UNIFORM MIXING TIME FOR RANDOM WALK ON LAMPLIGHTER GRAPHS

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Suppose that $\mathcal{G}$ is a finite, connected graph and X is a lazy random walk on $\mathcal{G}$. The lamplighter chain $X^\diamond$ associated with X is the random walk on the wreath product $\mathcal{G}^\diamond = \mathbf{Z}_2 \wr \mathcal{G}$, the graph whose vertices consist of pairs $(\underline{f}, x)$ where f is a labeling of the vertices of $\mathcal{G}$ by elements of $\mathbf{Z}_2 = \{0, 1\}$ and x is a vertex in $\mathcal{G}$. There is an edge between $(\underline{f}, x)$ and $(\underline{g}, y)$ in $\mathcal{G}^\diamond$ if and only if x is adjacent to y in $\mathcal{G}$ and $f_z = g_z$ for all $z \neq x, y$. In each step, $X^\diamond$ moves from a configuration $(\underline{f}, x)$ by updating x to y using the transition rule of X and then sampling both f_x and f_y according to the uniform distribution on $\mathbf{Z}_2$; f_z for $z \neq x, y$ remains unchanged. We give matching upper and lower bounds on the uniform mixing time of $X^\diamond$ provided $\mathcal{G}$ satisfies mild hypotheses. In particular, when $\mathcal{G}$ is the hypercube $\mathbf{Z}_2^d$, we show that the uniform mixing time of $X^\diamond$ is $\Theta(d2^d)$. More generally, we show that when $\mathcal{G}$ is a torus $\mathbf{Z}_n^d$ for $d \geq 3$, the uniform mixing time of $X^\diamond$ is $\Theta(dn^d)$ uniformly in n and d. A critical ingredient for our proof is a concentration estimate for the local time of the random walk in a subset of vertices.

1. Introduction. Suppose that $\mathcal{G}$ is a finite graph with vertices $V(\mathcal{G})$ and edges $E(\mathcal{G})$, respectively. Let $\mathcal{X}(\mathcal{G}) = \{f \colon V(\mathcal{G}) \to \mathbf{Z}_2\}$ be the set of markings of $V(\mathcal{G})$ by elements of $\mathbf{Z}_2$. The wreath product $\mathbf{Z}_2 \wr \mathcal{G}$ is the graph whose vertices are pairs $(\underline{f}, x)$ where $\underline{f} \in \mathcal{X}(\mathcal{G})$ and $x \in V(\mathcal{G})$. There is an edge between $(\underline{f}, x)$ and $(\underline{g}, y)$ if and only if $\{x, y\} \in E(\mathcal{G})$ and $f_z = g_z$ for all $z \notin \{x, y\}$. Suppose that P is a transition matrix for a Markov chain on $\mathcal{G}$. The lamplighter walk $X^\diamond$ (with respect to the transition matrix P) is the Markov chain on $\mathcal{G}^\diamond$ which moves from a configuration $(\underline{f}, x)$ by

1. picking y adjacent to x in $\mathcal{G}$ according to P, then
2. updating each of the values of f_x and f_y independently according to the uniform measure on $\mathbf{Z}_2$.

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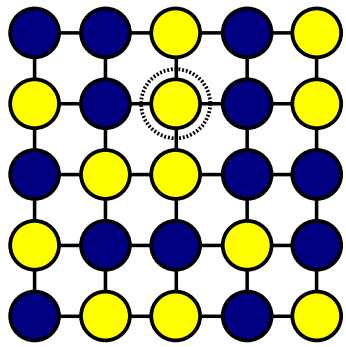

FIG 1. *A typical configuration of the lamplighter over a* 5×5 *planar grid. The colors indicate the state of the lamps and the dashed circle gives the position of the lamplighter.*

The lamp states at all other vertices in $\mathcal{G}$ remain fixed. It is easy to see that if P is ergodic and reversible with stationary distribution π_P then the unique stationary distribution of $X^\diamond$ is the product measure

$$\pi\big((\underline{f}, x)\big) = \pi_P(x) 2^{-|\mathcal{G}|},$$

and $X^\diamond$ is itself reversible. In this article, we will be concerned with the special case that P is the transition matrix for the *lazy random walk* on $\mathcal{G}$ in order to avoid issues of periodicity. That is, P is given by

$$P(x,y) = \begin{cases} \frac{1}{2} \text{ if } x = y, \\ \frac{1}{2d(x)} \text{ if } \{x,y\} \in E(\mathcal{G}), \end{cases} \tag{1.1}$$

for $x, y \in V(\mathcal{G})$ and where $d(x)$ is the degree of x.

1.1. *Main Results.* Let P be the transition kernel for the lazy random walk on a finite, connected graph $\mathcal{G}$ with stationary distribution π. The ϵ-*uniform mixing time* of $\mathcal{G}$ is given by

$$t_{\mathrm{u}}(\mathcal{G}, \varepsilon) = \min\left\{ t \geq 0 : \max_{x,y \in V(\mathcal{G})} \left| \frac{P^t(x,y) - \pi(y)}{\pi(y)} \right| \leq \epsilon \right\}. \tag{1.2}$$

Throughout, we let $t_{\mathrm{u}}(\mathcal{G}) = t_{\mathrm{u}}(\mathcal{G}, \frac{1}{2e})$. The main result of this article is a general theorem which gives matching upper and lower bounds of $t_{\mathrm{u}}(\mathcal{G}^\diamond)$ provided $\mathcal{G}$ satisfies several mild hypotheses. One important special case of this result is the hypercube $\mathbf{Z}_2^d$ and, more generally, tori $\mathbf{Z}_n^d$ for $d \geq 3$. These examples are sufficiently important that we state them as our first theorem.

THEOREM 1.1. *There exists constants $C_1, C_2 > 0$ such that*

$$C_1 \leq \frac{t_{\mathrm{u}}((\mathbf{Z}_2^d)^\diamond)}{d2^d} \leq C_2 \text{ for all } d.$$

More generally,

$$C_1 \leq \frac{t_{\mathrm{u}}((\mathbf{Z}_n^d)^\diamond)}{dn^{d+2}} \leq C_2 \text{ for all } n \geq 2 \text{ and } d \geq 3.$$

Prior to this work, the best known bound [11] for $t_{\mathrm{u}}((\mathbf{Z}_2^d)^\diamond)$ was

$$C_1 d2^d \leq t_{\mathrm{u}}((\mathbf{Z}_2^d)^\diamond) \leq C_2 \log(d) d2^d$$

for $C_1, C_2 > 0$.

In order to state our general result, we first need to review some basic terminology from the theory of Markov chains. The *relaxation time* of P is

$$t_{\mathrm{rel}}(\mathcal{G}) = \frac{1}{1-\lambda_2} \tag{1.3}$$

where λ_2 is the second largest eigenvalue of P. The *maximal hitting time* of P is

$$t_{\mathrm{hit}}(\mathcal{G}) = \max_{x,y \in V(\mathcal{G})} \mathbf{E}_x[\tau_y], \tag{1.4}$$

where τ_y denotes the first time t that $X(t) = y$ and $\mathbf{E}_x$ stands for the expectation under the law in which $X(0) = x$. The Green's function $G(x, y)$ for P is

$$G(x,y) = \mathbf{E}_x \left[\sum_{t=0}^{t_{\mathrm{u}}(\mathcal{G})} \mathbf{1}_{\{X(t)=y\}} \right] = \sum_{t=0}^{t_{\mathrm{u}}(\mathcal{G})} P^t(x,y), \tag{1.5}$$

i.e. the expected amount of time X spends at y up to the uniform mixing time t_{u} given $X(0) = x$. For each $1 \leq n \leq |\mathcal{G}|$, we let

$$G^*(n) = \max_{\substack{S \subseteq V(\mathcal{G}) \\ |S|=n}} \max_{z \in S} \sum_{y \in S} G(z,y). \tag{1.6}$$

This is the maximal expected time X spends in a set $S \subseteq V(\mathcal{G})$ of size n before the uniform mixing time. This quantity is related to the hitting time of subsets of $V(\mathcal{G})$.

Finally, recall that $\mathcal{G}$ is said to be vertex transitive if for every $x, y \in V(\mathcal{G})$ there exists an automorphism φ of $\mathcal{G}$ with $\varphi(x) = y$. Our main result requires the following hypothesis.

ASSUMPTION 1.2. *$\mathcal{G}$ is a finite, connected, vertex transitive graph and X is a lazy random walk on $\mathcal{G}$. There exists constants $K_1, K_2, K_3 > 0$ such that*

(A) $t_{\mathrm{hit}}(\mathcal{G}) \le K_1|\mathcal{G}|$,
(B) $2K_2(5/2)^{K_2}(\max_{y:y\sim x} G(x,y))^{K_2} \le \exp\left(-\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right)$,
(C) $G^*(n^*) \le K_3(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|)/(\log n^*)$

where $n^* = 4K_2 t_{\mathrm{u}}(\mathcal{G})/\min_{y:y\sim x} G(x,y)$ *for* $x, y \in V(\mathcal{G})$ *adjacent.*

REMARK 1.3. *Note that such constants always exists for a given fixed graph $\mathcal{G}$. However, to get the Theorem 1.5 to hold for a given* family of graphs*, the constant K_1, K_2, K_3 need to be uniform in the family of graphs under consideration. Thus, to get Theorem 1.1, the constants need to be uniform in n and d.*

REMARK 1.4. *We need to write $\max_{y:y\sim x} G(x,y)$ in B and $\min_{y:y\sim x} G(x,y)$ in the definition of n^* since the graph $\mathcal{G}$ is only assumed to be vertex transitive and not edge-transitive, thus $G(x,y)$ may vary along the neighbors of a vertex x.*

These assumptions are rather technical: (B) and (C) express some sort of local transience criterion. The gap left by Peres-Revelle in [11] is only there if $t_{\mathrm{rel}}(\mathcal{G}) = o(t_{\mathrm{mix}}(\mathcal{G}))$ and $\log|\mathcal{G}| = o(t_{\mathrm{mix}}(\mathcal{G}))$ both hold. Thus, the right hand side of criterion (B) is tending to 0 as $|\mathcal{G}| \to \infty$ in cases of our main interest, and the criterion says that the Green function should behave in a similar manner. The criterion connects two error terms later in the proofs. Assumption (C) will serve to be able to establish a Binomial domination argument in the lemmas below for covering the last few left-out vertices, and establishes that the extra time we will give for the Binomial domination is still less then the total time in which we want to mix the chain.

The general theorem is:

THEOREM 1.5. *Let $\mathcal{G}$ be any graph satisfying Assumption 1.2. There exists constants C_1, C_2 depending only on K_1, K_2, K_3 such that*

$$C_1 \le \frac{t_{\mathrm{u}}(\mathcal{G}^\diamond)}{|\mathcal{G}|\big(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|\big)} \le C_2 \tag{1.7}$$

The lower bound is proved in [11, Theorem 1.4]. The proof of the upper bound is based on the observation from [11] that the uniform distance to

stationarity can be related to $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ where $\mathcal{U}(t)$ is the set of vertices in $\mathcal{G}$ which have not been visited by X by time t. Indeed, suppose that $\underline{f}$ is any initial configuration of lamps, let $\underline{F}(t)$ be the state of the lamps at time t, and let $\underline{g}$ be an arbitrary lamp configuration. Let W be the set of vertices where $\underline{f} \neq \underline{g}$. Let $\mathcal{C}(t) = V(\mathcal{G}) \setminus \mathcal{U}(t)$ be the set of vertices which have been visited by X by time t. With $\mathbf{P}_{(\underline{f},x)}$ the probability under which $X^\diamond(0) = (\underline{f}, x)$, we have that

$$\mathbf{P}_{(\underline{f},x)}[\underline{F}(t) = \underline{g} | \mathcal{C}(t)] = 2^{-|\mathcal{C}(t)|} \mathbf{1}_{\{W \subseteq \mathcal{C}(t)\}}.$$

Since the probability of the configuration $\underline{g}$ under the uniform measure is $2^{-|\mathcal{G}|}$, we therefore have

$$\frac{\mathbf{P}_{(\underline{f},x)}[\underline{F}(t) = \underline{g}]}{2^{-|\mathcal{G}|}} = \mathbf{E}_{(\underline{f},x)}\left[2^{|\mathcal{U}(t)|} \mathbf{1}_{\{W \subseteq \mathcal{C}(t)\}}\right]. \tag{1.8}$$

The right hand side is clearly bounded from above by $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ (the initial lamp configuration and position of the lamplighter no longer matters). On the other hand, we can bound (1.8) from below by

$$\mathbf{P}_{(\underline{f},x)}[W \subseteq \mathcal{C}(t)] \geq \mathbf{P}[|\mathcal{U}(t)| = 0] \geq 1 - (\mathbf{E}\left[2^{|\mathcal{U}(t)|}\right] - 1).$$

Consequently, to bound $t_{\mathrm{u}}(\mathcal{G}^\diamond, \varepsilon)$ it suffices to compute

$$\min\{t \geq 0 : \mathbf{E}[2^{|\mathcal{U}(t)|}] \leq 1 + \epsilon\} \tag{1.9}$$

since the amount of time it requires for X to subsequently uniformly mix after this time is negligible.

In order to establish (1.9), we will need to perform a rather careful analysis of the process by which $\mathcal{U}(t)$ is decimated by X. The key idea is to break the process of coverage into two different regimes, depending on the size of $\mathcal{U}(t)$. The main ingredient to handle the case when $\mathcal{U}(t)$ is large is the following concentration estimate of the local time

$$\mathcal{L}_S(t) = \sum_{s=0}^{t} \mathbf{1}_{\{X(s) \in S\}}$$

for X in $S \subseteq V(\mathcal{G})$.

Proposition 1.6. *Assume that the second largest eigenvalue of P, $\lambda_2 \geq \frac{1}{2}$ and fix $S \subseteq V(\mathcal{G})$. Then for $C_0 = 1/50$, we have that*

$$\mathbf{P}_\pi\left[\mathcal{L}_S(t) \leq t\frac{\pi(S)}{2}\right] \leq \exp\left(-C_0 t \frac{\pi(S)}{t_{\mathrm{rel}}(\mathcal{G})}\right). \tag{1.10}$$

Proposition 1.6 is a corollary of [8, Theorem 1]; we consider this sufficiently important that we state it here. By invoking Green's function estimates, we are then able to show that the local time is not concentrated on a small subset of S. Since the total time of ours is limited by $C|\mathcal{G}|(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|)$, these estimates give us a small error probability only as long as the set which is not yet covered is large enough. Thus, we will handle the case when $\mathcal{U}(t)$ is small separately, via an estimate (Lemma 3.5) of the hitting time $\tau_S = \min\{t \geq 0 : X(t) \in S\}$ of S.

1.2. *Previous Work.* Suppose that μ, ν are probability measures on a finite measure space. Recall that the *total variation distance* between μ, ν is given by

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_A |\mu(A) - \nu(A)| = \frac{1}{2}\sum_x |\mu(x) - \nu(x)|. \tag{1.11}$$

The ϵ-*total variation mixing time* of P is

$$t_{\mathrm{mix}}(\mathcal{G}, \varepsilon) = \min\left\{t \geq 0 : \max_{x \in V(\mathcal{G})} \|P^t(x, \cdot) - \pi\|_{\mathrm{TV}} \leq \epsilon\right\}. \tag{1.12}$$

Let $t_{\mathrm{mix}}(\mathcal{G}) = t_{\mathrm{mix}}(\mathcal{G}, \frac{1}{2e})$. It was proved [11, Theorem 1.4] by Peres and Revelle that if $\mathcal{G}$ is a regular graph such that $t_{\mathrm{hit}}(\mathcal{G}) \leq K|\mathcal{G}|$, there exists constants C_1, C_2 depending only on K such that

$$C_1|\mathcal{G}|(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|) \leq t_{\mathrm{u}}(\mathcal{G}^\diamond) \leq C_2|\mathcal{G}|(t_{\mathrm{mix}}(\mathcal{G}) + \log|\mathcal{G}|).$$

These bounds fail to match in general. For example, for the hypercube $\mathbf{Z}_2^d$, $t_{\mathrm{rel}}(\mathbf{Z}_2^d) = \Theta(d)$ [9, Example 12.15] while $t_{\mathrm{mix}}(\mathbf{Z}_2^d) = \Theta(d \log d)$ [9, Theorem 18.3]. Theorem 1.5 says that the lower from [11, Theorem 1.4] is sharp.

Before we proceed to the proof of Theorem 1.5, we will mention some other work on mixing times for lamplighter chains. The mixing time of $\mathcal{G}^\diamond$ was first studied by Häggström and Jonasson in [6] in the case of the complete graph K_n and the one-dimensional cycle $\mathbf{Z}_n$. Their work implies a total variation cutoff with threshold $\frac{1}{2}t_{\mathrm{cov}}(K_n)$ in the former case and that there is no cutoff in the latter. Here, $t_{\mathrm{cov}}(\mathcal{G})$ for a graph $\mathcal{G}$ denotes the expected number of steps required by the lazy random walk to visit every site in $\mathcal{G}$. The connection between $t_{\mathrm{mix}}(\mathcal{G}^\diamond)$ and $t_{\mathrm{cov}}(\mathcal{G})$ is explored further in [11], in addition to developing the relationship between the relaxation time of $\mathcal{G}^\diamond$ and $t_{\mathrm{hit}}(\mathcal{G})$, and $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ and $t_{\mathrm{u}}(\mathcal{G}^\diamond)$. The results of [11] include a proof of total variation cutoff for $\mathbf{Z}_n^2$ with threshold $t_{\mathrm{cov}}(\mathbf{Z}_n^2)$. In [10], it is shown that $t_{\mathrm{mix}}((\mathbf{Z}_n^d)^\diamond) \sim \frac{1}{2}t_{\mathrm{cov}}(\mathbf{Z}_n^d)$ when $d \geq 3$ and more generally that $t_{\mathrm{mix}}(\mathcal{G}_n^\diamond) \sim$

$\frac{1}{2}t_{\mathrm{cov}}(\mathcal{G}_n)$ whenever $(\mathcal{G}_n)$ is a sequence of graphs satisfying some uniform local transience assumptions.

The mixing time of $X^\diamond = (\underline{F}, X)$ is typically dominated by the first coordinate $\underline{F}$ since the amount of time it takes for X to mix is negligible compared to that required by $X^\diamond$. We can sample from $\underline{F}(t)$ by:

1. sampling the range $\mathcal{C}(t)$ of the lazy random walk run for time t, then
2. marking the vertices of $\mathcal{C}(t)$ by iid fair coin flips.

Determining the mixing time of $X^\diamond$ is thus typically equivalent to computing the threshold t where the corresponding marking becomes indistinguishable from a uniform marking of $V(\mathcal{G})$ by iid fair coin flips. This in turn can be viewed as a statistical test for the uniformity of the uncovered set $\mathcal{U}(t)$ of X — if $\mathcal{U}(t)$ exhibits any sort of non-trivial systematic geometric structure then $X^\diamond(t)$ is not mixed. This connects this work to the literature on the geometric structure of the last visited points by the random walk [1–3, 10].

1.3. *Outline.* The remainder of this article is structured as follows. In Section 2, we will give the proof of Theorem 1.1 by checking the hypotheses of Theorem 1.5. Next, in Section 3 we will collect a number of estimates regarding the amount of X spends in and requires to cover sets of vertices in $\mathcal{G}$ of various sizes. Finally, in Section 4, we will complete the proof of Theorem 1.5.

2. Proof of Theorem 1.1. We are going to prove Theorem 1.1 by checking the hypotheses of Theorem 1.5. We begin by noting that by [9, Corollary 12.12] and [9, Section 12.3.1], we have that

$$t_{\mathrm{rel}}(\mathbf{Z}_n^d) = \Theta(dn^2). \tag{2.1}$$

By [4, Example 2, Page 2155], we know that $t_{\mathrm{u}}(\mathbf{Z}_n) = O(n^2)$. Hence by [5, Theorem 2.10], we have that

$$t_{\mathrm{u}}(\mathbf{Z}_n^d) = O((d\log d)n^2). \tag{2.2}$$

The key to checking parts (A)–(C) of Assumption 1.2 are the Green's function estimates which are stated in Proposition 2.2 (low dimension) and Proposition 2.6 (high dimension). In order to establish these we will need to prove several intermediate technical estimates. We begin by recording the following facts about the transition kernel P for the lazy random walk on a vertex transitive graph $\mathcal{G}$. First, we have that

$$P^t(x,y) \le P^t(x,x) \text{ for all } x, y. \tag{2.3}$$

To see this, we note that for t even, the Cauchy-Schwarz inequality and the semigroup property imply

$$P^t(x,y) = \sum_z P^{t/2}(x,z)P^{t/2}(z,y) \le \sqrt{P^t(x,x)P^t(y,y)} = P^t(x,x).$$

The inequality and final equality use the vertex transitivity of $\mathcal{G}$ so that $P(x,z) = P(z,x)$ and $P(x,x) = P(y,y)$. To get the same result for t odd, one just applies the same trick used in the proof of [9, Proposition 10.18(ii)]. Moreover, by [9, Proposition 10.18], we have that

$$P^t(x,x) \le P^s(x,x) \text{ for all } s \le t. \tag{2.4}$$

The main ingredient in the proof of Proposition 2.2, our low dimension Green's function estimate, is the following bound for the return probability of the lazy random walk on $\mathbf{Z}^d$.

Lemma 2.1. *Let $P(x,y;\mathbf{Z}^d)$ denote the transition kernel for the lazy random walk on $\mathbf{Z}^d$. For all $t \ge 1$, we have that*

$$P^t(x,x;\mathbf{Z}^d) \le \sqrt{2}\left(\frac{4d}{\pi}\right)^{d/2}\frac{1}{t^{d/2}} + e^{-t/8}. \tag{2.5}$$

Proof. To prove the lemma we first give an upper bound on the transition probabilities for a (non-lazy) simple random walk Y on $\mathbf{Z}^d$. One can easily give an exact formula for the return probability of Y to the origin of $\mathbf{Z}^d$ in $2t$ steps by counting all of the possible paths from 0 back to 0 of length $2t$ (here and hereafter, $P_{\mathrm{NL}}(x,y;\mathbf{Z}^d)$ denotes the transition kernel of Y):

$$\begin{aligned} P_{\mathrm{NL}}^{2t}\left(x,x;\mathbf{Z}^d\right) &= \sum_{n_1+\cdots+n_d=t} \frac{(2t)!}{(n_1!)^2(n_2!)^2\cdots(n_d!)^2}\cdot\frac{1}{(2d)^{2t}} \\ &= \frac{1}{(2d)^{2t}}\binom{2t}{t}\sum_{n_1+\cdots+n_d=t}\left(\frac{t!}{n_1!n_2!\cdots n_d!}\right)^2 \end{aligned}$$

We can bound the sum above as follows, using the multinomial theorem in the second step:

$$\begin{aligned} P_{\mathrm{NL}}^{2t}\left(x,x;\mathbf{Z}^d\right) &\le \frac{1}{(2d)^{2t}}\binom{2t}{t}\left(\max_{n_1+\cdots+n_d=t}\frac{t!}{n_1!\cdots n_d!}\right)\sum_{n_1+\cdots+n_d=t}\frac{t!}{n_1!\cdots n_d!} \\ &\le \frac{1}{(2d)^{2t}}\binom{2t}{t}\frac{t!}{[(\lfloor t/d\rfloor)!]^d}\cdot d^t. \end{aligned}$$

Applying Stirling's formula to each term above, we consequently arrive at

$$P_{\mathrm{NL}}^{2t}\left(x,x;\mathbf{Z}^d\right) \leq \frac{\sqrt{2}}{(2\pi)^{d/2}} \cdot \frac{d^{d/2}}{t^{d/2}} \tag{2.6}$$

We are now going to deduce from (2.6) a bound on the return probability for the lazy random walk X on $\mathbf{Z}^d$. We note that we can couple X and Y so that X is a random time change of Y: $X(t) = Y(N_t)$ where $N_t = \sum_{i=0}^t \xi_i$ and the (ξ_i) are iid with $\mathbf{P}[\xi_i = 0] = \mathbf{P}[\xi_i = 1] = \frac{1}{2}$ and are independent of Y. Note that N_t is distributed as a binomial random variable with parameters t and $1/2$. Thus,

$$\begin{aligned} P^t(x,x;\mathbf{Z}^d) &= \sum_{i=0}^{t/2} P_{\mathrm{NL}}^{2i}(x,x;\mathbf{Z}^d)\mathbf{P}(N_t = 2i) \\ &\leq \mathbf{P}(N_t < t/4) + \sqrt{2}\left(\frac{4d}{\pi}\right)^{d/2} \frac{1}{t^{d/2}}, \end{aligned}$$

where in the second term we used the monotonicity of the upper bound in (2.6) in t. The first term can be bounded from above by using the Hoeffding inequality. This yields the term $e^{-t/8}$ in (2.5). □

Throughout the rest of this section, we let $|x-y|$ denote the L^1 distance between $x, y \in \mathbf{Z}_n^d$.

Proposition 2.2. *Let $G(x,y)$ denote the Green's function for the lazy random walk on $\mathbf{Z}_n^d$. For each $\delta \in (0,1)$, there exists constants $C_1, C_2, C_3 > 0$ independent of n, d for $d \geq 3$ such that*

$$\begin{aligned} G(x,y) \leq \frac{C_1}{d}\left(\frac{4d}{\pi}\right)^{d/2} |x-y|^{1-d/2} + C_2(d\log d)\left(\frac{4d}{\pi}\right)^{d/2} n^{2-d(1-\delta/2)} \\ + C_3\big(d^2\log d\big)n^2 e^{-n^\delta/2} \end{aligned}$$

for all $x, y \in \mathbf{Z}_n^d$ distinct.

Proof. Fix $\delta \in (0,1)$. We first observe that the probability that there is a coordinate in which the random walk wraps around the torus within $t < n^2$ steps can be estimated by using Hoeffding's inequality and a union bound by

$$d \cdot \mathbf{P}(Z(t) > n) = de^{-\frac{n^2}{2t}}$$

where $Z(t)$ is a one dimensional simple random walk on $\mathbf{Z}$. Let $k = |x - y|$. Applying (2.3) and (2.4) in the second step, and estimating the probability of wrapping around in time $n^{2-\delta}$ in the third term, we see that

$$G(x,y) = \sum_{t=k}^{t_{\mathrm{u}}} P^t(x,y) \leq \sum_{t=k}^{n^{2-\delta}} P^t(x,x;\mathbf{Z}^d) + t_{\mathrm{u}} P^{n^{2-\delta}}(x,x;\mathbf{Z}^d) + d t_{\mathrm{u}} e^{-\frac{n^\delta}{2}}. \tag{2.7}$$

We can estimate the first term on the right hand side above using Lemma 2.1, yielding the first term in the assertion of the lemma. Applying Lemma 2.1 again, we see that there exists a constant C_2 which does not depend on n, d such that the second term in the right side of (2.7) is bounded by

$$C_2 (d \log d) \left(\frac{4d}{\pi}\right)^{d/2} n^{2-d(1-\delta/2)}. \tag{2.8}$$

Indeed, the factor $(d \log d) n^2$ comes from (2.2) and the other factor comes from Lemma 2.1. Combining proves the lemma. □

Proposition 2.2 is applicable when n is much larger than d. We now turn to state some preliminary estimates for Proposition 2.6. This Proposition will give us an estimate for the Green's function that we will use when d is large. Here comes the first ingredient:

Lemma 2.3. *Suppose that X is a lazy random walk on $\mathbf{Z}_n^d$ for $d \geq 8$ and that $|X(0)| = k \leq \frac{d}{8}$. For each $j \geq 0$, let τ_j be the first time t that $|X(t)| = j$. There exists a constant $C_k > 0$ depending only on k such that $\mathbf{P}[\tau_0 < \tau_{2k}] \leq C_k d^{-k}$. If, instead, $|X(0)| = 1$, then there exists a universal constant $p > 0$ such that $\mathbf{P}[\tau_0 < \tau_2] \geq p$.*

Proof. It clearly suffices to prove the result when X is non-lazy. Assume that $|X(t)| = j \in \{k, \ldots, 2k\}$. It is obvious that the probability that $|X|$ moves to $j+1$ in its next step is at least $1 - \frac{2k}{d}$. The reason is that the probability that the next coordinate to change is one of the coordinates of $X(t)$ whose value is 0 is at least $1 - \frac{2k}{d}$. Similarly, the probability that $|X|$ next moves to $j-1$ is at most $\frac{2k}{d}$. Consequently, the first result of the lemma follows from the Gambler's ruin problem (see, for example, [9, Section 17.3.1]). The second assertion of the lemma follows from the same argument. □

LEMMA 2.4. *Assume that $k \in \mathbf{N}$ and that $d = 2k \vee 3$. Suppose that X is a lazy random walk on $\mathbf{Z}^d$ and that $|X(0)| = 2k$. Let τ_k be the first time t that $|X(t)| = k$. There exists $p_k > 0$ depending only on k such that $\mathbf{P}[\tau_k = \infty] \geq p_k > 0$.*

PROOF. This is an easy corollary of the transience of the random walk on $\mathbf{Z}^d$ for $d \geq 3$. □

The next lemma is the corresponding version for the torus.

LEMMA 2.5. *Assume that $k \in \mathbf{N}$ and $d \geq 2k \vee 3$. Suppose that X is a lazy random walk on $\mathbf{Z}_n^d$ and that $|X(0)| = 2k$. Let τ_k be the first time t that $|X(t)| = k$. There exists $p_k, c_k > 0$ depending only on k such that $\mathbf{P}[\tau_k > c_k d n^2] \geq p_k > 0$.*

PROOF. We first assume that $d = 2k \vee 3$. It follows from Lemma 2.4 that there exists a constant $p_{k,1} > 0$ depending only on k such that $\mathbf{P}[\tau_k > \tau_{n/4}] \geq p_{k,1}$. The central limit theorem (see [7, Chapter 2]) implies that there exists constants $c_{k,1}, p_{k,2} > 0$ such that the probability that a random walk on $\mathbf{Z}_n^d$ moves more than distance $\frac{n}{4}$ in time $c_{k,1}n^2$ is at most $1 - p_{k,2}$. Combining implies the result for $d = 2k \vee 3$.

Now we suppose that $d \geq 2k \vee 3$. Let $(X_1(t), \ldots, X_d(t))$ be the coordinates of $X(t)$. By re-ordering if necessary, we may assume without loss of generality that $X_{2k+1}(0), \ldots, X_d(0) = 0$. Let $Y(t) = (X_1(t), \ldots, X_{2k}(t))$. Then Y is a random walk on $\mathbf{Z}_n^{2k}$. Clearly, $|Y(0)| = 2k$ because $X(0)$ cannot have more than $2k$ non-zero coordinates. For each j, let τ_j^Y be the first time t that $|Y(t)| = j$. Then $\tau_k^Y \leq \tau_k$. For each t, let N_t denote the number of steps that X takes in the time interval $\{1, \ldots, t\}$ in which one of its first $2k$ coordinates is changed (in other words, N_t is the number of steps taken by Y). The previous paragraph implies that $\mathbf{P}[N_{\tau_k^Y} \geq c_{k,1}n^2] \geq p_{k,3} > 0$ for a constant $p_{k,3} > 0$ depending only on k. Since the probability that the first $2k$ coordinates are changed in any step is k/d (recall that X is lazy), the final result holds from a simple large deviations estimate. □

Now we are ready to state and prove our estimate of $G(x, y)$ when d is large.

PROPOSITION 2.6. *Suppose that $d \geq 8$. Let $G(x, y)$ denote the Green's function for the lazy random walk on $\mathbf{Z}_n^d$. For each $k \in \mathbf{N}$ with $k \leq \frac{d}{8}$, there exists a constant $C_k > 0$ which does not depend on n, d such that*

$$G(x, y) \leq \frac{C_k}{d^k} \text{ for all } x, y \in \mathbf{Z}_n^d \text{ with } |x - y| \geq k.$$

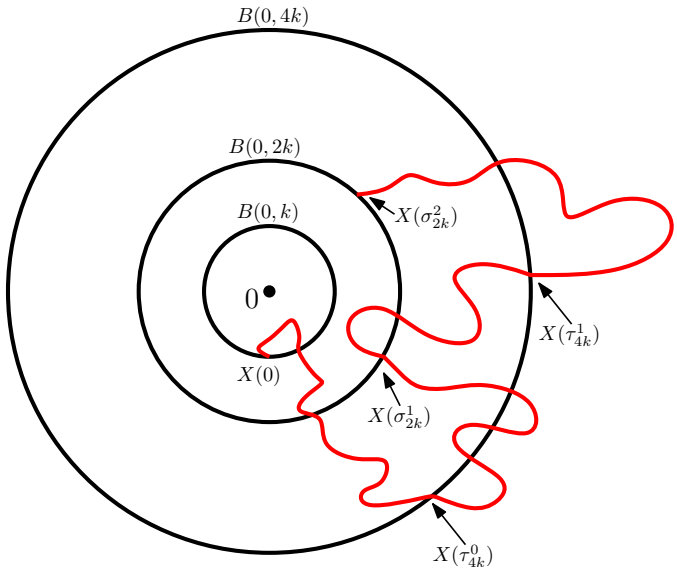

FIG 2. *Assume that $d \geq 8$ and that $k \in \mathbf{N}$ with $d \geq 8k$. Let X be a lazy random walk on $\mathbf{Z}^d_n$ and that $X(0) = x$ with $|x-y| = k$. In Proposition 2.6, we show that $G(x,y) \leq C_k d^{-k}$ where $C_k > 0$ is a constant depending only on k . By translation, we may assume without loss of generality that $|x| = k$ and $y = 0$. The idea of the proof is to first invoke Lemma 2.3 to show that X escapes to $\partial B(0,4k)$ with probability at least $1 - C_{k,1}d^{-k}$. We then decompose the path of X into successive excursions $\{X(\sigma^j_{2k}), \ldots, X(\tau^j_{4k}), \ldots, X(\sigma^{j+1}_{2k})\}$ between $\partial B(0,2k)$ back to itself through $\partial B(0,4k)$. By Lemma 2.3, we know that each excursion hits 0 with probability bounded by $C_{2k,1}d^{-2k}$ and Lemma 2.5 implies that each excursion takes length $c_k d n^2$ with probability at least $p_k > 0$. Consequently, the result follows from a simple stochastic domination argument.*

PROOF. See Figure 2 for an illustration of the proof. By translation, we may assume without loss of generality that $y = 0$; let $k = |x|$. Let τ_0 be the first time t that $|X(t)| = 0$. The strong Markov property implies that

$$G(x,y) \leq \mathbf{P}[\tau_0 \leq t_{\mathrm{u}}])G(x,x).$$

Consequently, it suffices to show that for each $k \in \mathbf{N}$, there exists constants $C_k, C_0 > 0$ such that

$$\mathbf{P}[\tau_0 \leq t_{\mathrm{u}}] \leq \frac{C_k}{d^k} \text{ and} \tag{2.9}$$

$$G(x,x) \leq C_0. \tag{2.10}$$

We will first prove (2.9); the proof of (2.10) will be similar.

Let N be a geometric random variable with success probability $C_{2k}d^{-2k}$ where C_{2k} is the constant from Lemma 2.3. Let (ξ_j) be a sequence of independent random variables with $\mathbf{P}[\xi_j = c_{2k}dn^2] = p_{2k}$ and $\mathbf{P}[\xi_j = 0] = 1-p_{2k}$ where c_{2k}, p_{2k} are the constants from Lemma 2.5 independent of N. We claim that τ_0 is stochastically dominated from below by $\sum_{j=1}^{N\zeta-1}\xi_j$ where ζ is independent of N and (ξ_j) with $\mathbf{P}[\zeta = 0] = C_k d^{-k} = 1 - \mathbf{P}[\zeta = 1]$. Indeed, to see this we will decompose the trajectory of the walk to excursions between the boundaries of the L_1- balls $B(0, 2k)$ and $B(0, 4k)$ as follows: let $\sigma_k^0 = 0$ and let τ_{4k}^0 be the first time t that $|X(t)| = 4k$. For each $j \geq 1$, we inductively let σ_{2k}^j be the first time t after τ_{4k}^{j-1} that $|X(t)| = 2k$ and let τ_{4k}^j be the first time t after σ_{2k}^j that $|X(t)| = 4k$. Let $\mathcal{F}_t$ be the filtration generated by X. Lemma 2.3 implies that the probability that X hits 0 in $\{\sigma_{2k}^j, \ldots, \tau_{4k}^j\}$ given $\mathcal{F}_{\sigma_{2k}^j}$ is at most $C_{2k}d^{-2k}$ for each $j \geq 1$ where $C_{2k} > 0$ only depends on $2k$. This leads to the success probability in the definition of N above. The factor ζ is to take into account the probability that X reaches distance $2k$ before hitting 0. Moreover, Lemma 2.5 implies that $\mathbf{P}[\sigma_{2k}^j - \tau_{4k}^{j-1} \geq c_{2k}dn^2 | \mathcal{F}_{\tau_{4k}^j}] \geq p_{2k}$. This leads to the definition of the (ξ_j) above. This implies our claim.

To see (2.9) from our claim, an elementary calculation yields that

$$\mathbf{P}[N\zeta \leq C_{2k}^{-1}d^k] \leq \mathbf{P}[N \leq C_{2k}^{-1}d^k \text{ or } \zeta = 0] \leq 2d^{-k} + C_k d^{-k}.$$

We also note that

$$\mathbf{P}\left[\sum_{j=1}^{m}\xi_j \leq \frac{p_{2k}c_{2k}mn^2}{2}\right] \leq e^{-cm}$$

for some constant $c > 0$. Combining these two observations along with a union bound implies (2.9) (with a larger constant C_k than in the one in Lemma 2.3). To see (2.10), we apply a similar argument using the second assertion of Lemma 2.3. □

Now that we have proved both Proposition 2.2 and Proposition 2.6, we are ready to check the criteria of Assumption 1.2.

2.1. *Part* (A). By [9, Proposition 1.14] with $\tau_x^+ = \min\{t \geq 1 : X(t) = x\}$, we have that $\mathbf{E}_x[\tau_x^+] = |\mathbf{Z}_n^d|$. Applying Proposition 2.6, we see that there exists constants $d_0, r > 0$ such that if $d \geq d_0$, then

$$G(x, y) \leq 1/2 \text{ for all } |x - y| \geq r. \tag{2.11}$$

Proposition 2.2 implies that there exists n_0 such that if $n \geq n_0$ and $3 \leq d < d_0$ then (2.11) likewise holds, possibly by increasing r (clearly, part (A) holds when $d \leq d_0$ and $n \leq n_0$; note also that we may assume without loss of generality that d_0, n_0 are large enough so that the diameter of the graph is at least $2r$). Let τ_r be the first time t that $|X(t) - X(0)| = r$. We observe that there exists $\rho_0 = \rho_0(r) > 0$ such that

$$\mathbf{P}_x[\tau_r < \tau_x^+] \geq \rho_0 \tag{2.12}$$

uniform in n, d since in each time step there are d directions in which $X(t)$ increases its distance from $X(0)$. By combining (2.11) with (2.12), we see that $\mathbf{P}_x[\tau_x^+ \geq t_{\mathrm{u}}(\mathcal{G})] \geq \rho_1 > 0$ uniform $d \geq d_0$. Let $\mathcal{F}_t$ be the filtration generated by X. We consequently have that

$$\begin{aligned}\mathbf{E}_x[\tau_x^+] &\geq \mathbf{E}_x[\tau_x^+ \mathbf{1}_{\{\tau_x^+ \geq t_{\mathrm{u}}(\mathcal{G})\}}] = \mathbf{E}_x\big[\mathbf{E}_x[\tau_x^+ | \mathcal{F}_{t_{\mathrm{u}}(\mathcal{G})}]\mathbf{1}_{\{\tau_x^+ \geq t_{\mathrm{u}}(\mathcal{G})\}}\big] \\ &\geq \mathbf{E}_x\big[\mathbf{E}_{X(t_{\mathrm{u}}(\mathcal{G}))}[\tau_x]\mathbf{1}_{\{\tau_x^+ \geq t_{\mathrm{u}}(G)\}}\big] \geq \rho_1 \left(1 - \frac{1}{2e}\right) \mathbf{E}_\pi[\tau_x].\end{aligned}$$

That is, with $\rho_2 := \rho_1(1 - \frac{1}{2e}) > 0$, there exists a ρ_2 uniform in $d \geq d_0$ such that $|Z_n^d| = \mathbf{E}_x[\tau_x^+] \geq \rho_2 \mathbf{E}_\pi[\tau_x]$. Combining this fact with [9, Lemma 10.2] stating that $t_{\mathrm{hit}}(G) \leq 2 \max_w \mathbf{E}_\pi(\tau_w)$ holds for any irreducible Markov chain, we arrive at $t_{\mathrm{hit}}(\mathbf{Z}_n^d) \leq K_1 |\mathbf{Z}_n^d|$ where $K_1 = 2/\rho_2$ is a uniform constant.

Remark 2.7. *There is another proof of Part A which is based on eigenfunctions. In particular, we know that*

$$t_{\mathrm{hit}}(\mathbf{Z}_n^d) \leq 2\mathbf{E}_\pi[\tau_x] = 4 \sum_i \frac{1}{1 - \lambda_i}$$

where the λ_i are the eigenvalues of the simple random walk on $\mathbf{Z}_n^d$ distinct from 1; the extra factor of 2 in the final equality accounts for the laziness of the chain. The λ_i can be computed explicitly using [9, Lemma 12.11] and the form of the λ_i when $d = 1$ which are given in [9, Section 12.3]. The assertion follows by performing the summation which can be accomplished by approximating it by an appropriate integral.

2.2. *Part* (B). It follows from Proposition 2.6 that there exist constants $C > 0$ and $d_0 \geq 3$ such that

$$G(x, y) \leq \frac{C}{d} \text{ for } x, y \in \mathbf{Z}_n^d \text{ with } |x - y| = 1 \tag{2.13}$$

provided $d \geq d_0$. Consequently, there exists $K \in \mathbf{N}$ which does not depend on $d \geq d_0$ such that

$$2K(5/2)^K G^K(x,y) = O\left(2K\left(\frac{5/2}{d}\right)^K\right) \tag{2.14}$$

It follows by combining (2.1) and (2.2) that we have that

$$\frac{t_{\mathrm{u}}(\mathbf{Z}_n^d)}{t_{\mathrm{rel}}(\mathbf{Z}_n^d)} = O(\log d). \tag{2.15}$$

Combining (2.14) with (2.15) shows that part (B) of Assumption 1.2 is satisfied provided we take $K_2 = K$ large enough. Moreover, (2.15) clearly holds if $3 \leq d < d_0$ by Proposition 2.2.

2.3. *Part* (C). We first note that it follows from (2.1), (2.2), Proposition 2.2, and Proposition 2.6 that there exists constants $C > 0$ such that n^* for $\mathbf{Z}_n^d$ is at most $Cd^2n^2 \log d$ for all $d \geq 3$. Thus, to check Assumption (C), we need to show that there exists $K_3 > 0$ such that

$$G^*(n^*) \leq K_3\left(\frac{dn^2 + d\log n}{\log d + \log n}\right). \tag{2.16}$$

We are going to prove the result by considering the regimes of $d \leq \sqrt{\log n}$ and $d > \sqrt{\log n}$ separately.

Case 1: $d < \sqrt{\log n}$.

From (2.16) it is enough to show that $G^*(n^*) \leq Kdn^2/\log n$. We can bound $G^*(n^*)$ in this case as follows. Let $D = (d\log d \log n)^{1/(\frac{1}{2}d-1)}$. By Proposition 2.2, we can bound from above the expected amount of time that X starting at 0 in $\mathbf{Z}_n^d$ spends in the L^1 ball of radius D by summing radially as follows:

$$\sum_{k=1}^{D}\sum_{y:|y|=k} G(0,y) \leq \sum_{k=1}^{D}\frac{C_1}{d}\left(\frac{4d}{\pi}\right)^{d/2} k^{1-d/2}\cdot 2d(2k)^{d-1}$$

$$\leq C_1\left(\frac{16d}{\pi}\right)^{d/2}\sum_{k=1}^{D}k^{d/2} \leq \frac{C_2}{d}\left(\frac{16d}{\pi}\right)^{d/2} D^{1+d/2} \leq C_3 n(d\log d\log n)^5$$

for constants $C_1, C_2, C_3 > 0$, where we used that $d^{d/2} \leq n$. We also note that $2d(2k)^{d-1}$ is the size of the L^∞ ball of radius k. The exponent 5 comes from the inequality

$$\frac{\frac{1}{2}d+1}{\frac{1}{2}d-1} \leq 5 \text{ for all } d \geq 3.$$

We can estimate $G^*(n^*)$ by dividing between the set of points which have distance at most D to 0 and those whose distance to 0 exceeds D, and then using the previous estimate on the first set and Proposition 2.2) on the latter:

$$\begin{aligned} G^*(n^*) \leq & C_3 n(d \log d \log n)^5 + C_4 D^{1-\frac{1}{2}d} n^* \\ \leq & C_3 n(d \log d \log n)^5 + \frac{C_4 \cdot C d^2 n^2 \log d}{d \log d \log n}, \end{aligned}$$

where $C_4 > 0$ is a constant and we recall that $C > 0$ is the constant from the definition of n^*. This implies the desired result.

Case 2: $d \geq \sqrt{\log n}$.

In this case, we are going to employ Proposition 2.6 to bound $G^*(n^*)$. The number of points which have distance at most k to 0 is clearly $1 + (2d)^k$. Consequently, by Proposition 2.6, we have that

$$\begin{aligned} G^*(n^*) \leq & \left(C_0 + \sum_{k=1}^{3} C_k d^{-k} (2d)^k \right) + C_4 d^{-4} n^* \\ \leq & C_5 + \frac{C_6 (\log d) n^2}{d^2} \end{aligned}$$

for some constants $C_5, C_6 > 0$. Since $d^2 \geq \log n$, this is clearly dominated by the right hand side of (2.16) (with a large enough constant), which completes the proof in this case.

□

3. Coverage Estimates. Now we turn to collect some preliminary lemmas for the proof of the general Theorem 1.5.

Throughout, we assume that $\mathcal{G}$ is a finite, connected, vertex transitive graph and X is a lazy random walk on $\mathcal{G}$ with transition matrix P and stationary measure π. For $S \subseteq V(\mathcal{G})$, we let $\mathcal{C}_S(t)$ be the set of vertices in S visited by X by time t and let $\mathcal{U}_S(t) = S \setminus \mathcal{C}_S(t)$ be the subset of S which X has not visited by time t. Recall the notation $\mathcal{C}(t) = \mathcal{C}_{V(\mathcal{G})}(t)$ and $\mathcal{U}(t) = \mathcal{U}_{V(\mathcal{G})}(t)$. As before, we will use $\mathbf{P}_x, \mathbf{E}_x$ to denote the probability measure and expectation under which $X(0) = x$. Likewise, we let $\mathbf{P}_\pi, \mathbf{E}_\pi$ correspond to the case that X is initialized at stationarity. The purpose of this section is to develop a number of estimates which will be useful for determining the amount of time required by X in order to cover subsets S of $V(\mathcal{G})$. We consider two different regimes depending on the size of S. If S is large, we will estimate the amount of time it takes for X to visit $t_{\mathrm{u}}(\mathcal{G})$

distinct vertices in S. If S is small, we will estimate the amount of time it takes for X to visit $1/2$ of the vertices in S.

3.1. *Large Sets.* In this subsection, we will prove that the amount of time it takes for X to visit $t^* = t_{\mathrm{u}}(\mathcal{G})$ distinct elements of a large set of vertices $S \subseteq V(\mathcal{G})$ by time $t = \text{constant} t^*/\pi(S)$ is stochastically dominated by a geometric random variable whose parameter depends on $t^*/t_{\mathrm{rel}}(\mathcal{G})$. The main result is:

Proposition 3.1. *Assume X satisfies part (B) of Assumption 1.2 with constants K_2. Let $S \subseteq V(\mathcal{G})$ consist of at least $2K_2 t_{\mathrm{u}}(\mathcal{G})/\min_{y:y\sim x} G(x,y)$ elements for $x, y \in V(\mathcal{G})$ adjacent and let*

$$t = \frac{2(K_2+2)t_{\mathrm{u}}(\mathcal{G})}{\pi(S)}.$$

There exists a universal constant $C > 0$ such that for every $x \in V(\mathcal{G})$, we have that

$$\mathbf{P}_x\left[\mathcal{C}_S(t) \le t_{\mathrm{u}}(\mathcal{G})\right] \le \exp\left(-C\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right).$$

Recall that $\mathcal{L}_S(t) = \sum_{s=0}^{t} \mathbf{1}_{\{X(s)\in S\}}$ is the amount of time that X spends in S up to time t. The proof of Proposition 3.1 consists of several steps. First we show that $\mathcal{L}_S(t)$ is large with high probability. This is done in Proposition 1.6, which we will deduce from [8, Theorem 1] shortly, and states that the probability that $\mathcal{L}_S(t)$ is less than $1/2$ its mean is exponentially small in t. Then, in order to show that X visits many vertices in S, we need to rule out the possibility of X concentrating most of its local time in a small subset of S. This is accomplished in Lemma 3.2. We now proceed to the proof of Proposition 1.6.

Proof of Proposition 1.6. We rewrite the event

$$\left\{\mathcal{L}_S(t) \le t\frac{\pi(S)}{2}\right\} = \left\{\sum_{s=0}^{t} f(X_s) > t\left(1-\pi(S)+\frac{\pi(S)}{2}\right)\right\} \tag{3.1}$$

where $f(x) = \mathbf{1}_{S^c}(x)$. Let $\epsilon = \pi(S)/2$ and $\mu = \mathbf{E}_\pi[f(X(t))] = 1-2\epsilon$. The case $\epsilon \ge 1/4$ follows immediately from [8, Equation 3] in the statement of [8, Theorem 1], so we will only consider the case $\epsilon \in (0, 1/4)$ here. Let $\overline{\mu} = 1-\mu = 2\epsilon$. For $x \in (0,1)$, let

$$I(x) = -x\log\left(\frac{\mu+\overline{\mu}\lambda_0}{1-2\overline{x}/(\sqrt{\Delta}+1)}\right) - \overline{x}\log\left(\frac{\overline{\mu}+\mu\lambda_0}{1-2x/(\sqrt{\Delta}+1)}\right)$$

where $\overline{x} = 1 - x$ and

$$\Delta = 1 + \frac{4\lambda_0 x\overline{x}}{\mu\overline{\mu}(1-\lambda_0)^2}. \tag{3.2}$$

For $x \in [\mu, \mu+\epsilon] = [1-2\epsilon, 1-\epsilon]$, $\epsilon \in (0, 1/4)$, and $\lambda_0 \geq 1/2$, we note that

$$\frac{1/2}{(1-\lambda_0)^2} \leq \Delta \leq \frac{20}{(1-\lambda_0)^2} \tag{3.3}$$

By [8, Theorem 1] and using the representation (3.1), we have that

$$\mathbf{P}_\pi\left[\mathcal{L}_S(t) \leq t\epsilon\right] \leq \exp(-I(\mu+\epsilon)t).$$

Since $I(\mu) = I'(\mu) = 0$ and $I''(x) = (\sqrt{\Delta}x\overline{x})^{-1}$ (see [8, Appendix B]), we can write

$$I(\mu+\epsilon) = \int_\mu^{\mu+\epsilon}\int_\mu^x \frac{1}{\sqrt{\Delta}y\overline{y}}dydx \tag{3.4}$$

where $\overline{y} = 1 - y$. Inserting the bounds from (3.3), we thus see that the right side of (3.4) admits the lower bound

$$\frac{1-\lambda_0}{\sqrt{20}}\int_{1-2\epsilon}^{1-\epsilon}\int_{1-2\epsilon}^x \frac{1}{2\epsilon}dydx \geq \frac{(1-\lambda_0)\epsilon}{16\sqrt{5}}$$

for all $\epsilon \in (0, 1/4)$ and $\lambda_0 \geq \frac{1}{2}$. □

As in the proof of Lemma 2.1, we couple X with a non-lazy random walk Y so that $X(t) = Y(N_t)$ where $N_t = \sum_{i=0}^t \xi_i$ and the (ξ_i) are iid with $\mathbf{P}[\xi_i = 0] = \mathbf{P}[\xi_i = 1] = \frac{1}{2}$ and are independent of Y. We let $\mathcal{L}_S^Y(t)$ denote the amount of time that $Y|_{[0,N_t]}$ spends in S (note that this differs slightly from the definition of $\mathcal{L}_x^Y(t)$ which appeared in Section 2). In other words, $\mathcal{L}_S^Y$ is the amount of that X spends in S by time t, not including those times where X does not move. The next lemma gives a lower bound on the probability that the number $\mathcal{C}_S(t)$ of distinct vertices X visits in a given set $S \subseteq V(\mathcal{G})$ by time t is proportional to $\mathcal{L}_S^Y(t)$. The lower bound for this probability will be given in terms of the Green's function $G(x,y)$ for X. Recall its definition from (1.5). Since X is a lazy random walk, we also have that

$$G(x,y) \leq G(x,x) \text{ for all } x, y \in V(\mathcal{G}). \tag{3.5}$$

This is a consequence of (2.3).

LEMMA 3.2. *Fix $S \subseteq V(\mathcal{G})$. For each positive integer k and t^*, we have that*

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) < \frac{\mathcal{L}_S^Y(t) - t^*}{k}\right] \leq \frac{t\pi(S)q^k(t)}{t^*}, \tag{3.6}$$

where

$$q(t) = \max_{y:y\sim x} G(x,y) + (1+(2e)^{-1})\frac{t - t_{\mathrm{u}}(\mathcal{G})}{|\mathcal{G}|}\mathbf{1}_{\{t>t_{\mathrm{u}}(\mathcal{G})\}}. \tag{3.7}$$

PROOF. We want to show that the number of distinct vertices covered is large, thus, we have to exclude cases where the random walk is concentrated on a small fraction of the set S. Thus, we will exclude all the points from our counting which are visited more than k times and see that the number of these is still small in expectation.

For $t \geq t_{\mathrm{u}}(\mathcal{G})$, we have $P^t(x,y) \leq (1+(2e)^{-1})\pi(y)$ by the definition of $t_{\mathrm{u}}(\mathcal{G})$. Thus by a union bound, to return to x more than once (since the chain Y is non-lazy)

$$\mathbf{P}_x[\mathcal{L}_x^Y(t) > 1] \leq q(t).$$

Hence by the strong Markov property, the probability that a vertex x is visited more than k times can be bounded from below by

$$\mathbf{P}_x[\mathcal{L}_x^Y(t) > k] \leq q^k(t).$$

Observe

$$\mathbf{P}_\pi[\tau_x = s] \leq \mathbf{P}_\pi[Y_s = x] \leq \pi(x). \tag{3.8}$$

Let

$$\mathcal{L}_{S,k}^Y(t) = \sum_{x\in S} \mathcal{L}_x^Y(t)\mathbf{1}_{\{\mathcal{L}_x^Y(t)>k\}}$$

be the total time that Y spends at points in S which it visits more than k times by time N_t. By (3.8), we have that

$$\mathbf{E}_\pi[\mathcal{L}_{S,k}^Y(t)] \leq \sum_{x\in S}\sum_{s=0}^{t} \mathbf{P}_\pi[\tau_x = s]q^k(t) \leq t\pi(S)q^k(t).$$

Applying Markov's inequality we have that

$$\mathbf{P}_\pi\left[\mathcal{L}_{S,k}^Y(t) \geq t^*\right] \leq \frac{\mathbf{E}_\pi[\mathcal{L}_{S,k}^Y(t)]}{t^*} \leq \frac{t\pi(S)q^k(t)}{t^*}$$

Observe

$$\mathcal{C}_S(t) = \sum_{x\in S} \mathbf{1}_{\{\mathcal{L}_x^Y(t)\geq 1\}} \geq \sum_{x\in S} \left(\mathbf{1}_{\{\mathcal{L}_x^Y(t)\geq 1\}} - \mathbf{1}_{\{\mathcal{L}_x^Y(t)>k\}}\right) \geq \frac{\mathcal{L}_S^Y(t) - \mathcal{L}_{S,k}^Y(t)}{k}.$$

Thus, if the non-lazy chain has visited less than t^* points at least k times, then the total number of points visited must be at least $\frac{\mathcal{L}_S^Y(t)-t^*}{k}$, i.e.

$$\left\{\mathcal{L}_{S,k}^Y(t) < t^*\right\} \subseteq \left\{\mathcal{C}_S(t) \geq \frac{\mathcal{L}_S^Y(t) - t^*}{k}\right\}.$$

We arrive at

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) \geq \frac{\mathcal{L}_S^Y(t) - t^*}{k}\right] \geq 1 - \mathbf{P}_\pi\left[\mathcal{L}_{S,k}^Y(t) \geq t^*\right] \geq 1 - \frac{t\pi(S)q^k(t)}{t^*},$$

which completes the proof of the lemma. □

Proposition 1.6 gives a lower bound on the probability $\mathcal{L}_S(t)$ is proportionally lower than its expectation, Lemma 3.2 gives a lower bound on the probability X visits less than a positive fraction of $\mathcal{L}_S^Y(t) - t^*$ vertices in S by time t, and standard large deviations estimates bound the probability that $\mathcal{L}_S^Y(t)$ is proportionally smaller than $\mathcal{L}_S(t)$. By combining these two lemmas, we obtain the following result, which gives a lower bound on the rate at which X covers vertices in S.

Lemma 3.3. *Fix $S \subseteq V(\mathcal{G})$. Then for any positive integer t^* and k we have*

$$\begin{aligned}\mathbf{P}_\pi\left[\mathcal{C}_S(t) \leq \frac{t\pi(S)/8 - t^*}{k}\right] \\ \leq \exp\left(-C_0 t\frac{\pi(S)}{t_{\mathrm{rel}}(\mathcal{G})}\right) + \exp\left(-\frac{1}{16}t\pi(S)\right) + \frac{t\pi(S)q^k(t)}{t^*}\end{aligned} \tag{3.9}$$

where the constant C_0 is as in Proposition 1.6 and the function q is as in (3.7).

Proof. On the event $\{\mathcal{L}_S^Y(t) > \frac{t\pi(S)}{8}\}$ we have $\{\mathcal{C}_S(t) \leq \frac{t\pi(S)/8-t^*}{k}\} \subseteq \{\mathcal{C}_S(t) \leq \frac{\mathcal{L}_S^Y(t)-t^*}{k}\}$. Thus, we can write the following union bound:

$$\begin{aligned}\mathbf{P}_\pi\left[\mathcal{C}_S(t) \leq \frac{t\pi(S)/8 - t^*}{k}\right] &\leq \mathbf{P}_\pi\left[\mathcal{L}_S^Y(t) \leq \frac{t\pi(S)}{8}\right] \\ &\quad + \mathbf{P}_\pi\left[\mathcal{C}_S(t) \leq \frac{\mathcal{L}_S^Y(t) - t^*}{k}\right]\end{aligned}$$

We can bound the second term from above by Lemma 3.2. For the first term we move back to the lazy chain to get the following bound from above

$$\mathbf{P}_\pi\left[\mathcal{L}_S^Y(t)\leq\frac{t\pi(S)}{8}\right]\leq\mathbf{P}_\pi\left[\mathcal{L}_S(t)\leq\frac{t\pi(S)}{2}\right]+\mathbf{P}_\pi\left[\mathcal{L}_S(t)>\frac{t\pi(S)}{2},\mathcal{L}_S^Y(t)\leq\frac{t\pi(S)}{8}\right].$$

We can bound the first term using Proposition 1.6. Conditionally on $\{\mathcal{L}_S(t) > \frac{t}{2}\pi(S)\}$, we note that $\{\mathcal{L}_S^Y(t) \leq \frac{t}{8}\pi(S)\}$ occurs if X stays in place for at least $\frac{3t}{8}\pi(S)$ time steps. Consequently, standard large deviations estimates imply that the second term above is bounded by $\exp(-\frac{1}{16}t\pi(S))$. □

We can now easily complete the proof of Proposition 3.1 by ignoring the first $t_{\mathrm{u}}(\mathcal{G})$ units of time in order to reduce to the stationary case, then apply Assumption 1.2 in order to match the error terms in Lemma 3.3.

Proof of Proposition 3.1. Now we set $t^* = t_{\mathrm{u}}(\mathcal{G})$. We first observe that

$$\mathbf{P}_x\left[\mathcal{C}_S(t)\leq t^*\right]\leq(1+(2e)^{-1})\mathbf{P}_\pi[\mathcal{C}_S(t-t_{\mathrm{u}}(\mathcal{G}))\leq t^*].$$

With $\widetilde{t} = 2K_2t^*/\pi(S)$ and using $|S| \geq 2K_2t^*/\min_{y:y\sim x} G(x,y)$ for $x, y \in V(\mathcal{G})$ adjacent, we see that

$$\max_{y:y\sim x} G(x,y)\leq q(\widetilde{t})\leq\frac{5}{2}\max_{y:y\sim x} G(x,y).$$

Combining this with part (B) of Assumption 1.2 implies

$$\frac{\widetilde{t}\pi(S)q^{K_2}(\widetilde{t})}{t^*}\leq 2K_2q^{K_2}(\widetilde{t})\leq\exp\left(-\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right). \tag{3.10}$$

Applying Lemma 3.3 gives the result. Note that we only needed to set $t^* = t_{\mathrm{u}}(\mathcal{G})$ to match the error terms, i.e if part (B) of Assumption 1.2 holds with a better rate than we can increase the size of the set t^*. □

3.2. *Small Sets.* We will now give an upper bound on the rate at which X covers 1/2 the elements of a set of vertices $S \subseteq V(\mathcal{G})$, provided $|S|$ is sufficiently small.

Proposition 3.4. *Fix $S \subseteq V(\mathcal{G})$, let $s = |S|$, and assume that*

$$t_{\mathrm{u}}(\mathcal{G})\leq\frac{|\mathcal{G}|}{4s}.$$

There exists constants $C_2, C_3 > 0$ such that

$$\mathbf{P}_x\left[\mathcal{C}_S\left(C_2|\mathcal{G}|G^*(s)\right)\leq\frac{s}{2}\right]\leq\exp(-C_3s)$$

for all $x \in V(\mathcal{G})$.

The main step in the proof of Proposition 3.4 is the next lemma, which gives an upper bound on the hitting time for S. Its proof is based on the following observation. Suppose that $S \subseteq V(\mathcal{G})$ and $\tau_S = \min\{t \geq 0 : X(t) \in S\}$. Let Z be a non-negative random variable with $Z\mathbf{1}_{\{\tau_S > t\}} = 0$ and $\mathbf{E}_x[Z\mathbf{1}_{\{\tau_S \leq t\}}] > 0$. Then we have that

$$\mathbf{P}_x[\tau_S \leq t] = \frac{\mathbf{E}_x[Z]}{\mathbf{E}_x[Z|\tau_S \leq t]}. \tag{3.11}$$

We will take Z to be the amount of time X spends in S.

Lemma 3.5. *Fix $S \subseteq V(\mathcal{G})$ and let $s = |S|$. Assume that*

$$t_{\mathrm{u}}(\mathcal{G}) \leq \frac{|\mathcal{G}|}{2s}.$$

There exists a universal constant $\rho_0 > 0$ such that $x \in V(\mathcal{G})$ we have

$$\mathbf{P}_x\left[\tau_S \leq \frac{|\mathcal{G}|}{s}\right] \geq \frac{\rho_0}{G^*(s)}.$$

Proof. Let us introduce $E = \left\{\tau_S \leq \frac{|\mathcal{G}|}{s}\right\}$. Observe that

$$\mathbf{P}_x[E] \geq \frac{\mathbf{E}_x\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)\right]}{\mathbf{E}_x\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)|E\right]}$$

We can bound the numerator from below as follows:

$$\begin{aligned}\mathbf{E}_x\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)\right] &\geq (1-(2e)^{-1})\mathbf{E}_\pi\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s} - t_{\mathrm{u}}(\mathcal{G})\right)\right] \\ &\geq (1-(2e)^{-1})\pi(S)\left(\frac{|\mathcal{G}|}{s} - t_{\mathrm{u}}(\mathcal{G})\right) \geq \frac{1}{4}.\end{aligned} \tag{3.12}$$

Let $\mathcal{L}_S(u,t) = \mathcal{L}_S(t) - \mathcal{L}_S(u-1)$ be the number of times in the set $\{u, \ldots, t\}$ that X spends in S. Then we can express the denominator as the sum

$$\begin{aligned}&\mathbf{E}_x\left[\mathcal{L}_S\left(\tau_S, \tau_S + t_{\mathrm{u}}(\mathcal{G})\right)|E\right] + \mathbf{E}_x\left[\mathcal{L}_S\left(\tau_S + t_{\mathrm{u}}(\mathcal{G}) + 1, \frac{|\mathcal{G}|}{s}\right)|E\right] \\ =:&D_1 + D_2.\end{aligned}$$

We have

$$D_2 \leq (1+(2e)^{-1})\mathbf{E}_\pi\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)\right] \leq 2.$$

We will now bound D_1. By the strong Markov property, we have that

$$D_1 \leq \max_{z\in S} \mathbf{E}_z[\mathcal{L}_S(t_{\mathrm{u}}(\mathcal{G}))] = \max_{z\in S} \mathbf{E}_z \sum_{t=0}^{t_{\mathrm{u}}(\mathcal{G})} \mathbf{1}_{\{X(t)\in S\}}$$
$$= \max_{z\in S} \sum_{y\in S} G(z,y) \leq G^*(s).$$

Note that $G^*(s) > 1/2$ always holds since the chain is lazy and $G(x,x)$ is an element of the summation, thus with a smaller constant ρ_0 we can omit the estimate for D_2 in the denominator. Putting everything together completes the proof. □

The remainder of the proof of Proposition 3.4 is based on a simple stochastic domination argument, and goes as follows.

Proof of Proposition 3.4. Let $C_2 > 0$; we will fix its precise value at the end of the proof. That X visits at least $s/2$ points in S by the time $C_2|\mathcal{G}|G^*(s/2)$ with probability exponentially close to 1 in s follows from a simple large deviation estimate of a binomial random variable. Namely, we run the chain for $C_2G^*(s)s/2$ rounds, each of length $2|\mathcal{G}|/s$. We let $S_0 = S$ and inductively let $S_i = S_{i-1} \setminus \{x\}$ if X hits x in the ith round for $i \geq 1$. If $|S_i| \geq s/2$, the hypotheses of Lemma 3.5 hold. In this case, the probability that X hits a point in S_i in the ith round is at least $\rho_0/G^*(s_i) \geq \rho_0/G^*(s) > 0$ for every i. Thus by stochastic domination, we have that

$$\mathbf{P}[\mathcal{C}_S(C_2|\mathcal{G}|G^*(s)) < s/2] \leq \mathbf{P}\left[Z < s/2\right]$$

where $Z \sim \mathrm{BIN}(\frac{C_2}{2}G^*(s)s, \rho_0/G^*(s))$. By picking C_2 large enough ($C_2 > 2/\rho_0$ will do, say) and applying the Chernoff bound, we see that

$$\mathbf{P}\left[\mathcal{C}_S(C_2|\mathcal{G}|G^*(s)) < s/2\right] \leq \exp(-C_3 s) \tag{3.13}$$

for some constant C_3 (one can check that $C_3 = \frac{1}{8}$ suffices). This estimate also holds if $s = 1$. In this case we cover the point with constant probability in $C_2|\mathcal{G}|$ steps. □

4. Proof of Theorem 1.5. We will now work towards completing the proof of Theorem 1.5 by applying the results of the previous section to describe the process by which X covers $V(\mathcal{G})$. Recall that $\mathcal{U}(t)$ is the set of vertices of $\mathcal{G}$ which X has not visited by time t. We will study the process of coverage in two different regimes: before and after $\mathcal{U}(t)$ contains at least n^* vertices (recall the definition of n^* from part (C) of Assumption 1.2).

To this end, we let

$$r = \max\{i : |\mathcal{G}| - it_{\mathrm{u}}(\mathcal{G}) \geq n^*\},$$
$$\widetilde{r} = \lfloor \log_2(|\mathcal{G}| - rt_{\mathrm{u}}(\mathcal{G}))\rfloor$$

and

$$\begin{aligned} s_i &= |\mathcal{G}| - it_{\mathrm{u}}(\mathcal{G}), \quad i = 0, \dots, r, \\ s_{r+i} &= \left\lfloor \frac{s_r}{2^i} \right\rfloor \quad i = 1, \dots, \widetilde{r} - 1, \\ s_{r+\widetilde{r}} &= 0. \end{aligned}$$

We also define the stopping times

$$T_i = \min\{t \geq 1 : |\mathcal{U}(t)| \leq s_i\}, \quad i = 1, \dots, r + \widetilde{r}.$$

Lemma 4.1. *There exists constants C_4, C_5 such that for each $1 \leq i \leq r$ and all $x \in V(\mathcal{G})$, we have that*

$$\mathbf{P}_x\left[|\mathcal{U}(t)| > s_i\right] \leq \exp\left(\frac{s_i}{t_{\mathrm{rel}}(\mathcal{G})}\left(C_4 \log|\mathcal{G}| - \frac{C_5}{|\mathcal{G}|}t\right)\right). \tag{4.1}$$

Proof. For each $i \in \{1, \dots, r\}$, we let

$$t_i = \frac{2(K_2 + 2)t_{\mathrm{u}}(\mathcal{G})|\mathcal{G}|}{s_i}$$

Proposition 3.1 implies that

$$\mathbf{P}_x[|\mathcal{U}(t + t_i)| \leq s_{i+1} \mid |\mathcal{U}(t)| \in (s_{i+1}, s_i]] \geq 1 - \exp\left(-C\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right).$$

Consequently, it follows that there exists independent variables $Z_j \sim \mathrm{GEO}(1-\exp(-Ct_{\mathrm{u}}(\mathcal{G})/t_{\mathrm{rel}}(\mathcal{G})))$ such that $T_j - T_{j-1}$ is stochastically dominated by $t_j Z_j$ for all $j \in \{1, \dots, r\}$.

Remark 4.2. *It becomes clear at this point why the uncovered set needs to be decimated in lengths $t_{\mathrm{u}}(\mathcal{G})$: from the definition of $t_i, i = 1$ and from the proof of Proposition 3.1 we know that $t_i > t_{\mathrm{u}}(\mathcal{G})$ is needed to re-mix sufficiently after each trial of collecting the given number of new points.*

For $\theta_i > 0$, we have that

$$\mathbf{P}_x[|\mathcal{U}(t)| > s_i] = \mathbf{P}_x[T_i > t] = \mathbf{P}_x\left[\sum_{j=1}^{i} T_j - T_{j-1} > t\right]$$

$$\leq e^{-\theta_i t} \prod_{j=1}^{i} \mathbf{E}_x[e^{\theta_i t_j Z_j}]. \tag{4.2}$$

Note that for every $\beta \in (0,1)$ there exists $\alpha = \alpha(\beta) > 0$ such the moment generating function of a GEO(p) random variable satisfies

$$\frac{pe^x}{1-(1-p)e^x} \leq e^{\alpha x} \text{ provided } (1-p)e^x \leq \beta. \tag{4.3}$$

Choosing

$$\theta_i = \frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_i t_{\mathrm{rel}}(\mathcal{G})}$$

we have that

$$\theta_i t_j = \frac{Ct_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})} \cdot \frac{t_j}{t_i} = \frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_{\mathrm{rel}}(\mathcal{G})} \cdot \frac{s_i}{s_j}.$$

Hence as $s_i \leq s_j$ for all $i, j \in \{1, \ldots, r\}$ with $j \leq i$, we have

$$\exp\left(\frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_{\mathrm{rel}}(\mathcal{G})} \cdot \frac{s_i}{s_j} - \frac{Ct_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right) \leq \exp\left(-\frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_{\mathrm{rel}}(\mathcal{G})}\right) \leq \exp(-C/2).$$

Let $\alpha = \alpha(e^{-C/2})$ as in (4.3). Consequently, we can bound the product of exponential moments in (4.2) by

$$\log \prod_{j=1}^{i} \mathbf{E}_x[e^{\theta_i t_j Z_j}] \leq \alpha \sum_{j=1}^{i} \theta_i t_j = \frac{\alpha C t_{\mathrm{u}}(\mathcal{G}) s_i}{2t_{\mathrm{rel}}(\mathcal{G})} \sum_{j=1}^{i} \frac{1}{s_j}$$

$$= \frac{\alpha C s_i}{2t_{\mathrm{rel}}(\mathcal{G})} \sum_{j=1}^{i} \frac{1}{|\mathcal{G}|/t_{\mathrm{u}}(\mathcal{G}) - j} \leq \frac{\alpha C s_i}{2t_{\mathrm{rel}}(\mathcal{G})} \log|\mathcal{G}|.$$

Inserting this expression into (4.2) gives (4.1). □

Now we turn to the estimates corresponding to covering of small sets as follows:

LEMMA 4.3. *There exists constants C_6, C_7 such that for all $1 \le i \le \widetilde{r}$ and $x \in V(\mathcal{G})$, we have that*

$$\begin{aligned} &\mathbf{P}_x\left[|\mathcal{U}(t)| > s_{r+i}\right] \\ \le &\mathbf{P}_x[|\mathcal{U}(t/2)| > s_r] + \exp\left(s_{r+i-1}\left(C_6 i - \frac{C_7}{|\mathcal{G}|G^*(n^*)}t\right)\right). \end{aligned} \tag{4.4}$$

PROOF. Let

$$q_{r+j} = C_2|\mathcal{G}|G^*(s_{r+j})$$

where C_2 is as in Proposition 3.4. Proposition 3.4 implies that

$$\mathbf{P}_x[|\mathcal{U}(t+q_{r+j})| \le s_{r+j+1} \mid |\mathcal{U}(t)| \in (s_{r+j+1}, s_{r+j}]] \ge 1 - \exp(-C_3 s_{r+j})$$

for $j \in \{1, \cdots, \widetilde{r}\}$. Consequently, there exists independent random variables $Z_{r+j} \sim \mathrm{GEO}(1 - \exp(-C_3 s_{r+j}))$ such that $T_{r+j} - T_{r+j-1}$ is stochastically dominated by $q_{r+j}Z_{r+j}$. We have that

$$\begin{aligned} &\mathbf{P}_x[|\mathcal{U}(t)| > s_{r+i}] = \mathbf{P}_x[T_{r+i} > t] \\ \le &\mathbf{P}_x\left[T_r > \frac{t}{2}\right] + \mathbf{P}_x\left[\sum_{j=1}^{i} T_{r+j} - T_{r+j-1} > \frac{t}{2}\right] =: I_1 + I_2 \end{aligned} \tag{4.5}$$

Using that $I_1 = \mathbf{P}[|\mathcal{U}(t/2)| > s_r]$ gives the first term in (4.4). We now turn to bound I_2. Fixing $\theta_{r+i} > 0$, we have

$$I_2 \le e^{-\theta_{r+i}t/2} \prod_{j=1}^{i} \mathbf{E}_x\left[e^{\theta_{r+i}q_{r+j}Z_{r+j}}\right]. \tag{4.6}$$

We see from (4.3) that here

$$\exp(\theta_{r+i}q_{r+j} - C_3 s_{r+j}) \le \exp(-C_3/2) =: \beta < 1$$

must hold for all $j \le i$. This is satisfied with the particular choice

$$\theta_{r+i} = \frac{C_3}{2C_2}\frac{s_{r+i}}{|\mathcal{G}|G^*(n^*)},$$

since $j \le i$ then $G^*(s_i) \le G^*(s_j) \le G^*(n^*)$. Thus by (4.3) there exists $\alpha = \alpha(\beta) > 0$ such that we can bound the exponential moments in (4.6) by

$$\log \prod_{j=1}^{i} \mathbf{E}_x\left[e^{\theta_{r+i}q_{r+j}Z_{r+j}}\right] \le \alpha\theta_{r+i}\sum_{j=1}^{i} q_{r+j} = \frac{\alpha C_3}{2} i s_{r+i}$$

Inserting this bound into (4.6) gives the second term in (4.4). □

LEMMA 4.4. *There are constants $C_8, C_9, C_{10} > 0$ such that for*

$$t = (1+a)C_8|\mathcal{G}|(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|)$$

and every $x \in V(\mathcal{G})$ we have

$$\mathbf{E}_x\left[2^{|\mathcal{U}(t)|}\right] \le 1 + C_9 \exp\left(-aC_{10}\log(n^*)\right). \tag{4.7}$$

PROOF. We can write

$$\mathbf{E}_x\left[2^{|\mathcal{U}(t)|}\right] \le 1 + \sum_{i=1}^{r+\widetilde{r}} 2^{s_{i-1}}\mathbf{P}\left[|\mathcal{U}(t)| > s_i\right].$$

For $i \le r$, we have that $s_{i-1} = s_i + t_{\mathrm{u}}(\mathcal{G})$. By Lemma 4.1, we have that

$$2^{s_i+t_{\mathrm{u}}(\mathcal{G})}\mathbf{P}[|\mathcal{U}(t)| > t] \le \exp\left((s_i + t_{\mathrm{u}}(\mathcal{G}))\log 2 + \frac{s_i}{t_{\mathrm{rel}}(\mathcal{G})}\left(C_4\log|\mathcal{G}| - \frac{C_5}{|\mathcal{G}|}t\right)\right).$$

By taking C_8 (in the definition of t) large enough, this is in turn bounded from above by

$$\exp\left(-as_i\left(1 + \frac{\log|\mathcal{G}|}{t_{\mathrm{rel}}(\mathcal{G})}\right)\right). \tag{4.8}$$

For $r+i \in \{r+1, \dots, r+\widetilde{r}\}$ we have from (4.4) that

$$\begin{aligned}
&2^{s_{r+i-1}}\mathbf{P}_x[|\mathcal{U}(t)| > s_{r+i}] \\
\le &2^{s_{r+i-1}}\mathbf{P}_x[|\mathcal{U}(t/2)| > s_r] + \exp\left(s_{r+i-1}\left((C_6 + \log 2)i - \frac{C_7}{|\mathcal{G}|G^*(n^*)}t\right)\right).
\end{aligned}$$

The first term admits the same bound as (4.8) with $i = r$, possibly by increasing C_8 if necessary. Using that $i \le \log_2|n^*|$, by increasing C_8 if necessary, from condition (C) it is easy to see that the second term admits the bound

$$\exp\left(-as_{r+i}\frac{\log|\mathcal{G}| + t_{\mathrm{rel}}(\mathcal{G})}{G^*(n^*)}\right). \tag{4.9}$$

Applying condition (C) again, we see that (4.9) is bounded from above by

$$\exp(-as_{r+i}\log(n^*)).$$

Putting together the estimates we get that for $i \in \{1 \dots \widetilde{r}\}$

$$\begin{aligned}
&2^{s_{r+i-1}}\mathbf{P}_x[|\mathcal{U}(t)| > s_{r+i}] \\
\le &\exp\left(-as_r\left(1 + \frac{\log|\mathcal{G}|}{t_{\mathrm{rel}}(\mathcal{G})}\right)\right) + \exp\left(-as_{r+i}\log(n^*)\right)
\end{aligned} \tag{4.10}$$

Summing (4.8) and (4.10) gives (4.7) (the dominant term in the summation comes from when $s_{r+i} = 1$) which proves the lemma. □

Proof of Theorem 1.5. This is a consequence of Lemma 4.4 and the relationship between $t_{\mathrm{u}}(\mathcal{G}^{\diamond})$ and $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ given in (1.9). □

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