log(1 + |x|) + \bar{d}(x,y)\big)\,. \tag{4.11}$$

Proof. From the definition (4.10), and the fact that

$$K|z(x)-z(y)| - c_1(K) \leqslant |x-y| \leqslant K|z(x)-z(y)| + c_1(K)\,,$$

for some constant $c_1(K)$, we see that it suffices to prove for $\mathbb{Z}^d$ an estimate similar to (4.11): $\mathbb{P}$–a.s. there exists $\kappa = \kappa(\xi) < \infty$ such that for all $z, z' \in \mathbb{Z}^d$:

$$|z-z'| \leqslant \kappa \left(1 + \log(1+|z|) + \bar{d}(z,z')\right)\,. \tag{4.12}$$

Combining the estimate of Proposition 4.2 and the Borel Cantelli argument shows that $\mathbb{P}$–a.s. there exists $n_0 = n_0(\xi) < \infty$ such that whenever $n \geqslant n_0$, $|z| \leqslant n$, then

$$|z-z'| \leqslant \frac{1}{2}\,\bar{d}(z,z')\,, \quad \text{for all } |z-z'| \geqslant \log n\,. \tag{4.13}$$

Let us verify that this implies (4.12). We may suppose first that $|z-z'| \geqslant \log(1+|z|)$. If $|z| \geqslant n_0$ then we may take $n = \lceil |z| \rceil$ and the claim follows from (4.13). Thus, assume that $|z| \leqslant n_0$. Clearly, we may further assume that $|z'| > n_0$, since otherwise $|z-z'| \leqslant 2n_0$. Therefore, if $|z-z'| \geqslant \log(1+|z'|)$ the claim follows from (4.13) by taking $n = \lceil |z'| \rceil$ and exchanging the roles of z and z'. In conclusion, the only case remaining is when $|z| \leqslant n_0, |z'| > n_0$ and $|z-z'| < \log(1+|z'|)$. But if t_0 is such that $\log(1+t) \leqslant t/2$ for all $t \geqslant t_0$, then we must have either $|z'| \leqslant t_0$ (and in this case $|z-z'| \leqslant n_0 + t_0$) or

$$|z'| \leqslant |z| + |z-z'| \leqslant |z| + \log(1+|z'|) \leqslant |z| + |z'|/2\,,$$

which implies $|z'| \leqslant 2n_0$. Therefore, $|z-z'| \leqslant |z| + |z'| \leqslant 3n_0$. This concludes the proof of (4.12). □

4.2. **Some uniform estimates.** The enlarged discrete holes allow us to obtain some useful estimates that we collect here. It is first convenient to extend our notation. Consider the map $\Psi(A)$ defined on finite subsets $A \subset \mathbb{Z}^d$ as $\Psi(A) = \cup_{z\in A} B(z)$. Then, given a hole $C = \Psi(C')$ ($C' \subset \mathbb{Z}^d$ being a discrete hole), its enlargement $\widetilde{C}$ is defined as $\Psi(\widetilde{C'})$, where $\widetilde{C'}$ is defined by (4.6). Note that we use the same notation $\widetilde{C}$ for the enlargement of a hole and of a discrete hole C (see Sec. 4.1). The kind of hole (discrete or not) we are handling will be clear from the contest. In this section we use a slightly different definition of the hole $G(x)$ with respect to the previous section. Namely, given $x \notin \mathfrak{C}^*_\infty$ we write now $G(x)$ for the unique hole C containing x. If $x \in \mathfrak{C}^*_\infty$, we set $G(x) = \emptyset$.

Given a hole C we call $[C]$ - the *class* of C - the union of all holes C' such that $\bar{d}(C,C') = 0$, i.e. such that there exists a chain of holes $C = C_0, \dots, C_m = C'$ such that $\widetilde{C}_i \cap \widetilde{C}_{i+1} \neq \emptyset$. We stress that $[C]$ is not the family of points $x \in \xi$ such that $\bar{d}(x,C) = 0$, in particular $[C] \cap \mathfrak{C}^*_\infty = \emptyset$.

Finally, we define

$$\Gamma_n = \sum_{i=1}^{n} \mathbb{I}_{\{X_i \notin \mathfrak{C}^*_\infty \cup [G(X_{i-1})]\}}\,.$$

Γ_n represents the number of jumps into a new class of holes up to time n (to be distinguished from the number of different classes of holes visited up to time n). As an example, suppose $X_0 \in \mathfrak{C}^*_\infty, X_1 \notin \mathfrak{C}^*_\infty$ and $X_2 \in \mathfrak{C}^*_\infty$. In this case, $\Gamma_2 = \Gamma_1 = 1$.

Let

$$T_1 = \inf\{n \geqslant 1 :\ X_n \in \mathfrak{C}^*_\infty\}\,, \tag{4.14}$$

and call $\Gamma = \Gamma_{T_1}$ the number of jumps into new classes before the return to $\mathfrak{C}^*_\infty$.

Lemma 4.4. *There exists $\delta > 0$ such that uniformly in $\xi \in \mathcal{N}$, $x \in \xi$:*

$$P_{x,\xi}(\Gamma \geqslant k) \leqslant (1-\delta)^k\,, \quad k \in \mathbb{N}\,. \tag{4.15}$$

Proof. Define the times at which the walk jumps out of a class of holes:

$$\tau_i = \inf\{n > \tau_{i-1} : \, X_n \notin [G(X_{\tau_{i-1}})]\}\,, \quad \tau_0 = 0\,. \tag{4.16}$$

Note that jumps within the cluster are included. From the strong Markov property applied to the stopping times τ_i, the claim (4.15) follows if one has the estimate: for some $\delta > 0$, for all i, uniformly in ξ it holds $P_{X_{\tau_i},\xi}\left(X_{\tau_{i+1}} \in \mathfrak{C}^*_\infty\right) \geqslant \delta$. Thus, it suffices to show that for some $\delta > 0$, uniformly in ξ and $x \in \xi$:

$$P_{x,\xi}\left(X_1 \in \mathfrak{C}^*_\infty \,|\, X_1 \notin [G(X_0)]\right) \geqslant \delta\,. \tag{4.17}$$

If $x = X_0 \in \mathfrak{C}^*_\infty$ the bound (4.17) is easy: here $[G(X_0)] = \emptyset$ and all we have to show is that $P_{x,\xi}\left(X_1 \in \mathfrak{C}^*_\infty\right) \geqslant \delta$; this follows from the fact that $w(x) \leqslant T(K, T_0)$ (see Lemma 2.1) and $r(x,y) \geqslant \delta_1 = \delta_1(K)$ for some $y \in \mathfrak{C}^*_\infty$, so that, with e.g. $\delta = \delta_1/T$.

$$P_{x,\xi}\left(X_1 \in \mathfrak{C}^*_\infty\right) \geqslant p(x,y) = \frac{r(x,y)}{w(x)} \geqslant \delta\,.$$

If $x = X_0 \notin \mathfrak{C}^*_\infty$, then $x \in C$ for some hole $C = G(X_0)$. Then

$$P_{x,\xi}\left(X_1 \in \mathfrak{C}^*_\infty \,|\, X_1 \notin [G(X_0)]\right) = \frac{P_{x,\xi}(X_1 \in \mathfrak{C}^*_\infty)}{P_{x,\xi}(X_1 \in \mathfrak{C}^*_\infty) + P_{x,\xi}(X_1 \notin \mathfrak{C}^*_\infty \cup [C])}$$

It is sufficient to prove that uniformly, for some $\delta > 0$:

$$P_{x,\xi}(X_1 \notin \mathfrak{C}^*_\infty \cup [C]) \leqslant \delta^{-1}\, P_{x,\xi}(X_1 \in \mathfrak{C}^*_\infty)\,. \tag{4.18}$$

To prove (4.18) we write

$$P_{x,\xi}(X_1 \notin \mathfrak{C}^*_\infty \cup [C]) = \sum_{C' :\, C' \notin [C]} \sum_{z' \in \mathbb{Z}^d : B(z') \subset C'} P_{x,\xi}(X_1 \in B(z'))\,, \tag{4.19}$$

where the sum is over holes C' in a different class than C. Let y denote the closest point $y \in \xi \cap \mathfrak{C}^*_\infty$ to x. Note that $P_{x,\xi}(X_1 \in \mathfrak{C}^*_\infty) \geqslant p(x,y) = r(y-x)/w(x)$.

If $z, \zeta \in \mathbb{Z}^d$ denote the vertices such that $x \in B(z)$ and $y \in B(\zeta)$, then $|x - y| \geqslant K|z - \zeta| - c_1(K)$. By construction, if $x' \in B(z')$ with $z' \in C'$ for some hole $C' \notin [C]$, then $|x - x'| \geqslant K|z - z'| - c_1(K)$ and $|z - z'| \geqslant 2|z - \zeta| - 2$. To justify the last inequality, let $C = \Psi(\underline{C})$, $\underline{C}$ being a discrete hole in $\mathbb{Z}^d$. The definition of ζ implies $|z - \zeta| \leqslant \mathrm{diam}_2(\underline{C})$. Since $C' \notin [C]$ and therefore $z' \notin \widetilde{\underline{C}}$,

$$|z - z'| \geqslant |z - \zeta| + \mathrm{diam}_2(\underline{C}) - 2 \geqslant 2|z - \zeta| - 2. \tag{4.20}$$

Next, observe that

$$n_{z'} \leqslant e^{|z - z'|^{\alpha/2}}\,, \tag{4.21}$$

since otherwise $z \in Q(z', R_{z'})$, cf. (2.3), which contradicts the fact that z, z' belong to distinct classes of holes. Note indeed that, since $x = X_0 \notin \mathfrak{C}^*_\infty$ and $x \in B(z)$, it must be $n_z \geqslant T_0$.

Define the function $\varphi : (0,\infty) \times \mathbb{N} \to (0,\infty)$ as $\varphi(a,m) = \sum_{v \in \mathbb{Z}^d :\, |v| \geqslant m} e^{-a\,|v|^\alpha}$. It is not hard to check that, for all fixed $a > 0, \alpha > 0$, as $m \to \infty$:

$$\varphi(a,m) = O(e^{-a\,m^\alpha} m^{d-\alpha})\,. \tag{4.22}$$

To bound (4.19) we write, for all $\varepsilon > 0$:

$$w(x)\, P_{x,\xi}(X_1 \notin \mathfrak{C}^*_\infty \cup [C]) \leqslant c(K) \sum_{z' \in \mathbb{Z}^d:\, |z-z'| \geqslant 2|z-\zeta|-2} n_{z'} e^{-K^\alpha |z-z'|^\alpha}$$
$$\leqslant c'(\varepsilon, K) \sum_{v \in \mathbb{Z}^d:\, |v| \geqslant 2|z-\zeta|-2} e^{-(1-\varepsilon)\, K^\alpha\, |v|^\alpha} = c'(\varepsilon, K) \varphi((1-\varepsilon)K^\alpha, 2|z-\zeta|-2)\,, \quad (4.23)$$

where we have used (4.21). On the other hand

$$w(x)\, P_{x,\xi}(X_1 \in \mathfrak{C}^*_\infty) \geqslant r(y-x) \geqslant c''(K)\, e^{-K^\alpha\, |z-\zeta|^\alpha}\,. \quad (4.24)$$

From (4.22), for all $\alpha, K > 0$, taking e.g. $\varepsilon > 0$ such that $(1-\varepsilon)2^\alpha > 1$, the ratio

$$\frac{\varphi((1-\varepsilon)K^\alpha, 2|z-\zeta|-2)}{e^{-K^\alpha\, |z-\zeta|^\alpha}}$$

is uniformly bounded in $z, \zeta \in \mathbb{Z}^d$. Using (4.23) and (4.24), this proves the uniform bound (4.17). □

Lemma 4.5. *For $c, \gamma \geqslant 1$, set $u_{\gamma,c}(t) = t^\gamma \exp[c(\log(t+1))^\alpha]$. Uniformly in ξ and $x \in \xi$:*

$$E_{x,\xi}\Big[\sup_{1 \leqslant j \leqslant T_1} u_{\gamma,c}(\bar d(x, X_j))\Big] < \infty\,. \quad (4.25)$$

Proof. With the definition (4.16) we have that $\Gamma = \max\{n \geqslant 0 :\ \tau_n < T_1\}$ and

$$\sup_{1 \leqslant j \leqslant T_1} \bar d(x, X_j) \leqslant \sum_{i=0}^{\Gamma} \bar d(X_{\tau_i}, X_{\tau_{i+1}})\,, \quad (4.26)$$

where Γ is defined as in Lemma 4.4 and $X_0 = x$.

Let us first suppose that $\alpha > 1$. In this case it is sufficient to show the claim with $u_{\gamma,c}(t)$ replaced by $\exp[c(\log(t+1))^\alpha]$, which is a convex function. Since, for every $N \in \mathbb{N}$:

$$\log\Big[1 + \sum_{i=0}^{N} \bar d(X_{\tau_i}, X_{\tau_{i+1}})\Big] \leqslant \log(N+1) + \log\Big[\frac{1}{N+1} \sum_{i=0}^{N} (1 + \bar d(X_{\tau_i}, X_{\tau_{i+1}}))\Big]\,,$$

simple estimates yield

$$\exp\Big\{2c\Big(\log\Big[1 + \sum_{i=0}^{N} \bar d(X_{\tau_i}, X_{\tau_{i+1}})\Big]\Big)^\alpha\Big\}$$
$$\leqslant \exp\Big\{c_1(\log(N+1))^\alpha\Big\}\, \frac{1}{N+1} \sum_{i=0}^{N} \exp\Big\{c_1(\log[1 + \bar d(X_{\tau_i}, X_{\tau_{i+1}})])^\alpha\Big\}\,, \quad (4.27)$$

for some constant $c_1 = c_1(\alpha, c)$. Suppose that, for some constant c_2, uniformly in ξ and $x \in \xi$ and $i \in \mathbb{N}$

$$E_{x,\xi} \exp\big\{c_1(\log[1 + \bar d(X_{\tau_i}, X_{\tau_{i+1}})])^\alpha\big\} \leqslant c_2\,. \quad (4.28)$$

Then, taking expectation in (4.27), using Schwarz' inequality:

$$\begin{aligned}
&E_{x,\xi}\Big[\exp\Big\{c\Big(\log\Big[1+\sum_{i=0}^{\Gamma}\bar d(X_{\tau_i},X_{\tau_{i+1}})\Big]\Big)^{\alpha}\Big\}\Big]\\
&\quad\leqslant\sum_{N=0}^{\infty}P_{x,\xi}(\Gamma=N)^{\frac12}\,E_{x,\xi}\Big[\exp\Big\{2c\Big(\log\Big[1+\sum_{i=0}^{N}\bar d(X_{\tau_i},X_{\tau_{i+1}})\Big]\Big)^{\alpha}\Big\}\Big]^{\frac12}\\
&\quad\leqslant\sqrt{c_2}\sum_{N=0}^{\infty}P_{x,\xi}(\Gamma=N)^{\frac12}\exp\Big\{c_1(\log(N+1))^{\alpha}\Big\}.
\end{aligned}$$

The last sum above is uniformly finite by Lemma 4.4. It remains to show the validity of (4.28). To this end, it suffices to show that, for some constant c_2, uniformly in ξ and $x\in\xi$:

$$E_{x,\xi}\exp\big\{c_1(\log[1+\bar d(x,X_{\tau_1})])^{\alpha}\big\}\leqslant c_2\,. \tag{4.29}$$

By summing over all possible ways of jumping out of the starting class of holes $[G(x)]$ one obtains that (4.29) follows from

$$E_{x,\xi}\left[\exp\big\{c_1(\log[1+\bar d(x,X_1)])^{\alpha}\big\}\;\middle|\;X_1\notin[G(x)]\right]\leqslant c_2\,. \tag{4.30}$$

Let x,y and z,ζ be as in the proof of Lemma 4.4, i.e. $x\in B(z)$, $y\in B(\zeta)$ and y is the closest point $y\in\xi\cap\mathfrak{C}^*_\infty$ to x. As in (4.24) we have

$$P_{x,\xi}(X_1\notin[G(x)])\geqslant P_{x,\xi}(X_1\in\mathfrak{C}_\infty)\geqslant c_3\,\frac{e^{-K^\alpha|z-\zeta|^\alpha}}{w(x)}\,, \tag{4.31}$$

for some constant c_3. Let $z'\in\mathbb{Z}^d$ be such that X_1 is in the K-box $B(z')$. Note that $\bar d(x,X_1)=0$ for all z' such that $|z-z'|\leqslant 2|z-\zeta|-2$. For other values of X_1 we simply bound $\bar d(x,X_1)\leqslant c_4|z-z'|$, for some $c_4=c_4(d,K,\alpha)$. Reasoning as in (4.23), to bound (4.30) we write, for all $\varepsilon>0$:

$$\begin{aligned}
&w_\alpha(x)\,E_{x,\xi}\left[\exp\big\{c_1(\log[1+\bar d(x,X_1)])^{\alpha}\big\}\;;\,X_1\notin[G(x)]\right]\\
&\quad\leqslant 1+c(K)\sum_{\substack{z'\in\mathbb{Z}^d:\\ |z-z'|\geqslant 2|z-\zeta|-2}} n_{z'}e^{-K^\alpha|z-z'|^\alpha}\,e^{c_5(\log[1+|z-z'|])^{\alpha}}\\
&\quad\leqslant 1+c'(\varepsilon,K)\sum_{\substack{v\in\mathbb{Z}^d:\\ |v|\geqslant 2|z-\zeta|-2}} e^{-(1-\varepsilon)\,K^\alpha\,|v|^\alpha}\\
&\quad= 1+c'(\varepsilon,K)\varphi((1-\varepsilon)K^\alpha,2|z-\zeta|-2)\,,
\end{aligned}\tag{4.32}$$

where we have used (4.21). From (4.32) and (4.31) we can conclude, as in the proof of Lemma 4.4, that the left hand side of (4.30) is uniformly bounded. This ends the proof of the case $\alpha>1$.

To prove the claim for $\alpha\leqslant 1$, observe that it is sufficient to prove the estimate (4.25) with $u_{\gamma,c}(t)$ replaced by t^γ. Here $\gamma\geqslant 1$ and t^γ is convex. Thus,

$$E_{x,\xi}\Big[\Big(\sum_{i=0}^{\Gamma}\bar d(X_{\tau_i},X_{\tau_{i+1}})\Big)^{\gamma}\Big]\leqslant\sum_{N=0}^{\infty}P_{x,\xi}(\Gamma=N)^{\frac12}\,N^{\gamma}\Big[\frac1N\sum_{i=0}^{N}E_{x,\xi}\Big(\bar d(X_{\tau_i},X_{\tau_{i+1}})^{2\gamma}\Big)\Big]^{\frac12}.$$

A uniform estimate of the expectation in the right hand side above can be obtained exactly as in the proof of (4.28). This ends the proof. □

We turn to a simple corollary of our previous results.

Lemma 4.6. *For every $p \geqslant 1$, there exists $c > 0$ such that uniformly in ξ and $x \in \xi$, and for all n:*

$$E_{x,\xi}\Big[\sup_{1 \leqslant j \leqslant T_n} \bar{d}(x, X_j)^p\Big] \leqslant c\, n^p\,. \tag{4.33}$$

Proof. Setting $\Delta_i := \sup_{1 \leqslant j \leqslant T_i - T_{i-1}} \bar{d}(X_{T_{i-1}}, X_{T_{i-1}+j})$, we have

$$\sup_{1 \leqslant j \leqslant T_n} \bar{d}(x, X_j)^p \leqslant n^{p-1} \sum_{i=1}^{n} \Delta_i^p\,.$$

The strong Markov property at time T_{i-1} together with the uniform estimate of Lemma 4.5 imply that for some $c > 0$, for all x and i one has $E_{x,\xi}\,[\Delta_i^p] \leqslant c$. □

4.3. **Some almost sure estimates.** We describe some more consequences of the estimates derived above.

Lemma 4.7. *$\mathbb{P}$–a.s., for every $p \geqslant 1$, there exists $\kappa = \kappa(p, \xi) < \infty$ such that for all $x \in \xi$, and for all $n \in \mathbb{N}$:*

$$E_{x,\xi}\Big[\sup_{1 \leqslant j \leqslant T_n} |X_j - x|^p\Big] \leqslant \kappa\big[1 + \log(1 + |x|)\big]^p n^p\,. \tag{4.34}$$

Proof. Using Corollary 4.3, we can write

$$|X_j - x|^p \leqslant \kappa\,\big[1 + \log(1 + |x|)\big]^p + \kappa\,\bar{d}(X_j, x)^p\,.$$

The conclusion then follows from Lemma 4.6. □

Lemma 4.8. *Take K, T_0 as in Proposition 4.2. Then, $\mathbb{P}$-almost surely, for all $x \in \xi$ $P_{x,\xi}(T_1 < \infty) = 1$.*

Proof. For every m, n, write

$$P_{x,\xi}(T_1 > m) = P_{x,\xi}(T_1 > m\,,\ S_n > m) + P_{x,\xi}(T_1 > m\,,\ S_n \leqslant m)\,,$$

where S_n denotes the first time $k \geqslant 1$ such that $|X_k|_\infty \geqslant n$.

Given $y \in \xi$, let $z(y) \in \mathbb{Z}^d$ be the unique point such that $y \in B(z)$. Fix $v \in \xi \setminus \mathfrak{C}_\infty^*$ such that $|z(v)| \leqslant n$. Then $z(v)$ is in a discrete hole (i.e. $z \notin C_\infty^*$). We know that ϑ dominates a supercritical Bernoulli field with large parameter p, and for the latter it is well known that a.s. holes intersecting $[-n, n]^d$ have diameter at most $O(\log n)$ (see [BP][Prop. 2.3]). By the stochastic domination, the same property still holds for the the discrete holes, which are the holes in ϑ. We claim that $w(v) \leqslant c' e^{c(\log n)^{\frac{\alpha}{2}}}$. To prove our claim, we write

$$w(v) = \sum_{\substack{y \in \xi: \\ |z(y) - z(v)| \leqslant n}} r(y - v) + \sum_{\substack{y \in \xi: \\ |z(y) - z(v)| > n}} r(y - x)\,.$$

For every $z \in \mathbb{Z}^d$ such that $|z - z(v)| \leqslant n$, if $n_z \geqslant T_0$ (i.e. if $B(z)$ is overcrowded) then it must be $n_z \leqslant e^{c(\log n)^{\frac{\alpha}{2}}}$ because any hole intersecting $[-n, n]^d$ has diameter $O(\log n)$. Taking n large enough that $T_0 \leqslant e^{c(\log n)^{\frac{\alpha}{2}}}$ we have

$$\sum_{\substack{y \in \xi: \\ |z(y) - z(v)| \leqslant n}} r(y - v) \leqslant \sum_{\substack{z \in \mathbb{Z}^d: \\ |z - z(v)| \leqslant n}} e^{c(\log n)^{\frac{\alpha}{2}}}\, e^{-c_1(K)|z - z(x)|^\alpha} \leqslant c_2 e^{c(\log n)^{\frac{\alpha}{2}}}\,.$$

On the other hand, if $|z(y)-z(v)|>n$, then $z(y)$ cannot belong to the same hole of $z(v)$, and therefore $n_z \leqslant e^{|z(y)-z(v)|^{\frac{\alpha}{2}}}$ whenever $n_z \geqslant T_0$. It follows that

$$\sum_{\substack{y\in\xi:\\ |z(y)-z(v)|>n}} r(y-v) \leqslant \sum_{\substack{z\in\mathbb{Z}^d:\\ |z-z(v)|>n}} T_0 e^{|z-z(v)|^{\frac{\alpha}{2}}}\, e^{-c_1(K)|z-z(v)|^{\alpha}} \leqslant c_3\,.$$

The above estimates trivially imply our claim.

Due to this claim and using again the fact that the hole containing v has diameter at most $O(\log n)$ we can estimate

$$P_{v,\xi}(X_1\in\mathfrak{C}^*_\infty) \geqslant \frac{e^{-c_4(\log n)^\alpha}}{w(v)} \geqslant e^{-c_5(\log n)^\alpha}\,.$$

From this observation we infer that

$$P_{x,\xi}(T_1>m\,,\ S_n>m) \leqslant \Big(1-e^{-c_5(\log n)^\alpha}\Big)^m \leqslant \exp\Big[-m\,e^{-c_5(\log n)^\alpha}\Big]\,. \tag{4.35}$$

On the other hand, by Markov's inequality

$$P_{x,\xi}(T_1>m\,,\ S_n\leqslant m) \leqslant P_{x,\xi}\Big(\sup_{1\leqslant j\leqslant T_1}|X_j|>n\Big) \leqslant n^{-1}E_{x,\xi}\Big(\sup_{1\leqslant j\leqslant T_1}|X_j|\Big)\,. \tag{4.36}$$

By Lemma 4.7 we conclude that

$$P_{x,\xi}(T_1>m\,,\ S_n\leqslant m) \leqslant c_x(\xi)\,n^{-1}\,, \tag{4.37}$$

for some $\mathbb{P}$–a.s. finite constant $c_x(\xi)$. Taking $m,n=n(m)$ such that $n(m)\to\infty$ and $m\exp[-(\log n(m))^\alpha]\to\infty$, as $m\to\infty$, (4.35) and (4.37) imply the conclusion. □

4.4. **Harmonicity with respect to the restricted random walk.**

Proposition 4.9. *Let $\Phi(\xi,x)=x+\chi(\xi,x)$. Then for $\mathbb{P}$–a.a. ξ and for any $x\in\xi\cap\mathfrak{C}^*_\infty$:*

$$\Phi(\tau_x\xi, X_{T_1}-x)\in L^1(P_{x,\xi})\,,\qquad E_{x,\xi}\big(\Phi(\tau_x\xi,X_{T_1}-x)\big)=0\,. \tag{4.38}$$

Proof. We recall that $x_i+\chi_i(\xi,x)\in\mathcal{H}^\perp_\nabla$, for each coordinate i. Hence, by Lemma 3.4, there exists a Borel subset $\mathcal{A}$ having $\mathbb{P}$–probability 1 such that for all $\xi\in\mathcal{A}$ and for all $z\in\xi$,

$$\sum_{y\in\xi} r(y-z)|\Phi(\tau_z\xi,y-z)|<\infty\,,\qquad \sum_{y\in\xi} r(y-z)\Phi(\tau_z\xi,y-z)=0\,.$$

This implies that the process $(M^\xi_n)_{n\geqslant 0}$ defined in terms of $(X_n)_{n\geqslant 0}$ as

$$M^\xi_0=0\,,\qquad M^\xi_n=\sum_{j=0}^{n-1}\Phi(\tau_{X_j}\xi,X_{j+1}-X_j)\quad\text{for } n\geqslant 1$$

is a martingale w.r.t. $P_{x,\xi}$. By shift covariance we have $M^\xi_n=\Phi(\tau_{X_0}\xi,X_n-X_0)$ for all $n\geqslant 0$. In particular, given $m\in\mathbb{N}$ and $\xi\in\mathcal{A}$, from the Optional Stopping Theorem, for any $m\in\mathbb{N}$, we have that $\Phi(\tau_x\xi,X_{T_1\wedge m}-x)\in L^1(P_{x,\xi})$ and

$$E_{x,\xi}\Phi(\tau_x\xi,X_{T_1\wedge m}-x)=0\,,\quad x\in\xi\,. \tag{4.39}$$

Since T_1 is a.s. finite (see Lemma 4.8), we have

$$\lim_{m\to\infty}\Phi(\tau_x\xi,X_{T_1\wedge m}-x)=\Phi(\tau_x\xi,X_{T_1}-x)\,,\qquad P_{x,\xi}\text{–a.s}\,,$$

In addition, using Lemma 7.2 below, we know that a.s. $|\Phi(\xi,y)| \leqslant c(\xi)\, u_{\gamma,c}(|y|)$ for suitable constants $c,\gamma>0$ and $c(\xi)<\infty$. We have

$$\big|\Phi(\tau_x\xi, X_{T_1\wedge m}-x)\big| \leqslant c_x(\xi)\, u_{\gamma,c}\Big(\max_{1\leqslant j\leqslant T_1}|X_j-x|\Big)\,, \qquad m\geqslant 1\,. \tag{4.40}$$

Therefore, Corollary 4.3 and Lemma 4.5 allow us to use the Dominated Convergence Theorem to conclude. □

Proposition 4.9 shows that $\Phi(\tau_x\xi, Y_n - x)$, with $Y_n = X_{T_n}$, is a martingale for every $x\in\xi\cap\mathfrak{C}^*_\infty$. Since $Y_t = X_{T_{N_t}}$ for an independent Poisson process with mean 1, Proposition 4.9 also implies

Corollary 4.10. *For $\mathbb{P}$–a.a. ξ, the process $\big(\Phi(\tau_z\xi, Y_t - z) : t\geqslant 0\big)$ is a continuous–time martingale w.r.t. to the law of the restricted random walk Y_t starting at $z\in\xi\cap\mathfrak{C}^*_\infty$.*

5. Heat kernel bound

In this section we prove the heat kernel bound (4.4). In order to avoid confusion, in this section we restore the convention to write Y_n for the discrete–time restricted RW and $\widetilde{Y}_t$ for the continuous–time restricted RW. The proof of the heat kernel bound (4.4) is divided in two parts: in the first one we derive a similar bound for a cut–off restricted random walk (see Proposition 5.1) by applying together the isoperimetric estimates of [CF1] and the method developed in [MoPe]. In the second part (see the proof of Proposition 5.3), we show that the above cut-off gives an approximation which is good enough to maintain the diffusive heat kernel bound. In particular, (4.4) follows immediately from Proposition 5.3.

5.1. **Cut–off of the restricted random walk.** We fix $L>0$ and introduce the discrete–time RW $\big(X^{(L)}_n : n\geqslant 0\big)$ on $\xi_L := \xi\cap[-L,L]^d$ jumping from x to y in ξ_L with probability

$$p^{(L)}(x,y) = \frac{r(y-x)}{w^{(L)}(x)}\,, \qquad w^{(L)}(x) = \sum_{z\in\xi_L} r(z-x)\,. \tag{5.1}$$

We call $C_L\subset\mathbb{Z}^d$ the largest connected component of the field ϑ (defined in (2.5)) inside $[-L,L]^d$. Then, we set $\mathfrak{C}_L = \cup_{z\in C_L} B(z)$ and $\zeta_L = \xi\cap\mathfrak{C}_L$. Let $Y^{(L)}_n$ be the restricted random walk associated to $X^{(L)}_n$ when visiting the good points ζ_L (similarly to the definition of Y_n as the restricted random walk associated to X_n when visiting the good points $\mathfrak{C}^*_\infty$). We define $\widetilde{Y}^{(L)}_t = Y^{(L)}_{N_t}$ where N_t is a Poisson process of parameter 1, independent of the random walk $Y^{(L)}_n$. The following heat kernel bound holds:

Proposition 5.1. *Take $L=L(t)=t^u$ with $u>1/2$. Then for $\mathbb{P}$–a.a. ξ*

$$\limsup_{t\to\infty}\ \max_{x,y\in\zeta_L} t^{d/2} P_{x,\xi}\big(\widetilde{Y}^{(L)}_t = y\big) < \infty\,. \tag{5.2}$$

The fact that $L(t)$ is polynomial in t in the above heat kernel estimate is essential. Indeed, because of the $\max_{x,y\in\zeta_L}$, one cannot expect the result to be true for functions $L(t)$ with an exponential growth in t, since it would contradict well known phenomena for the simple random walk on the supercritical percolation cluster [B].

Proof. Clearly, $\big(w^{(L)}(x)\,,\, x\in\xi_L\big)$ is a reversible measure for the random walk $X^{(L)}_n$. Let

$$\omega^{(L)}_{x,y} = P_{x,\xi}\big(Y^{(L)}_1 = y\big)\,, \qquad x,y\in\zeta_L\,.$$

Note that $w^{(L)}(x)\omega^{(L)}_{x,y} = w^{(L)}(y)\omega^{(L)}_{y,x}$ for all $x, y \in \zeta_L$, i.e. $\big(w^{(L)}(x)\,,\, x \in \zeta_L\big)$ is a reversible measure both for $Y^{(L)}_n$ and for $\widetilde{Y}^{(L)}_t$ (recall that $\omega^{(L)}_{x,y}$ coincides also with the probability rate of a jump of $\widetilde{Y}^{(L)}$ from x to y).

Let us denote by π_L the measure $w^{(L)}(x)$ on ζ_L and call $\varphi_L(t)$, $t > 0$, the isoperimetric profile of the RW $\widetilde{Y}^{(L)}$ w.r.t. π_L:

$$\varphi_L(t) := \inf\Big\{ I_U \,:\, U \subset \zeta_L,\ \pi_L(U) \leqslant \big(t \wedge \frac{1}{2}\big)\pi_L(\zeta_L)\Big\}\,,$$

where $I_U := \pi_L(U)^{-1} \sum_{\substack{x\in U,\\ y\in\zeta_L\setminus U}} \pi_L(x)\omega^{(L)}_{x,y}$. Note that due to the definition of ζ_L it holds

$$1 \leqslant \pi_L(x) \leqslant c\,, \qquad x \in \zeta_L \tag{5.3}$$

for some positive constant c independent of ξ and L (the upper bound follows from Lemma 2.1, the lower bound is trivial: $w^{(L)}(x) \geqslant r(0)$).

In order to estimate $P_{x,\xi}\big(\widetilde{Y}^{(L)}_t = y\big)$, $x, y \in \zeta_L$, we apply Theorem 13 in [MoPe] which states that, given $\varepsilon, t > 0$, if

$$t \geqslant \int_{4[\pi_L(x)\wedge\pi_L(y)]/\pi_L(\zeta_L)}^{4/\varepsilon} \frac{8du}{u\varphi^2_L(u)} \tag{5.4}$$

then

$$P_{x,\xi}\big(\widetilde{Y}^{(L)}_t = y\big) \leqslant \frac{\pi_L(y)}{\pi_L(\zeta_L)}(1+\varepsilon)\,. \tag{5.5}$$

To get a bound from below of the isoperimetric profile φ_L we observe that, given $x \neq y$ in ζ_L,

$$\pi_L(x)\omega^{(L)}_{x,y} \geqslant \pi_L(x)P_{x,\xi}(X^{(L)}_1 = y) = r(y-x) = m_L(x)\frac{r(y-x)}{m_L(x)}\,, \tag{5.6}$$

where $m_L(x) = \sum_{y\in\zeta_L} r(y-x)$. Since $m_L(x) \leqslant w^{(L)}(x)$, $m_L(x)$ satisfies a bound of the same form of (5.3). In particular,

$$\frac{m_L(U)}{m_L(\zeta_L)} \leqslant \kappa\frac{\pi_L(U)}{\pi_L(\zeta_L)}\,, \qquad \forall U \subset \zeta_L\,. \tag{5.7}$$

We stress that κ is a positive constant that does not depend on ξ, L.

Due to (5.6) and (5.7) we conclude that

$$\varphi_L(t) \geqslant \psi_L(\kappa t)\,, \qquad \forall t \in (0,1/2) : \kappa t \in (0,1/2)\,, \tag{5.8}$$

where ψ_L denotes the isoperimetric profile of the continuous–time random walk on ζ_L with generator

$$\mathcal{L}f(x) = \sum_{y\in\zeta_L} \frac{r(y-x)}{m_L(x)}(f(y)-f(x))\,, \qquad x \in \zeta_L\,,$$

with reversible measure m_L. We take $\gamma > 0$. The value will be fixed at the end. Due to assumption (H1) and [CF1, Lemma 2.1], there exists a constant $\delta > 0$ such that $\mathbb{P}$–a.s. it holds

$$\psi_L(u) \geqslant \delta \min\left\{\frac{1}{L^\gamma}, \frac{1}{u^{1/d}L}\right\}, \qquad 0 < u \leqslant 1/2,\ L \geqslant L_0(\xi) \tag{5.9}$$

for a suitable ξ–dependent constant $L_0(\xi)$. Since L diverges with t, we have $L \geqslant L_0(\xi)$ eventually in t. Let us choose $\varepsilon = L^d/t^{d/2}$. Due to our assumption $4/\varepsilon$ goes to 0 as $t \to \infty$. In particular, we can take t large enough that $4/\varepsilon \in (0,1/2)$ and $4\kappa/\varepsilon \in (0,1/2)$. This

together with (5.8) implies that $\varphi_L(u) \geqslant \psi_L(\kappa u)$ for u as in the r.h.s. of (5.4). In addition, note that

$$4\pi_L(z)/\pi_L(\zeta_L)^d \geqslant \tilde{c}\, L^{-d}\,, \qquad z \in \zeta_L \tag{5.10}$$

for some new constant $\tilde{c} > 0$. Since the bound (5.9) reads

$$\psi_L(u) \geqslant \begin{cases} \frac{\delta}{L^\gamma}\,, & \text{if } 0 < u \leqslant L^{-d(1-\gamma)}\,, \\ \frac{\delta}{u^{1/d}L}\,, & \text{if } u \geqslant L^{-d(1-\gamma)}\,, \end{cases}$$

taking γ small enough we can assume that $4t^{d/2} \gg L^{d\gamma}$, thus implying that (see (5.10))

$$\text{r.h.s. of (5.4)} \leqslant \int_{\tilde{c}L^{-d}}^{4t^{d/2}L^{-d}} \frac{8du}{u\psi_L^2(\kappa u)} = \int_{\kappa\tilde{c}L^{-d}}^{4\kappa t^{d/2}L^{-d}} \frac{8ds}{s\psi_L^2(s)} \leqslant$$
$$\frac{8L^{2\gamma}}{\delta^2}\int_{\kappa\tilde{c}L^{-d}}^{L^{-d+d\gamma}} s^{-1}ds + \frac{8L^2}{\delta^2}\int_{L^{-d+d\gamma}}^{4\kappa t^{d/2}L^{-d}} s^{2/d-1}ds = c(L^{2\gamma}+t)\,. \tag{5.11}$$

Taking γ small, we get $L^{2\gamma} < t$. At the cost of changing the definition of ε by setting $\varepsilon = c'\,L^d/t^{d/2}$ with c' small enough, we can assume that the last expression in (5.11) is smaller than t, thus implying (5.4) and therefore that

$$P_{x,\xi}(\widetilde{Y}_t^{(L)} = y) \leqslant \frac{c}{|\zeta_L|}(1 + L^d t^{-d/2})\,, \qquad x, y \in \zeta_L\,.$$

At this point the claim follows from the fact that $\mathbb{P}$–a.s. $|\zeta_L| \geqslant c_1 L^d$ for some positive constant c_1 and for all L. Indeed, defining C_L^* as the maximal connected component in $[-L, L]^d \cap \mathbb{Z}^d$ for the Bernoulli field $Z(p)$, it is known e.g. that if p is large enough, then a.s. $|C_L^*| \geqslant \frac{1}{2}\,L^d$ for all L sufficiently large. Due to the stochastic domination assumption (H1), the same holds for C_L as well. Since ξ, and therefore ζ_L, has at least one point in each box $B(z)$ with $z \in C_L$, it must be $|\zeta_L| \geqslant c_1 L^d$ for some $c_1 = c_1(K)$. This concludes the proof. □

5.2. **Comparison between $\widetilde{Y}_t$ and $\widetilde{Y}_t^{(L)}$.** We first define a coupling P_x between the random walks $X_n^{(L)}$ and X_n, starting at the same point x in ζ_L, as follows. We realize $(X_n : n \geqslant 0)$ starting at x, and call $\tau = \inf\{n \geqslant 1 : X_n \notin \xi_L\}$. Then we set $X_n^{(L)} = X_n$ for $n < \tau$, while on $[\tau, \infty)$ the random walk $X_n^{(L)}$ evolves independently from X_n with jump probabilities $p^{(L)}(\cdot,\cdot)$. To check the validity of the coupling, let A be the event that $X_n = X_n^{(L)}$ for $n \leqslant N$ (i.e. $A = \{N < \tau\}$). Note that, given $y, z \in \zeta_L$, the probability $P_x(X_{N+1}^{(L)} = z | X_N^{(L)} = y, A)$ can be written as

$$P_x(X_{N+1} = z | X_N = y, A) + P_x(X_{N+1} \notin \zeta_L | X_N = y, A)p^{(L)}(y,z)$$
$$= \frac{r(z-y)}{w(y)} + \frac{w(y) - w^{(L)}(y)}{w(y)}\frac{r(y-z)}{w^{(L)}(y)} = \frac{r(y-z)}{w^{(L)}(y)} = p^{(L)}(y,z)\,.$$

Introduce a Poisson process N_t of parameter 1 independent from $X_n^{(L)}$ and X_n, and therefore also from $Y_n^{(L)}$ and Y_n. Recall the continuous–time random walks $\widetilde{Y}_t^{(L)} = Y_{N_t}^{(L)}$, $\widetilde{Y}_t = Y_{N_t}$. We denote again by P_x the probability measure of the space where all the above processes are defined, and we write E_x for the associated expectation. An important

consequence of this coupling is the following observation. There exists $\varepsilon = \varepsilon(d) > 0$ such that, for $m \in \mathbb{N}$:

$$P_x\left(\exists\, n \leqslant m \,:\, Y_n \neq Y_n^{(L)}\right) \leqslant P_x\Big(\max_{1 \leqslant j \leqslant T_m} |X_j| > \varepsilon\, L\Big)\,. \tag{5.12}$$

Here T_m is, as usual, the time of the m-th visit to $\mathfrak{C}_\infty^*$ for the walk X_n. The above claim, in turn, is an immediate consequence of the following

Lemma 5.2. *There exists $\varepsilon = \varepsilon(d) > 0$ such that $\mathbb{P}$–a.s., for all sufficiently large L, it holds $\zeta_L \cap [-\varepsilon L, \varepsilon L]^d = \xi \cap \mathfrak{C}_\infty^* \cap [-\varepsilon L, \varepsilon L]^d$.*

Proof. Let us prove the equivalent statement for the corresponding white K-boxes. Namely, setting $\ell = L/K$, we want to prove

$$C_L \cap [-\varepsilon\ell, \varepsilon\ell]^d = C_\infty^* \cap [-\varepsilon\ell, \varepsilon\ell]^d\,. \tag{5.13}$$

By the stochastic domination assumption, and well known facts about Bernoulli percolation with large p (see e.g. [CF1, Proposition B.2]), we may assume that C_L coincides with the largest connected component of $[-\ell, \ell]^d \cap C_\infty^*$. Thus, the only thing that can go wrong in checking (5.13) is that there exist two vertices $x, y \in [-\varepsilon\ell, \varepsilon\ell]^d \cap C_\infty^*$ that are not connected within $[-\ell, \ell]^d$. Call F_ℓ this event. Using the stochastic domination assumption, and the fact that p is large, one can check that this event has exponentially (in ℓ) small probability for a suitable $\varepsilon > 0$. To see this, let $d_C(x, y)$ denote the graph distance of two vertices $x, y \in C_\infty^*$ in the graph C_∞^* (this is often called the chemical distance). From known estimates [AnPi], for $\gamma > 0$, if p is large, there exists $a > 0$ such that

$$\mathbb{P}\left(d_C(x, y) \geqslant (1+\gamma) d_1(x, y) \,|\, x, y \in C_\infty^*\right) \leqslant a^{-1}\, e^{-a\, d_1(x,y)}\,. \tag{5.14}$$

Let x, y be two vertices as in the event F_ℓ. Note the bounds $d_1(x, y) \leqslant d\varepsilon\ell$ and $d_C(x, y) \geqslant 2(1-\varepsilon)\ell$. Moreover, one can find y' such that $|y'|_\infty \leqslant 3\varepsilon\,\ell$, $4d\varepsilon\ell \geqslant d_1(x, y') \geqslant \varepsilon\ell$, and $d_C(x, y') \geqslant 2(1-2\varepsilon)\ell \geqslant 2(1-2\varepsilon)(4d\varepsilon)^{-1} d_1(x, y')$. Therefore, taking ε small enough, a union bound and (5.14) imply, for some constant c, the exponential bound

$$\mathbb{P}(F_\ell) \leqslant \sum_{|x| \leqslant \varepsilon\ell} \sum_{|y'| \leqslant 3\varepsilon\ell} a^{-1}\, e^{-a\,\varepsilon\,\ell} \leqslant c\; e^{-c^{-1}\,\varepsilon\,\ell}\,.$$

The identity (5.13) then follows from the Borel-Cantelli lemma. □

We can finally prove (4.4), an immediate consequence of

Proposition 5.3. *For $\mathbb{P}$–a.a. ξ,*

$$\limsup_{t\to\infty} \max_{\substack{x \in \xi \cap \mathfrak{C}_\infty^* : \\ |x|_\infty \leqslant t}} \sup_{y \in \xi \cap \mathfrak{C}_\infty^*} t^{d/2} P_x(\widetilde{Y}_t = y) < \infty\,. \tag{5.15}$$

Proof. Since $P_x(|N_t - t| \geqslant t/2) \leqslant e^{-ct}$ for some positive constant c, we can write

$$\Big|P_x(\widetilde{Y}_t = y) - \sum_{n=\lfloor t/2\rfloor}^{\lfloor 3t/2\rfloor} P_x(Y_n = y) P(N_t = n)\Big| \leqslant e^{-ct}\,. \tag{5.16}$$

We take $L(t) = t^u$ with $u > 1$. If $x \in \xi \cap \mathfrak{C}_\infty^*$ and $|x|_\infty \leqslant t$ then, for t large enough (independently from x), it holds $x \in \zeta_L$ ($L := L(t)$) by Lemma 5.2. We then consider the random walks $\widetilde{Y}_t^{(L)}$ and $Y_n^{(L)}$ starting at x. Reasoning as above we get that an expression

similar to (5.16) holds also for $\widetilde{Y}_t^{(L)}$ and $Y_n^{(L)}$. On the other hand, thanks to (5.12) we can use (for $n \leqslant \lfloor 3t/2 \rfloor$)

$$P_x(Y_n = y) \leqslant P_x\left(Y_n^{(L)} = y\right) + P_x\big(\max_{1 \leqslant j \leqslant T_{\lfloor 3t/2 \rfloor}} |X_j| > \varepsilon\, L\big)\,.$$

Together with the above observations, this gives:

$$\begin{aligned} P_x\left(\widetilde{Y}_t = y\right) &\leqslant e^{-ct} + \sum_{n=\lfloor t/2 \rfloor}^{\lfloor 3t/2 \rfloor} P_x(Y_n = y) P(N_t = n) \\ &\leqslant 2e^{-ct} + P_x\left(\widetilde{Y}_t^{(L)} = y\right) + P_x\big(\max_{1 \leqslant j \leqslant T_{\lfloor 3t/2 \rfloor}} |X_j| > \varepsilon\, L\big)\,. \quad (5.17) \end{aligned}$$

To bound the last expression, we use Markov inequality and Lemma 4.7:

$$\begin{aligned} P_x\left(\max_{1 \leqslant j \leqslant T_{\lfloor 3t/2 \rfloor}} |X_j| > \varepsilon\, L\right) &\leqslant (\varepsilon\, L)^{-1} E_x\left(\max_{1 \leqslant j \leqslant T_{\lfloor 3t/2 \rfloor}} |X_j|\right) \\ &\leqslant \kappa\, \varepsilon^{-1} L^{-1} \left(|x| + [1 + \log(1 + |x|)]\right) \lfloor 3t/2 \rfloor\,. \quad (5.18) \end{aligned}$$

If $L = t^u$, and $u > 1$, then we can assume $t \leqslant \varepsilon\, L$, and collecting (5.17), (5.18) and invoking Proposition 5.1 (using again Lemma 5.2) we get for $\mathbb{P}$–a.a. ξ that $t^{d/2} P_x(\widetilde{Y}_t = y) \leqslant c\left(1 + t^{d/2-u+2}\right)$ for all $x, y \in \xi \cap \mathfrak{C}_\infty^*$ such that $|x|_\infty \leqslant t$, for some finite constant $c = c(\xi)$. Taking e.g. $u = 2 + d/2$ concludes the proof. □

6. Expected distance bound

In this section we prove the distance estimate (4.5). Given $x, y \in \xi \cap \mathfrak{C}_\infty^*$ define the heat kernel by $q_t(x, y) = P_{x,\xi}(Y_t = y)/w(y)$. Given $\delta > 0$, define also

$$D = \sup_{\xi \in \mathcal{N}} \sup_{x \in \xi \cap \mathfrak{C}_\infty^*} \max_{1 \leqslant j \leqslant T_1} E_{x,\xi}\big[\bar{d}(x, X_j)^2\big] \quad (6.1)$$

$$M(x,t) = E_{x,\xi}\big[\bar{d}(x, Y_t)\big] = \sum_{y \in \xi \cap \mathfrak{C}_\infty^*} \bar{d}(x,y) q_t(x,y) w(y) \quad (6.2)$$

$$Q(x,t) = -E_{x,\xi}\big[\log q_t(x, Y_t)\big] = -\sum_{y \in \xi \cap \mathfrak{C}_\infty^*} q_t(x,y) w(y) \log q_t(x,y) \quad (6.3)$$

$$C_{\mathrm{vol}}(x,\delta) = \sup_{0<s<\delta} \left\{ s^d \sum_{y \in \xi \cap \mathfrak{C}_\infty^*} w(y) e^{-\bar{d}(x,y)s} \right\}. \quad (6.4)$$

(By continuity, $Q(x,0) = \log w(x)$.) By Lemma 4.6, we know that $D < \infty$.

Lemma 6.1. *For all x,*

$$M'(x,t)^2 \leqslant D Q'(x,t)\,, \qquad \forall t \geqslant 0\,, \quad (6.5)$$

$$M(x,t)^d \geqslant \exp\left\{-1 - C_{\mathrm{vol}}(x,\delta) + Q(x,t)\right\}, \qquad \text{if } M(x,t) > \delta^{-1}\,. \quad (6.6)$$

Proof. The proof of (6.6) is an adaptation of [B, Lemma 3.3]; see also [BP, Lemma 6.3(a)] for a similar argument. To prove (6.5), recall that $\omega_{x,y}$ denotes the jump rate of the restricted random walk (cf (4.2)). For (6.5), following [B, Proposition 3.4] almost exactly,

we use the triangle inequality for $\bar{d}$ and then Schwarz' inequality to arrive at:

$$M'(x,t)^2 \leqslant \frac{1}{2}\Big(\sum_{y,z} q_t(x,y)\omega_{y,z}\bar{d}(y,z)^2\Big)\times \\ \Big(\sum_{y,z}\omega_{y,z}\left(q_t(x,y)-q_t(x,z)\right)\left(\log q_t(x,y)-\log q_t(x,z)\right)\Big)\,, \quad (6.7)$$

where $\sum_{y,z}$ corresponds to $\sum_{y\in\xi\cap\mathfrak{C}^*_\infty}\sum_{z\in\xi\cap\mathfrak{C}^*_\infty}$.

The conclusion is now very different from [B] and the use of the distance $\bar{d}$ instead of the euclidean distance becomes crucial. We observe that by Lemma 4.6

$$\sum_{y,z} q_t(x,y)\omega_{y,z}\bar{d}(y,z)^2 = \sum_y q_t(x,y)\sum_z P_{y,\xi}(X_{T_1}=z)\bar{d}(y,z)^2 \\ = \sum_y P_{x,\xi}(Y_t=y)E_{y,\xi}\big[\bar{d}(y,X_{T_1})^2\big] \leqslant D\,,$$

so that the first factor inside the brackets in the last member of (6.7) is bounded by D. Exactly as in [B, Proposition 3.4], the second factor is equal to $2Q'(x,t)$. We therefore have $M'(x,t)^2 \leqslant DQ'(x,t)$, as desired. □

Lemma 6.2. *Take K, T_0 satisfying (4.7) for some $a>0$. For $\mathbb{P}$–a.a. ξ,*

$$C_{\mathrm{vol}}(x,\delta) \leqslant C(\xi)\delta^d[1+\log(1+|x|)]^d + C(\xi)e^{C(\xi)\delta}\,, \qquad x\in\xi\cap\mathfrak{C}^*_\infty \quad (6.8)$$

for some positive constant $C(\xi)$. In particular, for $\mathbb{P}$–a.a. ξ,

$$\max_{n\geqslant 1}\max_{\substack{x\in\xi\cap\mathfrak{C}^*_\infty:\\ |x|_\infty\leqslant n}}\sup_{t\geqslant n} C_{\mathrm{vol}}(x,1/\sqrt{t}) := C_1(\xi) < \infty\,. \quad (6.9)$$

Proof. The last bound (6.9) trivially follows from (6.8). Therefore, we concentrate on (6.8). Due to Lemma 2.1 and the definition of the random field ϑ, we know that $w(y)\leqslant c = c(K,T_0)$ for all $y\in\xi\cap\mathfrak{C}^*_\infty$. Moreover, we know that all K–boxes $B(z)$ with $z\in C^*_\infty$ are not overcrowded, i.e. $\xi(B(z))<T_0$. In particular, we can bound $\sum_{y\in\xi\cap B(z)} e^{-s|x-y|}$ from above by $T_0 e^{-sK(|v-z|-c(d))}$ if $x\in B(v)$. Let $\kappa=\kappa(\xi,K)$ be the positive constant appearing in Corollary 4.3. We define

$$W(x) = \big\{y\in\xi\cap\mathfrak{C}^*_\infty : |x-y|/\kappa \leqslant 2[1+\log(1+|x|)]\big\}\,.$$

Then, applying also Corollary 4.3, we conclude that

$$C_{\mathrm{vol}}(x,\delta) \leqslant c\sup_{0<s<\delta}\Big\{s^d|W(x)| + s^d\sum_{y\in\mathfrak{C}^*_\infty\setminus W(x)} e^{-\frac{s|x-y|}{2\kappa}}\Big\} \\ \leqslant C\delta^d[1+\log(1+|x|)]^d + Ce^{C\delta}\sup_{0<s<\delta}\big\{s^d\sum_{z\in\mathbb{Z}^d} e^{-s\frac{K|z|}{2\kappa}}\big\}\,,$$

for a positive constant C independent from x. The last term can be estimated by

$$c(d)C'\int_1^\infty s^d e^{-s\frac{Kr}{2\kappa}} r^{d-1}dr = c(d)C'\int_s^\infty e^{-y}y^{d-1}dy \leqslant c'(d)C''\,,$$

thus concluding the proof of (6.8). □

Let us now come back to the heat kernel estimate of Proposition 5.3. We know that $t^{d/2}\mathbb{P}_x^\xi(Y_t = y)$ is bounded from above uniformly as $t \geqslant 1$, $x, y \in \xi \cap \mathfrak{C}_\infty^*$ and $|x| \leqslant t$, for $\mathbb{P}$–a.a. ξ. Since $w(y) \geqslant r(0) = 1$ and $q_t(x,y) = \mathbb{P}_x^\xi(Y_t = y)/w(y)$, this implies that there exists a (finite) positive constant $A = A(\xi)$ (that we take larger than 1) such that

$$\sup_{t \geqslant |x| \vee 1} \sup_{y \in \xi \cap \mathfrak{C}_\infty^*} t^{d/2} q_t(x,y) \leqslant A\,, \qquad \forall x \in \xi \cap \mathfrak{C}_*^\infty\,. \tag{6.10}$$

Proposition 6.3. *Let $t(x) = |x| \vee e$ and set $T(x) = t(x)\log t(x) \vee \frac{c^2}{dD}$, where the positive constant $c = c(\xi)$ is the same appearing in (4.33) of Lemma 4.6 with $n = p = 1$. Then for $\mathbb{P}$–a.a. ξ there exists a constant $C_0(\xi) \geqslant 1$ such that*

$$M(x,t) \leqslant C_0(\xi)\sqrt{t} \qquad \forall x \in \xi \cap \mathfrak{C}_\infty^*\,,\ \forall t \geqslant T(x)\,. \tag{6.11}$$

Proof. We will follow [B, Proposition 3.4]. Since $C_0(\xi) \geqslant 1$, we can assume that $M(x,t) \geqslant \sqrt{t}$ otherwise we have nothing to prove. In this case, since $t \geqslant T(x) \geqslant |x|$, by (6.6) and (6.9) we can estimate

$$M(x,t)^d \geqslant \exp\{-1 - C_1(\xi) + Q(x,t)\}\,. \tag{6.12}$$

We will use this key lower bound at the end.

Following [B], we define $L(t) = \frac{1}{d}(Q(x,t) + \log A - \frac{d}{2}\log t)$ (recall (6.10)). Note that $L(t) \geqslant 0$ on $t \geqslant t(x)$ by (6.10). Then, we define

$$t_0 := \begin{cases} 1 & \text{if } L(t) \geqslant 0 \text{ on } (0, t(x)]\,, \\ \sup\{t \in (0,t(x)] \,:\, L(t) < 0\} & \text{otherwise}\,. \end{cases} \tag{6.13}$$

Note that in the second case, it must be $L(t_0) = 0$, i.e. $Q(x,t_0) = -\log A + \frac{d}{2}\log t_0$.

We claim that $M(x,t_0) \leqslant \sqrt{dDT(x)}$. To this aim, let us first assume that $L(t) \geqslant 0$ for all $t > 0$. Then $t_0 = 1$ and from the definition of $T(x)$ and Lemma 4.6 we deduce that

$$M(x,t_0) = E_{x,\xi}\big(\bar d(x, Y_1^{\text{c.t.}})\big) = E_{x,\xi}\Big[E_{x,\xi}\big(\bar d(x, Y_{N_1}^{\text{d.t.}})|N_1\big)\Big] \leqslant c\,E(N_1) = c \leqslant \sqrt{dDT(x)}\,,$$

where $(N_t)_{t \geqslant 0}$ is a Poisson process of parameter 1 independent from the discrete–time restricted random walk and were "c.t." and "d.t." mean continuous–time and discrete–time, respectively (to avoid ambiguity). Let us now assume that $L(t) < 0$ for some t (the second case of (6.13)). Since $L(t) \geqslant 0$ for $t \geqslant t(x)$ as already observed, it must be $t_0 \leqslant t(x)$. By (6.5) in Lemma 6.1 and Schwarz' inequality, $M(x,t_0)$ can be bounded from above by $\sqrt{t_0 D}\big(\int_0^{t_0} Q'(x,s)\,\mathrm{d}s\big)^{1/2}$. By continuity at both endpoints and using that $L(t_0) = 0$, this last expression is bounded from above by

$$\sqrt{t_0 D}\left(\frac{d}{2}\log t_0 - \log A - \log w(x)\right)^{1/2} \leqslant \sqrt{t_0 D d \log t_0} \leqslant \sqrt{DdT(x)}, \tag{6.14}$$

(recall that $A \geqslant 1$ and $w(x) \geqslant r(0) = 1$). This concludes the proof of our claim.

Since $Q'(x,t) = dL'(t) - d/(2t)$ and using (6.5), by the same computations as in [B] we get for all $t \geqslant t_0$

$$M(x,t) - M(x,t_0) \;\leqslant\; \sqrt{dD}\int_{t_0}^t \Big(\frac{1}{2s} + L'(s)\Big)^{1/2}\mathrm{d}s \;\leqslant\; \sqrt{2dDt} + L(t)\sqrt{dDt}\,. \tag{6.15}$$

Using now the bound $M(x,t_0) \leqslant \sqrt{dDT(x)}$ and $t \geqslant T(x)$ we conclude that

$$M(x,t) \leqslant \sqrt{dDT(x)} + \sqrt{2dDt} + L(t)\sqrt{dDt} \leqslant (1+\sqrt{2})\sqrt{dDt} + L(t)\sqrt{dDt}. \tag{6.16}$$

Conversely, because $\sqrt{t} \leqslant M(x,t)$, we can apply (6.12) to find that

$$M(x,t) \geqslant C_2(\xi) e^{L(t)} \sqrt{t} \tag{6.17}$$

for some positive constant C_2 depending on ξ. Combining these last two equations (6.16), (6.17), and eliminating the common $\sqrt{t}$, we see that

$$e^{L(t)} \leqslant \big[\sqrt{dD}/C_2(\xi)\big](1+\sqrt{2}+L(t)) \tag{6.18}$$

Since $e^y \geqslant 1 + y + \frac{1}{2}\, y^2$ for all $y \geqslant 0$, the above formula implies that $L(t)^2 \leqslant a + bL(t)$ for suitable constants $a = a(\xi)$, $b = b(\xi)$. This last bound implies that $L(t) \leqslant C_3(\xi)$. Coming back to (6.16) we get (6.11). □

6.1. **Proof of** (4.5)**.** We have now all the tools to prove (4.5). We take K, T_0 satisfying Corollary 4.3 and we define $b_n = n^\gamma$ with $\gamma \in (1,2)$. If $t \geqslant b_n$ with n large enough, then $t \geqslant T(x)$ for all $x \in \xi \cap \mathfrak{C}_\infty^*$ such that $|x|_\infty \leqslant n$. In particular, applying Proposition 6.3 we conclude that for $\mathbb{P}$–a.a. ξ,

$$\limsup_{n\uparrow\infty} \max_{\substack{x\in\xi\cap\mathfrak{C}_\infty^*:\\ |x|_\infty \leqslant n}} \sup_{t\,\geqslant\, b_n} \frac{E_{x,\xi}\big[\bar{d}(Y_t,x)\big]}{\sqrt{t}} < \infty\,. \tag{6.19}$$

We now apply Corollary 4.3, to estimate

$$\frac{|Y_t - x|}{\sqrt{t}} \leqslant \kappa n^{-\gamma/2}[1+\log(1+n)] + \kappa \frac{\bar{d}(x,Y_t)}{\sqrt{t}}\,,$$

The above bound together with (6.19) trivially implies (4.5). □

7. Sublinearity of the corrector

This section is devoted to the proof of Theorem 3.6.

7.1. **Preliminary bounds.** We start with a polynomial estimate on the size of the corrector for points within the cluster $\mathfrak{C}_\infty$. Note that we are now working with the cluster of occupied boxes $\mathfrak{C}_\infty$ and not with the (smaller) cluster of white boxes $\mathfrak{C}_\infty^*$. We will come back to the latter towards the end of this section.

Lemma 7.1. *For $\theta > d+1$,*

$$\lim_{n\to\infty} n^{-\theta} \max_{\substack{x,y\in\xi\cap\mathfrak{C}_\infty\\ |x|_\infty,|y|_\infty\,\leqslant\, n}} \big|\chi(\tau_x\xi, y-x)\big| = 0\,, \quad \mathbb{P}\text{–a.s.} \tag{7.1}$$

Proof. For any $x, y \in \xi \cap \mathfrak{C}_\infty$ with $|x|_\infty, |y|_\infty \leqslant n$ there exists a path $x = x_0, \dots, x_m = y \in \xi$, with x_i and x_{i+1} belonging to adjacent K–boxes B_i, B_{i+1} on $\mathfrak{C}_\infty$. For a fixed $\lambda < \infty$, let $E_{\lambda,n} \subset \mathcal{N}$, denote the event that, for any $x, y \in \xi \cap \mathfrak{C}_\infty$ with $|x|_\infty, |y|_\infty \leqslant n$, there exists such a path with the additional property that $\max_i |x_i|_\infty \leqslant \lambda\, n$.

Note that there exists $\delta = \delta(K) > 0$ such that $r(x_{i+1} - x_i) \geqslant \delta$ for every i (cf. (1.3)). Therefore, using the shift–covariance property we get

$$|\chi(\tau_x\xi, y-x)| \leqslant \delta^{-1} \sum_{i=0}^{m-1} r(x_{i+1}-x_i)|\chi(\tau_{x_i}\xi, x_{i+1}-x_i)|\,. \tag{7.2}$$

We shall write $B \subset B(\ell)$ when a K–box B is contained in the $|\cdot|_\infty$–box in $\mathbb{R}^d$ centered at the origin, of size ℓ. Thus on $E_{\lambda,n}$ we can estimate

$$\mathcal{R}_n(\xi) := \max_{\substack{x,y\in\xi\cap\mathfrak{C}_\infty \\ |x|_\infty,|y|_\infty\leqslant n}} \big|\chi(\tau_x\xi, y-x)\big| \leqslant \delta^{-1} \sum_{B\subset B(\lambda n)} \sum_{x\in\xi\cap B} \sum_{y\in\xi} r(y-x)|\chi(\tau_x\xi, y-x)|$$
$$= \delta^{-1} \sum_{\substack{x\in\xi: \\ |x|_\infty\leqslant\lambda n}} \sum_{y\in\xi} r(y-x)|\chi(\tau_x\xi, y-x)| = \delta^{-1} \sum_{\substack{x\in\xi: \\ |x|_\infty\leqslant\lambda n}} g(\tau_x\xi)\,, \quad (7.3)$$

where the function $g : \mathcal{N}_0 \to [0,\infty)$ is defined as $g(\xi) = \sum_{z\in\xi} r(z)|\chi(\xi, z)|$. Thus,

$$\mathbb{P}\Big[\mathcal{R}_n\, 1_{\{E_{\lambda,n}\}} \geqslant n^\theta\Big] \leqslant n^{-\theta}\, \mathbb{E}\Big[\mathcal{R}_n\,;\, E_{\lambda,n}\Big] \leqslant \delta^{-1}\, n^{-\theta}\, \mathbb{E}\Big[\sum_{\substack{x\in\xi: \\ |x|_\infty\leqslant\lambda n}} g(\tau_x\xi)\Big]\,. \quad (7.4)$$

Applying now the Campbell identity (2.1), with the notation (3.1) we can write the last expectation as

$$\rho(2\lambda n)^d \mathbb{E}_0(g(\xi)) \leqslant \rho(2\lambda n)^d \mathbb{E}_0(w(0))^{1/2} \|\chi\|_{L^2(\mu)} < \infty\,. \quad (7.5)$$

Using this bound in (7.4) we obtain $\mathbb{P}\big[\mathcal{R}_n\, 1_{\{E_{\lambda,n}\}} \geqslant n^\theta\big] \leqslant C\, n^{d-\theta}$, for some finite constant $C = C(\lambda,\rho,\delta)$. In conclusion,

$$\mathbb{P}\left[\mathcal{R}_n \geqslant n^\theta\right] \leqslant \mathbb{P}\left[\mathcal{R}_n\, 1_{\{E_{\lambda,n}\}} \geqslant n^\theta\right] + \mathbb{P}[E^c_{\lambda,n}] \leqslant C\, n^{d-\theta} + \mathbb{P}[E^c_{\lambda,n}]\,.$$

If λ is sufficiently large, the same argument used in the proof of Lemma 5.2 shows that $\mathbb{P}[E^c_{\lambda,n}]$ is exponentially decaying in n. Taking $\theta > d+1$, the Borel-Cantelli lemma implies that $n^{-\theta}\mathcal{R}_n \to 0$, $\mathbb{P}$–almost surely. □

Recall the definition of the positive exponent α (cf. (1.3)). The next lemma extends the estimate of Lemma 7.1 to the case where $y \notin \mathfrak{C}_\infty$. For $\alpha \leqslant 1$ this remains polynomial. When $\alpha > 1$ the bound is of the form $\exp((\log n)^\alpha)$. To unify the notation we use the function

$$u_{\gamma,c}(t) = t^\gamma \exp\big[c(\log(t+1))^\alpha\big]$$

introduced in Lemma 4.5.

Lemma 7.2. *There exist constants γ, c such that for $\mathbb{P}$–a.a. ξ and for all $n \geqslant n_0(\xi)$,*

$$\max_{\substack{x\in\xi\cap\mathfrak{C}_\infty, |x|\leqslant n \\ y\in\xi\,,\, |y|\leqslant n}} |\chi(\tau_x\xi, y-x)| \leqslant u_{\gamma,c}(n)\,. \quad (7.6)$$

Proof. As in the proof of Proposition 7.1, we get

$$\mathbb{E}\Big[\sum_{x\in\xi:|x|\leqslant n} \sum_{y\in\xi} |\chi(\tau_x\xi, y-x)|r(y-x)\Big] \leqslant c\, n^d\,, \quad (7.7)$$

for some constant c. From (7.7) and Markov's inequality, the Borel-Cantelli lemma shows that $\mathbb{P}$–a.s. for all n large enough:

$$\max_{x\in\xi:\,|x|\leqslant n} \sum_{y\in\xi} |\chi(\tau_x\xi, y-x)|r(y-x) \leqslant n^{d+2}\,. \quad (7.8)$$

Now, take $x, y \in \xi$ such that $|x|, |y| \leqslant n$ with n large. If $x \in \xi \cap \mathfrak{C}_\infty$, from Lemma 7.1, $|\chi(\tau_x\xi, z-x)| \leqslant n^\theta$ for all $z \in \xi\cap\mathfrak{C}_\infty$, $|z| \leqslant n$. However, a.s. there exists $z \in \xi\cap\mathfrak{C}_\infty$, $|z| \leqslant n$, such that $|z-y| \leqslant C\log n$ (the distance of y from $\mathfrak{C}_\infty$ cannot be larger than $C\log n$).

Thus, the claim follows by writing $|\chi(\tau_x\xi, y-x)| \leqslant |\chi(\tau_x\xi, z-x)| + |\chi(\tau_z\xi, y-z)|$ and using (7.8) on the obvious bound

$$|\chi(\tau_z\xi, y-z)| \leqslant \sup_{x\in\xi:\,|x|\leqslant n} e^{(C\log n)^\alpha} \sum_{y'\in\xi} |\chi(\tau_x\xi, y'-x)| r(y'-x)\,. \qquad \square$$

7.2. **Sublinearity along a given direction in $\mathfrak{C}_\infty$.** Let us fix a coordinate vector e (i.e. $e\in\mathbb{Z}^d$, $|e|_1=1$). Recall the notation $B_z = B(z)$, for the K-box at $z\in\mathbb{Z}^d$. Omitting the dependence on e, we set $n_0(\xi)=0$ and define inductively

$$n_{i+1}(\xi) = \min\{j > n_i(\xi) \,:\, B_{je}\subset\mathfrak{C}_\infty\}\,.$$

By property (A) in Section 2.3.1, for K large, the above maps are well–defined $\mathbb{P}$–a.s.

We introduce the space $\Omega = \{\xi : B_0\subset\mathfrak{C}_\infty(\xi)\}\times B_0^{\mathbb{N}}$. A generic element of this space is denoted by $\omega = \big(\xi, (v_i \,:\, i\in\mathbb{N})\big)$. On the space Ω we define a probability measure $\mathbb{P}_*$ (depending on the coordinate vector e) as follows: the marginal distribution of ξ is given by $\mathbb{P}(\cdot|B_0\subset\mathfrak{C}_\infty)$ while, conditioned to ξ, the sequence $(v_i \,:\, i\in\mathbb{N})$ is determined by choosing independently for each index i a point w_i with uniform probability in $B_{n_i(\xi)e}\cap\xi$ and then setting

$$v_i = w_i - n_i(\xi)Ke\,, \tag{7.9}$$

so that $v_i\in B_0$. Trivially, by (7.9), knowing $\omega = \big(\xi, (v_i \,:\, i\in\mathbb{N})\big)$ the points w_i are univocally determined, hence we write $w_i = w_i(\omega)$. Below we write $\mathbb{E}_*$ for the expectation w.r.t. $\mathbb{P}_*$. We point out that the space Ω is an example of the bridge spaces mentioned in the Introduction. The following key result is a consequence of assumption (H2):

Lemma 7.3. *Consider the map $\mathcal{T}:\Omega\to\Omega$ defined as*

$$\mathcal{T}\big(\xi, (v_i \,:\, i\in\mathbb{N})\big) = \big(\tau_{n_1(\xi)Ke}\xi, (v_{i+1} \,:\, i\in\mathbb{N})\big)\,.$$

Then $\mathbb{P}_*$ is ergodic and stationary w.r.t. the transformation $\mathcal{T}$.

Proof. Consider the space Θ with probability measure $P := P^{(K,e)}$ involved in assumption (H2). Define the subset $\mathcal{W}\subset\Theta$ as

$$\mathcal{W} := \big\{\big(\xi, (a_i : i\in\mathbb{Z})\big)\in\Theta \,:\, B_0\subset\mathfrak{C}_\infty\big\}$$

Then $P(\mathcal{W}) = \mathbb{P}(B_0\subset\mathfrak{C}_\infty) > 0$.

Recall the transformation $\tau:\Theta\to\Theta$ introduced in assumption (H2). It is invertible and ergodic w.r.t. P (by assumption (H2)). It is simple to check that it is measure preserving (using the stationarity of $\mathbb{P}$).

Let $F:\Theta\to\mathbb{N}\cup\{\infty\}$ be defined as

$$F(\vartheta) = \min\{k\geqslant 1 \,:\, \tau^k\vartheta\in\mathcal{W}\} = n_1(\xi)\,, \qquad \vartheta = \big(\xi, (a_i : i\in\mathbb{Z})\big)\in\Theta$$

and set $\mathcal{S}(\vartheta) = \tau^{F(\vartheta)}(\vartheta)$ if $F(\vartheta)<\infty$ (define $\mathcal{S}$ arbitrarily on the event $\{F(\vartheta)=\infty\}$, having zero P–probability). By the above observations all the conditions of Lemma 3.3 in [BB] are satisfied. In particular, we get that $\mathcal{S}$ (restricted to $\mathcal{W}$) is a measure preserving and ergodic transformation with respect to $P(\cdot|\mathcal{W})$.

Consider now the map $\pi:\mathcal{W}\to\Omega$ mapping $(\xi, (a_i : i\in\mathbb{Z}))$ to $\big(\xi, (a_{n_i(\xi)} : i\in\mathbb{N})\big)$. Note that

$$\mathcal{T}(\pi(\vartheta)) = \pi(\mathcal{S}(\vartheta)) \qquad \forall\vartheta\in\mathcal{W}\,. \tag{7.10}$$

Take $A\subset\Omega$ measurable such that $\mathcal{T}(A)=A$. Due to (7.10), $\mathcal{S}(\pi^{-1}(A)) = \pi^{-1}(A)$ and therefore $P(\pi^{-1}(A)|\mathcal{W})\in\{0,1\}$ by the ergodicity of $\mathcal{S}$ w.r.t. $P(\cdot|\mathcal{W})$. Since $\mathbb{P}_*(A) = P(\pi^{-1}(A)|\mathcal{W})$, we conclude that $\mathbb{P}_*(A)\in\{0,1\}$. $\square$

We define the vector function $\tilde\chi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ as

$$\tilde\chi(\omega, x) = \chi(\tau_{v_0}\xi, x - v_0)\,, \qquad \omega = \big(\xi, (v_i \,:\, i \in \mathbb{N})\big)\,. \tag{7.11}$$

Note that from the shift–covariance property of χ (cf. Lemma 3.5)

$$\tilde\chi(\omega, x) - \tilde\chi(\omega, y) = \chi(\tau_y\xi, x - y)\,, \qquad \forall \xi \in \mathcal{N}\,,\ \forall x, y \in \xi\,. \tag{7.12}$$

Lemma 7.4. *$\mathbb{E}_*(|\tilde\chi(\omega, w_1)|) < \infty$ and $\mathbb{E}_*(\tilde\chi(\omega, w_1)) = 0$, where w_1 is defined as in (7.9).*

Proof. Let us first show that $\mathbb{E}_*(|\tilde\chi(\omega, w_1)|) < \infty$. For $\xi \in \{B_0 \subset \mathfrak{C}_\infty\}$, we define $d(\xi)$ as the length of the minimal path in the cluster $C_\infty(\xi) \subset \mathbb{Z}^d$ from 0 to $n_1(\xi)e$. Note that $d(\xi) \geqslant n_1(\xi)$. Then, setting $g_k(\xi) = \sum_{y\in\xi} r(y)|\chi(\xi, y)|^k$, by the same arguments leading to (7.4) and applying twice Schwarz' inequality, we get

$$\mathbb{E}_*(|\tilde\chi(\omega, w_1)|) = \sum_{j=1}^\infty \mathbb{E}_*(|\tilde\chi(\omega, w_1)|\,;\, d(\xi) = j) \leqslant$$

$$\frac{1}{\delta}\sum_{j=1}^\infty \mathbb{E}\Big[\sum_{x\in\xi:|x|_\infty \leqslant Kj} g_1(\tau_x\xi)\,;\, d(\xi) = j \,\Big|\, B_0 \subset \mathfrak{C}_\infty\Big] \leqslant \frac{1}{\delta\mathbb{P}(B_0 \subset \mathfrak{C}_\infty)^{1/2}} \times$$

$$\sum_{j=1}^\infty \mathbb{P}\Big[d(\xi) = j \,\Big|\, B_0 \subset \mathfrak{C}_\infty\Big]^{1/2} \mathbb{E}\Big[\sum_{x\in\xi:|x|_\infty \leqslant Kj} g_0(\tau_x\xi)\Big]^{1/4} \mathbb{E}\Big[\sum_{x\in\xi:|x|_\infty \leqslant Kj} g_2(\tau_x\xi)\Big]^{1/4}.$$

Using the Campbell identity as in (7.5), we get that the last two expectations are bounded by $C(K,d)j^d\|\chi\|_{L^2(\mu)}$ and $C(K,d)j^d$, respectively. Finally, due to property (A) in Section 2.3.1 and standard facts in percolation theory (see for example Lemma 4.4 in [BB]), $\mathbb{P}\big[d(\xi) \geqslant j | B_0 \subset \mathfrak{C}_\infty\big] \leqslant e^{-aj}$ for some positive constant $a = a(K,d)$. Collecting the above bounds we get that $\mathbb{E}_*(|\tilde\chi(\omega, w_1)|) < \infty$.

We know that χ is the $L^2(\mu)$–limit of a sequence χ_n of functions of the form $\chi_n(\xi, x) = G_n(\tau_x\xi) - G_n(\xi)$, where $G_n : \mathcal{N}_0 \to \mathbb{R}^d$ is bounded and measurable. Since $(1,1)_\mu < \infty$, we derive that $\chi_n \in L^1(\mu)$ and that $\|\chi_n - \chi\|_{L^1(\mu)}$ goes to zero as $n \to \infty$. Repeating the above computations with $\tilde\chi$ replaced by $\tilde\chi - \tilde\chi_n$ we conclude that $\mathbb{E}_*\big(|\tilde\chi(\omega, w_1) - \tilde\chi_n(\omega, w_1)|\big)$ goes to zero as $n \to \infty$. In particular, $\lim_{n\to\infty} \mathbb{E}_*(\tilde\chi_n(\omega, w_1)) = \mathbb{E}_*(\tilde\chi(\omega, w_1))$. On the other hand, we can write

$$\mathbb{E}_*(\tilde\chi_n(\omega, w_1)) = \mathbb{E}_*(\chi_n(\tau_{v_0}\xi, w_1 - v_0)) = \mathbb{E}_*(G_n(\tau_{w_1}\xi)) - \mathbb{E}_*(G_n(\tau_{v_0}\xi))\,.$$

Setting $\omega = \big(\xi, (v_i \,:\, i \in \mathbb{N})\big)$ and $F_n(\omega) = G_n(\tau_{v_0}\xi)$, we can write $G_n(\tau_{w_1}\xi) = F_n(\mathcal{T}\omega)$. Hence, the conclusion follows from Lemma 7.3. □

Lemma 7.5. *With the notation of Eq. (7.9), one has*

$$\lim_{k\to\infty} \frac{\tilde\chi(\omega, w_k)}{k} = 0\,, \qquad \mathbb{P}_*\text{–a.s.} \tag{7.13}$$

Proof. Let $\omega = \big(\xi, (v_i : i \in \mathbb{N})\big)$. Since $v_0 = w_0$, it holds $\tilde\chi(\omega, w_0) = \chi(\tau_{v_0}\xi, 0) = 0$. Hence, we can write $\tilde\chi(\omega, w_k) = \sum_{j=0}^{k-1}\big(\tilde\chi(\omega, w_{j+1}) - \tilde\chi(\omega, w_j)\big)$. Applying now (7.12), we get $\tilde\chi(\omega, w_k) = \sum_{j=0}^{k-1} \chi(\tau_{w_j}\xi, w_{j+1} - w_j)$. On the other hand, since $\mathcal{T}^j\omega = \big(\tau_{n_j(\xi)Ke}\xi, (v_{j+i} : i \in \mathbb{N})\big)$, we can write

$$\chi(\tau_{w_j}\xi, w_{j+1} - w_j) =$$

$$\chi\Big(\tau_{v_j}(\tau_{n_j(\xi)Ke}\xi), v_{j+1} + n_1(\tau_{n_j(\xi)Ke}\xi)Ke - v_j\Big) = \tilde\chi\big(\mathcal{T}^j\omega, w_1(\mathcal{T}^j\omega)\big)\,. \tag{7.14}$$

Hence, $\tilde\chi(\omega, w_k) = \sum_{j=0}^{k-1} \tilde\chi\big(\mathcal{T}^j\omega, w_1(\mathcal{T}^j\omega)\big)$. The conclusion follows from the ergodicity stated in Lemma 7.3 and the results of Lemma 7.4. □

Next, we state a simple corollary of the above lemma, which is the starting point of our further investigations. In order to stress the dependence of the map $n_i(\cdot)$ from the vector e we write $n_i^{(e)}$. Below, $\mathbb{N}_+ = \{1, 2, \dots\}$.

Corollary 7.6. *Given a vector $e \in \mathbb{Z}^d$ with $|e|_1 = 1$, for $\mathbb{P}(\cdot|B_0 \subset \mathfrak{C}_\infty)$–a.a. ξ there exists a random sequence of points $(w_k^{(e)} : k \in \mathbb{N}_+)$ such that $w_k^{(e)} \in \xi \cap B(n_k^{(e)}(\xi)e)$, $k \in \mathbb{N}_+$ and*

$$\lim_{k\to\infty} \max_{x_0 \in \xi\cap B_0} \frac{|\chi(\tau_{x_0}\xi, w_k^{(e)} - x_0)|}{k} = 0\,. \tag{7.15}$$

7.3. **Sublinearity on average in $\mathfrak{C}_\infty$.** In this section we derive from Corollary 7.6 the sublinearity on average of the corrector field on $\xi \cap \mathfrak{C}_\infty$. A similar problem is attacked by Berger and Biskup in Section 5.2 in [BB] for the random walk on the supercritical percolation cluster. Their method cannot be applied directly to our context, and the adaptation of the geometric construction in [BB] would lead to a tremendous technical effort. We propose here a different construction, based on a two-scale argument, which allows us to give a self–contained treatment of the problem. The two scales refer to the fact that below the cluster C_∞ is considered at scale K (i.e. C_∞^K) and at scale mK (i.e. C_∞^{mK}), where the key point is that the cluster C_∞^{mK} can be made arbitrarily dense in $\mathbb{Z}^d$ by taking m large. Our target is to prove the following result:

Proposition 7.7. *For each $\varepsilon_0 > 0$, for $\mathbb{P}(\cdot|B_0 \subset \mathfrak{C}_\infty)$–a.a. $\xi \in \mathcal{N}$ and for all $x_0 \in \xi \cap B_0$, it holds*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x\in\xi\cap\mathfrak{C}_\infty\,:|x|_\infty \leqslant n} \mathbb{I}\{|\chi(\tau_{x_0}\xi, x - x_0)| \geqslant \varepsilon_0 n\} = 0\,. \tag{7.16}$$

For the reader's convenience, we isolate from the proof some technical lemmata.

Call $\mathbb{B} := \{e \in \mathbb{Z}^d : |e|_1 = 1\}$ and $\Lambda_s := [-s, s]^d$. Recall the definition of the random field σ^K. In order to stress the dependence on K, we write here B_x^K, C_∞^K and $\mathfrak{C}_\infty^K$.

Given positive numbers C, ε and $m \in \mathbb{N}_+$, we consider the Borel subsets $\mathcal{A}_{C,\varepsilon,m}$ and $\mathcal{A}_{C,m}$ in $\mathcal{N}$ defined as the family of $\xi \in \mathcal{N}$ satisfying properties (P1) and (P2), respectively:

(P1) for all $e \in \mathbb{B}$ and $N \in \mathbb{N}_+$, if $j \in \mathbb{N}_+$ satisfies $B_{je}^{mK} \subset \mathfrak{C}_\infty^{mK} \cap \Lambda_{mKN}$ (i.e. $je \in C_\infty^{mK} \cap \Lambda_N$) then there exists a point $x \in B_{je}^{mK} \cap \xi$ such that

$$\max_{x_0\in\xi\cap B_0^{mK}} |\chi(\tau_{x_0}\xi, x - x_0)| \leqslant C + \varepsilon N\,.$$

(P2) for any $x, x' \in \xi \cap B_0^{mK}$ one has

$$|\chi(\tau_x\xi, x' - x)| \leqslant C.$$

Let us fix $\varepsilon, \delta \in (0, 1)$. Thanks to property (A) (see Subsection 2.3.1), we can fix once for all m so large that

$$\mathbb{P}(0 \in C_\infty^{mK}) \geqslant 1 - \delta\,. \tag{7.17}$$

We have stated Corollary 7.6 working with the K–partition of $\mathbb{R}^d$, but trivially the conclusion remains valid if we work with the mK–partition (recall property (A)). In particular,

having fixed ε and m, we can find C large enough that $\mathbb{P}(\mathcal{A}_{C,\varepsilon,m}|0 \in C_\infty^{mK}) \geqslant 1-\delta$. Taking C large we also have $\mathbb{P}(\mathcal{A}_{C,m}|0 \in C_\infty^{mK}) \geqslant 1-\delta$. In particular,

$$\mathbb{P}(\mathcal{A}_{C,\varepsilon,m} \cap \mathcal{A}_{C,m}|0 \in C_\infty^{mK}) \geqslant 1-2\delta\,. \tag{7.18}$$

Given an integer ν with $1 \leqslant \nu \leqslant d$, we call

$$\Lambda_n^\nu := \Lambda_n \cap \{x \in \mathbb{R}^d \,:\, x_i = 0\ \forall i > \nu\}\,.$$

Lemma 7.8. *For $\mathbb{P}$–a.a. ξ there exists $n_0 = n_0(\xi)$, depending also on $C, m, \varepsilon, \delta$, such that for all $\nu : 1 \leqslant \nu \leqslant d$ and for all $n \geqslant n_0$ one has*

$$\frac{1}{|\Lambda_n^\nu \cap \mathbb{Z}^d|} \sum_{x \in \Lambda_n^\nu \cap C_\infty^{mK}} \mathbb{I}(\tau_{mKx}\xi \in \mathcal{A}_{C,\varepsilon,m} \cap \mathcal{A}_{C,m}) \geqslant 1-3\delta\,. \tag{7.19}$$

Proof. By the ergodicity assumption (H2) and the bounds (7.17) and (7.18), we have $\mathbb{P}$–a.s. that

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n^\nu \cap \mathbb{Z}^d|} \sum_{x \in \Lambda_n^\nu \cap C_\infty^{mK}} \mathbb{I}(\tau_{mKx}\xi \in \mathcal{A}_{C,\varepsilon,m} \cap \mathcal{A}_{C,m}) =$$

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n^\nu \cap \mathbb{Z}^d|} \sum_{x \in \Lambda_n^\nu \cap \mathbb{Z}^d} \mathbb{I}(\tau_{mKx}\xi \in \mathcal{A}_{C,\varepsilon,m} \cap \mathcal{A}_{C,m} \cap \{0 \in C_\infty^{mK}\}) =$$

$$\mathbb{P}(\mathcal{A}_{C,\varepsilon,m} \cap \mathcal{A}_{C,m}|0 \in C_\infty^{mK})\mathbb{P}(0 \in C_\infty^{mK}) \geqslant (1-2\delta)(1-\delta) > 1-3\delta\,. \quad \square$$

Suppose now that ξ satisfies (7.19) for all $n \geqslant n_0(\xi)$ (below we take $n \geqslant n_0(\xi)$). Call

$$G_n^\nu := \{x \in \Lambda_n^\nu \cap C_\infty^{mK} \,:\, \tau_{mKx}\xi \in \mathcal{A}_{C,\varepsilon,m} \cap \mathcal{A}_{C,m}\} \subset \mathbb{Z}^d\,.$$

By (7.19), one has

$$|G_n^\nu|/|\Lambda_n^\nu \cap \mathbb{Z}^d| \geqslant 1-3\delta\,. \tag{7.20}$$

Given $a \in \mathbb{Z}^d$ and $1 \leqslant \nu \leqslant d$, we set $a^\nu = (a_1, a_2, \dots, a_\nu, 0, \dots, 0)$. Then we define

$$\mathbb{G}_n^\nu = \left\{x \in \Lambda_n \cap \mathbb{Z}^d \,:\, x^\nu \in G_n^\nu\right\},$$
$$\mathbb{G}_n = \left\{x \in \Lambda_n \cap \mathbb{Z}^d \,:\, x^\nu \in G_n^\nu \ \ \forall \nu : 1 \leqslant \nu \leqslant d\right\}.$$

Trivially, $\mathbb{G}_n = \cap_{\nu=1}^d \mathbb{G}_n^\nu$. Moreover, by (7.20) it holds $|\mathbb{G}_n^\nu|/|\Lambda_n \cap \mathbb{Z}^d| \geqslant 1-3\delta$ and therefore, applying De Morgan's law,

$$|\mathbb{G}_n|/|\Lambda_n \cap \mathbb{Z}^d| \geqslant 1-3d\delta\,. \tag{7.21}$$

Lemma 7.9. *Suppose that ξ satisfies (7.19) and take $n \geqslant n_0(\xi)$. If $x \in \xi \cap B_a^{mK}$ with $a \in \mathbb{G}_n$, then there exists $x^{(1)} \in \xi \cap B_{a^1}^{mK}$ such that $|\chi(\tau_x\xi, x^{(1)} - x)| \leqslant Cd + \varepsilon dn$.*

Proof. Since $a \in G_n^d$ and $a^{d-1} \in G_n^{d-1} \subset C_\infty^{mK}$, by property (P1) applied to $\tau_{mKa}\xi$ with $N = n$ we know that there exists $x^{(d-1)} \in \xi \cap B_{a^{d-1}}^{mK}$ such that

$$|\chi(\tau_x\xi, x^{(d-1)} - x)| \leqslant C + \varepsilon n, .$$

Repeating the above argument, we obtain that there exist points $x^{(i)}$, $2 \leqslant i \leqslant d$, such that: $x^{(d)} = x$, $x^{(i)} \in \cap B_{a^i}^{mK}$ and $|\chi(\tau_{x^{(i)}}\xi, x^{(i-1)} - x^{(i)})| \leqslant C + \varepsilon n$. In particular, by the shift covariance property,

$$|\chi(\tau_x\xi, x^{(1)} - x)| \leqslant \sum_{i=2}^d |\chi(\tau_{x^{(i)}}\xi, x^{(i-1)} - x^{(i)})| \leqslant dC + d\varepsilon n\,. \quad \square$$

Lemma 7.10. *Suppose that ξ satisfies (7.19) and take $n \geqslant n_0(\xi)$. Then, for all $x, y \in \xi \cap (\cup_{a\in\mathbb{G}_n} B_a^{Km})$, it holds $|\chi(\tau_x\xi, y-x)| \leqslant 4dC + 3d\varepsilon n$.*

Proof. Suppose that $x \in \xi \cap B_a^{mK}$ with $a \in \mathbb{G}_n$ and $y \in \xi \cap B_b^{mK}$ with $b \in \mathbb{G}_n$. Take $x^{(1)}$, $y^{(1)}$ as in Lemma 7.9. By shift covariance,

$$\begin{aligned}\chi(\tau_x\xi, y-x) &= \chi(\tau_x\xi, x^{(1)}-x) + \chi(\tau_{x^{(1)}}\xi, y-x^{(1)}) \\ &= \chi(\tau_x\xi, x^{(1)}-x) + \chi(\tau_{x^{(1)}}\xi, y^{(1)}-x^{(1)}) + \chi(\tau_{y^{(1)}}\xi, y-y^{(1)})\,.\end{aligned}$$

Again by the shift covariance (see (3.6)), it holds $\chi(\tau_{y^{(1)}}\xi, y-y^{(1)}) = -\chi(\tau_y\xi, y^{(1)}-y)$. Hence, by the bound in Lemma 7.9 and the analogous estimate for y and $y^{(1)}$,

$$|\chi(\tau_x\xi, y-x)| \leqslant 2dC + 2d\varepsilon n + |\chi(\tau_{x^{(1)}}\xi, y^{(1)}-x^{(1)})|\,. \tag{7.22}$$

On the other hand, $a^1 \in G_n^1$ and $b^1 \in C_\infty^{mK}$. By property (P1) applied to $\tau_{mKa^1}\xi$, we conclude that there exists $\bar{y} \in B_{b^1}^{mK}$ such that $|\chi(\tau_{x^{(1)}}\xi, \bar{y}-x^{(1)})| \leqslant C + \varepsilon n$, while by property (P2) applied to $\tau_{mKb^1}\xi$ we get that $|\chi(\tau_{\bar{y}}\xi, y^{(1)}-\bar{y})| \leqslant C$. The thesis then follows observing that the shift covariance implies the identity

$$\chi(\tau_{x^{(1)}}\xi, y^{(1)}-x^{(1)}) = \chi(\tau_{x^{(1)}}\xi, \bar{y}-x^{(1)}) + \chi(\tau_{\bar{y}}\xi, y^{(1)}-\bar{y})\,. \qquad \square$$

We are finally able to conclude:

Proof of Proposition 7.7. We need to show that, given $\varepsilon_0, \delta_0 > 0$, for $\mathbb{P}(\cdot|B_0 \subset \mathfrak{C}_\infty)$–a.a. $\xi \in \mathcal{N}$ and for all $x_0 \in \xi \cap B_0$,

$$\lim_{s\to\infty} \frac{1}{|\Lambda_s \cap \mathbb{Z}^d|} \sum_{\substack{x\in\xi\cap\mathfrak{C}_\infty:\\ |x|\leqslant s}} \mathbb{I}\{|\chi(\tau_{x_0}\xi, x-x_0)| \geqslant \varepsilon_0 s\} \leqslant \delta_0\,. \tag{7.23}$$

Fix L such that

$$K^{-d}\mathbb{E}\big(\xi(B_0^K); \xi(B_0^K) \geqslant L\big) \leqslant \delta_0/2\,. \tag{7.24}$$

Take $\varepsilon := \varepsilon_0/(8d)$ and take $\delta > 0$ small enough that

$$\delta < \frac{\mathbb{P}(0 \in C_\infty^K)}{6d} \wedge \frac{\delta_0 K^d}{6dL}\,, \tag{7.25}$$

Set $r := \lceil s/K \rceil$ and $n := \lceil r/m \rceil$. Then choose first m and after that C as in the above construction, i.e. choose m large enough to assure (7.17), after that choose C large enough to assure (7.18). Finally, take s large enough that $r \geqslant mn_0(\xi)$, where $n_0(\xi)$ is as in Lemma 7.8 (this is meaningfull for $\mathbb{P}$–a.a. ξ, and in particular for $\mathbb{P}(\cdot|B_0 \subset \mathfrak{C}_\infty)$–a.a. ξ).

By ergodicity, $|\Lambda_r^1|^{-1}\sum_{j\in\Lambda_r^1} \mathbb{I}(je_1 \in C_\infty^K)$ converges to $p := \mathbb{P}(0 \in C_\infty^K)$ as $r \to \infty$. In particular, for r large enough (we write $r \geqslant r_1(\xi)$) the above average is larger than $p/2$. Hence, at the cost of losing a set of zero $\mathbb{P}$–probability and taking $r \geqslant r_1(\xi) \vee mn_0(\xi)$, we can assume that

$$|\{j \in \Lambda_r^1\,,\ je_1 \in C_\infty^K\}| \geqslant (2r+1)p/2\,. \tag{7.26}$$

Call π the projection of $\mathbb{Z}^d$ on its first coordinate axis $\mathbb{Z}e_1$, namely $\pi(x) = x^1 = (x_1, 0, \dots, 0)$. Note that $\pi(\mathbb{G}_n) \subset \mathbb{G}_n$ and, due to (7.21),

$$|\pi(\mathbb{G}_n)| \geqslant |\mathbb{G}_n|/(2n+1)^{d-1} \geqslant (1-3d\delta)(2n+1)\,.$$

In particular,

$$|\{j \in \Lambda_r^1 \,:\, \lfloor j/m \rfloor e_1 \in \mathbb{G}_n\}| \geqslant (1-3d\delta)(2n+1)m \geqslant (1-3d\delta)(2r+1)\,. \tag{7.27}$$

Since by (7.25)

$$(1-3d\delta)(2r+1)+(2r+1)p/2=(2r+1)(1-3d\delta+p/2)>2r+1\,,$$

we get that the set in the l.h.s. of (7.26) and the set in the l.h.s. of (7.27) must intersect. Hence, there exists $j\in\Lambda_r^1$ such that $B^K_{je_1}\subset\mathfrak{C}^K_\infty$ and $B^K_{je_1}\subset B^{mK}_a$ for some $a\in\mathbb{G}_n$.

Thanks to Corollary 7.6 on scale K, $\mathbb{P}(\cdot|B_0\subset\mathfrak{C}_\infty)$-a.s. there exists $x\in\xi$ in the above box $B^K_{je_1}$ such that

$$|\chi(\tau_{x_0}\xi,x-x_0)|\leqslant C(\xi)+\varepsilon r\,,\qquad\forall x_0\in\xi\cap B_0\,.\tag{7.28}$$

Note that $x\in\xi\cap(\cup_{a\in\mathbb{G}_n}B^{mK}_a)$. For all other points $y\in\xi\cap(\cup_{a\in\mathbb{G}_n}B^{mK}_a)$, by (7.28), Lemma 7.10 and the choice $\varepsilon=\varepsilon_0/8d$ we have

$$\begin{aligned}|\chi(\tau_{x_0}\xi,y-x_0)|\leqslant|\chi(\tau_{x_0}\xi,x-x_0)|+|\chi(\tau_x\xi,y-x)|&\leqslant(C(\xi)+\varepsilon r)+(4dC+3drm^{-1}\varepsilon)\\&\leqslant C'(\xi)+4dr\varepsilon\leqslant C'(\xi)+\varepsilon_0 r/2\leqslant C'(\xi)+\varepsilon_0 s/2\,.\end{aligned}\tag{7.29}$$

In particular, for s large enough (7.29) is smaller than $\varepsilon_0 s$.

Then we can bound

$$\begin{aligned}\sum_{\substack{y\in\xi\cap\mathfrak{C}^K_\infty:\\|y|_\infty\leqslant s}}\mathbb{I}\big\{|\chi(\tau_{x_0}\xi,y-x_0)|\geqslant\varepsilon_0 s\big\}&\leqslant\sum_{z\in\Lambda_r\cap\mathbb{Z}^d}\xi(B^K_z)\mathbb{I}(\xi(B^K_z)\geqslant L)\\&+\sum_{\substack{a\in\Lambda_n\cap\mathbb{Z}^d:\\a\notin\mathbb{G}_n}}\sum_{\substack{z\in\mathbb{Z}^d:\\B^K_z\subset B^{mK}_a}}\xi(B^K_z)\mathbb{I}\big(\xi(B^K_z)\leqslant L\big)=:|\Lambda_s\cap\mathbb{Z}^d|\big(A_1(s)+A_2(s)\big)\,.\end{aligned}\tag{7.30}$$

Using (7.24) and the ergodicity in assumption (H2), at the cost of removing a set of zero $\mathbb{P}$–probability, we have

$$\lim_{s\to\infty}A_1(s)=K^{-d}\mathbb{E}\big(\xi(B^K_0);\xi(B^K_0)\geqslant L\big)\leqslant\delta_0/2\,.\tag{7.31}$$

On the other hand,

$$A_2(s)\leqslant\frac{1}{|\Lambda_s\cap\mathbb{Z}^d|}\sum_{\substack{a\in\Lambda_n\cap\mathbb{Z}^d:\\a\notin\mathbb{G}_n}}\sum_{\substack{z\in\mathbb{Z}^d:\\B^K_z\subset B^{mK}_a}}L\leqslant\frac{Lm^d}{|\Lambda_s\cap\mathbb{Z}^d|}(|\Lambda_n\cap\mathbb{Z}^d|-|\mathbb{G}_n|)$$

Since $s\sim nKm$, by (7.21) we have that $\overline{\lim}_{s\to\infty}A_2(s)\leqslant LK^{-d}3d\delta$, which is smaller than $\delta_0/2$ by our choice of δ (cf. (7.25)). Coming back to (7.30), we get the thesis. □

7.4. **Sublinearity on average in $\mathfrak{C}^*_\infty$.** We now need to come back to the set of good points $\xi\cap\mathfrak{C}^*_\infty$.

Lemma 7.11. *If $\mathcal{A}\subset\mathcal{N}$ is a measurable set such that $\mathbb{P}(\mathcal{A}|B_0\subset\mathfrak{C}_\infty)=1$, then $\mathbb{P}(\mathcal{A}|B_0\subset\mathfrak{C}^*_\infty)=1$.*

Proof. Since $\mathfrak{C}^*_\infty\subset\mathfrak{C}_\infty$, the set $\mathcal{B}:=\{B_0\subset\mathfrak{C}^*_\infty\}\setminus\mathcal{A}$ is contained by the set $\mathcal{D}:=\{B_0\subset\mathfrak{C}_\infty\}\setminus\mathcal{A}$. Therefore we have the following sequence of implications

$$\begin{aligned}\mathbb{P}(\mathcal{A}|B_0\subset\mathfrak{C}^*_\infty)<1\Rightarrow\mathbb{P}(\mathcal{B}|B_0\subset\mathfrak{C}^*_\infty)>0\Rightarrow\mathbb{P}(\mathcal{B})>0\Rightarrow\mathbb{P}(\mathcal{D})>0\\\Rightarrow\mathbb{P}(\mathcal{D}|B_0\subset\mathfrak{C}_\infty)>0\Rightarrow\mathbb{P}(\mathcal{A}|B_0\subset\mathfrak{C}_\infty)<1\,,\end{aligned}\tag{7.32}$$

showing the contrapositive. □

From Proposition 7.7 and Lemma 7.11 we easily obtain the following

Corollary 7.12. *Given $\varepsilon > 0$, for $\mathbb{P}(\cdot | B_0 \subset \mathfrak{C}^*_\infty)$–a.a. $\xi \in \mathcal{N}$ and for all $x_0 \in \xi \cap B_0$,*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x\in\xi\cap\mathfrak{C}^*_\infty : |x| \leqslant n} \mathbb{I}\{|\chi(\tau_{x_0}\xi, x - x_0)| \geqslant \varepsilon n\} = 0\,. \tag{7.33}$$

7.5. **Strong sublinearity in $\mathfrak{C}^*_\infty$.**

Lemma 7.13. *For each $\varepsilon > 0$, for $\mathbb{P}(\cdot | B_0 \subset \mathfrak{C}^*_\infty)$–a.a. ξ and for all $x_0 \in \xi \cap B_0$,*

$$\lim_{n\to\infty} \frac{1}{n} \max_{\substack{x\in\xi\cap\mathfrak{C}^*_\infty : \\ |x|_\infty \leqslant n}} |\chi(\tau_{x_0}\xi, x - x_0)| = 0\,. \tag{7.34}$$

Proof. Let us define

$$R_n(\xi) = \max_{x_0\in\xi\cap B_0} \max_{\substack{z\in\xi\cap\mathfrak{C}^*_\infty : \\ |z|_\infty \leqslant n}} |\chi(\tau_{x_0}\xi, z - x_0)|\,. \tag{7.35}$$

Due to Lemma 7.1 and Lemma 7.11, for $\theta > d + 1$

$$\lim_{n\to\infty} n^{-\theta} R_n = 0\,, \qquad \mathbb{P}(\cdot | B_0 \subset \mathfrak{C}^*_\infty)\text{–a.s.} \tag{7.36}$$

Following an idea of Y. Peres, we only need to prove a recursive bound of the form: for each $\varepsilon, \delta > 0$, there exists an a.s. finite random variable $n_0 = n_0(\xi, \varepsilon, \delta)$ such that

$$R_n \leqslant \varepsilon n + \delta R_{5n}\,, \qquad n \geqslant n_0\,. \tag{7.37}$$

From (7.37), using the input (7.36), it is easy to conclude; see the explanation after [BP, Lemma 5.1].

We turn to the proof of (7.37). We take $\xi \in \mathcal{N}$ such that $\{B_0 \subset \mathfrak{C}^*_\infty\}$ and satisfying (7.36). Moreover, assume ξ and $b_n = o(n^2)$ satisfy (4.5) and (4.4) of Proposition 4.1. Take x_0, z such that $R_n(\xi) = |\chi(\tau_{x_0}\xi, z - x_0)|$, $x_0 \in \xi \cap B_0$, $z \in \xi \cap \mathfrak{C}^*_\infty$, $|z|_\infty \leqslant n$. Similarly to [BP], take $t = t(n) \geqslant b_{4n} \vee n$ (we will specify the function $t(n)$ at the end). Fix positive constants C_1, C_2 such that the expressions $\max_x \sup_t$ in (4.5) and (4.4) are bounded by C_1 and C_2, respectively, if n is large enough, that is $n \geqslant n_*(\xi)$ for a suitable constant $n_*(\xi)$. Take $n \geqslant K \vee n_*(\xi)$. Finally, define the stopping time

$$S_n := \inf\{t \geqslant 0 : |Y_t - z|_\infty \geqslant 2n\}\,.$$

Due to Corollary 4.10 and the Optional Stopping Theorem we can write

$$E_{z,\xi}\Big[\chi(\tau_z\xi, Y_{t\wedge S_n} - z) + Y_{t\wedge S_n} - z\Big] = \chi(\tau_z\xi, 0) = 0\,. \tag{7.38}$$

By the shift–covariance property we can write

$$\chi(\tau_{x_0}\xi, z - x_0) = \chi(\tau_{x_0}\xi, Y_{t\wedge S_n} - x_0) - \chi(\tau_z\xi, Y_{t\wedge S_n} - z)\,. \tag{7.39}$$

Combining (7.38) and (7.39) we get

$$\chi(\tau_{x_0}\xi, z - x_0) = E_{z,\xi}\Big[\chi(\tau_{x_0}\xi, Y_{t\wedge S_n} - x_0) + Y_{t\wedge S_n} - z\Big]\,, \tag{7.40}$$

thus implying that $R_n(\xi) \leqslant E_{z,\xi}\Big[\big|F(\xi, Y_{t\wedge S_n})\big|\Big]$ where

$$F(\xi, Y_{t\wedge S_n}) := \chi(\tau_{x_0}\xi, Y_{t\wedge S_n} - x_0) + Y_{t\wedge S_n} - z\,. \tag{7.41}$$

Let us introduce the event $A = \{S_n < t\,,\ |Y_{S_n} - z|_\infty > 4n\}$. Using (7.36) we can bound

$$E_{z,\xi}\Big[\big|F(\xi, Y_{t\wedge S_n})\big|\,;\,A\Big] \leqslant \sum_{k=4}^{\infty} E_{z,\xi}\Big[\big|F(\xi, Y_{t\wedge S_n})\big|\,;\,A\,;\,|Y_{S_n} - z|_\infty \in [kn, (k+1)n)\Big]$$
$$\leqslant \sum_{k=4}^{\infty}\big\{C(\xi)k^\theta n^\theta + (k+1)n\big\}P_{z,\xi}\Big[|Y_{S_n} - z|_\infty \in [kn,(k+1)n)\,,\ S_n < t\Big]$$
$$\leqslant C' n^\theta \sum_{k=4}^{\infty} k^\theta P_{z,\xi}\Big[|Y_{S_n} - z|_\infty \in [kn,(k+1)n)\,,\ S_n < t\Big]\,. \quad (7.42)$$

Recall that Y_t is the continuous time version of the discrete time walk $n \mapsto X_{T_n}$. If $|Y_{S_n} - z|_\infty \geqslant kn$, then up to time $t > S_n$ there must have been a jump of size at least $(k-2)n$ starting from some point x such that $|x|_\infty \leqslant 3n$. Taking $t = \vartheta n^2$, with $\vartheta > 0$, and using Lemma 4.7, we can estimate for any $k \geqslant 4$, $p \geqslant 1$:

$$P_{z,\xi}\big[|Y_{S_n} - z|_\infty \in [kn,(k+1)n)\,,\ S_n < t\big]$$
$$\leqslant \sum_{j=1}^{\infty} P(N_t = j) P_{z,\xi}\big[\max_{1\leqslant i\leqslant j} |X_{T_i} - X_{T_{i-1}}|_\infty \geqslant (k-2)n;\ |X_{T_{i-1}}|_\infty \leqslant 3n\big]$$
$$\leqslant \sum_{j=1}^{\infty} P(N_t = j) j\,((k-2)n)^{-p} \max_{1\leqslant i\leqslant j} E_{z,\xi}\big[|X_{T_i} - X_{T_{i-1}}|_\infty^p;\ |X_{T_{i-1}}|_\infty \leqslant 3n\big]$$
$$\leqslant \vartheta n^2((k-2)n)^{-p} \max_{|x|_\infty \leqslant 3n} E_{x,\xi}\big[|X_{T_1} - x|_\infty^p\big]$$
$$\leqslant \vartheta C k^{-p} n^{-p+2} (\log n)^p\,,$$

where $C = C(\xi)$ is a constant. Coming back to (7.42) and taking $p > 2 + \theta$, we get

$$E_{z,\xi}\Big[\big|F(\xi, Y_{t\wedge S_n})\big|\,;\,A\Big] \leqslant \vartheta C n^{\theta+2-p}(\log n)^p \sum_{k=4}^{\infty} k^{\theta-p} \leqslant (\varepsilon/2)n, \quad (7.43)$$

for all n large enough.

The above expectation, coming from the presence of long jumps, does not appear in [BP]. On the other hand, using Proposition 4.1 and taking $t = \vartheta n^2$ with $\vartheta > 0$, the control of $E_{z,\xi}\big[\big|F(\xi, Y_{t\wedge S_n})\big|\,;\,A^c\big]$ can be obtained by the same computations in the proof of Lemma 5.1 in [BP]. As a final result, one gets (7.37). □

7.6. **Proof of Theorem 3.6.** We now have most of the tools needed to prove the sublinearity of the corrector field stated in Theorem 3.6. First, we need to link the Palm distribution to the probability measures used above. To this end we introduce a bridge probability space. We call $\mathbb{Q}_0$ the distribution on $\mathcal{N}_0$ of the point process ξ defined in this way: pick a configuration $\tilde\xi \in \mathcal{N}$ with law $\mathbb{P}(\cdot|\tilde\xi(B_0) \geqslant 1)$, pick a point $v_0 \in \tilde\xi \cap B_0$ with uniform probability, and set $\xi = \tau_{v_0}\tilde\xi$.

Lemma 7.14. *Let $\mathcal{A} \subset \mathcal{N}_0$ be a measurable subset. Then $\mathbb{Q}_0(\mathcal{A}) = 0, 1$ if and only if $\mathbb{P}_0(\mathcal{A}) = 0, 1$, respectively.*

Proof. By taking the complements, it is enough to prove that $\mathbb{Q}_0(\mathcal{A}) = 1$ if and only if $\mathbb{P}_0(A) = 1$. Consider the measurable set $\mathcal{B} := \{\xi \in \mathcal{N} : \tau_x\xi \in \mathcal{A}\ \forall x \in \xi \cap B_0\}$. By the

Campbell identity (2.1) with $f(x,\xi) = \mathbb{I}(x \in B_0\,;\, \xi \in \mathcal{A}\}$ we have

$$\mathbb{P}_0(\mathcal{A}) = \frac{1}{\rho K^d}\int_{\mathcal{N}} \mathbb{P}(d\xi)\int_{B_0}\xi(dx)\mathbb{I}(\tau_x\xi\in\mathcal{A}) \leqslant \frac{1}{\rho K^d}\int_{\mathcal{N}} \mathbb{P}(d\xi)\int_{B_0}\xi(dx)1 = 1\,,$$

which implies that $\mathbb{P}_0(\mathcal{A}) = 1$ if and only if $\mathbb{P}(\mathcal{B}) = 1$.

Similarly, by definition of $\mathbb{Q}_0$ we have

$$\mathbb{Q}_0(\mathcal{A}) = \frac{1}{\mathbb{P}(\xi(B_0)\geqslant 1)}\int_{\{\xi:\xi(B_0)\geqslant 1\}}\mathbb{P}(d\xi)\frac{\int_{B_0}\xi(dx)\mathbb{I}(\tau_x\xi\in\mathcal{A})}{\xi(B_0)}$$
$$\leqslant \frac{1}{\mathbb{P}(\xi(B_0)\geqslant 1)}\int_{\{\xi:\xi(B_0)\geqslant 1\}}\mathbb{P}(d\xi)\frac{\int_{B_0}\xi(dx)1}{\xi(B_0)} = 1\,,$$

which implies that $\mathbb{Q}_0(\mathcal{A}) = 1$ if and only if $\mathbb{P}(\mathcal{B}) = 1$. □

Thanks to the above lemma, to prove Theorem 3.6 it suffices to show that for $\mathbb{P}(\cdot|\xi(B_0)\geqslant 1)$–a.a. ξ,

$$\lim_{n\to\infty}\frac{1}{n}\max_{x_0\in\xi\cap B_0}\max_{\substack{x\in\xi:\\ |x|_\infty\leqslant n}}|\chi(\tau_{x_0}\xi, x-x_0)| = 0\,. \tag{7.44}$$

The plan of the proof is the following: we first improve Lemma 7.13 by passing from $\mathbb{P}(\cdot|B_0\subset\mathfrak{C}^*_\infty)$ to $\mathbb{P}(\cdot|\xi(B_0)\geqslant 1)$ (see Lemma 7.16); after that we remove the restriction $x\in\mathfrak{C}^*_\infty$ which appears in Lemma 7.16 by applying the Optional Stopping Theorem.

We fix a coordinate vector e and define the map $n_*:\mathcal{N}\to\mathbb{N}_+ := \{1,2,\dots\}$ as follows:

$$n_*(\xi) = \min\{n\in\mathbb{N}_+\,:\, B_{ne}\subset\mathfrak{C}^*_\infty\}\,. \tag{7.45}$$

By assumption (H1), the map is well defined $\mathbb{P}$–a.s.

Lemma 7.15. *Call P the law on $\mathcal{N}$ of the point process $\tau_{n_*(\xi)Ke}\xi$, where $\xi\in\mathcal{N}$ is sampled with probability $\mathbb{P}(\cdot|\xi(B_0)\geqslant 1)$. Then, for any measurable subset $\mathcal{A}\subset\mathcal{N}$,*

$$\mathbb{P}(\mathcal{A}\,|\,B_0\subset\mathfrak{C}^*_\infty) = 1 \Rightarrow P(\mathcal{A}) = 1\,.$$

Proof. Given a bounded measurable function $f:\mathcal{N}\to\mathbb{R}$ we can write

$$E_P(f) = \int\mathbb{P}\big(d\xi\,|\,\xi(B_0)\geqslant 1\big)f(\tau_{n_*(\xi)Ke}\xi) = \frac{\mathbb{E}\left[f(\tau_{n_*(\xi)Ke}\xi);\xi(B_0)\geqslant 1\right]}{\mathbb{P}(\xi(B_0)\geqslant 1)}\,.$$

Moreover,

$$\mathbb{E}\left[f(\tau_{n_*(\xi)Ke}\xi);\xi(B_0)\geqslant 1\right] = \sum_{m=1}^\infty\mathbb{E}\left[f(\tau_{mKe}\xi);\xi(B_0)\geqslant 1; n_*(\xi)=m\right]$$
$$= \sum_{m=1}^\infty\mathbb{E}\big[f(\tau_{mKe}\xi)\,;\,\xi(B_0)\geqslant 1\,;B_{ke}\not\subset\mathfrak{C}^*_\infty\ \forall k: 1\leqslant k\leqslant m-1\,,\ B_{me}\subset\mathfrak{C}^*_\infty\big]\,. \tag{7.46}$$

Due to the stationarity of $\mathbb{P}$ the last expression can be written as

$$\sum_{m=1}^\infty\mathbb{E}\big[f(\xi)\,;\,\xi(B_{-me})\geqslant 1\,;B_{ke}\not\subset\mathfrak{C}^*_\infty\ \forall k: 1-m\leqslant k\leqslant -1\,;B_0\subset\mathfrak{C}^*_\infty\big]$$
$$= \mathbb{E}\big[f(\xi)\hat{n}(\xi)\,;B_0\subset\mathfrak{C}^*_\infty\big]\,, \tag{7.47}$$

where we define

$$n_-(\xi) = \max\{j \leqslant -1 : B_{je} \subset \mathfrak{C}^*_\infty\},$$
$$\hat{n}(\xi) = \sharp\{j : n_-(\xi) \leqslant j \leqslant -1,\ \xi(B_{je}) \geqslant 1\}.$$

Collecting the above observations we get

$$E_P(f) = \frac{\mathbb{E}\,[f(\xi)\hat{n}(\xi)\,; B_0 \subset \mathfrak{C}^*_\infty]}{\mathbb{P}(\xi(B_0) \geqslant 1)} = \mathbb{E}\,[f(\xi)\hat{n}(\xi)\,;\, B_0 \subset \mathfrak{C}^*_\infty \,|\, \xi(B_0) \geqslant 1]\,. \tag{7.48}$$

This proves that

$$P(d\xi) = \hat{n}(\xi)\mathbb{I}(B_0 \subset \mathfrak{C}^*_\infty)\mathbb{P}(d\xi \,|\, \xi(B_0) \geqslant 1) = \hat{n}(\xi)\frac{\mathbb{I}(B_0 \subset \mathfrak{C}^*_\infty)}{\mathbb{P}(\xi(B_0) \geqslant 1)}\mathbb{P}(d\xi)\,. \tag{7.49}$$

Since P is a probability measure it must hold $1 = \mathbb{E}\,[\hat{n}(\xi); B_0 \subset \mathfrak{C}^*_\infty]\,/\mathbb{P}[\xi(B_0) \geqslant 1]$. Take $\mathcal{A} \subset \mathcal{N}$ measurable and satisfying $\mathbb{P}(\mathcal{A} \,|\, B_0 \subset \mathfrak{C}^*_\infty) = 1$. This implies that $\mathbb{I}(\xi \in \mathcal{A})\mathbb{I}(B_0 \subset \mathfrak{C}^*_\infty) = \mathbb{I}(B_0 \subset \mathfrak{C}^*_\infty)$ $\mathbb{P}$–a.s. In particular, we have

$$P(\mathcal{A}) = \frac{\mathbb{E}\big[\hat{n}(\xi)\mathbb{I}(B_0 \subset \mathfrak{C}^*_\infty)\mathbb{I}(\xi \in \mathcal{A})\big]}{P(\xi(B_0) \geqslant 1)} = \frac{\mathbb{E}\big[\hat{n}(\xi)\mathbb{I}(B_0 \subset \mathfrak{C}^*_\infty)\big]}{P(\xi(B_0) \geqslant 1)}\,,$$

while we have already shown that the last member equals 1. □

Recall the definition (7.35) of R_n.

Lemma 7.16. *For $\mathbb{P}(\cdot|\xi(B_0) \geqslant 1)$–a.a. ξ, it holds $\lim_{n\to\infty} R_n/n = 0$.*

Proof. Recall the definition of the function $n_*(\xi)$ given in (7.45). As a byproduct of Lemma 7.13 and Lemma 7.15, we get for $\mathbb{P}\big(\cdot|\xi(B_0) \geqslant 1\big)$–a.a. ξ that, for any $x' \in \xi \cap B_{n_*(\xi)e}$, it holds

$$\lim_{n\to\infty} \frac{1}{n} \max_{\substack{y \in \xi \cap \mathfrak{C}^*_\infty \\ |y - n_*(\xi)Ke|_\infty \leqslant 2n}} \big|\chi\big(\tau_{x'}\xi, y - x'\big)\big| = 0 \tag{7.50}$$

(the above choice of $2n$ instead of n is due to later applications). On the other hand, by the shift covariance, given $x_0 \in \xi \cap B_0$, $x \in \xi \cap \mathfrak{C}^*_\infty$ and $x' \in \xi \cap B_{n_*(\xi)e}$, we can write

$$\chi(\tau_{x_0}\xi, x - x_0) = \chi(\tau_{x_0}\xi, x' - x_0) + \chi(\tau_{x'}\xi, x - x')\,. \tag{7.51}$$

Since, given ξ and $x \in \xi \cap \mathfrak{C}^*_\infty$ with $|x|_\infty \leqslant n$, it holds $|n_*(\xi)Ke - x|_\infty \leqslant 2n$ for n large enough, one can apply (7.50) with $y = x$. From (7.50) and (7.51) we then obtain

$$\lim_{n\to\infty} \frac{1}{n} \max_{x_0 \in \xi \cap B_0} \max_{\substack{x \in \xi \cap \mathfrak{C}^*_\infty \\ |x|_\infty \leqslant n}} \big|\chi(\tau_{x_0}\xi, x - x_0)\big| = 0\,, \tag{7.52}$$

which corresponds to the thesis. □

Let us finally conclude. Take $x_0 \in B_0 \cap \xi$ and $x \in \xi$ with $|x|_\infty \leqslant n$. From the Optional Stopping Theorem (cf. Proposition 4.9) we know that

$$x + \chi(\tau_{x_0}\xi, x - x_0) = E_{x,\xi}\left[X_{T_1} + \chi(\tau_{x_0}\xi, X_{T_1} - x_0)\right]. \tag{7.53}$$

Take $\varepsilon < 1$ and let $\mathcal{S}_k = \{(k-1)n^\varepsilon \leqslant |X_{T_1} - x| < kn^\varepsilon\}$, $k = 1, 2, \ldots$. Recalling (7.35), we can estimate

$$|\chi(\tau_{x_0}\xi, x - x_0)| \leqslant n^\varepsilon + R_{2n} + \sum_{k=2}^{\infty}(kn^\varepsilon + R_{n+kn^\varepsilon})P_{x,\xi}(\mathcal{S}_k)\,. \tag{7.54}$$

Lemma 7.16 gives us the estimate $R_n = o(n)$, and therefore we see that the desired conclusion follows if we can show that a.s. $\sum_{k=2}^{\infty}(n+kn^{\varepsilon})P_{x,\xi}(\mathcal{S}_k) = o(n)$. This, in turn, follows from Lemma 4.7 and the fact that $|x|_\infty \leqslant n$. Indeed, for every $k \geqslant 1$, and $p \geqslant 1$:

$$P_{x,\xi}(\mathcal{S}_{k+1}) \leqslant P_{x,\xi}(|X_{T_1} - x| \geqslant kn^{\varepsilon}) \leqslant \kappa\,(kn^{\varepsilon})^{-p}\,(\log n)^p\,.$$

Taking $p > 3$ we get the desired bound.

8. Proof of Theorem 1.2 in the presence of energy marks

Let us suppose now that the function $u(E_x, E_y)$ is non trivial. In this case, the environment of the random walk is given by $\omega = \big\{(x, E_x) : x \in \xi\big\}$ and corresponds to a marked simple point process. We refer to [FSS, Section 2] for detailed definitions and references. Here we simply recall that stationarity and ergodicity of the point process ξ automatically extend to $\mathbb{P}$. Moreover, the Campbell identity remains valid in the marked case (with suitable changes).

We fix some notation. We write $\tilde{\mathcal{N}}$ for the state space of the marked point process ω and, given $v \in \mathbb{R}^d$, we define the translation $\tau_v\omega$ as

$$\tau_v\omega := \big\{(x - v, E_x) : x \in \xi\big\}\,, \qquad \omega = \big\{(x, E_x) : x \in \xi\big\}\,.$$

Let us suppose that $\mathbb{P}$ is an ergodic stationary marked simple point process with finite second moment. As already mentioned assumption (H1) is the same, while assumption (H2) has to be slightly modified as follows:

(H2) for each $K > 0$ and for each vector $e \in \mathbb{Z}^d$ with $|e|_1 = 1$, consider the product probability space $\Theta := \tilde{\mathcal{N}} \times [([0, K)^d \times \mathbb{R}) \cup \{\partial\}]^{\mathbb{Z}}$ whose elements $\big(\omega, (a_i : i \in \mathbb{Z})\big)$ are sampled as follows: choose $\omega = \big\{(x, E_x) : x \in \xi\big\}$ with law $\mathbb{P}$, afterwards choose independently for each index i a point $b_i \in \xi \cap B(ie)$ with uniform probability and set $a_i := (b_i - iKe, E_{b_i})$ (if $\xi \cap B(ie) = \emptyset$, set $a_i = \partial$). We assume that the resulting law $P^{(K,e)}$ on $\tilde{\mathcal{N}} \times [([0, K)^d \times \mathbb{R}) \cup \{\partial\})]^{\mathbb{Z}}$ is ergodic w.r.t. the transformation

$$\tau : \big(\omega, (a_i : i \in \mathbb{Z})\big) \to \big(\tau_{Ke}\omega, (a_{i+1} : i \in \mathbb{Z})\big)\,. \tag{8.1}$$

The proof that the marked PPP satisfies the new assumption (H2) is similar to the non–marked case. The proof of Theorem 1.2 in the presence of the energy marks can be obtained by a straightforward extension of the proof presented in the non–marked case. Indeed, the presence of the energy marks is rather painless since the weights $e^{-u(E_x,E_y)}$ are uniformly bounded from above and from below by some positive constants. Furthermore, the following covariant property, implicitly used in the non–marked case, holds: writing $p_\omega(x, y)$ for the jump probability of X_n in the environment ω, then $p_\omega(x, y) = p_{\tau_v\omega}(x - v, y - v)$ for all $v \in \xi$.

Appendix A. Strong invariance principle for Mott random walk on diluted lattices

In this appendix we discuss the quenched invariance principle for diluted lattices. The proof differs from the one of Theorem 1.2 in few points (mainly related to ergodicity) that we comment below. In order to simplify the notation, we disregard the energy marks (all the arguments can be easily adapted to the marked case).

We start with a lattice Γ (or crystal, cf. [AM]): Γ is a locally finite set $\Gamma \subset \mathbb{R}^d$ such that for a suitable basis $v_1, v_2, \ldots, v_d$ of $\mathbb{R}^d$, it holds

$$\Gamma - x = \Gamma \qquad \forall x \in G := \big\{z_1v_1 + z_2v_2 + \cdots + z_dv_d \,:\, z_i \in \mathbb{Z}^2 \ \ \forall i\big\}\,. \tag{A.1}$$

Let Δ be the elementary cell defined as $\{t_1v_1 + t_2v_2 + \cdots + t_dv_d : 0 \leqslant t_i < 1 \ \forall i\}$. (Note that both the group G and the cell Δ depend on the basis $v_1, v_2, \dots, v_d$.)

Let $\omega = (\omega_x : x \in \Gamma)$ be a site Bernoulli percolation on Γ with parameter $p \in (0,1]$. For each $u \in \Gamma \cap \Delta$ we call P_u the law on $\mathcal{N}_0$ of the random point process given by $\{0\} \cup \{x - u : x \in \Gamma,\ \omega_x = 1\}$ and we consider $\mathbb{P}_0 = \frac{1}{|\Delta \cap \Gamma|} \sum_{u \in \Delta \cap \Gamma} P_u$. As proved in [FM], $\mathbb{P}_0$ does not depend on the basis $v_1, \dots, v_d$ and on the fundamental cell Δ, moreover $\mathbb{P}_0$ is indeed the Palm distribution of the stationary point process with law $\mathbb{P}$ realized as $\{x - V : x \in \Gamma,\ \omega_x = 1\}$ where V is a random vector uniformly distributed in the fundamental cell Δ, independent from the field ω. Finally, we call P the law of the point process $\{x \in \Gamma : \omega_x = 1\}$.

It is simple to check that both the discrete–time and the continuous–time Mott random walks are well defined $\mathbb{P}_0$–a.s., $\mathbb{P}$–a.s. and P–a.s. Moreover, as in Theorem 1.2 and Corollary 1.3, proving the invariance principle for $\mathbb{P}_0$–a.a. ξ with starting point $x_0 = 0$, one automatically gets the strong invariance principle.

The corrector field is defined as in Section 3. By applying a linear isomorphism, we can assume that the basis $v_1, \dots, v_d$ coincides with the coordinate basis of $\mathbb{Z}^d$ and therefore that $\Delta = [0,1)^d$. We restrict to $K \in \mathbb{N}_+$. Then under P, the point processes $B_K(z) \cap \tau_{Kz}\xi$ with $z \in \mathbb{Z}^d$ are i.i.d. In particular, P is stationary and ergodic w.r.t. the translation τ_{Kv_i}. Moreover, sampling ξ with law P, the random field $\sigma^K(\xi)$ is a Bernoulli random field, supercritical if K is taken large enough. Define C_∞ as its unique infinite cluster and define $\mathfrak{C}_\infty$ as before. In the definition of the law $\mathbb{P}_*$ on the space Ω given in Section 7.2, replace $\mathbb{P}$ with P. With this trick, $\mathbb{P}_*$ remains ergodic and stationary w.r.t. to the map $\mathcal{T}$ defined in Section 7.2 and one can prove the sublinearity of the corrector field along a given direction. At this point, substituting $\mathbb{P}$ with P, the proof of the quenched invariance principle follows the same main steps of the proof of Theorem 1.2, even with huge simplications. Indeed, working with diluted lattices overcrowded regions are absent. In particular, taking T_0 large enough, the field ϑ^{K,T_0} coincides with σ^K.

Appendix B. Miscellanea

We start with a key technical lemma:

Lemma B.1. *Let $\mathbb{P}_0$ be the Palm distribution associated to a stationary simple point process $\mathbb{P}$ with finite density.*

(i) Suppose that $\rho_2 = \mathbb{E}\left(\xi([0,1)^d)^k\right) < \infty$, and let $f : \mathbb{R}^d \times \mathcal{N}_0 \to \mathbb{R}$ be a measurable function satisfying $\mathbb{E}_0\Big[\sum_{x\in\xi} |f(x,\xi)|\Big] < \infty$. Then $\mathbb{E}\Big[\sum_{x\in\xi} |f(-x,\tau_x\xi)|\Big] < \infty$ and

$$\mathbb{E}_0\Big[\sum_{x\in\xi} f(x,\xi)\Big] = \mathbb{E}\Big[\sum_{x\in\xi} f(-x,\tau_x\xi)\Big]\,. \tag{B.1}$$

(ii) Let n be a nonnegative integer such that $\rho_{n+1} < \infty$. Then

$$\mathbb{E}_0\Big[\big(\sum_{x\in\xi} e^{-\gamma|x|^\alpha}\big)^n\Big] < \infty\,, \qquad \forall \alpha, \gamma > 0\,. \tag{B.2}$$

Proof. Part (ii) can be derived generalizing the proof Lemma 2 of [FSS]. The proof of part (i) uses some arguments taken from the proof of Lemma 1 (i) in [FSS]. We give some more details. Without loss of generality we can assume that $f \geqslant 0$. We define $F(\xi) = \sum_{x\in\xi} f(x,\xi)$ and $G(\xi) = \sum_{x\in\xi} f(-x,\tau_x\xi)$. Note that, given $u \in \xi$, it holds $F(\tau_u\xi) = \sum_{y\in\xi} f(y-u,\tau_u\xi)$ and $G(\tau_u\xi) = \sum_{y\in\xi} f(u-y,\tau_y\xi)$. In particular,

taking $L > 0$ and setting $\Lambda_L = [-L/2, L/2]^d$, by the Campbell identity we can write $\mathbb{E}_0[F] = A(L) + B(L)$ and $\mathbb{E}_0[G] = A(L) + C(L)$ where

$$A(L) = \frac{1}{\rho L^d}\mathbb{E}\Big[\sum_{u\in\xi\cap\Lambda_L}\sum_{y\in\xi\cap\Lambda_L} f(y-u,\tau_u\xi)\Big] = \frac{1}{\rho L^d}\mathbb{E}\Big[\sum_{y\in\xi\cap\Lambda_L}\sum_{u\in\xi\cap\Lambda_L} f(u-y,\tau_y\xi)\Big],$$

$$B(L) = \frac{1}{\rho L^d}\mathbb{E}\Big[\sum_{u\in\xi\cap\Lambda_L}\sum_{y\in\xi\setminus\Lambda_L} f(y-u,\tau_u\xi)\Big],$$

$$C(L) = \frac{1}{\rho L^d}\mathbb{E}\Big[\sum_{u\in\xi\cap\Lambda_L}\sum_{y\in\xi\setminus\Lambda_L} f(u-y,\tau_y\xi)\Big].$$

Since

$$\sum_{u\in\xi\cap\Lambda_L}\sum_{y\in\xi\setminus\Lambda_L} f(y-u,\tau_u\xi) \leqslant \sum_{u\in\xi\cap\Lambda_L}\sum_{x\in\tau_u\xi\setminus\Lambda_{2L}} f(x,\tau_u\xi)\,, \tag{B.3}$$

$$\sum_{u\in\xi\cap\Lambda_L}\sum_{y\in\xi\setminus\Lambda_L} f(u-y,\tau_y\xi) \leqslant \sum_{u\in\xi\cap\Lambda_L}\sum_{x\in\tau_u\xi\setminus\Lambda_{2L}} f(-x,\tau_x\tau_u\xi)\,, \tag{B.4}$$

by Campbell's identity we can bound $B(L)$ and $C(L)$ from above by $\mathbb{E}_0\big[\sum_{x\in\xi\setminus\Lambda_{2L}} f(x,\xi)\big]$ and $\mathbb{E}_0\big[\sum_{x\in\xi\setminus\Lambda_{2L}} f(-x,\tau_x\xi)\big]$, respectively. Suppose for the moment that $f(x,\xi)$ is bounded and $f(x,\xi) = 0$ if $|x|_\infty \geqslant \ell$ for some positive ℓ. This assures that all the above expectations are bounded (we invoke part (ii) and the assumption $\rho_2 < \infty$). In particular, by the Dominated Convergence Theorem, we conclude that $B(L)$ and $C(L)$ go to zero as $L \to \infty$. As a consequence, it holds $\mathbb{E}_0[F] = \mathbb{E}_0[G]$, which is simply the thesis in point (i). On the other hand, given a general nonnegative function f and a constant $\ell > 0$, we can define the cutoff

$$f_\ell(x,\xi) = \begin{cases} f(x,\xi) & \text{if } |x|_\infty \leqslant \ell\,,\ f(x,\xi) \leqslant \ell\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Then the thesis holds for f_ℓ (by what was proved above) and extends to f by the Monotone Convergence Theorem. □

Lemma B.2. *Given a measurable subset $\mathcal{A}_0 \subset \mathcal{N}_0$, define $\mathcal{A} \subset \mathcal{N}$ as*

$$\mathcal{A} = \{\xi \in \mathcal{N} \,:\, \tau_x\xi \in \mathcal{A}_0 \ \ \forall x \in \xi\}\,.$$

Then $\mathbb{P}_0(\mathcal{A}_0) = 1$ if and only if $\mathbb{P}(\mathcal{A}) = 1$.

Proof. Given $L > 0$ we set $\Lambda_L = [-L, L]^d$ and we apply the Campbell identity (2.1) to the function $f(x,\xi) := \mathbb{I}(x \in \Lambda_L\,;\, \xi \in \mathcal{A}_0)$:

$$(2L)^d = (2L)^d\mathbb{P}_0(\mathcal{A}_0) = \rho^{-1}\mathbb{E}\Big[\sum_{x\in\xi\cap\Lambda_L} \mathbb{I}(\tau_x\xi \in \mathcal{A}_0)\Big] \leqslant \rho^{-1}\mathbb{E}\Big[\xi(\Lambda_L)\Big] = (2L)^d\,.$$

Hence, all members in the above expression must be equal. In particular, $\mathbb{P}$–a.s. it holds $\tau_x\xi \in \mathcal{A}_0$ for all $x \in \xi \cap \Lambda_L$. Using the arbitrariness of L, we conclude. □

Lemma B.3. *Suppose that $\mathbb{P}$ is the law of a stationary ergodic marked simple point process with finite second moment, or a marked diluted lattice. Then, both for $\mathbb{P}$ and for $\mathbb{P}_0$–a.a. ω, the DTRW and the CTRW are well defined for any starting point $x_0 \in \xi$.*

Proof. First, we point out that Lemma B.2 holds also in the marked case and the proof is very similar. By the assumption of finite second moment (recall that diluted lattices have finite moments of all orders) it holds $\mathbb{E}_0[w(0)] < \infty$. This implies that for $\mathbb{P}_0$–a.a. ω and for all $x \in \xi$ it holds $w(x) < \infty$. By Lemma B.2, the same property is fulfilled $\mathbb{P}$–a.e. As a consequence, the DTRW is well defined. The claim for the CTRW follows from [FSS, Prop. 10] and Lemma B.2 (diluted lattices can be treated apart since due to the uniform density bounds the proof becomes trivial). □

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