On the limiting velocity of random walks in mixing random environment

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Abstract

We consider random walks in strong-mixing random Gibbsian environments in $\mathbb{Z}^d, d \geq 2$. Based on regeneration arguments, we will first provide an alternative proof of Rassoul-Agha's conditional law of large numbers (CLLN) for mixing environment [8]. Then, using coupling techniques, we show that there is at most one nonzero limiting velocity in high dimensions ($d \geq 5$).

1 Introduction

An *environment* is an element $\omega = \{\omega(x,e)\}_{x\in\mathbb{Z}^d,|e|=1}$ of $\Omega = \mathcal{M}^{\mathbb{Z}^d}$, where $\mathcal{M}$ is the space of probability measures on $\{e \in \mathbb{Z}^d : |e| = 1\}$ and $|\cdot|$ denotes the Euclidean norm. The random walk in the environment $\omega \in \Omega$ started at x is the canonical Markov chain (X_n) on $(\mathbb{Z}^d)^{\mathbb{N}}$, with state space $\mathbb{Z}^d$ and law P^x_ω specified by

$$\begin{aligned} &P^x_\omega\{X_0 = x\} = 1, \\ &P^x_\omega\{X_{n+1} = y + e | X_n = y\} = \omega(y,e), \quad e \in \mathbb{Z}^d, |e| = 1. \end{aligned}$$

Let P be a stationary (with respect to the shifts in $\mathbb{Z}^d$) probability measure on Ω. The joint law of the environment and the walks is denoted by $\mathbf{P}^x = P \otimes P^x_\omega$. We also write $\mathbf{P}^o$ as $\mathbf{P}$, where o denotes the origin. We say that the random environment is *iid* if P is a product measure. We say that P is *uniformly elliptic* if there is a constant $\kappa \in (0, 1/2d)$ such that P-almost surely,

$$\omega(o,e) > \kappa \text{ for all } e \in \mathbb{Z}^d \text{ with } |e| = 1.$$

For any vector $\ell \in S^{d-1}$, we let

$$A_\ell = \{\lim_{n\to\infty} X_n \cdot \ell = \infty\}.$$

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In recent years, much progress has been made in the study of the limiting velocity $\lim_{n\to\infty} X_n/n$ of random walks in random iid environment, see [12] for a survey. For one-dimensional RWRE, the law of large numbers (LLN) is well known (see [10]). For $d \geq 2$, a conditional law of large numbers (CLLN) is proved in [11, 13] (see [12, Theorem 3.2.2] for the full version), which states that $\mathbf{P}$-almost surely, for any direction ℓ,

$$\lim_{n\to\infty} \frac{X_n \cdot \ell}{n} = v_\ell 1_{A_\ell} - v_{-\ell} 1_{A_{-\ell}} \tag{CLLN}$$

for some deterministic constants v_ℓ and $v_{-\ell}$ (we set $v_\ell = 0$ if $\mathbf{P}(A_\ell) = 0$). Moreover, for $d = 2$, the LLN follows from combining the CLLN and Zerner and Merkl's 0-1 law [14] for two-dimensional RWRE: for any direction ℓ,

$$\mathbf{P}(A_\ell) \in \{0, 1\}.$$

When $d \geq 3$, the 0-1 law and the LLN are among the main open questions in the study of RWRE. Nevertheless, in high dimension ($d \geq 5$), Berger [1] showed that the limiting velocity can take at most one non-zero value, i.e.,

$$v_\ell v_{-\ell} = 0. \tag{1}$$

The purpose of this paper is to extend the CLLN and Berger's result (1) to the case when the environments on different sites are allowed to be dependent. Of special interest is the environment that is produced by a Gibbsian particle system (which we call the *Gibbsian environment*) and satisfies Dobrushin-Shlosman's strong-mixing condition IIIc in [4, page 378], see [6, 7, 2, 3, 8] for related works. For the definition of the Gibbsian environment and the strong-mixing condition [6, (6.1)], see [6, pages 1454-1455]. An important feature of this model is that the influence of the environments in remote locations decays exponentially as the distance grows.

In [6], assuming a ballisticity condition (Kalikow's condition) which implies that the event of escape in a direction has probability 1, Rassoul-Agha proved the LLN for the strong-mixing Gibbsian environment, using the invariant measure of the "environment viewed from the point of view of the particle" process. In [8], Rassoul-Agha also obtained a CLLN for the strong-mixing Gibbsian environment, under an analyticity condition (see Hypothesis (M) in [8]). Comets and Zeitouni proved the LLN for environments with a weaker cone-mixing assumption ($\mathcal{A}1$) in [2], but under some conditions about ballisticity and the uniform integrability of the regeneration times (see ($\mathcal{A}5$) in [2]).

Our first purpose is to prove the CLLN for random walks in the strong-mixing Gibbsian environment. Display (2) in Theorem 2 is a minor extension of Rassoul-Agha's CLLN in [8], in which he assumes slightly more than strong-mixing. Yet, our proof is very different from the proof in [8] , which is based on a large deviation principle in [7]. The main contribution of our

proof of (2) is a new definition of the regeneration structure, which enables us to divide a random path in the mixing environment into "almost iid" parts. With this regeneration structure, we will use the "ϵ-coins" introduced in [2] and coupling arguments to prove the CLLN. This regeneration structure will also be used in the proof of (3).

Our second main result (3) is an extension of Berger's result (1) from the iid case to the strong-mixing case. In [1], assuming that $\mathbf{P}(A_\ell) > 0$ for a direction ℓ, Berger coupled the iid environment ω with a transient (in the direction ℓ) environment $\tilde{\omega}$ and a "backward path", such that $\tilde{\omega}$ and ω coincide in the locations off the path. Using heat kernel estimates for random walks with iid increments, he showed that if $v_\ell v_{-\ell} > 0$ and $d \geq 5$, then with positive probability, the random walks in $\tilde{\omega}$ is transient to the $-\ell$ direction without intersecting the backward path, which contradicts $\tilde{\omega}$ being transient in the direction ℓ. The difficulties in applying this argument to mixing environments are that the regeneration slabs are not iid, and that unlike the iid case, the environments visited by two disjoint paths are not independent. To overcome these difficulties, we will construct an environment (along with a path) that is "very transient" in ℓ, and show that the ballistic walks in the opposite direction ($-\ell$) will move further and further away from the given path (see Figure 2 in Section 5). The key ingredient here is a heat kernel estimate, which we will obtain in Section 4 using coupling arguments.

We now describe our main results. Recall first the definition of an r-Markov environment (see [3]).

Definition 1. *For $r \geq 1$, let $\partial_r V = \{x \in \mathbb{Z}^d \setminus V : d(x, V) \leq r\}$ be the r-boundary of $V \subset \mathbb{Z}^d$. A random environment (P, Ω) on $\mathbb{Z}^d$ is called r-Markov if for any finite $V \subset \mathbb{Z}^d$,*

$$P\big((\omega_x)_{x\in V} \in \cdot|\mathcal{F}_{V^c}\big) = P\big((\omega_x)_{x\in V} \in \cdot|\mathcal{F}_{\partial_r V}\big),\ P\text{-a.s.},$$

where $d(\cdot,\cdot)$ denotes the l^1-distance and $\mathcal{F}_\Lambda := \sigma(\omega_x : x \in \Lambda)$.

We say that an r-Markov environment P satisfies condition $(*)$ if there exist constants $\gamma, C < \infty$ such that for all finite subsets $\Delta \subset V \subset \mathbb{Z}^d$ with $d(\Delta, V^c) \geq r$, and $A \subset V^c$,

$$\frac{\mathrm{d}P\big((\omega_x)_{x\in\Delta} \in \cdot|\eta\big)}{\mathrm{d}P\big((\omega_x)_{x\in\Delta} \in \cdot|\eta'\big)} \leq \exp\big(C \sum_{x\in A, y\in\Delta} e^{-\gamma d(x,y)}\big) \qquad (*)$$

for P-almost all pairs of configurations $\eta, \eta' \in \mathcal{M}^{V^c}$ which agree on $V^c \setminus A$. Here

$$P\big((\omega_x)_{x\in\Delta} \in \cdot|\eta\big) := P\big((\omega_x)_{x\in\Delta} \in \cdot|\mathcal{F}_{V^c}\big)\big|_{(\omega_x)_{x\in V^c}=\eta}.$$

We remark that r and γ are used as parameters of the environment throughout the article.

By Lemma 9 in [6], the strong-mixing Gibbsian environment satisfies $(*)$. Obviously, every finite-range dependent environment also satisfies $(*)$.

Our main theorem is:

Theorem 2. *Assume that P is uniformly elliptic and satisfies $(*)$. Then there exist two deterministic constants $v_+, v_- \geq 0$ and a vector ℓ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_+ \ell 1_{A_\ell} - v_- \ell 1_{A_{-\ell}}, \tag{2}$$

and $v_+ = v_- = 0$ if $\mathbf{P}(A_\ell \cup A_{-\ell}) < 1$. Moreover, if $d \geq 5$, then there is at most one non-zero velocity. That is,

$$v_+ v_- = 0. \tag{3}$$

We remark here that for the finite-range dependent case, the CLLN is proved in [12].

The structure of this paper is as follows. In Section 2, we prove a refined version of [13, Lemma 3]. With this combinatorial result, we will prove the CLLN in Section 3, using coupling arguments. In Section 4, using coupling, we obtain heat kernel estimates, which is later used in Section 5 to show the uniqueness of the non-zero limiting velocity.

Throughout the paper, we assume that the environment is uniformly elliptic and satisfies $(*)$. We use c, C to denote finite positive constants that depend only on the dimension d and the environment measure P (and implicitly, on the parameters κ, r and γ of the environment). They may differ from line to line. We denote by $c_1, c_2, \ldots$ positive constants which are fixed throughout, and which depend only on d and the measure P. Let $\{e_1, \ldots, e_d\}$ be the natural basis of $\mathbb{Z}^d$.

2 A combinatorial lemma and its consequences

In this section we consider the case that $\mathbf{P}(\overline{\lim}_{n\to\infty} X_n \cdot e_1/n > 0) > 0$. We will adapt the arguments in [13] and prove that with positive probability, the number of visits to the i-th level $\mathcal{H}_i = \mathcal{H}_i(X_0) := \{x : x \cdot e_1 = X_0 \cdot e_1 + i\}$ grows slower than Ci^2. An important ingredient of the proof is a refinement of a combinatorial lemma of Zerner [13, Lemma 3] about deterministic paths.

We say that a sequence $\{x_i\}_{i=0}^{k-1} \in (\mathbb{Z}^d)^k$, $2 \leq k \leq \infty$, is a *path* if $|x_i - x_{i-1}| = 1$ for $i = 1, \cdots, k-1$. For $i \geq 0$ and an infinite path $X_\cdot = \{X_n\}_{n=0}^\infty$ such that $\sup_n X_n \cdot e_1 = \infty$, let

$$T_i = \inf\{n \geq 0 : X_n \in \mathcal{H}_i\}.$$

For $0 \leq i < j$ and $k \geq 1$, let $T_{i,j}^1 := T_i$ and define recursively

$$T_{i,j}^{k+1} = \inf\{n \geq T_{i,j}^k : X_n \in \mathcal{H}_i \text{ and } n < T_j\} \in \mathbb{N} \cup \{\infty\}.$$

That is, $T^k_{i,j}$ is the time of the k-th visit to $\mathcal{H}_i$ before hitting $\mathcal{H}_j$. Let

$$N_{i,j} = \sup\{k : T^k_{i,j} < \infty\}$$

be the total number of visits to $\mathcal{H}_i$ before hitting $\mathcal{H}_j$.

As in [13], for $i \geq 0, l \geq 1$, let

$$h_{i,l} = T^{N_{i,i+l}}_{i,i+l} - T_i$$

denote the time spent between the first and the last visits to $\mathcal{H}_i$ before hitting $\mathcal{H}_{i+l}$. For $m, M, a \geq 0$ and $l \geq 1$, set

$$H_{m,l} = \sum_{i=0}^{l-1} N_{m+i,m+l}/(i+1)^2$$

and

$$E_{M,l}(a) = \frac{\#\{0 \leq m \leq M : h_{m,l} \leq a \text{ and } H_{m,l} \leq a\}}{M+1}.$$

Note that $E_{M,l}(a)$ decreases in l and increases in a.

The following lemma is a minor adaptation of [13, Lemma 3].

Lemma 3. *For any path $X.$ with $\overline{\lim}_{n\to\infty} X_n \cdot e_1/n > 0$,*

$$\sup_{a\geq 0} \inf_{l\geq 1} \overline{\lim}_{M\to\infty} E_{M,l}(a) > 0. \tag{4}$$

Proof: Since $\overline{\lim}_{n\to\infty} n/T_n = \overline{\lim}_{n\to\infty} X_n \cdot e_1/n > 0$, there exist an increasing sequence $(n_k)_{k=0}^{\infty}$ and $\delta < \infty$ such that

$$T_{n_k} < \delta n_k \text{ for all } k.$$

Thus for any m such that $n_k/2 \leq m \leq n_k$,

$$T_m \leq 2\delta m. \tag{5}$$

Set $M_k = \lceil n_k/2 \rceil$. Then for all k and $1 < l < \lfloor n_k/2 \rfloor$,

$$\begin{aligned} \sum_{m=0}^{M_k} H_{m,l} &= \sum_{i=0}^{l-1} \Big(\sum_{m=0}^{M_k} N_{m+i,m+l} \Big)/(i+1)^2 \\ &\leq \sum_{i=0}^{l-1} T_{M_k+l}/(i+1)^2 \overset{(5)}{\leq} 4\delta(M_k+l). \end{aligned} \tag{6}$$

By the same argument as in Page 193-194 of [13], we will show that there exist constants $c_1, c_2 > 0$ such that

$$\inf_{l\geq 1} \overline{\lim}_{k\to\infty} \frac{\#\{0 \leq m \leq M_k : h_{m,l} \leq c_1\}}{M_k+1} > c_2. \tag{7}$$

Indeed, if (7) fails, then for any $u > 0$,

$$\overline{\lim_{k\to\infty}} \frac{\#\{0 \le m \le M_k, h_{m,l} \le u\}}{M_k + 1} \longrightarrow 0$$

as $l \to \infty$ (note that the right side is decreasing in l). Hence, one can find a sequence $(l_i)_{i\ge 0}$ with $l_{i+1} > l_i, l_0 = 0$, such that for all $i \ge 0$,

$$\overline{\lim_{k\to\infty}} \frac{\#\{0 \le m \le M_k, h_{m,l_{i+1}} \le 6\delta l_i\}}{M_k + 1} < \frac{1}{3}. \tag{8}$$

On the other hand, for $i \ge 0$

$$\begin{aligned}
&\overline{\lim_{k\to\infty}} \frac{\#\{0 \le m \le M_k, h_{m,l_i} \ge 6\delta l_i\}}{M_k + 1} \\
&\le \overline{\lim_{k\to\infty}} \frac{1}{(M_k+1)6\delta l_i} \sum_{m=0}^{M_k} (T_{m+l_i} - T_m) \\
&\le \overline{\lim_{k\to\infty}} \frac{l_i T_{M_k+l_i}}{6\delta l_i (M_k+1)} \overset{(5)}{\le} \frac{1}{3}.
\end{aligned} \tag{9}$$

By (8) and (9) , for any $i \ge 0$,

$$\overline{\lim_{k\to\infty}} \frac{\#\{0 \le m \le M_k, h_{m,l_{i+1}} > h_{m,l_i}\}}{M_k + 1} \ge \frac{1}{3}. \tag{10}$$

Therefore, for any $j \ge 1$, noting that

$$\sum_{i=0}^{j-1} 1_{h_{m,l_{i+1}} > h_{m,l_i}} \le N_{m,m+l_j} \le H_{m,l_j},$$

we have

$$\begin{aligned}
\frac{j}{3} &\overset{(10)}{\le} \overline{\lim_{k\to\infty}} \sum_{i=0}^{j-1} \frac{\#\{0 \le m \le M_k, h_{m,l_{i+1}} > h_{m,l_i}\}}{M_k + 1} \\
&\le \overline{\lim_{k\to\infty}} \frac{1}{M_k+1} \sum_{m=0}^{M_k} H_{m,l_j} \overset{(6)}{\le} 4\delta,
\end{aligned}$$

which is a contradiction if j is large. This proves (7).

It follows from (7) that, for any $l \ge 1$, there is a subsequence (M'_k) of (M_k) such that

$$\frac{\#\{0 \le m \le M'_k : h_{m,l} \le c_1\}}{M'_k + 1} > c_2$$

for all k. Letting $c_3 = 9\delta/c_2$, we have that when k is large enough,

$$\frac{1}{M'_k+1} \sum_{m=0}^{M'_k} 1_{h_{m,l} \le c_1, H_{m,l} > c_3} \le \frac{1}{c_3(M'_k+1)} \sum_{m=0}^{M'_k} H_{m,l} \overset{(6)}{\le} \frac{c_2}{2}.$$

Hence for any $l > 1$ and large k,

$$E_{M'_k,l}(c_1 \vee c_3) \geq \frac{1}{M'_k+1} \sum_{m=0}^{M'_k} 1_{h_{m,l} \leq c_1, H_{m,l} \leq c_3}$$
$$= \frac{1}{M'_k+1} \sum_{m=0}^{M'_k} (1_{h_{m,l} \leq c_1} - 1_{h_{m,l} \leq c_1, H_{m,l} > c_3}) \geq \frac{c_2}{2}.$$

This shows the lemma, and what is more, with explicit constants. □

For $i \geq 0$, let $N_i = \lim_{j\to\infty} N_{i,j}$ denote the total number of visits to $\mathcal{H}_i$. With Lemma 3, one can deduce that with positive probability, $N_i \leq C(i+1)^2$ for all $i \geq 0$:

Theorem 4. *If* $\mathbf{P}(\overline{\lim}_{n\to\infty} X_n \cdot e_1/n > 0) > 0$, *then there exists a constant* c_5 *such that*

$$\mathbf{P}(R = \infty) > 0,$$

where R *is the stopping time defined by*

$$R = R_{e_1}(X_\cdot, c_5)$$
$$:= \inf\{n \geq 0 : \sum_{i=0}^{n} 1_{X_i \in \mathcal{H}_j} > c_5(j+1)^2 \text{ for some } j \geq 0\} \wedge D,$$

and $D := \inf\{n \geq 1 : X_n \cdot e_1 \leq X_0 \cdot e_1\}$.

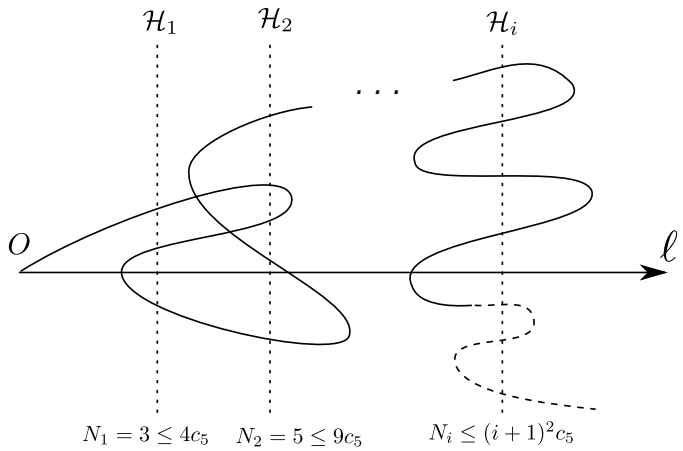

Figure 1: On $\{R = \infty\}$, the path visits the i-th level no more than $c_5(i+1)^2$ times, and never visits the left-half space.

Note that for any $L > 0$ and a path $(X_i)_{i=0}^{\infty}$ with $X_0 = o$,

$$\sum_{\substack{y: y\cdot e_1 \le -L \\ 0 \le i \le R}} e^{-\gamma d(y, X_i)} \le \sum_{j=0}^{\infty} (\#\text{visits to } \mathcal{H}_j \text{ before time } R) e^{-\gamma(j+L)}$$

$$\le C \sum_{j=0}^{\infty} c_5 (j+1)^2 e^{-\gamma(j+L)} \le C e^{-\gamma L}. \tag{11}$$

Hence on the event $\{R = \infty\}$, by (11) and $(*)$, the trajectory $(X_i)_{i=0}^{\infty}$ is "almost independent" with the environments $\{\omega_x : x \cdot e_1 \le -L\}$ when L is large. See Figure 1. This fact will be used in our definition of the regeneration times in the Section 3.

To prove Theorem 4, we need the following lemma. Recall that r, γ are parameters of the environment measure P. Let S be a countable set of finite paths. With abuse of notation, we also use S as the synonym for the event

$$\bigcup_{(x_i)_{i=0}^N \in S} \{X_i = x_i \text{ for } 0 \le i \le N\}. \tag{12}$$

Lemma 5. *Let $a > 0$ and $A \subset \Lambda \subset \mathbb{Z}^d$. Suppose $S \neq \emptyset$ is a countable set of finite paths $x. = (x_i)_{i=0}^N, N < \infty$ that satisfy $d(x., \Lambda) \ge r$ and*

$$\sum_{y \in A, 0 \le i \le N} e^{-\gamma d(y, x_i)} \le a.$$

Then, P-almost surely,

$$\exp(-Ca) \le \frac{E_P[P_\omega(S) | \omega_x : x \in \Lambda]}{E_P[P_\omega(S) | \omega_x : x \in \Lambda \setminus A]} \le \exp(Ca). \tag{13}$$

Proof: We shall first show that for any $(x_i)_{i=0}^N \in S$, P-almost surely,

$$\begin{aligned} &E_P[P_\omega(X_i = x_i, 0 \le i \le N) | \omega_y : y \in \Lambda] \\ &\le \exp(Ca) E_P[P_\omega(X_i = x_i, 0 \le i \le N) | \omega_y : y \in \Lambda \setminus A]. \end{aligned} \tag{14}$$

Note that when Λ^c is a finite subset of $\mathbb{Z}^d$, (14) is an easy consequence of $(*)$. For general Λ, we let

$$\Lambda_n = \Lambda \cup \{x : |x| \ge n\}.$$

When n is sufficiently big, $(*)$ implies that

$$\frac{E_P[P_\omega(X_i = x_i, 0 \le i \le N) | \omega_y : y \in \Lambda_n]}{E_P[P_\omega(X_i = x_i, 0 \le i \le N) | \omega_y : y \in \Lambda_n \setminus A]} \le \exp(Ca).$$

Since $\Lambda_n \downarrow \Lambda$ as $n \to \infty$, (14) follows by taking $n \to \infty$ in the above inequality.

Summing over all $(x_i)_{i=0}^N \in S$ on both sides of (14), we conclude that P-almost surely,

$$E_P[P_\omega(S)|\omega_y : y \in \Lambda] \le \exp(Ca) E_P[P_\omega(S)|\omega_y : y \in \Lambda \setminus A].$$

The upper bound of (13) is proved. The lower bound follows likewise. □

Now we can prove the theorem. Our proof is a modification of the proof of Theorem 1 in [13]:

Proof of Theorem 4: It follows by Lemma 3 that there exists a constant $c_4 > 0$ such that

$$\mathbf{P}(\inf_{l\ge 1} \overline{\lim_{M\to\infty}} E_{M,l}(c_4) > 0) > 0. \tag{15}$$

For $l > r$, $k \ge 0$ and $z \in \mathbb{Z}^d$ with $z \cdot e_1 = r$, let $B_{m,l}(z,k,c)$ denote the event

$$\{N_{m+r,m+l} = k, X_{T^k_{m+r,m+l}} = X_{T_m} + z, H_{m+r,l-r} \le c\}.$$

Note that on the event $\{h_{m,l} \le c_4 \text{ and } H_{m,l} \le c_4\}$, we have

$$\begin{aligned} T^{N_{m+r,m+l}}_{m+r,m+l} - T_m &\le h_{m,l} + \sum_{i=0}^{r} N_{m+i,m+l} \\ &\le c_4 + \sum_{i=0}^{r} (i+1)^2 c_4 \le (1+r)^3 c_4, \end{aligned}$$

and

$$\begin{aligned} H_{m+r,l-r} &\le \sum_{i=0}^{l-r-1} (r+1)^2 N_{m+r+i,m+l}/(r+i+1)^2 \\ &\le (r+1)^2 c_4 =: c_5. \end{aligned}$$

Hence $\{h_{m,l} \le c_4 \text{ and } H_{m,l} \le c_4\} \subset \bigcup_{|z|,k\le (r+1)^3 c_4} B_{m,l}(z,k,c_5)$, and

$$\lim_{l\to\infty} \overline{\lim_{M\to\infty}} E_{M,l}(c_4) \le \sum_{|z|,k\le (r+1)^3 c_4} \overline{\lim_{l\to\infty}} \overline{\lim_{M\to\infty}} \frac{1}{M+1} \sum_{m=0}^{M} 1_{B_{m,l}(z,k,c_5)}.$$

Thus by (15), for some k_0 and z_0 with $z_0 \cdot e_1 = r$,

$$\mathbf{P}(\overline{\lim_{l\to\infty}} \overline{\lim_{M\to\infty}} \frac{1}{M+1} \sum_{m=0}^{M} 1_{B_{m,l}(z_0,k_0,c_5)} > 0) > 0. \tag{16}$$

In what follows, we write $B_{m,l}(z_0,k_0,c_5)$ simply as $B_{m,l}$.

For any $l > r$ and any fixed $i \le l-1$, let $m_j = m_j(l,i) := i + jl$, i.e. $(m_j)_{j\ge 0}$ is the class of residues of $i (\mathrm{mod}\ l)$. Now take any $j \in \mathbb{N}$. Observe that for any event $E = \{1_{B_{m_{j-1},l}} = \cdot, \ldots, 1_{B_{m_0,l}} = \cdot\}$ and $x \in \mathcal{H}_{m_j}$,

$$\begin{aligned} &P_\omega(\{X_{T_{m_j}} = x\} \cap E \cap B_{m_j,l}) \\ &\quad \le P_\omega(\{X_{T_{m_j}} = x\} \cap E) P^{x+z_0}_\omega (D > T_{l-r}, H_{0,l-r} \le c_5). \end{aligned} \tag{17}$$

Moreover, for any $x \in \mathcal{H}_{m_j}$, there exists a countable set S of finite paths $(x_i)_{i=0}^N$ that satisfy $m_j + r \le x_i \cdot e_1 \le m_j + l$ and $\#\{k \le N : x_k \in \mathcal{H}_i(x_0)\} \le c_5(i+1)^2$ for $0 \le i \le N$, such that

$$\begin{aligned}&\{X_0 = x + z_0, D > T_{l-r}, H_{0,l-r} \le c_5\}\\ &= \bigcup_{(x_i)_{i=0}^N \in S} \{X_i = x_i \text{ for } 0 \le i \le N\}.\end{aligned}$$

Noting that (by the same argument as in (11)) for any $(x_i)_{i=0}^N \in S$,

$$\sum_{\substack{y: y\cdot e_1 \le m_j \\ i \le N}} e^{-\gamma d(y,x_i)} \le Ce^{-\gamma r},$$

by Lemma 5 we have

$$\begin{aligned}&E_P[P_\omega^{x+z_0}(D > T_{l-r}, H_{0,l-r} \le c_5) | \omega_y : y \cdot e_1 \le m_j]\\ &\le \exp(Ce^{-\gamma r}) \mathbf{P}(D > T_{l-r}, H_{0,l-r} \le c_5).\end{aligned}$$

Thus for $j \ge 0$ and $l > r$,

$$\begin{aligned}&\mathbf{P}(E \cap B_{m_j,l})\\ &\overset{(17)}{\le} \sum_{x \in \mathcal{H}_{m_j}} E_P\big[P_\omega(\{X_{T_{m_j}} = x\} \cap E) P_\omega^{x+z_0}(D > T_{l-r}, H_{0,l-r} \le c_5)\big]\\ &\le \exp(Ce^{-\gamma r}) \sum_{x \in \mathcal{H}_{m_j}} \mathbf{P}(\{X_{T_{m_j}} = x\} \cap E) \mathbf{P}(D > T_{l-r}, H_{0,l-r} \le c_5)\\ &= C\mathbf{P}(E)\mathbf{P}(D > T_{l-r}, H_{0,l-r} \le c_5).\end{aligned}$$

Hence, for any $j \ge 0$ and $l > r$,

$$\mathbf{P}(1_{B_{m_j,l}} = 1 | 1_{B_{m_{j-1},l}}, \dots, 1_{B_{m_0,l}}) \le C\mathbf{P}(D > T_{l-r}, H_{0,l-r} \le c_5),$$

which implies that $\mathbf{P}$-almost surely,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=0}^{n-1} 1_{B_{m_j,l}} \le C\mathbf{P}(D > T_{l-r}, H_{0,l-r} \le c_5). \tag{18}$$

Therefore, $\mathbf{P}$-almost surely,

$$\begin{aligned}\overline{\lim_{l\to\infty}} \overline{\lim_{M\to\infty}} \frac{1}{M+1} \sum_{m=0}^{M} 1_{B_{m,l}} &\le \overline{\lim_{l\to\infty}} \frac{1}{l} \sum_{i=0}^{l-1} \overline{\lim_{M\to\infty}} \frac{l}{M+1} \sum_{\substack{0 \le m \le M \\ m \bmod l = i}} 1_{B_{m,l}}\\ &\overset{(18)}{\le} \lim_{l\to\infty} C\mathbf{P}(D > T_{l-r}, H_{0,l-r} \le c_5)\\ &= C\mathbf{P}(D = \infty, \sum_{i=0}^{\infty} N_i/(i+1)^2 \le c_5).\end{aligned}$$

This and (16) yield $\mathbf{P}(D = \infty, \sum_{i=0}^{\infty} N_i/(i+1)^2 \le c_5) > 0$. The theorem follows. □

3 The conditional law of large numbers

In this section we will prove the conditional law of large numbers (2), using regeneration times and coupling. Given the dependence structure of the environment, we want to define regeneration times in such a way that what happens after a regeneration time has little dependence on the past. To this end, we will use the "ϵ-coins" trick introduced in [2] and the stopping time R to define the regeneration times. Intuitively, at a regeneration time, the past and the future movements have nice properties. That is, the walker has walked straight for a while without paying attention to the environment, and his future movements have little dependence on his past movements.

We define the ϵ-coins $(\epsilon_{i,x})_{i\in\mathbb{N},x\in\mathbb{Z}^d} =: \epsilon$ to be iid random variables with distribution Q such that

$$Q(\epsilon_{i,x} = 1) = d\kappa \text{ and } Q(\epsilon_{i,x} = 0) = 1 - d\kappa.$$

For fixed ω, ϵ, $P^x_{\omega,\epsilon}$ is the law of the Markov chain (X_n) such that $X_0 = x$ and that for any $e \in \mathbb{Z}^d$ such that $|e| = 1$,

$$P^x_{\omega,\epsilon}(X_{n+1} = z + e|X_n = z) = \frac{1_{\epsilon_{n,z}=1}}{2d} + \frac{1_{\epsilon_{n,z}=0}}{1 - d\kappa}[\omega(z, z + e) - \frac{\kappa}{2}].$$

Note that the law of $X_\cdot$ under $\bar{P}^x_\omega = Q \otimes P^x_{\omega,\epsilon}$ coincides with its law under P^x_ω. Sometimes we also refer to $P^x_{\omega,\epsilon}(\cdot)$ as a measure on the sets of paths, without indicating the specific random path. Denote by $\bar{\mathbf{P}} = P \otimes Q \otimes P^o_{\omega,\epsilon}$ the law of the triple $(\omega, \epsilon, X_\cdot)$.

Now we define the regeneration times in the direction e_1. Let L be a fixed number which is sufficiently large. Set $R_0 = 0$. Define inductively for $k \geq 0$:

$$\begin{aligned} S_{k+1} = \inf\{n \geq R_k : X_{n-L} \cdot e_1 > \max\{X_m \cdot e_1 : m < n - L\}, \\ \epsilon_{n-i,X_{n-i}} = 1, X_{n-i+1} - X_{n-i} = e_1 \text{ for all } 1 \leq i \leq L\}, \\ R_{k+1} = R \circ \theta_{S_{k+1}} + S_{k+1}, \end{aligned}$$

where θ_n denotes the time shift of the path, i.e., $\theta_n X = (X_{n+i})_{i=0}^\infty$.

Let

$$K = \inf\{k \geq 1 : S_k < \infty, R_k = \infty\}$$

and $\tau_1 = \tau_1(e_1, \epsilon, X_\cdot) := S_K$. For $k \geq 1$, the (L-)regeneration times are defined inductively by

$$\tau_{k+1} = \tau_1 \circ \theta_{\tau_k} + \tau_k.$$

By similar argument as in [2, Lemma 2.2], we can show:

Lemma 6. *If* $\mathbf{P}(\lim_{n\to\infty} X_n \cdot e_1/n = 0) < 1$, *then*

$$\mathbf{P}(A_{e_1} \cup A_{-e_1}) = 1. \tag{19}$$

Moreover, on A_{e_1}, τ_i*'s are* $\bar{\mathbf{P}}$*-almost surely finite.*

Proof: If $\mathbf{P}(\lim_{n\to\infty} X_n \cdot e_1/n = 0) < 1$, then

$$\mathbf{P}(\overline{\lim_{n\to\infty}} X_n \cdot e_1/n > 0) > 0 \quad \text{or} \quad \mathbf{P}(\overline{\lim_{n\to\infty}} X_n \cdot (-e_1)/n > 0) > 0.$$

Without loss of generality, assume that

$$\mathbf{P}(\overline{\lim_{n\to\infty}} X_n \cdot e_1/n > 0) > 0.$$

It then follows from Theorem 4 that $\mathbf{P}(R = \infty) > 0$. We want to show that $R_k = \infty$ for all but finitely many k's.

For $k \geq 0$,

$$\begin{aligned}
&\bar{\mathbf{P}}(R_{k+1} < \infty) \\
&= \bar{\mathbf{P}}(S_{k+1} < \infty, R \circ \theta_{S_{k+1}} < \infty) \\
&= \sum_{n,x} \bar{\mathbf{P}}(S_{k+1} = n, X_n = x, R \circ \theta_n < \infty) \\
&= \sum_{n,x} E_{P\otimes Q}\big[P_{\omega,\epsilon}(S_{k+1} = n, X_n = x) P^x_{\omega,\theta^n\epsilon}(R < \infty)\big],
\end{aligned}$$

where $\theta^n\epsilon$ denotes the time shift of the coins ϵ, i.e. $(\theta^n\epsilon)_{i,x} = \epsilon_{n+i,x}$. Note that $P_{\omega,\epsilon}(S_{k+1} = n, X_n = x)$ and $P^x_{\omega,\theta^n\epsilon}(R < \infty)$ are independent under the measure Q, since the former is a function of ϵ's before time n, and the latter involves ϵ's after time n. It then follows by induction that

$$\begin{aligned}
&\bar{\mathbf{P}}(R_{k+1} < \infty) \\
&= \sum_{n,x} E_P\big[\bar{P}_\omega(S_{k+1} = n, X_n = x)\bar{P}^x_\omega(R < \infty)\big] \\
&= \sum_{n,x} E_P\big[\bar{P}_\omega(S_{k+1} = n, X_n = x) E_P[\bar{P}^x_\omega(R < \infty)|\omega_y : y \cdot e_1 \leq x \cdot e_1 - L]\big] \\
&\overset{(11),\text{Lemma 5}}{\leq} \bar{\mathbf{P}}(R_k < \infty) \exp(e^{-cL}) \bar{\mathbf{P}}(R < \infty) \\
&\leq [\exp(e^{-cL}) \bar{\mathbf{P}}(R < \infty)]^{k+1},
\end{aligned}$$

where we used in the second equality the fact that $\bar{P}_\omega(S_{k+1} = n, X_n = x)$ is $\sigma(\omega_y : y \cdot e_1 \leq x \cdot e_1 - L)$-measurable. Hence, by taking L sufficiently large and by the Borel-Cantelli Lemma, $\bar{\mathbf{P}}$-almost surely, $R_k = \infty$ except for finitely many values of k.

Let $\mathcal{O}_{e_1}$ denote the event that the signs of $X_n \cdot e_1$ change infinitely many often. It is easily seen that (by the ellipticity of the environment)

$$\mathbf{P}(\mathcal{O}_{e_1} \cup A_{e_1} \cup A_{-e_1}) = 1$$

and

$$\mathcal{O}_{e_1} \subset \{\sup_n X_n \cdot e_1 = \infty\}.$$

However, on $\{\sup_n X_n \cdot e_1 = \infty\}$, given that R_k is finite, S_{k+1} is also finite. Hence τ_1 is $\bar{\mathbf{P}}$-almost surely finite on $\{\sup_n X_n \cdot e_1 = \infty\}$, and so are the regeneration times $\tau_2, \tau_3 \ldots$. Therefore,

$$\mathbf{P}(\mathcal{O}_{e_1}) = \bar{\mathbf{P}}(\mathcal{O}_{e_1} \cap \{\tau_1 < \infty\}).$$

Since $\mathcal{O}_{e_1} \cap \{\tau_1 < \infty\} = \emptyset$, we get $\mathbf{P}(\mathcal{O}_{e_1}) = 0$. This gives (19). □

When $\mathbf{P}(R = \infty) > 0$, we let

$$\hat{\mathbf{P}}(\cdot) := \bar{\mathbf{P}}(\cdot | R = \infty).$$

The following proposition is a consequence of Lemma 5.

Proposition 7. *Assume $\mathbf{P}(R = \infty) > 0$. Let $l > r$ and $\Lambda \subset \{x : x \cdot e_1 < -r\}$. Then for any $A \subset \Lambda \cap \{x : x \cdot e_1 < -l\}$ and $k \in \mathbb{N}$,*

$$\exp(-Ce^{-\gamma l}) \leq \frac{E_P\big[\bar{P}_\omega\big((X_i)_{i=0}^{\tau_k} \in \cdot, R = \infty\big) | \omega_y : y \in \Lambda \setminus A\big]}{E_P\big[\bar{P}_\omega\big((X_i)_{i=0}^{\tau_k} \in \cdot, R = \infty\big) | \omega_y : y \in \Lambda\big]} \leq \exp(Ce^{-\gamma l}). \tag{20}$$

Furthermore, for any $k \in \mathbb{N}$ and $n \geq 0$, $\hat{\mathbf{P}}$-almost surely,

$$\exp(-e^{-cL}) \leq \frac{\hat{\mathbf{P}}\big((X_{\tau_n+i} - X_{\tau_n})_{i=0}^{\tau_{n+k}-\tau_n} \in \cdot | X_{\tau_n}\big)}{\hat{\mathbf{P}}\big((X_i)_{i=0}^{\tau_k} \in \cdot\big)} \leq \exp(e^{-cL}). \tag{21}$$

Proof: First, we shall prove (20). By the definition of the regeneration times, for any finite path $x_\cdot = (x_i)_{i=0}^N, N < \infty$, there exists an event $G_{x_\cdot} \in \sigma(\epsilon_{i,X_i}, X_i : i \leq N)$ such that $G_{x_\cdot} \subset \{R > N\}$ and

$$\{(X_i)_{i=0}^{\tau_k} = (x_i)_{i=0}^N, R = \infty\} = G_{x_\cdot} \cap \{R \circ \theta_N = \infty\}.$$

(For example, when $k = 1$, we let

$$G_{x_\cdot} = \bigcup_{j=1}^{\infty} \{(X_i)_{i=0}^N = (x_i)_{i=0}^N, S_j = N, R > N\}.$$

Then $\{(X_i)_{i=0}^{\tau_1} = (x_i)_{i=0}^N, R = \infty\} = G_{x_\cdot} \cap \{R \circ \theta_N = \infty\}$.)

For $n \in \mathbb{N}$, we let

$$E_n := G_{x_\cdot} \cap \{R \circ \theta_N \geq n\}.$$

Note that $E_n \in \sigma(\epsilon_{i,X_i}, X_i : i \leq N + n)$ can be interpreted (in the sense of (12)) as a set of paths with lengths $\leq N + n$. Also note that $E_n \subset \{R > N + n\}$. Then by Lemma 5 and (11), we have

$$\exp(-Ce^{-\gamma l}) \leq \frac{E_P\big[\bar{P}_\omega\big(E_n\big) | \omega_y : y \in \Lambda \setminus A\big]}{E_P\big[\bar{P}_\omega\big(E_n\big) | \omega_y : y \in \Lambda\big]} \leq \exp(Ce^{-\gamma l}).$$

(20) follows by letting $n \to \infty$.

Next, we shall prove (21). Let $x \in \mathbb{Z}^d$ be any point that satisfies

$$\bar{\mathbf{P}}(X_{\tau_n} = x) > 0.$$

By the definition of the regeneration times, for any $m \in \mathbb{N}$, there exists an event $G_m^x \in \sigma\{\epsilon_{i,X_i}, X_i : i \le m\}$ such that $\bar{P}_\omega(G_m^x)$ is $\sigma(\omega_y : y \cdot e_1 \le x \cdot e_1 - L)$-measurable, and

$$\{\tau_n = m, X_m = x, R = \infty\} = G_m^x \cap \{R \circ \theta_m = \infty\}.$$

Thus

$$\begin{aligned}
&\bar{\mathbf{P}}\big((X_{\tau_n+i} - X_{\tau_n})_{i=0}^{\tau_{n+k}-\tau_n} \in \cdot, X_{\tau_n} = x, R = \infty\big) \\
&= \sum_m \bar{\mathbf{P}}\big((X_{\tau_n+i} - X_{\tau_n})_{i=0}^{\tau_{n+k}-\tau_n} \in \cdot, \tau_n = m, X_m = x, R = \infty\big) \\
&= \sum_m E_P\big[\bar{P}_\omega(G_m^x)\bar{P}_\omega^x((X_i - x)_{i=0}^{\tau_k} \in \cdot, R = \infty)\big] \\
&\overset{(20)}{\le} \exp(Ce^{-\gamma L}) \sum_m \bar{\mathbf{P}}(G_m^x)\bar{\mathbf{P}}\big((X_i)_{i=0}^{\tau_k} \in \cdot, R = \infty\big). \qquad (22)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bar{\mathbf{P}}(X_{\tau_n} = x, R = \infty) &= \sum_m E_P[\bar{P}_\omega(G_m^x)\bar{P}_\omega^x(R = \infty)] \\
&\overset{(20)}{\ge} \exp(-Ce^{-\gamma L}) \sum_m \bar{\mathbf{P}}(G_m^x)\bar{\mathbf{P}}(R = \infty). \qquad (23)
\end{aligned}$$

By (22) and (23), we have (note that L is sufficiently big)

$$\hat{\mathbf{P}}\big((X_{\tau_n+i} - X_{\tau_n})_{i=0}^{\tau_{n+k}-\tau_n} \in \cdot | X_{\tau_n} = x\big) \le \exp(e^{-cL})\hat{\mathbf{P}}\big((X_i)_{i=0}^{\tau_k} \in \cdot\big).$$

The right side of (21) is proved. The left side of (21) follows likewise. □

The next lemma describes the dependency of a regeneration on its remote past. It is a version of Lemma 2.2 in [3]. (The denominator is omitted in the last equality in [3, page 101], which is corrected here, see the equality in (25).)

Set $\tau_0 = 0$. Denote the truncated path between τ_{n-1} and $\tau_n - L$ by

$$P_n = (P_n^i)_{0 \le i \le \tau_n - \tau_{n-1} - L} := (X_{i+\tau_{n-1}} - X_{\tau_{n-1}})_{0 \le i \le \tau_n - \tau_{n-1} - L}.$$

Set

$$\begin{aligned}
W_n &= (\omega_{x+X_{\tau_{n-1}}})_{x \in P_n} =: \omega_{X_{\tau_{n-1}}+P_n}, \\
F_n &= X_{\tau_n} - X_{\tau_{n-1}}, \\
J_n &= (P_n, W_n, F_n, \tau_n - \tau_{n-1}).
\end{aligned}$$

For $i \geq 0$, let $h_{i+1}(\cdot|j_i, \dots, j_1) := \hat{\mathbf{P}}(J_{i+1} \in \cdot|J_i, \dots, J_1)|_{J_i=j_i,\dots,J_1=j_1}$ denote the transition kernel of (J_n). Note that when $i = 0$, $h_{i+1}(\cdot|j_i, \dots, j_1) = h_1(\cdot|\emptyset) = \hat{\mathbf{P}}(J_1 \in \cdot)$.

Lemma 8. *Assume $\mathbf{P}(R = \infty) > 0$, $0 \leq k \leq n$. Then $\hat{\mathbf{P}}$-almost surely,*

$$\exp\left(-e^{-c(k+1)L}\right) \leq \frac{h_{n+1}(\cdot|J_n, \dots, J_1)}{h_{k+1}(\cdot|J_n, \dots, J_{n-k+1})} \leq \exp\left(e^{-c(k+1)L}\right). \tag{24}$$

Proof: For $j_m = (p_m, w_m, f_m, t_m), m = 1, \dots n$, let

$$\begin{aligned} &\bar{x}_m := f_1 + \cdots + f_m, \\ &\bar{t}_m := t_1 + \cdots + t_m, \\ &B_{p_1,\dots,p_m} := \{R = \infty, P_i = p_i \text{ for all } i = 1, \dots, m\}, \\ \text{and} \quad &\omega_{p_1,\dots,p_m} := (\omega_{\bar{x}_{i-1}+p_i})_{i=1}^m. \end{aligned}$$

First, we will show that for any $1 \leq k \leq n$,

$$h_{k+1}(\cdot|j_k, \dots, j_1) = \frac{E_P\left[\bar{P}_\omega^{\bar{x}_k}(J_1 \in \cdot, R = \infty)|\omega_{p_1,\dots,p_k}\right]}{E_P\left[\bar{P}_\omega^{\bar{x}_k}(R = \infty)|\omega_{p_1,\dots,p_k}\right]}\Big|_{\omega_{p_1,\dots,p_k}=(w_i)_{i=1}^k}. \tag{25}$$

By the definition of the regeneration times, there exists an event

$$G_{p_1,\dots,p_k} \in \sigma(X_{i+1}, \epsilon_{i,X_i}, 0 \leq i \leq \bar{t}_k - 1)$$

such that

$$B_{p_1,\dots,p_k} = G_{p_1,\dots,p_k} \cap \{R \circ \theta_{\bar{t}_k} = \infty\}. \tag{26}$$

On the one hand, for any $\sigma(J_k, \dots, J_1)$-measurable function $g(J_k, \dots, J_1)$,

$$\begin{aligned} &E_{\bar{\mathbf{P}}}\left[h_{k+1}(\cdot|J_k, \dots, J_1)g(J_k, \dots, J_1)1_{B_{p_1,\dots,p_k}}\right] \\ &= E_{\bar{\mathbf{P}}}\left[g1_{B_{p_1,\dots,p_k}}1_{J_{k+1}\in\cdot}\right] \\ &= E_P[g1_{B_{p_1,\dots,p_k}}\bar{P}_\omega(J_{k+1} \in \cdot, B_{p_1,\dots,p_k})] \\ &\overset{(26)}{=} E_P\left[g1_{B_{p_1,\dots,p_k}}\bar{P}_\omega(G_{p_1,\dots,p_k})\bar{P}_\omega^{\bar{x}_k}(J_1 \in \cdot, R = \infty)\right]. \end{aligned} \tag{27}$$

On the other hand, we also have

$$\begin{aligned} &E_{\bar{\mathbf{P}}}\left[h_{k+1}(\cdot|J_k, \dots, J_1)g(J_k, \dots, J_1)1_{B_{p_1,\dots,p_k}}\right] \\ &= E_P\left[h_{k+1}(\cdot|J_k, \dots, J_1)g1_{B_{p_1,\dots,p_k}}\bar{P}_\omega(B_{p_1,\dots,p_k})\right] \\ &\overset{(26)}{=} E_P\left[h_{k+1}(\cdot|J_k, \dots, J_1)g1_{B_{p_1,\dots,p_k}}\bar{P}_\omega(G_{p_1,\dots,p_k})\bar{P}_\omega^{\bar{x}_k}(R = \infty)\right]. \end{aligned} \tag{28}$$

Comparing (27) and (28) and observing that on $B_{p_1,\dots,p_k}$, $\bar{P}_\omega(G_{p_1,\dots,p_k})$ and all functions of $J_1, \dots, J_k$ are $\sigma(\omega_y : y \in \bar{x}_{i-1} + p_i, i \leq k)$-measurable , we obtain that on $B_{p_1,\dots,p_k}$, P-almost surely,

$$h_{k+1}(\cdot|J_k, \dots, J_1) = \frac{E_P\left[\bar{P}_\omega^{\bar{x}_k}(J_1 \in \cdot, R = \infty)|\omega_{\bar{x}_{i-1}+p_i}, i \leq k\right]}{E_P\left[\bar{P}_\omega^{\bar{x}_k}(R = \infty)|\omega_{\bar{x}_{i-1}+p_i}, i \leq k\right]}.$$

Noting that

$$B_{p_1,\dots,p_k} \cap \{\omega_{p_1,\dots,p_k} = (w_i)_{i=1}^k\} = \{J_i = j_i, 1 \le i \le k\},$$

(25) is proved.

Next, we will prove the lower bound in (24).

When $n \ge k \ge 1$, by formula (25) and (20), we have

$$\begin{aligned}
&h_{n+1}(\cdot|j_n,\dots,j_1)\\
&= \frac{E_P[\bar{P}_\omega^{\bar{x}_n}(J_1 \in \cdot, R=\infty)|\omega_{p_1,\dots,p_n}]}{E_P\big[\bar{P}_\omega^{\bar{x}_n}(R=\infty)|\omega_{p_1,\dots,p_n}\big]}\Bigg|_{\omega_{p_1,\dots,p_n}=(w_i)_{i=0}^n}\\
&\le \frac{\exp(Ce^{-\gamma(k+1)L})E_P[\bar{P}_\omega^{\bar{x}_n}(J_1 \in \cdot, R=\infty)|\omega_{\bar{x}_{i-1}+p_i}, n-k+1 \le i \le n]}{\exp(-Ce^{-\gamma(k+1)L})E_P[\bar{P}_\omega^{\bar{x}_n}(R=\infty)|\omega_{\bar{x}_{i-1}+p_i}, n-k+1 \le i \le n]}\\
&\qquad \Big|_{\omega_{p_1,\dots,p_n}=(w_i)_{i=0}^n}\\
&= \exp(2Ce^{-\gamma(k+1)L})\frac{E_P[\bar{P}_\omega^{\bar{x}_n-\bar{x}_{n-k}}(J_1 \in \cdot, R=\infty)|\omega_{p_{n-k+1},\dots,p_n}]}{E_P[\bar{P}_\omega^{\bar{x}_n-\bar{x}_{n-k}}(R=\infty)|\omega_{p_{n-k+1},\dots,p_n}]}\\
&\qquad \Big|_{\omega_{p_{n-k+1},\dots,p_n}=(w_i)_{i=n-k+1}^n}\\
&\overset{(25)}{=} \exp(2Ce^{-\gamma(k+1)L})h_{k+1}(\cdot|j_n,\dots,j_{n-k+1}),
\end{aligned} \tag{29}$$

where we used the translation invariance of the measure P in the last but one equality.

When $k = 0$ and $n \ge 1$, by formula (25) and (20),

$$\begin{aligned}
h_{n+1}(\cdot|j_n,\dots,j_1) &\le \frac{\exp(Ce^{-\gamma L})E_P[\bar{P}_\omega^{\bar{x}_n}(J_1 \in \cdot, R=\infty)]}{\exp(-Ce^{-\gamma L})E_P[\bar{P}_\omega^{\bar{x}_n}(R=\infty)]}\\
&= \exp(2Ce^{-\gamma L})\hat{\mathbf{P}}(J_1 \in \cdot)\\
&= \exp(2Ce^{-\gamma L})h_1(\cdot|\emptyset).
\end{aligned} \tag{30}$$

When $k = n = 0$, (24) is trivial. Hence combining (29) and (30), the lower bound in (24) follows as we take L sufficiently big. The upper bound follows likewise. □

Lemma 9. *Suppose that a sequence of non-negative random variables (X_n) satisfies*

$$a \le \frac{\mathrm{d}P(X_{n+1} \in \cdot|X_1,\dots,X_n)}{\mathrm{d}\mu} \le b$$

for all $n \ge 1$, where $a \le 1 \le b$ are constants and μ is a probability measure. Let $m_\mu \le \infty$ be the mean of μ. Then almost surely,

$$am_\mu \le \varliminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i \le \varlimsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i \le bm_\mu. \tag{31}$$

Before giving the proof, let us recall the "splitting representation" of random variables:

Proposition 10. *[9, Page 94] Let ν and μ be probability measures. Let X be a random variable with law ν. If for some $a \in (0,1)$,*

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \geq a,$$

then, enlarging the probability space if necessary, we can find independent random variables Δ, π, Z such that

i) Δ is Bernoulli with parameter $1-a$, i.e., $P(\Delta = 1) = 1-a$, $P(\Delta = 0) = a$;

ii) π is of law μ, and Z is of law $(\nu - a\mu)/(1-a)$;

iii) $X = (1-\Delta)\pi + \Delta Z$.

Proof of Lemma 9:
By Proposition 10, enlarging the probability space if necessary, there are random variables $\Delta_i, \pi_i, Z_i, i \geq 1$, such that for any $i \in \mathbb{N}$,

- Δ_i is Bernoulli with parameter $(1-a)$, and π_i is of law μ;
- Δ_i, π_i and Z_i are mutually independent;
- (Δ_i, π_i) is independent of $\sigma(\Delta_k, \pi_k, Z_k : k < i)$;
- $X_i = (1-\Delta_i)\pi_i + \Delta_i Z_i$.

Note that since X_i's are supported on $[0,\infty)$, $\pi_i \geq 0$ and $Z_i \geq 0$ for all $i \in \mathbb{N}$. Thus by the law of large numbers, almost surely,

$$\varliminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \geq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(1-\Delta_i)\pi_i = a m_\mu.$$

This proves the first inequality of (31).

If $m_\mu = \infty$, the last inequality of (31) is trivial. Assume that $m_\mu < \infty$. Let $(\tilde{\Delta}_i)_{i\geq 1}$ be an iid Bernoulli sequence with parameter $1 - b^{-1}$ such that every $\tilde{\Delta}_i$ is independent of all the X_n's. By a similar splitting procedure, we can construct non-negative random variables $\tilde{\pi}_i, \tilde{Z}_i, i \geq 1$, such that $(\tilde{\pi}_i)_{i\geq 1}$ are iid with law μ, and

$$\tilde{\pi}_i = (1-\tilde{\Delta}_i)X_i + \tilde{\Delta}_i \tilde{Z}_i.$$

Let $Y_i = (1 - b^{-1} - \tilde{\Delta}_i)X_i 1_{X_i \leq i}$, we will first show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = 0. \tag{32}$$

By Kronecker's Lemma, it suffices to show that

$$\sum_{i=1}^{\infty} \frac{Y_i}{i} \text{ converges.}$$

Observe that $(\sum_{i=1}^{n} Y_i/i)_{n\in\mathbb{N}}$ is a martingale sequence. Moreover, for all $n \in \mathbb{N}$,

$$\begin{aligned} E\big(\sum_{i=1}^{n} \frac{Y_i}{i}\big)^2 = \sum_{i=1}^{n} EY_i^2/i^2 &\le \sum_{i=1}^{\infty} EX_i^2 1_{X_i\le i}/i^2 \\ &\le b\sum_{i=1}^{\infty} E\tilde{\pi}_i^2 1_{\tilde{\pi}_i\le i}/i^2 \\ &= b\int_0^{\infty} x^2(\sum_{i\ge x}\frac{1}{i^2})\,\mathrm{d}\mu \\ &\le C\int_0^{\infty} x\,\mathrm{d}\mu = Cm_\mu < \infty. \end{aligned}$$

By the L^2-martingale convergence theorem, $\sum Y_i/i$ converges a.s. and in L^2. This proves (32).

Since

$$\sum_i P(Y_i \ne (1-b^{-1}-\tilde{\Delta}_i)X_i) \le \sum_i P(X_i > i) \le b\sum_i P(\pi_1 > i) \le bm_\mu < \infty,$$

by the Borel-Cantelli lemma, it follows from (32) that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(1-b^{-1}-\tilde{\Delta}_i)X_i = 0, \text{a.s..}$$

Hence almost surely,

$$m_\mu = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\tilde{\pi}_i \ge \overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=1}^{n}(1-\tilde{\Delta}_i)X_i = \overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=1}^{n} b^{-1}X_i.$$

The last inequality of (31) is proved. □

Theorem 11. *There exist two deterministic numbers $v_{e_1}, v_{-e_1} \ge 0$ such that $\mathbf{P}$-almost surely,*

$$\lim_{n\to\infty}\frac{X_n\cdot e_1}{n} = v_{e_1}1_{A_{e_1}} - v_{-e_1}1_{A_{-e_1}}. \tag{33}$$

Moreover, if $v_{e_1} > 0$, then $E_{\hat{\mathbf{P}}}\tau_1 < \infty$ and $\mathbf{P}(A_{e_1}\cup A_{-e_1}) = 1$.

Proof: We only consider the nontrivial case that $\mathbf{P}(\lim X_n \cdot e_1/n = 0) < 1$, which by Lemma 6 implies $\mathbf{P}(A_{e_1} \cup A_{-e_1}) = 1$. Without loss of generality, assume $\mathbf{P}(\overline{\lim}_{n\to\infty} X_n \cdot e_1/n > 0) > 0$. We will show that on A_{e_1},

$$\lim_{n\to\infty} X_n \cdot e_1/n = v_{e_1} > 0, \ \mathbf{P}\text{-a.s.}.$$

By (21) and Lemma 9, we obtain that $\mathbf{P}(\cdot|A_{e_1})$-almost surely,

$$\begin{aligned}\exp(-e^{-cL})E_{\hat{\mathbf{P}}}X_{\tau_1} \cdot e_1 &\le \varliminf_{n\to\infty} \frac{X_{\tau_n} \cdot e_1}{n} \\ &\le \varlimsup_{n\to\infty} \frac{X_{\tau_n} \cdot e_1}{n} \le \exp(e^{-cL})E_{\hat{\mathbf{P}}}X_{\tau_1} \cdot e_1, \qquad (34)\\ \exp(-e^{-cL})E_{\hat{\mathbf{P}}}\tau_1 &\le \varliminf_{n\to\infty} \frac{\tau_n}{n} \le \varlimsup_{n\to\infty} \frac{\tau_n}{n} \le \exp(e^{-cL})E_{\hat{\mathbf{P}}}\tau_1. \qquad (35)\end{aligned}$$

Note that (34), (35) hold even if $E_{\hat{\mathbf{P}}}X_{\tau_1} \cdot e_1 = \infty$ or $E_{\hat{\mathbf{P}}}\tau_1 = \infty$. But it will be shown later that under our assumption, both of them are finite.

We claim that

$$E_{\hat{\mathbf{P}}}X_{\tau_1} \cdot e_1 < \infty. \qquad (36)$$

To see this, let $\Theta := \{i : X_{\tau_k} \cdot e_1 = i \text{ for some } k \in \mathbb{N}\}$. Since τ_i's are finite on A_{e_1}, there exist (recall that $\tau_0 = 0$) a sequence $(k_n)_{n\in\mathbb{N}}$ such that $X_{\tau_{k_n}} \cdot e_1 \le n < X_{\tau_{k_n+1}} \cdot e_1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} k_n = \infty$. Hence for $n \ge 1$,

$$\frac{\sum_{i=1}^n 1_{i\in\Theta}}{n} \le \frac{k_n + 1}{X_{\tau_{k_n}} \cdot e_1}, \qquad \hat{\mathbf{P}}\text{-a.s.}.$$

Then, $\hat{\mathbf{P}}$-a.s.,

$$\varlimsup_{n\to\infty} \frac{\sum_{i=1}^n 1_{i\in\Theta}}{n} \le \varlimsup_{n\to\infty} \frac{n}{X_{\tau_n} \cdot e_1}.$$

Let $B_k = \{\epsilon_{k,X_k} = 0, X_{k+1} - X_k = e_1, \epsilon_{k+i,X_{k+i}} = 1, X_{k+i+1} - X_{k+i} = e_1 \text{ for all } 1 \le i \le L\}$. Then

$$\bar{P}_\omega(B_k) \ge (d\kappa)^L(1 - d\kappa)(\frac{\kappa}{2})(\frac{1}{2d})^L \overset{1\ge 2d\kappa}{>} (\frac{\kappa}{2})^{L+2}.$$

Observe that by the definition of the regeneration times, for $n > L + 1$,

$$\begin{aligned}&\{T_{n-L-1} = k, X_k = x - (L+1)e_1, R > k\} \cap B_k \cap \{R \circ \theta_{k+L+1} = \infty\} \\ &\subset \{R = \infty, n \in \Theta, T_n = k + L + 1, X_{T_n} = x\}.\end{aligned}$$

Hence for $n > L + 1$,

$$\begin{aligned}&\hat{\mathbf{P}}(n \in \Theta) \\ &\ge \sum_{k\in\mathbb{N}, x\in\mathcal{H}_n} \hat{\mathbf{P}}(B_k \cap \{T_{n-L-1} = k, X_k = x - (L+1)e_1, R \circ \theta_{k+L+1} = \infty\}) \\ &\ge \sum_{k\in\mathbb{N}, x\in\mathcal{H}_n} E_P\big[P_\omega\big(T_{n-L-1} = k, X_k = x - (L+1)e_1, R > k\big)(\frac{\kappa}{2})^{L+2} \\ &\qquad\qquad \times P_\omega^x(R = \infty)\big]/\mathbf{P}(R = \infty).\end{aligned}$$

Since by (20) and the translation invariance of P,

$$E_P\big[P^x_\omega(R=\infty)|\omega_y : y\cdot e_1 \le x\cdot e_1 - L - 1\big] \ge \exp(-e^{-cL})\mathbf{P}(R=\infty),$$

we have for $n > L+1$,

$$\begin{aligned}
&\hat{\mathbf{P}}(n\in\Theta)\\
&\ge (\frac{\kappa}{2})^{L+2}\exp(-e^{-cL})\sum_{k\in\mathbb{N},x\in\mathcal{H}_n}\mathbf{P}(T_{n-L-1}=k, X_k = x-(L+1)e_1, R>k)\\
&\ge (\frac{\kappa}{2})^{L+2}e^{-1}\mathbf{P}(R=\infty). \qquad (37)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{C}{E_{\hat{\mathbf{P}}}X_{\tau_1}\cdot e_1} &\overset{(34)}{\ge} E_{\hat{\mathbf{P}}}\varlimsup_{n\to\infty}\frac{n}{X_{\tau_n}\cdot e_1} \ge E_{\hat{\mathbf{P}}}\varlimsup_{n\to\infty}\frac{\sum_{i=1}^n 1_{i\in\Theta}}{n}\\
&\ge \varlimsup_{n\to\infty} E_{\hat{\mathbf{P}}}\frac{\sum_{i=1}^n 1_{i\in\Theta}}{n}\\
&\overset{(37)}{\ge} (\frac{\kappa}{2})^{L+2}e^{-1}\mathbf{P}(R=\infty) > 0.
\end{aligned}$$

This gives (36).

Now we can prove the theorem. By (34) and (35),

$$\begin{aligned}
\exp(-2e^{-cL})\frac{E_{\hat{\mathbf{P}}}X_{\tau_1}\cdot e_1}{E_{\hat{\mathbf{P}}}\tau_1} &\le \varliminf_{n\to\infty}\frac{X_{\tau_n}\cdot e_1}{\tau_{n+1}}\\
&\le \varlimsup_{n\to\infty}\frac{X_{\tau_{n+1}}\cdot e_1}{\tau_n} \le \exp(2e^{-cL})\frac{E_{\hat{\mathbf{P}}}X_{\tau_1}\cdot e_1}{E_{\hat{\mathbf{P}}}\tau_1}, \qquad (38)
\end{aligned}$$

$\mathbf{P}(\cdot|A_{e_1})$-almost surely. Further, by the fact that $|X_i| \le i$ and the obvious inequalities

$$\varliminf_{n\to\infty}\frac{X_{\tau_n}\cdot e_1}{\tau_{n+1}} \le \varliminf_{n\to\infty}\frac{X_n\cdot e_1}{n} \le \varlimsup_{n\to\infty}\frac{X_n\cdot e_1}{n} \le \varlimsup_{n\to\infty}\frac{X_{\tau_{n+1}}\cdot e_1}{\tau_n},$$

we have that

$$\varlimsup_{n\to\infty}\Big|\frac{X_n\cdot e_1}{n} - \frac{E_{\hat{\mathbf{P}}}X_{\tau_1}\cdot e_1}{E_{\hat{\mathbf{P}}}\tau_1}\Big| \le \exp(2e^{-cL}) - 1,\ \mathbf{P}(\cdot|A_{e_1})\text{-a.s.}$$

Therefore, $\mathbf{P}(\cdot|A_{e_1})$-almost surely,

$$\lim_{n\to\infty}\frac{X_n\cdot e_1}{n} = \lim_{L\to\infty}\frac{E_{\hat{\mathbf{P}}}X_{\tau_1^{(L)}}\cdot e_1}{E_{\hat{\mathbf{P}}}\tau_1^{(L)}} := v_{e_1},$$

where τ_1 is written as $\tau_1^{(L)}$ to indicate that it is an L-regeneration time. Moreover, our assumption $\mathbf{P}(\varlimsup_{n\to\infty} X_n\cdot e_1/n > 0) > 0$ implies that $v_{e_1} > 0$ and (by (38))

$$E_{\hat{\mathbf{P}}}\tau_1 < \infty.$$

Our proof is complete. □

If $v_{e_1} > 0$, then it follows by (35) that

$$E_{\hat{\mathbf{P}}}\tau_n \le CnE_{\hat{\mathbf{P}}}\tau_1 < \infty. \tag{39}$$

Observe that although Theorem 11 is stated for e_1, the previous arguments, if properly modified, still work if one replaces e_1 with any $z \in \mathbb{R}^d \backslash \{o\}$. So Theorem 11 is true for the general case. That is, for any $z \neq o$, there exist two deterministic constants $v_z, v_{-z} \ge 0$ such that

$$\lim_{n\to\infty} \frac{X_n \cdot z}{n} = v_z 1_{A_z} - v_{-z} 1_{A_{-z}}$$

and that $\mathbf{P}(A_z \cup A_{-z}) = 1$ if $v_z > 0$. Then, by the same argument as in [5, page 1112], one concludes that the limiting velocity $\lim_{n\to\infty} X_n/n$ can take at most two antipodal values. This proves display (2) of Theorem 2.

4 Heat kernel estimate

The following heat kernel estimates are crucial for the proof of the uniqueness of the non-zero velocity in the next section. Although in the mixing case we don't have iid regeneration slabs, we know that (by Lemma 8) a regeneration slab has little dependence on its remote past. This allows us to use coupling techniques to get the same heat kernel estimates as in [1]:

Theorem 12 (Heat kernel estimate)**.** *Assume $v_{e_1} > 0$. For $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we let*

$$Q(n,x) := \hat{\mathbf{P}}(x \textit{ is visited in } [\tau_{n-1}, \tau_n)).$$

Then, for any $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$,

$$\hat{\mathbf{P}}(X_{\tau_n} = x) \le Cn^{-d/2}, \tag{40}$$

$$\sum_{x\in\mathbb{Z}^d} Q(n,x)^2 \le C(E_{\hat{\mathbf{P}}}\tau_1)^2 n^{-d/2}. \tag{41}$$

By Lemma 8, we have for $n \ge 2$ and $1 \le k \le n-1$, $\hat{\mathbf{P}}$-almost surely,

$$\begin{aligned}\frac{h_{k+1}(\cdot|J_{n-1},\dots,J_{n-k})}{h_k(\cdot|J_{n-1},\dots,J_{n-k+1})} &= \frac{h_{k+1}(\cdot|J_{n-1},\dots,J_{n-k})}{h_n(\cdot|J_{n-1},\dots,J_1)} \frac{h_n(\cdot|J_{n-1},\dots,J_1)}{h_k(\cdot|J_{n-1},\dots,J_{n-k+1})} \\ &\ge \exp(-e^{-c(k+1)L} - e^{-ckL}) \\ &\ge 1 - e^{-ckL}\end{aligned} \tag{42}$$

for large L. Hence for $n \ge 2$ and $1 \le k \le n-1$, we can define a (random) probability measure $\zeta_{n,k}^{J_{n-1},\dots,J_{n-k}}$ that satisfies

$$\begin{aligned}&h_{k+1}(\cdot|J_{n-1},\dots,J_{n-k}) \\ &\quad = e^{-ckL}\zeta_{n,k}^{J_{n-1},\dots,J_{n-k}}(\cdot) + (1-e^{-ckL})h_k(\cdot|J_{n-1},\dots,J_{n-k+1}).\end{aligned} \tag{43}$$

To prove Theorem 12, we will construct in Section 4.1 a sequence of random variables $(\tilde{J}_i, i \in \mathbb{N})$ (in a larger probability space) such that for any $n \in \mathbb{N}$,

$$(\tilde{J}_1, \ldots, \tilde{J}_n) \sim \hat{\mathbf{P}}(J_1 \in \cdot, \ldots, J_n \in \cdot), \tag{44}$$

and $\tilde{J}_i$'s can be decomposed into random variables that have good independence properties (See Remark 13.). Here "$X \sim \mu$" means "X is of law μ".

4.1 Construction of the $\tilde{J}_i$'s

Our construction consists of three steps:
Step 1. We let $\tilde{J}_1, \tilde{J}_{2,1}, \tilde{\Delta}_{2,1}$ be independent random variables such that

$$\tilde{J}_1 \sim h_1(\cdot|\emptyset), \quad \tilde{J}_{2,1} \sim h_1(\cdot|\emptyset)$$

and $\tilde{\Delta}_{2,1}$ is Bernoulli with parameter e^{-cL}. Let $\tilde{Z}_{2,1}$ be independent of $\sigma(\tilde{J}_{2,1}, \tilde{\Delta}_{2,1})$ such that

$$P(\tilde{Z}_{2,1} \in \cdot|\tilde{J}_1) = \zeta_{2,1}^{\tilde{J}_1}(\cdot).$$

Setting $\tilde{J}_2 := (1 - \tilde{\Delta}_{2,1})\tilde{J}_{2,1} + \tilde{\Delta}_{2,1}\tilde{Z}_{2,1}$, by (43) we have

$$(\tilde{J}_1, \tilde{J}_2) \sim \hat{\mathbf{P}}(J_1 \in \cdot, J_2 \in \cdot).$$

Step 2. For $n \geq 3$, assume we have constructed $\tilde{J}_1$ and $(\tilde{J}_{i,1}, \tilde{\Delta}_{i,j}, \tilde{Z}_{i,j}, 1 \leq j < i \leq n-1)$ such that

$$(\tilde{J}_1, \ldots, \tilde{J}_{n-1}) \sim \hat{\mathbf{P}}(J_1 \in \cdot, \ldots, J_{n-1} \in \cdot),$$

where for $2 \leq j \leq i \leq n-1$,

$$\tilde{J}_{i,j} := (1 - \tilde{\Delta}_{i,j-1})\tilde{J}_{i,j-1} + \tilde{\Delta}_{i,j-1}\tilde{Z}_{i,j-1}$$

and

$$\tilde{J}_i := \tilde{J}_{i,i}.$$

Then, we define $\tilde{J}_{n,1}$ and $(\tilde{\Delta}_{n,k}, \tilde{Z}_{n,k}, 1 \leq k < n)$ to be random variables such that, *conditioning* on the values of $\tilde{J}_1$ and $(\tilde{J}_{i,1}, \tilde{\Delta}_{i,j}, \tilde{Z}_{i,j}, 1 \leq j < i < n)$,

- $(\tilde{J}_{n,1}, \tilde{\Delta}_{n,k}, \tilde{Z}_{n,k}, 1 \leq k \leq n-1)$ are conditionally independent;
- The conditional distribution of $\tilde{J}_{n,1}$ is $h_1(\cdot|\emptyset)$;
- For $1 \leq k \leq n-1$, the conditional distribution of $\tilde{Z}_{n,k}$ is $\zeta_{n,k}^{\tilde{J}_{n-1},\ldots,\tilde{J}_{n-k}}(\cdot)$, and $\tilde{\Delta}_{n,k}$ is Bernoulli with parameter e^{-ckL}.

Step 3. For $2 \le k \le n$, set

$$\tilde{J}_{n,k} := (1 - \tilde{\Delta}_{n,k-1})\tilde{J}_{n,k-1} + \tilde{\Delta}_{n,k-1}\tilde{Z}_{n,k-1},$$
$$\tilde{J}_n := \tilde{J}_{n,n}.$$

Then (by (43)) almost surely,

$$P(\tilde{J}_{n,k} \in \cdot | \tilde{J}_{n-1}, \ldots, \tilde{J}_1) = h_k(\cdot | \tilde{J}_{n-1}, \ldots, \tilde{J}_{n-k+1}). \tag{45}$$

It follows immediately that

$$(\tilde{J}_1, \ldots, \tilde{J}_n) \sim \hat{\mathbf{P}}(J_1 \in \cdot, \ldots, J_n \in \cdot).$$

Therefore, by induction, we have constructed $(\tilde{J}_i, i \in \mathbb{N})$ such that (44) holds for all $n \in \mathbb{N}$.

In what follows, with abuse of notation, we will identify $\tilde{J}_i$ with J_i and simply write $\tilde{J}_{i,j}, \tilde{\Delta}_{i,j}, \tilde{Z}_{i,j}$ as $J_{i,j}, \Delta_{i,j}$ and $Z_{i,j}$, $1 \le j < i$. We still use $\hat{\mathbf{P}}$ to denote the law of the random variables in the enlarged probability space.

Remark 13. *To summarize, we have introduced random variables $J_{i,j}, \Delta_{i,j}, Z_{i,j}$, $1 \le j < i$ such that for any $n \ge 2$,*

$$\begin{aligned}
&J_{n,2} = (1 - \Delta_{n,1})J_{n,1} + \Delta_{n,1}Z_{n,1},\\
&\ldots,\\
&J_{n,n-1} = (1 - \Delta_{n,n-2})J_{n,n-2} + \Delta_{n,n-2}Z_{n,n-2},\\
&J_n = (1 - \Delta_{n,n-1})J_{n,n-1} + \Delta_{n,n-1}Z_{n,n-1}.
\end{aligned}$$

Intuitively, we flip a sequence of "coins" $\Delta_{n,n-1}, \ldots, \Delta_{n,1}$ to determine whether $J_1, \ldots, J_{n-1}$ are in the "memory" of J_n. For instance, if

$$\Delta_{n,n-1} = \cdots = \Delta_{n,n-i} = 0,$$

then $J_n = J_{n,n-i}$ doesn't "remember" $J_1, \ldots, J_i$ (in the sense that

$$\hat{\mathbf{P}}(J_{n,n-i} \in \cdot | J_{n-1}, \ldots, J_1) = h_{n-i}(\cdot | J_{n-1}, \ldots, J_{i+1}).$$

See (45)*.).*

4.2 Proof of Theorem 12

For $1 < i \le n$, let $I_n(i)$ be the event that $\Delta_{i,i-1} = \ldots = \Delta_{i,1} = 0$ and $\Delta_{m,m-1} = \ldots = \Delta_{m,m-i} = 0$ for all $i < m \le n$. Let $I_n(1)$ be the event that $\Delta_{m,m-1} = 0$ for all $m = 2, \ldots, n$. Set $J_{1,1} = J_1$. Then on $I_n(i)$,

$$J_i = J_{i,1} \text{ and } J_m = J_{m,m-i} \text{ for all } i < m \le n. \tag{46}$$

Lemma 14. *For $n \geq 2$, let H be a nonempty subset of $\{1, \ldots, n\}$, and set*

$$M_n := \{1 \leq i \leq n : \text{the event } I_n(i) \text{ happens}\}.$$

Conditioning on the event $\{M_n = H\}$, the sequence $(J_i)_{i \in H}$ is iid $\sim \hat{\mathbf{P}}(J_1 \in \cdot)$ and independent of $(J_i)_{i \in \{1,\ldots,n\} \setminus H}$.

Proof of Lemma 14: It follows from our construction that for any $i \geq 1$, $J_{i,1}$ is independent of

$$\sigma(\Delta_{k,j}, 1 \leq j < k) \vee \sigma(J_l, 1 \leq l < i) \vee \sigma(J_{m,m-i}, m > i).$$

Since M_n is measurable with respect to $\sigma(\Delta_{k,j}, 1 \leq j < k)$, we get that $J_{i,1}$ is independent of

$$\sigma(M_n) \vee \sigma(J_l, 1 \leq l < i) \vee \sigma(J_{m,m-i}, m > i).$$

Hence, by (46), for any $i \in H$ and any appropriate measurable sets $(V_j)_{1 \leq j \leq n}$,

$$\begin{aligned} &\hat{\mathbf{P}}(J_j \in V_j, 1 \leq j \leq n | M_n = H) \\ &= \hat{\mathbf{P}}(J_{i,1} \in V_i) \hat{\mathbf{P}}(J_j \in V_j, 1 \leq j \leq n, j \neq i | M_n = H). \end{aligned}$$

Noting that $\hat{\mathbf{P}}(J_{i,1} \in V_i) = \hat{\mathbf{P}}(J_1 \in V_i)$ and by induction, we get

$$\begin{aligned} &\hat{\mathbf{P}}(J_j \in V_j, 1 \leq j \leq n | M_n = H) \\ &= \prod_{i \in H} \hat{\mathbf{P}}(J_{i,1} \in V_i) \hat{\mathbf{P}}(J_1 \in V_j, 1 \leq j \leq n, j \notin H | M_n = H). \end{aligned}$$

The lemma is proved. □

Proof of Theorem 12: By Lemma 14, for $i \in H \subset \{1, \ldots, n\}$,

$$\hat{\mathbf{P}}\big(X_{\tau_i} - X_{\tau_{i-1}} = (L+1)e_1 \pm e_j | M_n = H\big) = \hat{\mathbf{P}}(X_{\tau_1} = (L+1)e_1 \pm e_j) > 0$$

for all $j \in \{1, \ldots, d\}$, where the last inequality is due to ellipticity. Hence arguing as in [1, pages 736, 737], using Lemma 14 and the heat kernel estimate for bounded iid random walks in $\mathbb{Z}^d$, we get that for any $x \in \mathbb{Z}^d$,

$$\hat{\mathbf{P}}(\sum_{i \in H} X_{\tau_i} - X_{\tau_{i-1}} = x | M_n = H) \leq C|H|^{-d/2},$$

where $|H|$ is the cardinality of H. Hence, for any subset $H \subset \{1, \ldots, n\}$

such that $|H| \geq n/2$,

$$
\begin{aligned}
&\hat{\mathbf{P}}(X_{\tau_n} = x | M_n = H) \\
&= \sum_y \hat{\mathbf{P}}\big(\sum_{i\in H} X_{\tau_i} - X_{\tau_{i-1}} = x - y, \sum_{i\in\{1,\dots,n\}\setminus H} X_{\tau_i} - X_{\tau_{i-1}} = y | M_n = H\big) \\
&= \sum_y \Bigg[\hat{\mathbf{P}}\big(\sum_{i\in H} X_{\tau_i} - X_{\tau_{i-1}} = x - y | M_n = H\big) \\
&\qquad\qquad \times \hat{\mathbf{P}}\big(\sum_{i\in\{1,\dots,n\}\setminus H} X_{\tau_i} - X_{\tau_{i-1}} = y | M_n = H\big)\Bigg] \\
&\leq Cn^{-d/2}, \qquad (47)
\end{aligned}
$$

where we used Lemma 14 in the second equality.

On the other hand,

$$
\begin{aligned}
|M_n| &\geq n - \Bigg(\sum_{i=2}^n 1_{\Delta_{i,i-1}+\cdots+\Delta_{i,1}>0} + \sum_{i=1}^n \sum_{m=i+1}^n 1_{\Delta_{m,m-1}+\cdots+\Delta_{m,m-i}>0}\Bigg) \\
&= n - \sum_{i=2}^n 1_{\Delta_{i,i-1}+\cdots+\Delta_{i,1}>0} - \sum_{m=2}^n \sum_{i=1}^{m-1} 1_{\Delta_{m,m-1}+\cdots+\Delta_{m,m-i}>0} \\
&\geq n - 2\sum_{m=2}^n K_m,
\end{aligned}
$$

where $K_m := \sup\{1 \leq j < m : \Delta_{m,j} = 1\}$. Here we follow the convention that $\sup\emptyset = 0$. Since K_m's are independent, and for $m \geq 2$,

$$
\begin{aligned}
E_{\hat{\mathbf{P}}} e^{K_m} &= \sum_{j=0}^{m-1} e^j \hat{\mathbf{P}}(K_m = j) \\
&\leq \sum_{j=1}^{m-1} e^j \hat{\mathbf{P}}(\Delta_{m,j} = 1) + 1 \\
&\leq \sum_{j=1}^{\infty} e^j e^{-cjL} + 1 \to 1 \text{ as } L \to \infty,
\end{aligned}
$$

we take L to be large enough such that $E_{\hat{\mathbf{P}}} e^{K_m} \leq e^{1/8}$ for all $m \geq 2$ and so

$$
\begin{aligned}
\hat{\mathbf{P}}(|M_n| < n/2) &\leq \hat{\mathbf{P}}(K_2 + \cdots + K_n > n/4) \\
&\leq e^{-n/4} E_{\hat{\mathbf{P}}} e^{K_2+\cdots+K_n} \leq e^{-n/8}. \qquad (48)
\end{aligned}
$$

By (47) and (48), inequality (40) follows immediately.

Furthermore, since

$$
\begin{aligned}
&Q(n,x)\\
&= \sum_y \hat{\mathbf{P}}(X_{\tau_{n-1}} = y)\hat{\mathbf{P}}(x \text{ is visited in } [\tau_{n-1},\tau_n)|X_{\tau_{n-1}} = y)\\
&\overset{\text{Lemma 8}}{\leq} C\sum_y \hat{\mathbf{P}}(X_{\tau_{n-1}} = y)\hat{\mathbf{P}}((x-y) \text{ is visited during } [0,\tau_1)),
\end{aligned}
$$

by Hölder's inequality we have

$$
\begin{aligned}
&Q(n,x)^2\\
&\leq C\big[\sum_y \hat{\mathbf{P}}\big((x-y) \text{ is visited during } [0,\tau_1)\big)\big]\\
&\qquad\qquad \times \big[\sum_y \hat{\mathbf{P}}(X_{\tau_{n-1}} = y)^2\hat{\mathbf{P}}\big((x-y) \text{ is visited during } [0,\tau_1)\big)\big]\\
&\leq CE_{\hat{\mathbf{P}}}\tau_1 \sum_y \hat{\mathbf{P}}(X_{\tau_{n-1}} = y)^2\hat{\mathbf{P}}\big((x-y) \text{ is visited during } [0,\tau_1)\big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_x Q(n,x)^2\\
&\leq CE_{\hat{\mathbf{P}}}\tau_1 \sum_y \big[\hat{\mathbf{P}}(X_{\tau_{n-1}} = y)^2 \sum_x \hat{\mathbf{P}}\big((x-y) \text{ is visited during } [0,\tau_1)\big)\big]\\
&\leq C(E_{\hat{\mathbf{P}}}\tau_1)^2 \sum_y \hat{\mathbf{P}}(X_{\tau_{n-1}} = y)^2\\
&\overset{(40)}{\leq} C(E_{\hat{\mathbf{P}}}\tau_1)^2 n^{-d/2} \sum_y \hat{\mathbf{P}}(X_{\tau_{n-1}} = y) = C(E_{\hat{\mathbf{P}}}\tau_1)^2 n^{-d/2}.
\end{aligned}
$$

Theorem 12 is proved. □

5 The uniqueness of the non-zero velocity

In this section we will show that in high dimension ($d \geq 5$), there exists at most one non-zero velocity. The idea is the following. Consider two random walk paths: one starts at the origin, the other starts near the n-th regeneration position of the first path. By Levy's martingale convergence theorem, the second path is "more and more transient" as n grows (Lemma 16). On the other hand, by heat kernel estimates, when $d \geq 5$, two ballistic walks in opposite directions will grow further and further apart from each other (see Lemma 15), thus they are almost independent. This contradicts the previous fact that starting at the n-th regeneration point of the first path

will prevent the second path from being transient in the opposite direction.

Set $\delta = \delta(d) := \frac{d-4}{8(d-1)}$ (the reason of choosing this notation will become clear in (56).). For any finite path $y_\cdot = (y_i)_{i=0}^M, M < \infty$, define $A(y_\cdot, z)$ to be the set of paths $(x_i)_{i=0}^N, N \le \infty$ that satisfy

1) $x_0 = y_0 + z$;

2) $d(x_i, y_j) > (i \vee j)^\delta$ if $i \vee j > |z|/3$.

The motivation for the definition of $A(y_\cdot, z)$ is as follows. Note that for two paths $x_\cdot = (x_i)_{i=0}^N$ and $y_\cdot = (y_i)_{i=0}^M$ with $x_0 = y_0 + z$, if $i \vee j \le |z|/3$, then

$$d(x_i, y_j) \ge d(x_0, y_0) - d(x_0, x_i) - d(y_0, y_j) \ge |z| - i - j \ge |z|/3.$$

Hence, for $(x_i)_{i=0}^N \in A(y_\cdot, z)$,

$$\begin{aligned}\sum_{i \le N, j \le M} e^{-\gamma d(x_i, y_j)} &\le \sum_{0 \le i,j \le |z|/3} e^{-\gamma |z|/3} + \sum_{i \vee j > |z|/3} e^{-\gamma (i^\delta + j^\delta)/2} \\ &\le (\frac{|z|}{3})^2 e^{-\gamma |z|/3} + (\sum_{i=0}^\infty e^{-\gamma i^\delta/2})^2 < C. \end{aligned} \tag{49}$$

This gives us (by $(*)$) an estimate of the interdependence between $\sigma\big(\omega_x : x \in (x_i)_{i=0}^N\big)$ and $\sigma\big(\omega_x : x \in (y_i)_{i=0}^M\big)$.

In what follows, we use

$$\tau'_\cdot = \tau_\cdot(-e_1, \epsilon, X_\cdot)$$

to denote the regeneration times in the $-e_1$ direction. Assume that there are two opposite nonzero limiting velocities in directions e_1 and $-e_1$, i.e.,

$$v_{e_1} \cdot v_{-e_1} > 0.$$

We let $\check{\mathbf{P}}(\cdot) := \mathbf{P}(\cdot | R_{-e_1} = \infty)$.

Lemma 15. *Assume that there are two nonzero limiting velocities in direction e_1. We sample $(\epsilon, \tilde{X}_\cdot)$ according to $\hat{\mathbf{P}}$ and let $\tilde{\tau}_\cdot = \tau_\cdot(e_1, \epsilon, \tilde{X}_\cdot)$ denote its regeneration times. For $n \ge 1$, we let*

$$Y_\cdot^n = (Y_i^n)_{i=0}^{\tilde{\tau}_n} := (\tilde{X}_{\tilde{\tau}_n - i})_{i=0}^{\tilde{\tau}_n}$$

be the reversed path of $(\tilde{X}_i)_{i=0}^{\tilde{\tau}_n}$. If $|z|$ is large enough, $d \ge 5$ and $n \ge 1$, then

$$E_{\hat{\mathbf{P}}} \check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n} + z}\big(X_\cdot \in A(Y_\cdot^n, z)\big) > C > 0. \tag{50}$$

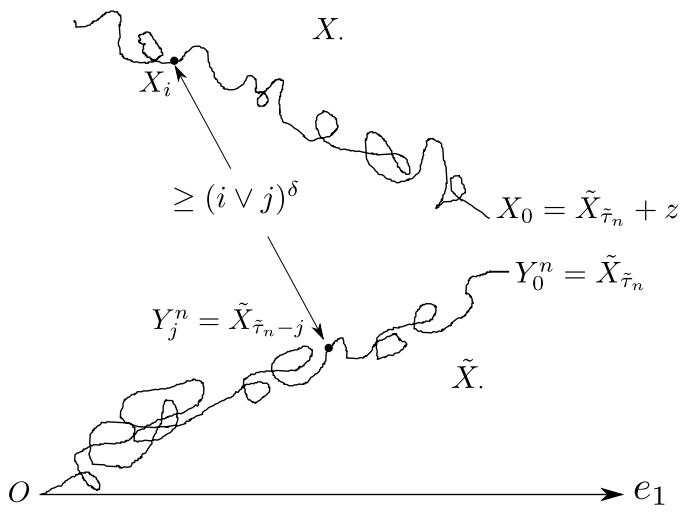

Figure 2: $X_. \in A(Y_.^n, z)$. When $i \vee j > |z|/3$, the distance between Y_j^n of the "backward path" and X_i is at least $(i \vee j)^\delta$.

Proof: Let

$$m_z := \lfloor |z|^{1/2} \rfloor.$$

Then

$$\begin{aligned}
&E_{\hat{\mathbf{P}}} \check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}\big(X_. \notin A(Y_.^n, z)\big) \\
&\leq E_{\hat{\mathbf{P}}} \check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(\tau'_{m_z} \geq |z|/3) + \hat{\mathbf{P}}(\tilde{\tau}_n - \tilde{\tau}_{n-m_z} \geq |z|/3) && (51) \\
&\quad + E_{\hat{\mathbf{P}}} \check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(d(X_i, Y_.^n) \leq i^\delta \text{ for some } i > \tau'_{m_z}) && (52) \\
&\quad + E_{\hat{\mathbf{P}}} \check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(d(\tilde{X}_{\tilde{\tau}_n - j}, X_.) \leq j^\delta \text{ for some } j > \tilde{\tau}_n - \tilde{\tau}_{n-m_z}). && (53)
\end{aligned}$$

We will first estimate (51). By the translation invariance of the environment measure,

$$\check{\mathbf{P}}^x(\tau'_{m_z} \geq |z|/3) = \check{\mathbf{P}}(\tau'_{m_z} \geq |z|/3) \text{ for any } x \in \mathbb{Z}^d.$$

Hence

$$E_{\hat{\mathbf{P}}} \check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(\tau'_{m_z} \geq |z|/3) = \check{\mathbf{P}}(\tau'_{m_z} \geq |z|/3) \leq \frac{3E_{\check{\mathbf{P}}}\tau'_{m_z}}{|z|} \overset{(39)}{\leq} C(E_{\check{\mathbf{P}}}\tau'_1)|z|^{-1/2}. \tag{54}$$

Similarly,

$$\hat{\mathbf{P}}(\tilde{\tau}_n - \tilde{\tau}_{n-m_z} \geq |z|/3) \overset{(21)}{\leq} \exp(e^{-cL})\hat{\mathbf{P}}(\tau_{m_z} \geq |z|/3) \leq C(E_{\hat{\mathbf{P}}}\tau_1)|z|^{-1/2}. \tag{55}$$

To estimate (52) and (53), for $i \geq 1, n \geq j \geq 1$, we let

$$Q'(i,x) = \check{\mathbf{P}}(x \text{ is visited in } [\tau'_{i-1}, \tau'_i)),$$
$$\tilde{Q}(j,x) = \hat{\mathbf{P}}(X_{\tau_n} + x \text{ is visited in} [\tau_{n-j}, \tau_{n-j+1})).$$

Note that by arguments that are similar to the proof of Theorem 12, one can also obtain the heat kernel estimate (40) for $Q'(i,x)$ and $\tilde{Q}(j,x)$. For $l > 0$, let $B(o,l) = \{x \in \mathbb{Z}^d : d(o,x) \leq l\}$. Recall the definition of the r-boundary in Definition 1. By the translation invariance of the environment measure,

$$\check{\mathbf{P}}^y(X_i = y + z) = \check{\mathbf{P}}(X_i = z) \text{ for any } y, z \in \mathbb{Z}^d \text{and } i \in \mathbb{N}.$$

Hence

$$\begin{aligned}
&E_{\hat{\mathbf{P}}}\check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(d(X_i, \tilde{X}_\cdot) \leq i^\delta \text{ for some } i > \tau'_{m_z}) \\
&\leq \sum_{i \geq m_z} \sum_{y \in \partial_1 B(o,i^\delta)} \sum_x E_{\hat{\mathbf{P}}}\big[\check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(\tilde{X}_{\tilde{\tau}_n} + z + x \text{ is visited in } [\tau'_i, \tau'_{i+1})) \\
&\qquad\qquad \times 1_{\tilde{X}_{\tilde{\tau}_n}+z+x+y \in Y^n_\cdot}\big] \\
&= \sum_{i \geq m_z} \sum_{y \in \partial_1 B(o,i^\delta)} \sum_x \check{\mathbf{P}}(x \text{ is visited in } [\tau'_i, \tau'_{i+1}))\hat{\mathbf{P}}(\tilde{X}_{\tilde{\tau}_n} + z + x + y \in Y^n_\cdot) \\
&= \sum_{i \geq m_z} \sum_{y \in \partial_1 B(o,i^\delta)} \sum_{j \leq n} \sum_x Q'(i,x)\tilde{Q}(j, x+z+y).
\end{aligned}$$

By the heat kernel estimates and Hölder's inequality,

$$\begin{aligned}
\sum_{j \leq n} \sum_x Q'(i,x)\tilde{Q}(j,x+z+y) &\leq \sqrt{\sum_x Q'(i,x)^2} \sum_{j \leq n} \sqrt{\sum_x \tilde{Q}(j,x+y)^2} \\
&\leq C(E_{\check{\mathbf{P}}}\tau'_1) i^{-d/4} \sum_{j \leq n} (E_{\hat{\mathbf{P}}}\tau_1) j^{-d/4} \\
&\overset{d \geq 5}{\leq} C i^{-d/4} E_{\check{\mathbf{P}}}\tau'_1 E_{\hat{\mathbf{P}}}\tau_1.
\end{aligned}$$

Thus

$$\begin{aligned}
&E_{\hat{\mathbf{P}}}\check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(d(X_i, \tilde{X}_\cdot) \leq i^\delta \text{ for some } i > \tau'_{m_z}) \\
&\leq C \sum_{i \geq m_z} \sum_{y \in \partial_1 B(o,i^\delta)} i^{-d/4} E_{\check{\mathbf{P}}}\tau'_1 E_{\hat{\mathbf{P}}}\tau_1 \\
&\leq C \sum_{i \geq m_z} i^{(d-1)\delta} i^{-d/4} E_{\check{\mathbf{P}}}\tau'_1 E_{\hat{\mathbf{P}}}\tau_1 \leq C|z|^{-(d-4)/8} E_{\check{\mathbf{P}}}\tau'_1 E_{\hat{\mathbf{P}}}\tau_1, \qquad (56)
\end{aligned}$$

where we used $d \geq 5$ and $\delta = \frac{d-4}{8(d-1)}$ in the last inequality. Similarly, we have

$$\begin{aligned}
&E_{\hat{\mathbf{P}}}\check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}(d(\tilde{X}_{\tilde{\tau}_n - j}, X_\cdot) \leq j^\delta \text{ for some } j > \tilde{\tau}_n - \tilde{\tau}_{n-m_z}) \\
&\leq C|z|^{-(d-4)/8} E_{\check{\mathbf{P}}}\tau'_1 E_{\hat{\mathbf{P}}}\tau_1. \qquad (57)
\end{aligned}$$

Combining (54), (55), (56) and (57), we conclude that

$$E_{\hat{\mathbf{P}}}\check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z}\big(X_\cdot \in A(Y_\cdot^n, z)\big) > C > 0,$$

if $|z|$ is large enough and $d \geq 5$. $\square$

Let

$$T^o = \inf\{i \geq 0 : X_i \cdot e_1 < 0\}.$$

For every fixed $\omega \in \Omega$ and $P^o_{\omega,\epsilon}$-almost every $X_\cdot$,

$$P^{X_n}_{\omega,\theta^n\epsilon}(T^o = \infty)1_{T^o>n} = P^o_{\omega,\epsilon}(T^o = \infty | X_1, \ldots, X_n),$$

and so by Levy's martingale convergence theorem,

$$\lim_{n\to\infty} P^{X_n}_{\omega,\theta^n\epsilon}(T^o = \infty)1_{T^o>n} = 1_{T^o=\infty}, \quad P^o_{\omega,\epsilon}\text{-almost surely.}$$

Hence, for $(\omega, \epsilon, \tilde{X}_\cdot)$ sampled according to $\hat{\mathbf{P}}$,

$$\lim_{n\to\infty} P^{\tilde{X}_{\tilde{\tau}_n}}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o = \infty) = 1, \quad \hat{\mathbf{P}}\text{-almost surely.}$$

It then follows by the dominated convergence theorem that

$$\lim_{n\to\infty} E_{\hat{\mathbf{P}}} P^{\tilde{X}_{\tilde{\tau}_n}}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o < \infty) = 0. \tag{58}$$

Lemma 16. *For any $z \in \mathbb{Z}^d$,*

$$\lim_{n\to\infty} E_{\hat{\mathbf{P}}} P^{\tilde{X}_{\tilde{\tau}_n}+z}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o < \infty) = 0. \tag{59}$$

Proof: For $n > |z|$, obviously

$$(\tilde{X}_{\tilde{\tau}_n} + z) \cdot e_1 > 0.$$

This together with ellipticity yields

$$P^{\tilde{X}_{\tilde{\tau}_n}}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o < \infty) \geq (\frac{\kappa}{2})^{|z|} P^{\tilde{X}_{\tilde{\tau}_n}+z}_{\omega,\theta^{\tilde{\tau}_n+|z|}\epsilon}(T^o < \infty).$$

Hence using (58),

$$\lim_{n\to\infty} E_{\hat{\mathbf{P}}} P^{\tilde{X}_{\tilde{\tau}_n}+z}_{\omega,\theta^{\tilde{\tau}_n+|z|}\epsilon}(T^o < \infty) = 0.$$

On the other hand, noting that $\{R > \tau_1\} = \{R = \infty\}$,

$$\begin{aligned}
&E_{\hat{\mathbf{P}}} P^{\tilde{X}_{\tilde{\tau}_n}+z}_{\omega,\theta^{\tilde{\tau}_n+|z|}\epsilon}(T^o < \infty) \\
&= \sum_{m,x} E_{P\otimes Q}[P^{x+z}_{\omega,\theta^{m+|z|}\epsilon}(T^o < \infty) P^o_{\omega,\epsilon}(R > \tau_1, \tau_n = m, X_m = x)]/\mathbf{P}(R = \infty) \\
&= \sum_{m,x} E_{P\otimes Q}[P^{x+z}_{\omega,\theta^m\epsilon}(T^o < \infty) P^o_{\omega,\epsilon}(R > \tau_1, \tau_n = m, X_m = x)]/\mathbf{P}(R = \infty) \\
&= E_{\hat{\mathbf{P}}} P^{\tilde{X}_{\tilde{\tau}_n}+z}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o < \infty),
\end{aligned}$$

where we used the independence (under Q) of $P^{x+z}_{\omega,\theta^m\epsilon}(T^o<\infty)$ and $P^o_{\omega,\epsilon}(R>\tau_1,\tau_n=m,X_m=x)$ in the second to last equality. The conclusion follows. $\square$

Proof of the uniqueness of the non-zero velocity when $d\geq 5$, as stated in Theorem 2: If the two antipodal velocities are both non-zero, we assume that

$$v_{e_1}\cdot v_{-e_1}>0.$$

Sample $(\omega,\epsilon_\cdot,\tilde{X}_\cdot)$ according to $\hat{\mathbf{P}}$. Henceforth, we take $z=z_0$ such that (50) holds and

$$z_0\cdot e_1<-L.$$

We will prove Theorem 2 by showing that

$$E_{\hat{\mathbf{P}}}P^{\tilde{X}_{\tilde{\tau}_n}+z_0}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o<\infty)>C \tag{60}$$

for all $n>|z_0|$, which contradicts with (59).

First, let $\mathcal{G}$ denote the set of finite paths $y_\cdot=(y_i)_{i=0}^M$ that satisfy $y_M=0, M<\infty$. Then

$$\begin{aligned}
&E_{\hat{\mathbf{P}}}P^{\tilde{X}_{\tilde{\tau}_n}+z_0}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}(T^o<\infty) \qquad (61)\\
&\geq E_{\hat{\mathbf{P}}}P^{\tilde{X}_{\tilde{\tau}_n}+z_0}_{\omega,\theta^{\tilde{\tau}_n}\epsilon}\big((X_i)_{i=0}^{T^o}\in A(Y^n_\cdot,z_0),T^o<\infty\big)\\
&=\sum_{y_\cdot=(y_i)_{i=0}^M\in\mathcal{G}}E_{\hat{\mathbf{P}}}[P^{y_0+z_0}_{\omega,\theta^M\epsilon}\big((X_i)_{i=0}^{T^o}\in A(y_\cdot,z_0),T^o<\infty\big)1_{Y^n_\cdot=y_\cdot}]\\
&=\frac{1}{\mathbf{P}(R=\infty)}\sum_{y_\cdot\in\mathcal{G}}\sum_{\substack{N<\infty\\(x_i)_{i=0}^N\in A(y_\cdot,z_0)}}E_{P\otimes Q}[P^{y_0+z_0}_{\omega,\theta^M\epsilon}\big((X_i)_{i=0}^{T^o}=x_\cdot\big)P_{\omega,\epsilon}(Y^n_\cdot=y_\cdot)].
\end{aligned}$$

By the definition of the regeneration times, for any finite path $y_\cdot=(y_i)_{i=0}^M$, there exists an event $G_{y_\cdot}$ such that $P_{\omega,\epsilon}(G_{y_\cdot})$ is $\sigma(\epsilon_{i,y_i},\omega_{y_j}:0\leq i\leq M,0\leq j\leq M-L)$-measurable and

$$\{Y^n_\cdot=y_\cdot\}=\{(\tilde{X}_i)_{i=0}^{\tilde{\tau}_n}=(y_{M-j})_{j=0}^M\}=G_{y_\cdot}\cap\{R\circ\theta_M=\infty\}.$$

Hence, for and any $y_\cdot=(y_i)_{i=0}^M\in\mathcal{G}$ and $x_\cdot=(x_i)_{i=0}^N\in A(y_\cdot,z_0)$, $N<\infty$,

$$\begin{aligned}
&E_{P\otimes Q}[P^{y_0+z_0}_{\omega,\theta^M\epsilon}\big((X_i)_{i=0}^{T^o}=x_\cdot,R_{-e_1}>N\big)P_{\omega,\epsilon}(Y^n_\cdot=y_\cdot)]\\
&=E_P[\bar{P}^{y_0+z_0}_\omega\big((X_i)_{i=0}^{T^o}=x_\cdot,R_{-e_1}>N\big)\bar{P}_\omega(G_{y_\cdot})\bar{P}^{y_0}_\omega(R=\infty)]\\
&\overset{(20)}{\geq}CE_P[\bar{P}^{y_0+z_0}_\omega\big((X_i)_{i=0}^{T^o}=x_\cdot,R_{-e_1}>N\big)\bar{P}_\omega(G_{y_\cdot})]\bar{\mathbf{P}}(R=\infty). \qquad (62)
\end{aligned}$$

where we used in the equality that $(\epsilon_{i,x})_{i\geq 0,x\in\mathbb{Z}^d}$ are iid and in the inequality the fact that

$$\bar{P}^{y_0+z_0}_\omega\big((X_i)_{i=0}^{T^o}=x_\cdot,R_{-e_1}>N\big)\bar{P}_\omega(G_{y_\cdot})$$

is $\sigma(\omega_v : v \cdot e_1 \le y_0 \cdot e_1 - L)$-measurable (note that $z_0 \cdot e_1 < -L$). Further, by Lemma 5 and (49), we have

$$\begin{aligned} &E_P[\bar{P}_\omega^{y_0+z_0}\big((X_i)_{i=0}^{T^o} = x., R_{-e_1} > N\big)\bar{P}_\omega(G_{y.})] \\ &\ge C\bar{\mathbf{P}}^{y_0+z_0}\big((X_i)_{i=0}^{T^o} = x., R_{-e_1} > N\big)\bar{\mathbf{P}}(G_{y.}). \end{aligned} \tag{63}$$

Note that

$$\begin{aligned} \bar{\mathbf{P}}(G_{y.})\bar{\mathbf{P}}(R=\infty) &\overset{(20)}{\ge} CE_P[\bar{P}_\omega(G_{y.})\bar{P}_\omega^{y_0}(R=\infty)] \\ &= C\bar{\mathbf{P}}(Y_.^n = y.) \ge C\hat{\mathbf{P}}(Y_.^n = y.). \end{aligned} \tag{64}$$

Therefore, by (61), (62) and (63),

$$\begin{aligned} &E_{\hat{\mathbf{P}}}P_{\omega,\theta^{\tilde{\tau}_n}\epsilon}^{\tilde{X}_{\tilde{\tau}_n}+z_0}(T^o < \infty) \\ &\ge C\sum_{y.\in\mathcal{G}} \sum_{\substack{N<\infty \\ (x_i)_{i=0}^N \in A(y.,z_0)}} \bar{\mathbf{P}}^{y_0+z_0}\big((X_i)_{i=0}^{T^o} = x., R_{-e_1} > N\big)\bar{\mathbf{P}}(G_{y.})\bar{\mathbf{P}}(R=\infty) \\ &\overset{(64)}{\ge} C\sum_{y.\in\mathcal{G}} \sum_{\substack{N<\infty \\ (x_i)_{i=0}^N \in A(y.,z_0)}} \bar{\mathbf{P}}^{y_0+z_0}\big((X_i)_{i=0}^{T^o} = x., R_{-e_1} > N\big)\hat{\mathbf{P}}(Y_.^n = y.) \\ &\ge CE_{\hat{\mathbf{P}}}\check{\mathbf{P}}^{\tilde{X}_{\tilde{\tau}_n}+z_0}(X_. \in A(Y_.^n, z_0)) \overset{\text{Lemma 15}}{>} C. \end{aligned}$$

Inequality (60) is proved. □

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