The generalized weighted probability measure on the symmetric group and the asymptotic behavior of the cycles

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Abstract

The goal of this paper is to analyse the asymptotic behaviour of the cycle process and the total number of cycles of weighted and generalized weighted random permutations which are relevant models in physics and which extend the Ewens measure. We combine tools from combinatorics and complex analysis (e.g. singularity analysis of generating functions) to prove that under some analytic conditions (on relevant generating functions) the cycle process converges to a vector of independent Poisson variables and to establish a central limit theorem for the total number of cycles. Our methods allow us to obtain an asymptotic estimate of the characteristic functions of the different random vectors of interest together with an error estimate, thus having a control on the speed of convergence. In fact we are able to prove a finer convergence for the total number of cycles, namely *mod-Poisson convergence*. From there we apply previous results on mod-Poisson convergence to obtain Poisson approximation for the total number of cycles as well as large deviations estimates.

Résumé

Keywords: symmetric group, weighted probability measure, cycle counts, total number cycles, mod-Poisson convergence, Poisson approximation

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Contents

1. Introduction

In this paper we are interested in finding the asymptotic behaviour of the cycle structure and the number of cycles of weighted random permutations which appear in mathematical biology and in theoretical physics. More precisely, we define the weighted and generalized weighted probability measures on the group of permutations S_n of order n as follows:

Definition 1.1. *Let $\sigma \in S_n$ be given. We write $C_j(\sigma)$ for the number of cycles of length j in the decomposition of σ as a product of disjoint cycles (see also Definition 2.2).*

1. *Let $\Theta = (\theta_m)_{m \geq 1}$ be given, with $\theta_j \geq 0$ for every $j \geq 1$. We define the generalized weighted measures as*

$$\mathbb{P}_\Theta [\sigma] := \frac{1}{h_n n!} \prod_{m=1}^{n} \theta_m^{C_m(\sigma)} \tag{1.1}$$

   with $h_n = h_n(\Theta)$ a normalization constant and $h_0 := 1$.
2. *More generally, we define the generalized weighted measures as follows. Let $F_m : \mathbb{N} \to \mathbb{R}_{>0}$ be given for $m \geq 1$ with $F_m(0) = 1$. We then define*

$$\mathbb{P}_F [\sigma] := \frac{1}{n! h_n(F)} \prod_{m=1}^{n} F_m\big(C_m(\sigma)\big) \tag{1.2}$$

   with $h_n(F)$ a normalization constant. It follows immediately from the definition that $\mathbb{P}_\Theta [.] = \mathbb{P}_F [.]$ with $F_m(k) = \theta_m^k$.

When $\theta_j = 1$ for all $j \geq 1$, the measure $\mathbb{P}_\Theta [.]$ is the uniform measure on permutations and these are well studied objects with a long history (see e.g. the monograph [1] for a complete account). In particular it was proven by Goncharov ([18]) and Shepp and Loyd ([26]) that the process of cycle counts converges in distribution to a Poisson process, that is as $n \to \infty$,

$$\left(C_1^{(n)}, C_2^{(n)}, \dots\right) \to^d (Z_1, Z_2, \dots), \tag{1.3}$$

where the Z_j are independent Poisson distributed random variables with $\mathbb{E}[Z_j] = \frac{1}{j}$. There have also been further studies to analyze the rate of convergence for

$$\left(C_1^{(n)}, \dots, C_b^{(n)}\right) \to^d (Z_1, \dots, Z_b), n \to \infty$$

where b is a fixed integer. The above can be interpreted as a result on small cycles. There exist as well results on large cycles, due to Kingman ([23]) and Vershik and Shmidt ([27]) who prove that the vector of renormalized and ordered length of the cycles converges in law to a Poisson-Dirichlet distribution with parameter 1. Moreover if one notes

$$K_{0n} = C_1^{(n)} + \dots + C_n^{(n)}$$

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the number of cycles, then the distribution of K_{0n} is well known and a central limit theorem can be proven:

$$\frac{K_{0n} - \log n}{\sqrt{\log n}} \to^d \mathcal{N}(0, 1),\tag{1.4}$$

where $\mathcal{N}(0, 1)$ stands for a standard Gaussian random variable. In fact one can prove Poisson approximation results for $K_{0n} - 1$ as well as large deviations estimates. For instance Hwang ([20]) showed that for $k \sim x \log n$,

$$\mathbb{P}[K_{0n} = k] = \frac{(\log n)^{k-1} \exp(-\log n)}{(k-1)!} \left(\frac{1}{\Gamma(1+r)} + O\left(\frac{k}{(\log n)^2} \right) \right),$$

where $r = (k-1)/\log n$.

Similar results exist if one considers the more general Ewens measure corresponding to $\theta_j = \theta > 0$ for all j in equation (1.1) defining $\mathbb{P}_\Theta[.]$. This measure was introduced by Ewens ([11]) to study population dynamics and has received much attention since. In particular (1.3) and (1.4) hold with $\mathbb{E}[Z_j] = \frac{\theta}{j}$ and $\log n$ replaced with $\theta \log n$ in the central limit theorem. Estimates on the rate of convergence as well as Poisson approximation results for K_{0n} area available as well.

The measure $\mathbb{P}_\Theta[.]$ is thus a natural extension of the Ewens measure and besides has a physical interpretation: indeed such a model appears in the study of the quantum Bose gas in statistical mechanics (see [6], [5] and [4]). There it is of interest to understand the structure of the cycles when the asymptotic behaviour of θ_j is fixed. The case where $\theta_j \to \theta$, i.e. asymptotically the Ewens case, was also considered in [8]. Another random variable of interest that we shall not consider in this paper is L_1, the length of the cycle that contains 1, which can be interpreted as giving the length of a typical cycle, has also been considered in some of the above mentioned papers. It appeared in these works that obtaining the convergence in distribution of the cycle process or the central limit theorem for the number of cycles is a challenging problem. Indeed, there does not exist something such as the Feller coupling for the random permutations under the measure $\mathbb{P}_\Theta[.]$, since in general these measures do not possess any compatibility property between the different dimensions. The main important property of $\mathbb{P}_\Theta[.]$ is that it is invariant on conjugacy classes, and we shall exploit this fact.

In a recent skillful paper, Ercolani and Ueltschi ([10]) have obtained, under a variety of assumptions on the asymptotic properties of the θ_j's, the convergence of the cycle process to a Poisson vector, as in (1.3), with this time $\mathbb{E}[Z_j] = \frac{\theta_j}{j}$. In some cases they obtained an equivalent for $\mathbb{E}[K_{0n}]$ and in some "degenerate cases" they proved that the total number of cycles converges in distribution to 1 or to 1 plus a Poisson random variable. But their method which is a subtle saddle point analysis of generating functions does not give any information on the asymptotic behaviour of the different random variables under consideration nor on the rate of convergence. On the other hand because the method is general, they are able to cover concrete cases corresponding to a large variety of assumptions on the asymptotic behaviour of the θ_j's.

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The goal of this paper is to bring a complementary point of view to the approach of Ercolani and Ueltschi ([10]) and to provide sufficient conditions on the θ_j's, or more precisely on the analyticity properties of the generating series of $(\theta_m)_{m \geq 1}$, under which one has (1.3), (1.4), estimates on the rate of convergence, as well as Poisson approximation and large deviations estimates for the total number of cycles K_{0n}.

Our approach is based on the so called singularity analysis of the generating series of $(\theta_n)_{n \geq 1}$ and is general enough to deal with the case of the more general measures $\mathbb{P}_F[.]$. The starting point is the well known relation

$$\sum_{n=1}^{\infty} h_n t^n = \exp(g(t)), \text{ with } g(t) = \sum_{n=1}^{\infty} \frac{\theta_n}{n} t^n$$

which relates the generating series of the sequence $(h_n)_{n \geq 1}$ to that of the sequence $(\theta_n)_{n \geq 1}$, and its generalized version for the measure $\mathbb{P}_F[.]$ (in fact these formulas will follow from some combinatorial lemmas which are useful to compute expectations or statistics-e.g. the characteristic function- under $\mathbb{P}_F[.]$). To obtain an asymptotic for h_n as well as an estimate for the characteristic functions of the different random variables under consideration, it will reveal crucial to extract precise asymptotic information with an error term on the coefficient of $[t^n][F(t, w)]$ for $F(t, w) = \exp(wg(t))S(t, w)$ where $S(t, w)$ is some holomorphic function in a domain containing $\{(t, w) \in \mathbb{C}^2; |t| \leq r, |w| \leq \hat{r}\}$ where r is the radius of convergence of the generating series $g(t)$. This will reveal possible if one makes some assumptions on the analyticity property of the generating series $g(t)$ together with assumptions on the nature of its singularities at the point r on the circle of convergence. Several results of this nature exist in the literature (see e.g. the monograph [16]) but the relevant ones for us are due to Hwang ([21] and [19]). Combining these results with some combinatorial lemmas, we are able to show that the cycle count process converges in law to a vector of Poisson process as in (1.3), with this time $\mathbb{E}[Z_j] = \frac{\theta_j r^j}{j}$, where we recall that r is the radius of convergence of the generating series $g(t)$ of the sequence $(\theta_n)_{n \geq 0}$. We also have a an estimate for the speed of convergence. We are also able to prove a central limit theorem for K_{0n}, as well as Poisson approximation results and large deviations estimates. In fact our methods allow us to prove a stronger convergence result than the central limit theorem, namely *mod-Poisson convergence*. This type of convergence together with mod-Gaussian convergence was introduced in [22] and further developed in [24]. It can be viewed as a higher order central limit theorem from which one can deduce many relevant information (in particular the central limit theorem). More precisely the paper is organized as follows:

- In Section 2 we establish some basic combinatorial lemmas and review some facts from complex analysis (e.g. singularity analysis theorems and the Lindelöf integral representation theorem) from which we deduce the asymptotic behaviour of the normalization constant h_n (under some assumptions on the generating function $g(t)$).

- In Section 3 we prove the Poisson convergence for the cycle process together with a rate of convergence;

- Section 4 is devoted to various limit theorems for the total number of cycles K_{0n};

- Section 5 contains some examples;

- In Section 6 we prove general limit theorems under the more general measure $\mathbb{P}_F [.]$. We shall illustrate these results with the example of exp-polynomial weights and a toy example of spatial random permutations which plays an important role in physics (see e.g. [7]). Here we consider the simpler case where the lattice is fixed. It is our hope that our methods can be adapted to deal with more complicated cases where the lattice is not fixed anymore.

2. Combinatorics and singularity analysis

2.1. Combinatorics of S_n and generating functions

We recall in this section some basic facts about S_n and partitions, and at the end of the section state a useful lemma to perform averages on the symmetric group. We only give here a very short overview and refer to [1] and [25] for more details.

We first analyse the conjugation classes of S_n since all probability measures and functions considered in this paper are invariant under conjugation, i.e. they are class functions. It is well known that the conjugation classes of S_n can be parametrized with partitions of n.

Definition 2.1. *A* partition λ *is a sequence of non-negative integers* $\lambda_1 \geq \lambda_2 \geq \cdots$ *eventually trailing to 0's, which we usually omit. We use the notation* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$. *The* length $\ell(\lambda)$ *of* λ *is the largest* ℓ *such that* $\lambda_\ell \neq 0$. *We define the* size $|\lambda| := \sum_m \lambda_m$. *We call* λ *a partition of* n *if* $|\lambda| = n$. *We use the notation*

$$\sum_{\lambda \vdash n}(..) := \sum_{\lambda \ partition \ of \ n} (..) \ and \ \sum_{\lambda}(..) := \sum_{\lambda \ partition} (..).$$

Let $\sigma \in S_n$ be arbitrary. We can write $\sigma = \sigma_1 \cdots \sigma_l$ with σ_i disjoint cycles of length λ_i. Since disjoint cycles commute, we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. We call the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ the *cycle-type* of σ. We write $\mathcal{C}_\lambda$ for the set of all $\sigma \in S_n$ with cycle type λ. One can show that two elements $\sigma, \tau \in S_n$ are conjugate if and only if σ and τ have the same cycle-type and that the C_λ are the conjugacy classes of S_n (see e.g. [25] for more details).

Definition 2.2. *Let* $\sigma \in S_n$ *be given with cycle-type* λ. *The cycle numbers* C_m *and the total number of cycles* K_{0n} *are defined as*

$$C_m = C_m^{(n)}(\sigma) := \# \{i; \lambda_i = m\} \tag{2.1}$$

and

$$K_{0n} := \sum_{m=1}^{n} C_m^{(n)}. \tag{2.2}$$

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The functions C_m and K_{0n} depend only on the cycle type and are thus class functions (i.e. they are constant on conjugacy classes).

All expectations in this paper have the form $\frac{1}{n!}\sum_{\sigma \in S_n} u(\sigma)$ for a certain class function u. Since u is constant on conjugacy classes, it is more natural to sum over all conjugacy classes. We thus need to know the size of each conjugacy class.

Lemma 2.3. *We have*

$$|\mathcal{C}_\lambda| = \frac{|S_n|}{z_\lambda} \text{ with } z_\lambda := \prod_{m=1}^{n} m^{c_m} c_m! \text{ and } c_m = c_m(\lambda) = \#\{\lambda_i; \lambda_i = m\}, \qquad (2.3)$$

and

$$\frac{1}{n!} \sum_{\sigma \in S_n} u(\sigma) = \sum_{\lambda \vdash n} u(\mathcal{C}_\lambda) \qquad (2.4)$$

for a class function $u : S_n \to \mathbb{C}$.

Proof. The first part can be found in [25] or in [9, chapter 39]. The second part follows immediately from the first part. $\qquad \square$

Given a sequence $(g_n)_{n \geq 1}$ of numbers, one can encode the information about this sequence into a formal power series called the generating series.

Definition 2.4. *Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. We then define the (ordinary) generating function of $(g_n)_{n \in \mathbb{N}}$ as the formal power series*

$$G(t) = G(g_n, t) = \sum_{n=0}^{\infty} g_n t^n. \qquad (2.5)$$

Definition 2.5. *Let $G(t) = \sum_{n=0}^{\infty} g_n t^n$ be a formal power series. We then define $[t^n][G] := g_n$, i.e. the coefficient of t^n in $G(t)$*

The reason why generating functions are useful is that it is often possible to write down a generating function without knowing g_n explicitly. Then one can try to use tools from analysis to extract information about g_n, for large n, from the generating function. It should be noted that there are several variants in the definition of generating series and we shall use several of them and still call all of them generating series without risk of confusion. We will also later replace (see Section 4) g_n by holomorphic functions $g_n(w)$. Such generating functions are then called bivariate generating functions. Again for simplicity we shall still call them generating functions.

We now introduce two generating functions (in the broad sense) which will play a crucial role in our study of random weighted permutations under the measure $\mathbb{P}_\Theta[(.)]$. For $\Theta = (\theta_n)_{n \geq 1}$, we set

$$g_\Theta(t) := \sum_{k=1}^{\infty} \frac{\theta_k}{k} t^k \text{ and } G_\Theta(t) := \exp\left(\sum_{k=1}^{\infty} \frac{\theta_k}{k} t^k\right) \qquad (2.6)$$

For now, $g_\Theta(t)$ and $G_\Theta(t)$ are just formal power series. We will see in Section 3 and in Section 4 that the asymptotic behaviour of $C_m^{(n)}$ and K_{0n} depend on the analytic properties of $g_\Theta(t)$, essentially because of the remarkable well known identity (that we will quickly derive below)

$$G_\Theta(t) = \sum_{n=0}^{\infty} h_n t^n. \tag{2.7}$$

One of the main tools in this paper to compute generating series is the following lemma (or cycle index theorem) of which we shall prove a more general form in Section 6 to deal with the more general measure $\mathbb{P}_F[.]$.

Lemma 2.6. *Let $(a_m)_{m \in \mathbb{N}}$ be a sequence of complex numbers. Then*

$$\sum_\lambda \frac{1}{z_\lambda} \left(\prod_{m=1}^{\ell(\lambda)} a_{\lambda_m} \right) t^{|\lambda|} = \sum_\lambda \frac{1}{z_\lambda} \left(\prod_{m=1}^{\infty} (a_m t^m)^{C_m} \right) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} a_m t^m \right) \tag{2.8}$$

with the same z_λ as in Lemma 2.3.
If one of the sums in (2.8) is absolutely convergent then so are the others.

Proof. The first equality follows immediately from the definition of C_m. The proof of the second equality in (2.8) can be found in [25] or can be directly verified using the definitions of z_λ and the exponential function. The last statement follows from dominated convergence. $\square$

We now use this lemma to show (2.7), i.e. $G_\Theta(t) = \sum_{n=0}^{\infty} h_n t^n$. We know from (1.1) that

$$h_n = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{m=1}^{n} \theta_m^{C_m} = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m}. \tag{2.9}$$

It now follows from lemma 2.6 that

$$\sum_{n=1}^{\infty} h_n t^n = \sum_\lambda \frac{1}{z_\lambda} t^{|\lambda|} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} = \exp\left(\sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m \right) = G_\Theta(t) \tag{2.10}$$

This proves (2.7).

2.2. Singularity analysis

If a generating function $g(t)$ is given then a natural question is: what is $[t^n][g]$ and what is the asymptotic behaviour of $[t^n][g]$. If $g(t)$ is holomorphic near 0 then one can use Cauchy's integral formula to do this. Unfortunately it is often difficult to compute the integral explicitly, but there exist several other results which allow to achieve this task. One such theorem is due to Hwang [19], and we prepare it with some preliminary definition and notation.
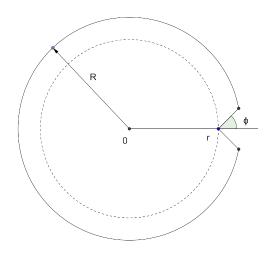
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Figure 1: Illustration of Δ_0

Definition 2.7. *Let $0 < r < R$ and $0 < \phi < \frac{\pi}{2}$ be given. We then define*

$$\Delta_0 = \Delta_0(r, R, \phi) = \{z \in \mathbb{C}; |z| < R, z \neq r, |\arg(z - r)| > \phi\} \tag{2.11}$$

Definition 2.8. *Let $g(t)$ and $\theta \geq 0, r > 0$ be given. We then call $g(t)$ of class $\mathcal{F}(r, \theta)$ if*

1. *there exists $R > r$ and $0 < \phi < \frac{\pi}{2}$ such that $g(t)$ is holomorphic in $\Delta_0(r, R, \phi)$,*
2. *there exists a constant K such that*

$$g(t) = \theta \log\left(\frac{1}{1 - t/r}\right) + K + O\left(t - r\right) \ \text{for } t \to r. \tag{2.12}$$

We shall use the shorter notation $g(t) \in \mathcal{F}(r, \theta)$.

We emphasize only the dependence on θ and r since the other constants do not appear in the main results. We now state an important theorem due to Hwang:

Theorem 2.9 (Hwang [19])**.** *Let $F(t, w) = e^{wg(t)} S(t, w)$ be given. Suppose that*

1. *Let $g(t)$ is of class $\mathcal{F}(r, \theta)$,*
2. *$S(t, w)$ is holomorphic in (t, w) for $|t| \leq r$ and $|w| \leq \hat{r}$ for some $\hat{r} > 0$, i.e. $S(t, w)$ is holomorphic in (t, w) in a domain containing the set $\{(t, w) \in \mathbb{C}^2; |t| \leq r, |w| \leq \hat{r}\}$.*

Then

$$[t^n]\left[F(t, w)\right] = \frac{e^{Kw} n^{w\theta - 1}}{r^n} \left(\frac{S(r, w)}{\Gamma(\theta w)} + O\left(\frac{1}{n}\right)\right) \tag{2.13}$$

uniformly for $|w| \leq \hat{r}$ and with the same K as in (2.12).

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Proof. The idea of the proof is to take a suitable Hankel contour and to estimate the integral over each piece. The details can be found in [19, chapter 5]. □

Remark:
We use most times $w = 1$ and S independent of w. One can also compute lower order error terms if one has more terms in the expansion of $g(t)$ near r.

A natural question at this point is: how can one prove that $g(t)$ is of class $\mathcal{F}(r, \theta)$? It is most times easy to compute the radius of convergence of $g(t)$, but it is not obvious how to show that $g(t)$ is holomorphic in some Δ_0. A way to achieve this is through Lindelöf's integral representation:

Theorem 2.10 (Lindelöf's integral representation). *Let $\phi(z)$ be a holomorphic function for $\mathrm{Re}(z) > 0$, satisfying*

$$|\phi(z)| < Ce^{A|z|} \text{ for } |z| \to \infty \text{ and } \mathrm{Re}(z) \geq \frac{1}{2} \text{ with some } A \in]0, \pi[, C > 0. \qquad (2.14)$$

We then have

$$g(t) := \sum_{k=1}^{\infty} \phi(k)(-t)^k = \frac{-1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \phi(z)t^z \frac{\pi}{\sin(\pi z)} \, dz. \qquad (2.15)$$

Furthermore, $g(t)$ can be holomorphically continued to the sector $-(\pi-A) < \arg(t) < (\pi-A)$.

Proof. The proof of this theorem is a straight forward use of the residue theorem and can be found instance in [13, Theorem 2]. □

In many situations Theorem 2.10 allows us to prove holomorphicity in a domain Δ_0, but does not give any information about the behaviour of $g(t)$ near the singularity. One way to compute the asymptotic behaviour of $g(t)$ near the singularity is to use the Mellin transform. We do not introduce here the Mellin transform since this would take us to far away from the topic of this paper. We would rather refer to [14] for an introduction and to [12, Section 3] for an application to the polylogarithm.

Theorem 2.10 is not always so easy to apply and the computation of the asymptotic behaviour near the singularity is often very difficult. An alternative approach is to combine singularity analysis with more elementary methods. The idea is to write $F = F_1 F_2$ in a way that we can apply singularity analysis to F_1 and can estimate the growth rate of $[t^n][F_2]$. One can then compute the coefficient $[t^n][F]$ directly and conclude with elementary analysis. This method is called the *convolution method*. We now introduce a class of functions which is suitable for this approach.

Definition 2.11. *Let $g(t)$ and $\theta \geq 0, r > 0, 0 < \gamma \leq 1$ be given. We then call $g(t)$ of class $e\mathcal{F}(r, \theta, \gamma)$ if there exists an analytic function g_0 in the disc $\{|z| < r\}$ such that*

$$g(t) = \theta \log\left(\frac{1}{1-t/r}\right) + g_0(t) \text{ for } t \to r \text{ with } [t^n][g_0] = O(r^{-n}n^{-1-\gamma}). \qquad (2.16)$$

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We can now state a singularity analysis theorem corresponding to the class $e\mathcal{F}(r,\theta,\gamma)$:

Theorem 2.12 (Hwang [21]). *Let $F(t,w) = e^{wg(t)}S(t,w)$ be given such that*

    1. $g(t)$ *is of class $e\mathcal{F}(r,\theta,\gamma)$,*
    2. $S(t,w)$ *is holomorphic in (t,w) for $|t| \leq r$ and $|w| \leq \hat{r}$ for some $\hat{r} > 0$.*

We then have

$$[t^n]\,[F(t,w)] = \frac{e^{wK}n^{w\theta-1}}{r^n}\frac{S(r,w)}{\Gamma(\theta w)} + R_n(w) \qquad (2.17)$$

with $K = g_0(r)$ and

$$R_n(w) = \begin{cases} O\left(\dfrac{n^{\theta\mathrm{Re}(w)-1-\gamma}\log(n)}{r^n}\right), & \text{if } \mathrm{Re}(w) \geq 0 \\ O\left(\dfrac{n^{-1-\gamma}}{r^n}\right), & \text{if } \mathrm{Re}(w) < 0. \end{cases} \qquad (2.18)$$

uniformly for bounded w.

Proof. See [21]. $\qquad\square$

We can now compute the asymptotic behaviour of h_n.

Lemma 2.13. *Let $\theta > 0$ and $g_\Theta(t)$ be of class $\mathcal{F}(r,\theta)$ or of class $e\mathcal{F}(r,\theta,\gamma)$. We then have*

$$h_n = \frac{e^K n^{\theta-1}}{r^n\Gamma(\theta)}\left(1 + O\left(\frac{1}{n}\right)\right) \qquad\qquad\qquad\text{if } g_\Theta(t) \in \mathcal{F}(r,\theta), \qquad (2.19)$$

$$h_n = \frac{e^K n^{\theta-1}}{r^n\Gamma(\theta)} + O\left(\frac{n^{\theta-1-\gamma}\log(n)}{r^n}\right) \qquad\qquad\text{if } g_\Theta(t) \in e\mathcal{F}(r,\theta). \qquad (2.20)$$

with the same K as in (2.12) resp. as in Theorem 2.12.

Proof. We have proven that $\sum_{n=0}^{\infty} h_n t^n = G_\Theta(t) = \exp(g_\Theta(t))$. We thus can apply Theorem 2.9 resp. Theorem 2.12 with $g(t) = g_\Theta(t)$ for $w = 1$ and $S(t,w) \equiv 1$. This proves the lemma. $\qquad\square$

Remark:
There exist several other versions of the Theorems 2.9 and Theorems 2.12. For example one can replace $\log(1 - t/r)$ by other functions (see [15]) or allow more than one singularity (see [16, chapter VI.5].

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3. Limit theorem for the cycle numbers

In this section we establish the convergence in distribution of the cycle process to a vector of Poisson random variables.

Theorem 3.1. *Let $b \in \mathbb{N}$ be fixed. We then have as formal power series*

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_{\Theta}\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] t^n = \exp\left(\sum_{m=1}^{b} \frac{\theta_m}{m}(e^{is_m}-1)t^m\right) G_{\Theta}(t). \tag{3.1}$$

If g_θ is of class $\mathcal{F}(\theta, r)$ with $\theta > 0$, then

$$\mathbb{E}_{\Theta}\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] = \exp\left(\sum_{m=1}^{b} \frac{\theta_m}{m}(e^{is_m}-1)r^m\right) + O\left(\frac{1}{n}\right). \tag{3.2}$$

If $g_{\Theta}(t)$ is of class $e\mathcal{F}(r, \theta, \gamma)$ with $\theta > 0$, then

$$\mathbb{E}_{\Theta}\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] = \exp\left(\sum_{m=1}^{b} \frac{\theta_m}{m}(e^{is_m}-1)r^m\right) + O\left(\frac{\log(n)}{n^\gamma}\right) \tag{3.3}$$

The convergence result now follows immediately from Theorem 3.1:

Corollary 3.2. *Let $\Theta = (\theta_m)_{m \in \mathbb{N}}$ be given and S_n be endowed with $\mathbb{P}_{\Theta}[(.)]$. Assume that $\theta > 0$ and $g_{\Theta}(t)$ is of class $\mathcal{F}(r, \theta)$ or of class $e\mathcal{F}(r, \theta, \gamma)$. We then have for each $b \in \mathbb{N}$*

$$\left(C_1^{(n)}, C_2^{(n)}, \cdots C_b^{(n)}\right) \xrightarrow{d} (Y_1, \cdots, Y_b) \tag{3.4}$$

with $Y_1, \cdots, Y_b$ independent Poisson distributed random variables with $\mathbb{E}[Y_m] = \frac{\theta_m}{m}r^m$. Let $c_1, \cdots, c_n$ be integers. We then have for $g_{\Theta} \in \mathcal{F}(r, \theta)$

$$\left|\mathbb{P}_{\Theta}\left[C_1^{(n)} = c_1, \cdots, C_b^{(n)} = c_b\right] - \mathbb{P}_{\Theta}[Y_1 = c_1, \cdots, Y_b = c_b]\right| = O\left(\frac{1}{n}\right)$$

and for $g_{\Theta} \in e\mathcal{F}(r, \theta, \gamma)$

$$\left|\mathbb{P}_{\Theta}\left[C_1^{(n)} = c_1, \cdots, C_b^{(n)} = c_b\right] - \mathbb{P}_{\Theta}[Y_1 = c_1, \cdots, Y_b = c_b]\right| = O\left(\frac{\log(n)}{n^\gamma}\right)$$

Proof. The first part follows immediately from Lévy's continuity theorem. To prove the second part, we use the Fourier inversion formula. Let $\psi(s_1, \cdots, s_b)$ be the characteristic function of $C_1^{(n)}, C_2^{(n)}, \cdots C_b^{(n)}$ and $\phi(s_1, \cdots, s_b)$ the characteristic function of $Y_1, \cdots, Y_b$. Equation (3.2) resp. (3.3) shows that $|\psi - \phi| = O\left(\frac{1}{n}\right)$ resp. $|\psi - \phi| = O\left(\frac{\log(n)}{n^\gamma}\right)$ with $O(.)$ independent of $s_1, \cdots, s_d$. On the other hand we have

$$\mathbb{P}_{\Theta}\left[C_1^{(n)} = c_1, \cdots, C_b^{(n)} = c_b\right] - \mathbb{P}_{\Theta}[Y_1 = c_1, \cdots, Y_b = c_b]$$

$$= \frac{1}{(2\pi)^b}\int_{[-\pi,\pi]^b}\left(\psi(s_1, \cdots, s_b) - \phi(s_1, \cdots, s_b)\right)e^{-\sum_{m=1}^{b} ic_m s_m}\, ds_1 \cdots ds_b.$$

This proves the corollary.

$\square$

Proof of Theorem 3.1. We first compute the generating function in (3.1). The factor h_n in (3.1) is necessary to use Lemma 2.6. We have

$$h_n \mathbb{E}_\Theta \left[\exp \left(i \sum_{m=1}^{b} s_m C_m^{(n)} \right) \right] = h_n \frac{1}{h_n} \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \exp \left(i \sum_{m=1}^{b} s_m C_m^{(n)} \right) \prod_{m=1}^{l(\lambda)} \theta_{\lambda_m}$$

$$= \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \left(\prod_{m=1}^{b} (\theta_m e^{is_m})^{C_m^{(n)}} \right) \left(\prod_{m=b+1}^{n} (\theta_m)^{C_m^{(n)}} \right) \quad (3.5)$$

We now apply Lemma 2.6 with $a_m = \begin{cases} \theta_m e^{is_m}, & \text{if } 1 \leq m \leq b, \\ \theta_m, & \text{if } b < m. \end{cases}$ We get

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta \left[\exp \left(i \sum_{m=1}^{b} s_m C_m^{(n)} \right) \right] t^n = \sum_{\lambda} \frac{1}{z_\lambda} t^{|\lambda|} \left(\prod_{m=1}^{b} (\theta_m e^{is_m})^{C_m^{(n)}} \right) \left(\prod_{m=b+1}^{l(\lambda)} (\theta_m)^{C_m^{(n)}} \right)$$

$$= \exp \left(\sum_{m=1}^{b} \frac{\theta_m e^{is_m}}{m} t^m + \sum_{m=b+1}^{\infty} \frac{\theta_m}{m} t^m \right)$$

$$= \exp \left(\sum_{m=1}^{\infty} \frac{\theta_m}{m} (e^{is_m} - 1) t^m \right) G_\Theta(t). \quad (3.6)$$

This proves (3.1). The proof of (3.2) is very similar to the proof of (3.3). The only difference is that one has to apply Theorem 2.9 if $g_\Theta(t) \in \mathcal{F}(r, \theta)$ and Theorem 2.12 if $g_\Theta(t) \in e\mathcal{F}(r, \theta, \gamma)$. We thus only prove (3.2).

The function $\sum_{m=1}^{b} \frac{\theta_m}{m} (e^{is_m} - 1) t^m$ is a polynomial in t and is therefore holomorphic on the whole complex plane. We can thus apply Theorem 2.9 with

$$S(t) = S(t, w) = \exp \left(\sum_{m=1}^{b} \frac{\theta_m}{m} (e^{is_m} - 1) t^m \right)$$

to obtain:

$$h_n \mathbb{E}_\Theta \left[\exp \left(i \sum_{m=1}^{b} s_m C_m^{(n)} \right) \right] = S(r) \frac{e^K n^{\theta-1}}{r^n \Gamma(\theta)} + O \left(\frac{n^{\theta-2}}{r^n} \right). \quad (3.7)$$

For $g_\Theta(t) \in e\mathcal{F}(r, \theta, \gamma)$ we get a similar expression. We have computed in Lemma 2.13 that

$$\frac{1}{h_n} \sim r^n \frac{\Gamma(\theta)}{e^K n^{\theta-1}} \left(1 + O \left(\frac{1}{n} \right) \right).$$

This together with (3.7) proves the theorem for $g_\Theta(t) \in \mathcal{F}(r, \theta)$. The argumentation for $g_\Theta(t) \in e\mathcal{F}(r, \theta, \gamma)$ is similar and we omit it. $\qquad \square$

4. The total number of cycles

We prove in this section a central limit theorem and mod-Poisson convergence for K_{0n}. From the mod-Poisson convergence we deduce Poisson approximation results for K_{0n} as well as large deviations estimates. As before we use generating functions.

Lemma 4.1. *We have for each $w \in \mathbb{C}$ as formal power series*

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_{\Theta} \left[\exp(wK_{0n}) \right] t^n = \sum_{n=0}^{\infty} h_n \mathbb{E}_{\Theta} \left[\exp\left(w \sum_{m=1}^{n} C_m \right) \right] t^n = \exp(e^w g_{\Theta}(t)). \tag{4.1}$$

If $g_{\Theta}(t)$ is of class $\mathcal{F}(r, \theta)$ with $\theta > 0$, then

$$\mathbb{E}_{\Theta} \left[\exp(wK_{0n}) \right] = n^{\theta(e^w-1)} e^{K(e^w-1)} \left(\frac{\Gamma(\theta)}{\Gamma(\theta e^w)} + O\left(\frac{1}{n} \right) \right) \tag{4.2}$$

with $O(.)$ uniform for bounded $w \in \mathbb{C}$. If $g_{\Theta}(t)$ is of class $e\mathcal{F}(r, \theta, \gamma)$ with $\theta > 0$, then

$$\mathbb{E}_{\Theta} \left[\exp(wK_{0n}) \right] = n^{\theta(e^w-1)} e^{K(e^w-1)} \frac{\Gamma(\theta)}{\Gamma(\theta e^w)} + O\left(\frac{n^{\max\{0, \theta(\mathrm{Re}(e^w)-1)\}} \log(n)}{n^{\gamma}} \right) \tag{4.3}$$

with $O(.)$ uniform for bounded $w \in \mathbb{C}$.

Proof. To prove the first part, one can use exactly the same argumentation as in (3.1). We thus omit the details. To prove the second part, we use Theorem 2.9 for $g_{\Theta}(t) \in \mathcal{F}(r, \theta)$ to obtain

$$[t^n] \left[\exp(e^w g_{\Theta}(t)) \right] = \frac{e^{Ke^w} n^{e^w \theta - 1}}{r^n} \left(\frac{1}{\Gamma(\theta e^w)} + O\left(\frac{1}{n} \right) \right) \tag{4.4}$$

with $O(.)$ uniform for bounded $w \in \mathbb{C}$. We thus get with Lemma 2.13

$$\mathbb{E}_{\Theta} \left[\exp(wK_{0n}) \right] = n^{e^w \theta - \theta} e^{K(e^w-1)} \left(\frac{\Gamma(\theta)}{\Gamma(\theta e^w)} + O\left(\frac{1}{n} \right) \right). \tag{4.5}$$

The argumentation for $g_{\Theta}(t) \in e\mathcal{F}(r, \theta, \gamma)$ is similar. $\square$

We can now prove a central limit theorem for K_{0n}.

Theorem 4.2. *Let $\theta > 0$ and $g_{\Theta}(t)$ be of class $\mathcal{F}(r, \theta)$ or of class $e\mathcal{F}(r, \theta, \gamma)$. We then have*

$$\frac{K_{0n} - \theta \log(n)}{\theta \sqrt{\log(n)}} \xrightarrow{d} \mathcal{N}(0, 1) \tag{4.6}$$

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Proof. We prove this theorem by showing

$$\mathbb{E}\left[\exp\left(\frac{isK_{0n}}{\sqrt{\log(n)}}\right)\right] \sim e^{is\theta\sqrt{\log(n)}}e^{-\theta s^2/2}, \tag{4.7}$$

and then applying Lévy's continuity theorem. Since $O(.)$ in (4.2) resp. (4.3) is uniform in w on compact sets, we can chose $w = e^{is/\sqrt{\log(n)}}$. We get

$$\mathbb{E}_{\Theta}\left[\exp\left(\frac{is}{\sqrt{\log(n)}}K_{0n}\right)\right] \sim n^{\theta\left(e^{\frac{is}{\sqrt{\log(n)}}}-1\right)} = \exp\left(\theta\log(n)\left(e^{\frac{is}{\sqrt{\log(n)}}}-1\right)\right)$$

$$= \exp\left(\theta\log(n)\left(\frac{is}{\sqrt{\log(n)}} - \frac{s^2}{2\log(n)} + O\left(\log^{-3/2}(n)\right)\right)\right)$$

$$\sim e^{is\theta\sqrt{\log(n)}}e^{-\theta s^2/2}.$$

$\square$

In fact K_{0n} converges in a stronger sense, namely in the mod-Poisson sense:

Definition 4.3. *We say that a sequence of random variables Z_n converges in the strong mod-Poisson sense with parameters λ_n if*

$$\lim_{n\to\infty}\exp\left(\lambda_n(1 - e^{is})\right)\mathbb{E}\left[e^{isZ_n}\right] = \Psi(s) \tag{4.8}$$

locally uniform for each $s \in \mathbb{R}$ and $\Psi(s)$ a continuous function with $\Psi(0) = 1$.

The mod-Poisson convergence is stronger than the normal convergence in Theorem 4.2 since mod-Poisson convergence implies normal convergence if $\lambda_n \to \infty$ (see [24, Proposition 2.4]). Mod-Gaussian convergence and mod-Poisson convergence were first introduced in [24] and [22]. Details on mod-Poisson and mod-Gaussian convergence and its use in number theory, probability theory and random matrix theory can be found in [24] and [22].

Theorem 4.4. *Let $\theta > 0$ and $g_\Theta(t)$ be of class $\mathcal{F}(r,\theta)$ or of class $e\mathcal{F}(r,\theta,\gamma)$. Then the sequence K_{0n} converges in the strong mod-Poisson sense with parameters $K + \theta\log(n)$ with limiting function $\frac{\Gamma(\theta)}{\Gamma(\theta e^{is})}$.*

Proof. This follows immediately from Lemma 4.1. $\square$

Remark:
In fact we could show mod-Poisson convergence with parameter $\theta\log n$ instead of $K + \theta\log n$; then the limiting function would be slightly modified.
It is natural in this context to approximate K_{0n} with a Poisson distribution with parameter $\lambda_n = K + \theta\log(n)$ (or simply $\lambda_n = \theta\log(n)$). To measure the distance between K_{0n} and a Poisson distribution, we introduce some distances between measures.

Definition 4.5. *Let X and Y be integer valued random variables with distributions μ and ν. We then define*

1. *the point metric*

$$d_{loc}(X,Y) := d_{loc}(\mu,\nu) := \sup_{j \in \mathbb{Z}} |\mu\{j\} - \nu\{j\}| \tag{4.9}$$

2. *the Kolmogorov distance*

$$d_K(X,Y) := d_K(\mu,\nu) := \sup_{j \in \mathbb{Z}} |\mu\{(-\infty,j]\} - \nu\{(-\infty,j]\}| \tag{4.10}$$

Lemma 4.6. *Let $P_{K+\theta \log(n)}$ be a Poisson distributed random variable with parameter $K + \theta \log(n)$. If g_Θ is of class $\mathcal{F}(r,\theta)$ then*

$$d_{loc}(K_{0n}, P_{K+\theta \log(n)}) \leq \frac{c_1}{\log(n)} \tag{4.11}$$

$$d_K(K_{0n}, P_{K+\theta \log(n)}) \leq \frac{c_2}{\sqrt{\log(n)}} \tag{4.12}$$

with $c_1 > 0, c_2 > 0$ independent of n. If g_Θ is of class $e\mathcal{F}(r,\theta,\gamma)$, then

$$d_{loc}(K_{0n}, P_{K+\theta \log(n)}) \leq \frac{c_3}{\log(n)} \tag{4.13}$$

Proof. Formulas (4.11) and (4.12) can be proven with Proposition 3.1 and Corollary 3.2 in [2] for $\chi = e^{(K+\theta \log(n))(e^{is}-1)}$, $\psi_\nu(s) = \frac{\Gamma(\theta)}{\Gamma(\theta e^{is})}$, $\psi_\mu(s) = 1$ and ϵ the error term in Lemma 4.1. We cannot prove (4.13) with Proposition 3.1 and Corollary 3.2 in [2], since the characteristic function of K_{0n} does not fulfil the requested conditions. But we can modify the method in [2]. We have by the Fourier inversion formula

$$\mu\{j\} - \nu\{j\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijs}(\phi_\mu(s) - \phi_\nu(s)) \, ds \tag{4.14}$$

with $\phi_\mu(s), \phi_\nu(s)$ the characteristic functions of μ and ν. The characteristic function of K_{0n} is given by (4.3) and the characteristic function of $P_{K+\theta \log(n)}$ is $\exp((K + \theta \log(n))(e^{is} - 1))$. We thus get

$$
\begin{aligned}
&|\mathbb{P}[K_{0n} = j] - \mathbb{P}[P_{\theta \log(n)} = j]| \\
&= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{-ijs} e^{(K+\theta \log(n))(e^{is}-1)} \left(\frac{\Gamma(\theta)}{\Gamma(\theta e^{is})} - 1 \right) + O\left(\frac{\log(n)}{n^\gamma} \right) \, ds \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(K+\theta \log(n))(\cos(s)-1)} \left| \frac{\Gamma(\theta)}{\Gamma(\theta e^{is})} - 1 \right| \, ds + O\left(\frac{\log(n)}{n^\gamma} \right) \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(K+\theta \log(n))s^2/5} \gamma |s| \, ds + O\left(\frac{\log(n)}{n^\gamma} \right)
\end{aligned}
$$

We are now in the same situation as in the proof of Proposition 2.1 in [2]. One can thus use exactly the same arguments to get the desired upper bound. We thus omit the details. $\square$

We now wish to deduce some large deviations estimates from the mod-Poisson convergence. For this we use results from some work in progress [3] which establishes links between mod-* convergence (e.g. mod-Poisson or mod-Gaussian convergence) and precise large deviations. More precisely the framework is as follows: we assume we are given a sequence of random variables X_n such that $\varphi_n(z) = \mathbb{E}[e^{zX_n}]$ exists in a strip $-\varepsilon < Re(z) < c$, with c and ε positive numbers. We assume that there exists an infinitely divisible distribution with moment generating function $\exp(\eta(z))$ and an analytic function $\phi(z)$ such that locally uniformly in z

$$\exp\left(-t_n\eta(z)\right)\varphi_n(z) \to \phi(z),\ n \to \infty, \tag{4.15}$$

for $-\varepsilon < Re(z) < c$ and some sequence t_n tending to infinity. We further assume that $\phi(z)$ does not vanish on the real part of the domain.

Theorem 4.7 ([3]). *Let (X_n) be a sequence of random variables which satisfies the assumptions above. Assume further that these and the corresponding infinitely divisible distribution have minimal lattice $\mathbb{N}$. Assume further that the rate of convergence in (4.15) is faster than any power of $1/t_n$. Let x be a real number such that $t_n x \in \mathbb{N}$ and such that there exists h with $\eta'(h) = x$. Noting $I(z) = hx - \eta(h)$ the Fenchel-Legendre transform, we have the following asymptotic expansion*

$$\mathbb{P}[X_n = xt_n] \sim \frac{\exp\left(-t_n I(x)\right)}{\sqrt{2\pi t_n \eta''(h)}}\left(\phi(h) + \frac{a_1}{t_n} + \frac{a_2}{t_n^2} + \dots\right), n \to \infty \tag{4.16}$$

where (4.16) has to be understood in the sense that for every integer N, one has

$$\mathbb{P}[X_n = xt_n] = \frac{\exp\left(-t_n I(x)\right)}{\sqrt{2\pi t_n \eta''(h)}}\left(\phi(h) + \frac{a_1}{t_n} + \dots + \frac{a_{N-1}}{t_n^{N-1}} + O\left(\frac{1}{t_n^N}\right)\right).$$

The coefficients (a_j) are real, depend only on h and can be computed explicitly.

We now apply the above result to the random variable $X_n = K_{0n} - 1$ in order to have a random variable distributed on $\mathbb{N}$. It is easy to see that in this case $\eta(z) = e^z - 1$, $h = \log x$ and $I(z) = z\log z - z + 1$. It follows from Theorem 4.4 and Lemma 4.1 that $K_{0n} - 1$ satisfies the assumptions of the above theorem with $\phi(z) = \dfrac{\Gamma(\theta)}{e^z\Gamma(\theta e^z)}$ and $t_n = K + \theta\log n$. Using Stirling's formula we obtain the following large deviations estimates (the case $K = 0$ and $\theta = 1$ corresponds to the result of Hwang [20]).

Theorem 4.8. *Let $X_n = K_{0n} - 1$. Let $x \in \mathbb{R}$ such that $t_n x \in \mathbb{N}$, where $t_n = K + \theta\log n$. We note $k = t_n x$. Then*

$$\mathbb{P}[X_n = xt_n] = e^{-t_n}\frac{t_n^k}{k!}\left(\frac{\Gamma(\theta)}{x\Gamma(\theta x)} + O\left(\frac{1}{\log t_n}\right)\right).$$

Remark:
In fact one could obtain an arbitrary long expansion in the above above theorem.

5. Some examples

In this section we consider some examples of sequences Θ and check if $g_\Theta(t)$ is of class $\mathcal{F}(r, \theta)$ or of class $e\mathcal{F}(r, \theta, \gamma)$.

5.1. Simple sequences

5.1.1. The Ewens measure

The simplest possible sequence is the constant sequence $\theta_m = \theta$. This case is known as Ewens measure and is well studied, see for example [1]. We have

$$g_\Theta(t) = \theta \log \left(\frac{1}{1-t} \right). \tag{5.1}$$

We thus have $g_\Theta(t) \in \mathcal{F}(1, \theta)$ and our results apply if $\theta > 0$.

5.1.2. The condition $\sum_{m=1}^{\infty} \frac{|\theta_m - \theta|}{m} < \infty$

We have for $t \to 1$ and $|t| \le 1$

$$
\begin{aligned}
g_\Theta(t) &= \sum_{m=1}^{\infty} \frac{\theta}{m} t^m + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} t^m = \theta \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} t^m \\
&= \theta \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m}(t^m - 1) \\
&= \theta \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} + o(1).
\end{aligned}
\tag{5.2}
$$

It is clear that $g_\Theta(t) \in e\mathcal{F}(1, \theta, \gamma)$ if we assume $|\theta_m - \theta| = O(m^{-\gamma})$. It is not clear from (5.2) if $g_\Theta(t) \in \mathcal{F}(1, \theta)$ or not, even when one can extend $g_\Theta(t)$ holomorphically to some $\Delta(1, R, \phi)$. If a holomorphic extension is available, then one can modify Theorem 2.9 by replacing $O(t-1)$ by $o(1)$ in the definition of $\mathcal{F}(1, \theta)$. The only difference in the result is that one has to replace the error term $O\left(\frac{1}{n}\right)$ in (2.13) by $o(1)$. We do not prove this here since one only has to do some minor changes in the proof of Theorem 2.9.

5.1.3. The condition $\sum_{m=1}^{\infty} |\theta_m - \theta| < \infty$

We have for $t \to 1$ and $|t| \le 1$

$$
\begin{aligned}
g_\Theta(t) &= \theta \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m}(t^m - 1) \\
&= \theta \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} + (t-1) \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m}\left(\sum_{k=0}^{m-1} t^k\right) \\
&= \theta \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^{\infty} \frac{\theta_m - \theta}{m} + O(t-1).
\end{aligned}
\tag{5.3}
$$

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As before $g_\Theta(t) \in e\mathcal{F}(1, \theta, \gamma)$ if we assume $|\theta_m - \theta| = O(m^{-\gamma})$. If $g_\Theta(t)$ can be holomorphically extended to some $\Delta(1, R, \phi)$, then (5.3) shows that $g_\Theta(t) \in \mathcal{F}(1, \theta)$ (if we have an asymptotic expansion near 1 in $\Delta(1, R, \phi)$).

5.1.4. The sequence $\theta_m = e^{-\alpha_m}$ and $\sum_{m=1}^\infty \frac{|\alpha_m - \alpha|}{m} < \infty$ or $\sum_{m=1}^\infty |\alpha_m - \alpha| < \infty$
Both conditions on α_m ensure that $|e^{-\alpha_m} - e^{-\alpha}| \leq C|\alpha_m - \alpha|$. One can use the same argumentation as above to see that in the first case

$$g_\Theta(t) = e^{-\alpha} \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^\infty \frac{e^{-\alpha_m} - e^{-\alpha}}{m} + o(1) \tag{5.4}$$

and in the second case

$$g_\Theta(t) = e^{-\alpha} \log\left(\frac{1}{1-t}\right) + \sum_{m=1}^\infty \frac{e^{-\alpha_m} - e^{-\alpha}}{m} + O(t-1). \tag{5.5}$$

This shows that we are in the same situation as in Section 5.1.2 and Section 5.1.3.

5.2. polylogarithm
Let $\theta_m = m^\delta$ with $\delta \neq 0$. We then have

$$\mathrm{Li}_{1+\delta} := g_\Theta(t) = \sum_{m=1}^\infty \frac{1}{m^{1+\delta}} t^m. \tag{5.6}$$

The functions $\mathrm{Li}_{1+\delta}$ are known as the polylogarithm. A simple computation shows that the convergence radius of $\mathrm{Li}_{1+\delta}$ is 1. It was shown by Ford in [17] with Theorem 2.10 that the polylogarithm can extend holomorphically to the whole complex plane split along the axis $\mathbb{R}_{\geq 1}$. The asymptotic behaviour of $\mathrm{Li}_{1+\delta}$ near 1 can be computed with the Mellin transform. This has been done in [12, Section 3]. We just state here the result.
The case $\delta = 0$ is trivial since we then have $\mathrm{Li}_1 = -\log\left(\frac{1}{1-t}\right)$.
If $\delta \in \{1, 2, 3, \cdots\}$, then

$$\mathrm{Li}_{1+\delta} = \zeta(\delta + 1) + O(t - 1) \tag{5.7}$$

and for $\delta \notin \{0, 1, 2, \cdots\}$

$$\mathrm{Li}_{1+\delta} = \Gamma(-\delta)(1 - t)^\delta + \zeta(\delta + 1) + O(t - 1) \tag{5.8}$$

For $\delta > 0$ we can apply Theorem 2.9, but the main term in the asymptotic expansion is 0. If $\delta < 0$ then $\mathrm{Li}_{1+\delta}$ is not of class $\mathcal{F}(r, \theta)$ nor of class $e\mathcal{F}(r, \theta, \gamma)$. This shows that we cannot apply Theorem 2.9 nor Theorem 2.12. Ercolani and Ueltschi have shown in [10] that K_{0n} converge for $\delta < 0$ in distribution to a shifted Poisson distribution (without re-normalization) and for $\delta > 0$ they have shown $\mathbb{E}[K_{0n}] \approx An^{\frac{\delta}{1+\delta}}$. It is not yet known if K_{0n} converge in distribution (after re-normalization). The method of Ercolani and Ueltschi bases also on generating functions, but they use the saddle point method to extract the asymptotic behaviour of $[t^n][g]$. In this way they can have weaker assumptions on the sequence Θ, but do not get information on the error term.

5.3. $\theta_m = \exp(cm^\theta)$

A simple computation shows that the convergence radius r of $g_\Theta(t)$ is

$$r = \begin{cases} 1 & \text{if } \theta < 1, \\ e^{-c} & \text{if } \theta = 1, \\ 0 & \text{if } \theta > 1 \text{ and } c > 0, \\ \infty & \text{if } \theta > 1 \text{ and } c < 0 \end{cases} \tag{5.9}$$

We can not apply our method for $\theta > 1$ and for $\theta = 1$ we have $g_\Theta(t) = -\log(1 - te^c)$. We thus restrict ourselves to $\theta < 1$.

Lemma 5.1. *For $\theta < 1$, the function $g_\Theta(t)$ can holomorphically extended to the whole complex plane split along the axis $\mathbb{R}_{\geq 1}$.*

Proof. As simple computation shows that $\left|\frac{\exp(cz^\theta)}{z}\right| \leq 2\exp(|c||z|^\theta)$ for $\mathrm{Re}(z) \geq \frac{1}{2}$. Since $\theta < 1$, we can find for each $A \in]0, \pi[$ a $C > 0$ with $\left|\frac{\exp(cz^\theta)}{z}\right| \leq C\exp(A|z|)$. This shows that we can apply Theorem 2.10 with $\phi = \frac{\exp(cz^\theta)}{z}$. $\qquad\square$

We now determine the asymptotic behaviour near 1.
Case $\theta < 0$
We have

$$g_\Theta(t) = \sum_{m=1}^\infty \exp(cm^\theta)\frac{t^m}{m} = \sum_{k=0}^\infty \sum_{m=1}^\infty \frac{c^k}{k!}m^{k\theta-1}t^m = \sum_{k=0}^\infty \frac{c^k}{k!}\mathrm{Li}_{1-k\theta}(t). \tag{5.10}$$

We know the expansion of each summand near 1, but we have to justify that we plug them in. We first look at the case $|t| \leq 1$. We use the same argumentation as in (5.7) to see that

$$\sum_{m=1}^\infty \frac{1}{m^{1-k\theta}}t^m = \sum_{m=1}^\infty \frac{1}{m^{1-k\theta}} + (t-1)\sum_{m=1}^\infty \frac{1}{m^{1-k\theta}}\left(\sum_{k=1}^m t^k\right)$$
$$= \sum_{m=1}^\infty \frac{1}{m^{1-k\theta}} + O\left((t-1)\sum_{m=1}^\infty \frac{1}{m^2}\right) \tag{5.11}$$

with $O(.)$ uniform for $-k\theta \geq 2$. We thus can this put into (5.10) and get for $|t| < 1$

$$g_\Theta(t) = \log\left(\frac{1}{1-t}\right) + K + O(t-1) \text{ for } t \to 1, |t| < 1 \tag{5.12}$$

One can now use the same argumentation as in [12, Section 3] with the Mellin transformation to see that the expansion is also valid for t in some Δ_0.

Case $c < 0, 0 < \theta < 1$
In this case it is easy to see that $g_\Theta(t) = K + O(1 - t)$ for $t \to 1$. We thus cannot apply

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Theorem 2.9 nor Theorem 2.12. This case was also considered by Ercolani and Ueltschi. They have shown that K_{0n} converge as in polylogarithm case in distribution to a shifted Poisson distribution.

Case $c > 0, 0 < \theta < 1$

The behaviour of K_{0n} in this case is an open question.

6. The generalized weighted measure

In this section we introduce the generalized weighted measure. This is defined as

Definition 6.1. *Let $F_m : \mathbb{N} \to \mathbb{R}_{>0}$ be given for $m \geq 1$ with $F_m(0) = 1$. We then define*

$$\mathbb{P}_F[\sigma] := \frac{1}{n! h_n(F)} \prod_{m=1}^{n} F_m\big(C_m(\sigma)\big) \tag{6.1}$$

with $h_n(F)$ a normalization constant.

It follows immediately from the definition that $\mathbb{P}_\Theta[.] = \mathbb{P}_F[.]$ with $F_m(k) = \theta_m^k$.
Our approach has been based so far on generating functions, and more especially on Lemma 2.6. It is obvious that Lemma 2.6 cannot be used anymore for general functions F_m, but we can prove a more general version of it:

Lemma 6.2. *Let $A_m : \mathbb{N} \to \mathbb{C}$ for $m \geq 1$ be given with $A_m(0) = 1$ for $m \geq 1$. We then have as formal power series*

$$\sum_{\lambda} \frac{1}{z_\lambda} t^{|\lambda|} \prod_{m=1}^{|\lambda|} \big(A_m(C_m)\big) = \prod_{m=1}^{\infty} \mathrm{EG}\left(A_m, \frac{t^m}{m}\right) \quad \text{with } \mathrm{EG}(A, t) := \sum_{k=0}^{\infty} \frac{A(k)}{k!} t^k. \tag{6.2}$$

with $C_m = C_m(\lambda)$ as in Definition 2.2

Proof. We have

$$\sum_{\lambda \vdash n} \frac{1}{z_\lambda} t^n \prod_{m=1}^{n} \big(A_m(C_m)\big) = \sum_{\substack{c_1, \cdots, c_n \in \mathbb{N} \\ \sum_{m=1}^{n} m c_m = n}} \frac{\prod_{m=1}^{n} A_m(c_m) t^{m c_m}}{\prod_{m=1}^{n} c_m! m^{c_m}}$$

$$= \sum_{\substack{c_1, \cdots, c_n \in \mathbb{N} \\ \sum_{m=1}^{n} m c_m = n}} \prod_{m=1}^{n} \frac{A_m(c_m)}{c_m!} \left(\frac{t^m}{m}\right)^{c_m}. \tag{6.3}$$

We get

$$\sum_{\lambda} \frac{1}{z_{\lambda}} t^{|\lambda|} \prod_{m=1}^{|\lambda|} A_m(C_m) = \sum_{\substack{(c_m)_{m=1}^{\infty}, \\ \sum_{m=1}^{\infty} mc_m < \infty}} \prod_{m=1}^{|\lambda|} \frac{A_m(c_m)}{c_m!} \left(\frac{t^m}{m}\right)^{c_m} = \prod_{m=1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{A_m(k)}{k!} \left(\frac{t^m}{m}\right)^k\right)$$

$$= \prod_{m=1}^{\infty} EG\left(A_m, \frac{t^m}{m}\right) \tag{6.4}$$

by definition of $EG(F, t)$. $\qquad \square$

As before we can write down the generating functions for the cycle numbers and for K_{0n}:

Theorem 6.3. *We have*

$$\sum_{n=0}^{\infty} t^n h_n(F) = \prod_{m=1}^{\infty} EG\left(F_m, \frac{t^m}{m}\right). \tag{6.5}$$

If $b \in \mathbb{N}$ and $s_1, \cdots, s_b \in \mathbb{R}$ are fixed, then

$$\sum_{n=0}^{\infty} t^n h_n(F) \mathbb{E}\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] = \prod_{m=1}^{b} EG\left(F_m, e^{is_m}\frac{t^m}{m}\right) \prod_{m=b+1}^{\infty} EG\left(F_m, \frac{t^m}{m}\right) \tag{6.6}$$

and for each $w \in \mathbb{C}$

$$\sum_{n=0}^{\infty} t^n h_n(F) \mathbb{E}\left[\exp\left(wK_{0n}\right)\right] = \prod_{m=1}^{\infty} EG\left(F_m, e^w\frac{t^m}{m}\right) \tag{6.7}$$

Proof. The identity (6.5) follows from (6.6) by choosing $s_1 = \cdots = s_b = 0$. It is thus enough to prove (6.6). We have

$$\sum_{n=0}^{\infty} h_n(F) \mathbb{E}_F\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] t^n = \sum_{\lambda} \frac{1}{z_{\lambda}} t^{|\lambda|} \exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right) \prod_{m=1}^{n} F_m(C_m)$$

$$= \sum_{\lambda} \frac{1}{z_{\lambda}} t^{|\lambda|} \left(\prod_{m=1}^{b} (e^{is_m})^{C_m^{(n)}} F_m(C_m^{(n)})\right) \left(\prod_{m=b+1}^{l(\lambda)} F_m(C_m^{(n)})\right) \tag{6.8}$$

We now use Lemma 6.2 with

$$A_m(k) = \begin{cases} (e^{is_m})^k F_m(k), & \text{if } 1 \le m \le b; \\ F_m(k), & \text{if } m > b. \end{cases} \tag{6.9}$$

A simple computation then shows that

$$EG\left(A_m, \frac{t^m}{m}\right) = EG\left(F_m, e^{is_m}\frac{t^m}{m}\right) \tag{6.10}$$

for $1 \le m \le b$. This proves (6.6). The proof of (6.7) is similar. $\qquad \square$

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We have thus found the generating functions in this general setting. To get the asymptotic behaviour as in Section 4 and Section 3, we need some analyticity assumptions. We now give here some examples

6.1. exp-polynomial weights

Suppose that a polynomial $P(t) = \theta t + \sum_{k=2}^{d} b_k t^k$ is given with $b_k \geq 0$ and $\theta > 0$. We then define F_m implicit by the equation

$$EG(F_m, t) = \exp(P(t)). \tag{6.11}$$

It is easy to see from (6.11) that $F_m(0) = 1$ and $F_m(c) \geq 0$. The functions F_m thus generate a probability measure on S_n. We do not need an explicit expression for F_m since we are only interested in the asymptotic behaviour of C_m and K_{0n}. We have for $|t| < 1$:

$$\sum_{n=0}^{\infty} t^n h_n = \prod_{m=1}^{\infty} \exp\left(P\left(\frac{t^m}{m}\right)\right) = \exp\left(\theta\left(\sum_{m=1}^{\infty} \frac{t^m}{m}\right) + \sum_{k=2}^{d} b_k \sum_{m=1}^{\infty} \left(\frac{t^m}{m}\right)^k\right)$$

$$= \exp\left(-\theta \log(1-t) + \sum_{k=2}^{d} b_k \mathrm{Li}_k(t^k)\right) \tag{6.12}$$

with $\mathrm{Li}_k(t) := \sum_{m=1}^{\infty} \frac{t^m}{m^k}$, which are known as polylogarithms. It was proven in [17] with Lindelöf's integral representation (see Theorem 2.10) that polylogarithms can be extended holomorphically to the whole complex plane slit along the axis $\mathbb{R}_{\geq 1}$. The asymptotic expansions of $\mathrm{Li}_k(t)$ as $t \to 1$ can be found for instance in [12, Section 3]. We have

$$\mathrm{Li}_2(t) = \frac{\pi^2}{3} + (1-t)\log(1-t) + O\left((1-t)^2\right) \tag{6.13}$$

$$\mathrm{Li}_k(t) = H_k + O(1-t) \quad \text{for} \quad k \in \mathbb{N}_{\geq 3}. \tag{6.14}$$

If $b_2 = 0$, then $-\theta \log(1-t) + \sum_{m=2}^{d} b_k \mathrm{Li}_k(t^k)$ is of class $\mathcal{F}(1, \theta)$ and we can apply Theorem 2.9. It is now obvious that we get in this situation the same asymptotic behaviour as for the Ewens measure. If $b_2 \neq 0$, one still can apply the method presented in this paper. In fact the results are the same, only the error terms are slightly worse.

6.2. Spatial random permutations

The origin of the measure in this section arises from a model for the Bose gas and has a connection to Bose-Einstein condensation. We give here only the definition and a very brief idea of the model behind, but avoiding further details. We refer to [7] for a more comprehensive overview.

We follow here the physicists notation and write $\theta_m = e^{-\alpha_m}$. Let $\xi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and Λ be a lattice in $\mathbb{R}^d$ be given. Assume that for all $k \in \Lambda$.

$$0 \leq e^{-\epsilon(k)} := \int_{\mathbb{R}^d} e^{-\xi(x)} e^{-2\pi i <k,x>} \, dx \quad \text{and} \quad \sum_{k \in \Lambda} e^{-\epsilon(k)} < \infty. \tag{6.15}$$

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We then define

$$
F_m(c) = F_m^{(\alpha,\xi)}(c) = \left(e^{-\alpha_m} \sum_{k\in\Lambda} e^{-\epsilon(k)m} \right)^c .
\tag{6.16}
$$

The functions $F_m^{(\alpha,\xi)}$ are well defined since there exists only finitely many $k \in \Lambda$ with $e^{-\epsilon(k)m} > 1$. As before, we use $h_n = h_n^{(\alpha,\xi)}$ for the normalization constant.

We now describe the physical model behind this. Let D be a fundamental domain of $\mathbb{R}^d/\Lambda$ and $x_1, \cdots, x_n$ be n particles in D. The function ξ plays the role to penalize certain configurations of the x_i. More precise, the probability of a given configuration is defined as

$$
\mathbb{P}_{n,dx}\,[\sigma, dx] = \frac{1}{h_n n!} \exp\left(-\sum_{m=1}^n \xi\big(x_m - x_{\sigma(m)}\big) - \sum_{m=1}^n C_m \alpha_m \right).
\tag{6.17}
$$

where dx is the normalized Lebesgue measure on D. The probability measure $\mathbb{P}_{n,dx}[.]$ now induces a probability measure $\mathbb{P}_n[.]$ on S_n (by averaging over D). It is not obvious, but it can be proven that $\mathbb{P}_n[.]$ and $\mathbb{P}_{\mathrm{F}}[.]$ are the same measures, see [7, Proposition 3.1].

We now compute the asymptotic behaviour of C_m and K_{0n} with respect to this measure. The lattice in [7] depends on n and is so chosen that the density $\rho := \frac{|D|}{n}$ is fix. We assume here that the lattice Λ is independent of n. The reason why we do this that we need in our approach that the weights are independent of n. We define

$$
g^{(\alpha)}(t) := \sum_{m=1}^\infty \frac{e^{-\alpha_m}}{m} t^m
\tag{6.18}
$$

$$
g^{(\alpha,\xi)}(t) := \sum_{k\in\Lambda^d} g^{(\alpha)}\big(e^{-\epsilon(k)}t\big) = \sum_{k\in\Lambda} \sum_{m=1}^\infty \frac{e^{-\alpha_m}}{m} t^m \big(e^{-\epsilon(k)}\big)^m
\tag{6.19}
$$

The functions $g^{(\alpha)}(t)$ and $g^{(\alpha,\xi)}(t)$ are both formal power series in t since by assumption $e^{-\epsilon(k)} \geq 0$ and the coefficient of each t^m is finite.

The function $g^{(\alpha)}(t)$ agrees with $g_\Theta(t)$ for $\Theta = (e^{-\alpha_m})_{m\in\mathbb{N}}$. The difference to the Section 3 and 4 is that $g^{(\alpha,\xi)}(t)$ will play role of $g_\Theta(t)$ and not $g^{(\alpha)}(t)$, i.e. the asymptotic behaviour of C_m and K_{0m} depends directly on the analytic properties of $g^{(\alpha,\xi)}$ (see below).

We use the same argumentation as in the previous sections to get the asymptotic behaviour of C_m and K_{0n}. We begin with the generating functions:

Theorem 6.4. *Assume that $\sum_{k\in\Lambda} e^{-\epsilon(k)}$ is convergent. We then have the following identities*

(as formal power series)

$$G^{(\alpha,\xi)}(t) := \sum_{n=0}^{\infty} t^n h_n^{(\alpha,\xi)} = \exp\left(g^{(\alpha,\xi)}(t)\right) \tag{6.20}$$

$$\sum_{n=0}^{\infty} t^n h_n^{(\alpha,\xi)} \mathbb{E}\left[e^{\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)}\right] = G^{(\alpha,\xi)}(t) \prod_{m=1}^{b} \exp\left((e^{is}-1)\frac{t^m}{m}\left(e^{-\alpha_m}\sum_{k\in\Lambda} e^{-\epsilon(k)m}\right)\right) \tag{6.21}$$

$$\sum_{n=0}^{\infty} h_n^{(\alpha,\xi)} \mathbb{E}\left[\exp\left(wK_{0n}\right)\right] t^n = \exp\left(wg^{(\alpha,\xi)}(t)\right). \tag{6.22}$$

Proof. We begin with (6.20). We have

$$\mathrm{EG}\left(F_m, \frac{t^m}{m}\right) = \sum_{c=0}^{\infty} \frac{F_m(c)}{c!}\left(\frac{t^m}{m}\right)^c = \sum_{c=0}^{\infty} \frac{1}{c!}\left(\frac{t^m}{m}e^{-\alpha_m}\sum_{k\in\Lambda} e^{-\epsilon(k)m}\right)^c \tag{6.23}$$

$$= \exp\left(\frac{t^m}{m}\left(e^{-\alpha_m}\sum_{k\in\Lambda} e^{-\epsilon(k)m}\right)\right).$$

Thus

$$\prod_{m=1}^{\infty} \mathrm{EG}\left(F_m, \frac{t^m}{m}\right) = \exp\left(\sum_{m=1}^{\infty} \frac{t^m}{m}\left(e^{-\alpha_m}\sum_{k\in\Lambda} e^{-\epsilon(k)m}\right)\right) \tag{6.24}$$

$$= \exp\left(\sum_{k\in\Lambda}\sum_{m=1}^{\infty} e^{-\alpha_m}\frac{(e^{-\epsilon(k)}t)^m}{m}\right) = \exp\left(\sum_{k\in\Lambda} g^\alpha(e^{-\epsilon(k)}t)\right)$$

The above expressions are formal power series since $g^{(\alpha,\xi)}(t)$ is a formal power series. This proves (6.20). We have

$$\mathrm{EG}\left(F_m, w\frac{t^m}{m}\right) = \exp\left(w\frac{t^m}{m}\left(e^{-\alpha_m}\sum_{k\in\Lambda} e^{-\epsilon(k)m}\right)\right) \tag{6.25}$$

The identities (6.21) and (6.22) thus follow immediately from Theorem 6.3. $\square$

The next step is to obtain the asymptotic behaviour of C_m and K_{0n}. For this we have to assume some analytic properties. In this context, it is natural to assume that $g^{(\alpha)}$ is of class $\mathcal{F}(\alpha, r)$.

Lemma 6.5. *Let $g^{(\alpha)}$ be of class $\mathcal{F}(\alpha, r)$. We define*

$$\widetilde{r} := \min\left\{e^{\epsilon(k)}, k \in \Lambda\right\} \quad \text{and} \quad A := \#\left\{k \in \Lambda, e^{\epsilon(k)} = \widetilde{r}\right\}. \tag{6.26}$$

Then $g^{(\alpha,\xi)}(t)$ is of class $\mathcal{F}(A\alpha, \widetilde{r}r)$.

Proof. We have by assumption that $g^{(\alpha)}$ is holomorphic in $\Delta_0(r, R, \phi)$ for some ϕ, R. Thus $g^{(\alpha)}(e^{-\epsilon(k)}t)$ is holomorphic in $e^{\epsilon(k)}\Delta_0(r, R, \phi)$. This shows that $g^{(\alpha,\xi)}(t)$ is a sum of functions holomorphic in $\widetilde{r}\Delta_0(r, R, \phi)$. We thus have to show that the sum is convergent. We have $|g^{(\alpha)}(t)| < \widehat{C}_1|t|$ for $|t| < r/2$ since by assumption $g^{(\alpha)}(0) = 0$. Let t in $\widetilde{r}\Delta_0(r, R, \phi)$ be fixed. Since the $\sum_{k \in \Lambda} e^{-\epsilon(k)}$ is convergent, there are only finitely many $k \in \Lambda^d$ with $|t|e^{-\epsilon(k)} > \frac{1}{2}r$. We thus get

$$|g^{(\alpha,\xi)}(t)| = \left| \sum_{k \in \Lambda} g^{(\alpha)}(e^{-\epsilon(k)}t) \right| \leq \widehat{C}_2 + \sum_{k \in \Lambda^d} \widehat{C}_1|e^{-\epsilon(k)}t| \leq \widehat{C}_2 + \widehat{C}_1|t| \sum_{k \in \Lambda} |e^{-\epsilon(k)}| < \infty. \quad (6.27)$$

This proves that $g^{(\alpha,\xi)}(t)$ is holomorphic in $\widetilde{r}\Delta_0(r, R, \phi)$. The other statements are clear. $\square$

We now apply Theorem 2.9 to obtain:

Theorem 6.6. *Let $g^{(\alpha)}$ be of class $\mathcal{F}(\alpha, r)$ and $\widetilde{r}, A$ be as in Lemma 6.5. We then have for each $b \in \mathbb{N}$*

$$\left(C_1^{(n)}, C_2^{(n)}, \cdots C_b^{(n)} \right) \xrightarrow{d} (Y_1, \cdots, Y_b) \quad (6.28)$$

with $Y_1, \cdots, Y_b$ independent Poisson distributed random variables with

$$\mathbb{E}[Y_m] = \frac{(r\widetilde{r})^m}{m} \left(e^{-\alpha_m} \sum_{k \in \Lambda} e^{-\epsilon(k)m} \right) \quad (6.29)$$

Proof. The proof of this theorem is the same as the proof of Theorem 3.1 One only has to use Theorem 6.4 and Lemma 6.5. We therefore omit the details. $\square$

Theorem 6.7. *Let $g^{(\alpha)}$ be of class $\mathcal{F}(\alpha, r)$ and $\widetilde{r}, A$ be as in Lemma 6.5. Then K_{0n} converges in the strong mod-Poisson sense with parameters $A\alpha \log(n)$ with limiting function $\frac{\Gamma(\alpha)}{\Gamma(\alpha e^{is})}$.*

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