# A Berry-Esseen bound with applications to vertex degree counts in the Erdős-Rényi random graph

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#### Abstract

Applying Stein's method, an inductive technique and size bias coupling yields a Berry-Esseen theorem for normal approximation without the usual restriction that the coupling be bounded. The theorem is applied to counting the number of vertices in the Erdős-Rényi random graph of a given degree.

## 1 Introduction

We present a new Berry-Esseen theorem for sums Y of dependent variables by combining Stein's method, size bias couplings, and the inductive technique of Bolthausen (1984) originally developed for the combinatorial central limit theorem. We apply the theorem to asses the accuracy of the normal approximation to the distribution of the number of vertices of degree d in the classical Erdős-Rényi (1959) random graph  $G_n$  having n vertices connected by independent edges with common success probability depending on n and a parameter  $\theta$ . Over the range of parameters considered the theorem yields a bound that is the same up to constants as the one obtained earlier by Barbour et al. (1989) for the weaker smooth function metric (19).

Stein's method (Stein (1972), Stein (1986)) often proceeds by coupling a random variable Y of interest to a related variable Y', using, for example, the method of exchangeable pairs, size bias couplings, or zero bias couplings; for an overview see Chen et al. (2010). The chief innovation here is the removal of an inconvenient restriction present in a number of results that provide Kolmogorov distance bounds using Stein's method, that the difference |Y - Y'| between Y and the coupled Y' be bounded almost surely by a constant. Through the use of an unbounded coupling, in Theorem 2.1 we are able to extend the previous work by Kordecki (1990) on the number of isolated, or degree zero, vertices of  $G_n$  to all positive degrees.

To describe Theorem 1.1, our general result, recall that for a nonnegative random variable Y with finite, nonzero mean  $\mu$  we say that  $Y^s$  has the Y-size bias distribution if

$$E[Yf(Y)] = \mu E[f(Y^s)]$$
 for all functions f for which these expectations exist. (1)

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In employing the size bias version of Stein's method (see Baldi et al. (1989), Goldstein and Rinott (1996) and Chen et al. (2010)), the goal is to construct, on the same space as Y, a variable  $Y^s$  with the Y-size bias distribution such that Y and  $Y^s$  are close is some sense. Previous applications of the size bias coupling technique for obtaining Berry-Esseen bounds by Stein's method, requiring that  $|Y^s - Y|$  be bounded, include Goldstein (2005), Goldstein and Penrose (2010) and Goldstein and Zhang (2011).

Let  $\mathbb{N} = \{1, 2, \ldots\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Our abstract framework consists of random elements indexed by  $n \geq n_0$  for some  $n_0 \in \mathbb{N}_0$  whose distributions  $\mathcal{L}_{\theta}(\cdot)$  depend on n, left implicit when clear from context, and a parameter  $\theta$  in a topological space  $\Theta_n$ . In our particular application the parameter  $\theta$  lies in a subset  $\Theta_n$  of the real numbers  $\mathbb{R}$  and interest centers on the distributions of the nonnegative random variables  $Y_n$  counting the number of degree  $d \in \mathbb{N}$  vertices of the Erdős-Rényi random graph  $G_n$ .

In general, when  $Y_n$  is a given sum of nontrivial exchangeable indicators, Lemma 1.1 below says, essentially, that to construct a variable  $Y_n^s$  with the  $Y_n$ -size bias distribution, one chooses an indicator uniformly and sets it to one if it was not so already, and then 'adjusts' the remaining indicators, if necessary, to have their original distribution given that the selected indicator is one. Applying Lemma 1.1 when  $Y_n$  counts the number of vertices of degree d in  $G_n$  results in the construction of Barbour et al. (1992), where nothing is changed if a uniformly chosen vertex already has degree d, and otherwise edges to the chosen vertex are added if the vertex has degree less than d, or removed if it has degree in excess of d. As it is possible that the chosen vertex has, say, n-1 edges, the resulting coupling fails to be bounded in n. Nevertheless, when there is only a small probability that a very large number of edges will need to be added or removed, the coupling can be controlled using moments on bounds  $K_n$  that satisfy  $|Y_n^s - Y_n| \leq K_n$ .

After coupling, the second ingredient in our method has an inductive flavor. We construct a variable  $V_n$  such that its distribution, conditional on a collection  $J_n$  of random elements, is that of  $Y_n$ , reduced in size by some 'small' amount  $L_n$ , with parameter  $\psi_{n,\theta}$  'close' to the original  $\theta$ . Formally, we require that

$$\mathcal{L}_{\theta}(V_n|J_n) = \mathcal{L}_{\psi_{n,\theta}}(Y_{n-L_n}) \tag{2}$$

hold on an event where the size of  $L_n$  is controlled, and that a bound  $B_n$  on the absolute difference  $|Y_n - V_n|$  not be 'too large.' As bounds to the normal for  $Y_n$  can be expressed in terms of quantities that include bounds to the normal for reduced versions of the same problem, a recursive inequality for the sought after bound can be produced.

In the graph degree problem  $V_n$  counts the number of degree d vertices in the graph obtained by removing a uniformly chosen vertex from  $G_n$ , along with all its incident edges, and the set  $J_n$  consists of the identity of the chosen vertex, and its degree. Conditional on  $J_n$  the graph that remains is an Erdős-Rényi graph on the reduced vertex set, with the same connectivity as before. As with the bound  $K_n$ , it is not required that  $B_n$  be almost surely bounded by a constant; though  $|Y_n - V_n|$  may be large in the graph degree problem, it is unlikely that it will be.

Tension exists in choosing the set  $J_n$  that appears in the conditioning equality (2). In order to reduce the larger problem to a smaller one so that induction may be applied, working conditionally we must be able to treat the bounds  $K_n$  and  $B_n$ , and the parameters of the reduced problem,  $L_n$  and  $\psi_{n,\theta}$ , as constants. Hence we require that these variables be measurable with respect to  $\mathcal{F}_n$ , the  $\sigma$ -algebra generated by the conditioning collection  $J_n$ .

Though this restriction necessitates that  $\mathcal{F}_n$  be large enough to contain, say, information on  $Y_n^s - Y_n$ , it must also be small enough so that, say,  $L_n$  and  $B_n$  are not too large, and that the conditioning 'leaves enough randomness' to yield a useful recursion for the ultimate bound.

At the heart of our main result, and Stein's method for normal approximation, is the characterization that Z is a standard normal random variable if and only if

$$E[Zf(Z)] = E[f'(Z)]$$

for all absolutely continuous functions f for which the above expectations exist. This characterization leads to the Stein equation, when, given a test function h on which to evaluate the difference Eh(W) - Eh(Z) between the expectation of the random variable W of interest and the standard normal Z, one solves

$$f'(w) - wf(w) = h(w) - Eh(Z)$$

for f. Using f, one evaluates this difference by substituting W for w, and takes expectation on the left hand side, rather than the right. Though we focus on manipulation of the Stein equation using the size bias coupling, many variations are possible, see Chen et al. (2010) for an overview.

Throughout, for  $n_0 \in \mathbb{N}$  and all  $n \geq n_0$  and  $\theta \in \Theta_n$  we let  $\mu_{n,\theta} = E_\theta Y_n$  and  $\sigma_{n,\theta}^2 = \operatorname{Var}_\theta(Y_n)$  indicate the mean and variance of  $Y_n$  under  $\mathcal{L}_\theta$ . The value  $r_{n,\theta}$  appearing in Theorem 1.1 is a function that determines the quality of the bound to the normal, while the sequence  $s_{n,\theta}$  is used to control  $L_n$ , and hence the size of the smaller subproblem  $V_n$  related to  $Y_n$ . Without further mention,  $\mu_{n,\theta}$ ,  $\sigma_{n,\theta}^2$  and  $r_{n,\theta}$  are assumed to be measurable in  $\theta \in \Theta_n$ , a condition satisfied for all natural examples, including the one considered here. To avoid repetition, the distribution of random variables indicated after  $\theta \in \Theta_n$  has been fixed is with respect to  $\mathcal{L}_\theta$ . The random variable Z will always denote the standard normal.

To familiarize the reader with the conditions of Theorem 1.1, towards the end of this section we present its application in the simple case where a bounded size bias coupling of  $Y_n^s$  to  $Y_n$  exists.

**Theorem 1.1** For some  $n_0 \in \mathbb{N}_0$  and all  $n \geq n_0$  and  $\theta \in \Theta_n$ , let  $Y_n$  be a nonnegative random variable with mean  $\mu_{n,\theta}$  and positive variance  $\sigma_{n,\theta}^2$ , and set

$$W_{n,\theta} = \frac{Y_n - \mu_{n,\theta}}{\sigma_{n,\theta}},\tag{3}$$

the standardized value of  $Y_n$ . Let  $r_{n,\theta}$  be positive for all  $n \geq n_0$  and  $\theta \in \Theta_n$ , and for all  $r \geq 0$ , letting

$$\Theta_{n,r} = \{ \theta \in \Theta_n : r_{n,\theta} \ge r \},\,$$

assume there exists  $r_1 > 0$  and  $n_1 \ge n_0$  such that

$$\max_{n_0 \le n < n_1} \sup_{\theta \in \Theta_{n,r_1}} r_{n,\theta} < \infty. \tag{4}$$

Further, suppose that for all  $n \ge n_1$  and  $\theta \in \Theta_{n,r_1}$ , there exist random variables  $Y_n^s, K_n, L_n$ ,  $\psi_{n,\theta}, V_n$  and  $B_n$  on the same space as  $Y_n$ , and a  $\sigma$ -algebra  $\mathcal{F}_n$ , generated by a collection of random elements  $J_n$ , such that the following conditions hold.

1. The random variable  $Y_n^s$  has the  $Y_n$ -size bias distribution, and

$$\Psi_{n,\theta} = \sqrt{Var_{\theta} \left( E_{\theta} (Y_n^s - Y_n | Y_n) \right)} \quad satisfies \quad \sup_{n \ge n_1, \theta \in \Theta_{n,r_1}} \frac{r_{n,\theta} \mu_{n,\theta} \Psi_{n,\theta}}{\sigma_{n,\theta}^2} < \infty. \tag{5}$$

2. The random variable  $K_n$  is  $\mathcal{F}_n$ -measurable,  $|Y_n^s - Y_n| \leq K_n$ , and

$$\sup_{n \ge n_1, \theta \in \Theta_{n,r_1}} \frac{r_{n,\theta} \mu_{n,\theta} E_{\theta}[(1+|W_{n,\theta}|) K_n^2]}{\sigma_{n,\theta}^3} < \infty, \tag{6}$$

with  $W_{n,\theta}$  as given in (3).

3. The random variable  $L_n$  takes values in  $\{0, 1, ..., n\}$ , there exists a positive integer valued sequence  $\{s_{n,\theta}\}_{n\geq n_1}$  satisfying  $n-s_{n,\theta}\geq n_0$ , the variables  $L_n$  and  $\psi_{n,\theta}$  are  $\mathcal{F}_n$ -measurable, there exists  $F_{n,\theta}\in\mathcal{F}_n$ ,  $F_{n,\theta}\subset\{L_n\leq s_{n,\theta}\}$  such that

$$\psi_{n,\theta} \in \Theta_{n-L_n} \quad and \quad \mathcal{L}_{\theta}(V_n|J_n) = \mathcal{L}_{\psi_{n,\theta}}(Y_{n-L_n}) \quad on \ F_{n,\theta}$$
 (7)

and

$$\sup_{n \ge n_1, \theta \in \Theta_{n, r_1}} \frac{r_{n, \theta}^2 \mu_{n, \theta}}{\sigma_{n, \theta}^3} E_{\theta} \left[ K_n^2 (1 - \mathbf{1}(F_{n, \theta})) \right] < \infty.$$
(8)

4. There exists  $\{c_1, c_2\} \subset (0, \infty)$  such that

$$\sigma_{n,\theta}^2 \le c_1 \sigma_{n-L_n,\psi_{n,\theta}}^2$$
 and  $r_{n,\theta} \le c_2 r_{n-L_n,\psi_{n,\theta}}$  on  $F_{n,\theta}$ .

5. The random variable  $B_n$  is  $\mathcal{F}_n$ -measurable,  $|Y_n - V_n| \leq B_n$  and

$$\sup_{n \ge n_1, \theta \in \Theta_{n,r_1}} \frac{r_{n,\theta}^2 \mu_{n,\theta} E_{\theta}[K_n^2 B_n]}{\sigma_{n,\theta}^4} < \infty. \tag{9}$$

- 6. Either
  - (a) there exists  $l_{n,0} \in \mathbb{N}_0$  such that  $P_{\theta}(L_n = l_{n,0}) = 1$  for all  $\theta \in \Theta_{n,r_1}$  or
  - (b) the set  $\Theta_{n,r_1}$  is a compact subset of  $\Theta_n$ , and the functions of  $\theta$

$$t_{n,\theta,l} = E_{\theta} \left( \frac{K_n^2}{EK_n^2} \mathbf{1}(L_n = l) \right), \quad l \in \{0, 1, \dots, n\}$$
 (10)

are continuous on  $\Theta_{n,r_1}$  for  $l \in \{0,1,\ldots,s_n\}$  where  $s_n = \sup_{\theta \in \Theta_{n,r_1}} s_{n,\theta}$ .

Then there exists a constant C such that for all  $n \geq n_0$  and  $\theta \in \Theta_n$ 

$$\sup_{z \in \mathbb{R}} |P_{\theta}(W_{n,\theta} \le z) - P(Z \le z)| \le C/r_{n,\theta}. \tag{11}$$

When higher moments exist a number of the conditions of the theorem may be verified using standard inequalities. In particular, by the Cauchy-Schwarz inequality a sufficient condition for (6) is

$$\sup_{n \ge n_1, \theta \in \Theta_{n,r_1}} \frac{r_{n,\theta} \mu_{n,\theta} k_{n,\theta,4}^{1/2}}{\sigma_{n,\theta}^3} < \infty \quad \text{where} \quad k_{n,\theta,m} = E_{\theta} K_n^m, \tag{12}$$

and, when  $F_{n,\theta} = \{L_{n,\theta} \leq s_{n,\theta}\}$  a sufficient condition for (8) is

$$\sup_{n \ge n_1, \theta \in \Theta_{n, r_1}} \frac{r_{n, \theta}^2 \mu_{n, \theta} k_{n, \theta, 4}^{\frac{1}{2}} l_{n, \theta, 2}^{\frac{1}{2}}}{\sigma_{n, \theta}^3 s_{n, \theta}} < \infty \quad \text{where} \quad l_{n, \theta, m} = E_{\theta} L_n^m,$$

since, additionally using the Markov inequality yields

$$E_{\theta}\left[K_{n}^{2}\mathbf{1}(L_{n}>s_{n,\theta})\right] \leq k_{n,\theta,4}^{1/2}P_{\theta}(L_{n}>s_{n,\theta})^{\frac{1}{2}} = k_{n,\theta,4}^{1/2}P_{\theta}(L_{n}^{2}>s_{n,\theta}^{2})^{\frac{1}{2}} \leq \frac{k_{n,\theta,4}^{\frac{1}{2}}l_{n,\theta,2}^{\frac{1}{2}}}{s_{n,\theta}}.$$

Similarly, a sufficient condition for (9) is

$$\sup_{n \ge n_1, \theta \in \Theta_{n,r_1}} \frac{r_{n,\theta}^2 \mu_{n,\theta} k_{n,\theta,4}^{\frac{1}{2}} b_{n,\theta,2}^{\frac{1}{2}}}{\sigma_{n,\theta}^4} < \infty \quad \text{where} \quad b_{n,\theta,m} = E_{\theta} B_n^m.$$
 (13)

A general prescription for size biasing a sum of nonnegative variables is given in Goldstein and Rinott (1996); specializing to exchangeable indicators yields the following result.

**Lemma 1.1** Let  $Y = \sum_{\alpha \in \mathcal{I}} X_{\alpha}$  be a finite sum of nontrivial exchangeable Bernoulli variables  $\{X_{\alpha}, \alpha \in \mathcal{I}\}$ , and suppose that for  $\alpha \in \mathcal{I}$  the variables  $\{X_{\beta}^{\alpha}, \beta \in \mathcal{I}\}$  have joint distribution

$$\mathcal{L}(X_{\beta}^{\alpha}, \beta \in \mathcal{I}) = \mathcal{L}(X_{\beta}, \beta \in \mathcal{I}|X_{\alpha} = 1).$$

Then

$$Y^{\alpha} = \sum_{\beta \in \mathcal{T}} X_{\beta}^{\alpha}$$

has the Y-size biased distribution  $Y^s$ , as does the mixture  $Y^I$  when I is a random index with values in  $\mathcal{I}$ , independent of all other variables.

*Proof:* First, fixing  $\alpha \in \mathcal{I}$ , we show that  $Y^{\alpha}$  satisfies (1). For given f,

$$E[Yf(Y)] = \sum_{\beta \in \mathcal{I}} E[X_{\beta}f(Y)] = \sum_{\beta \in \mathcal{I}} P[X_{\beta} = 1]E[f(Y)|X_{\beta} = 1].$$

As exchangeability implies that  $E[f(Y)|X_{\beta}=1]$  does not depend on  $\beta$ , we have

$$E[Yf(Y)] = \left(\sum_{\beta \in \mathcal{I}} P[X_{\beta} = 1]\right) E[f(Y)|X_{\alpha} = 1] = E[Y]E[f(Y^{\alpha})],$$

demonstrating the first result. The second follows easily using that  $Y^I$  is a mixture of random variables all of which have distribution  $Y^s$ .

Employing size bias couplings and Stein's method, Chen and Röllin (2010) prove a general result to compute bounds to the normal in the Waserstein metric. In particular, Corollary 2.2 and Construction 3A of Chen and Röllin (2010) yield

$$d_W(\mathcal{L}_{\theta}(W_{n,\theta}), \mathcal{L}(Z)) \le 0.8 \frac{\mu_{n,\theta} \Psi_{n,\theta}}{\sigma_{n,\theta}^2} + \frac{\mu_{n,\theta} k_{n,\theta,2}}{\sigma_{n,\theta}^3}.$$
(14)

To compare (14) with one conclusion of Theorem 1.1, as well as to familiarize the reader with the roles of some of the variables appearing in its formulation, we now consider its application in the simple case where a bounded size bias coupling exists, that is, when the bound  $K_n$  on  $|Y_n^s - Y_n|$  can be taken to be a constant, say  $k_n$ , almost surely. In such cases we set  $J_n$  to be the empty set, and note that any constant is measurable with respect to the trivial  $\sigma$ -algebra that  $J_n$  generates. Conditions 3 through 6 are easily satisfied in this case. In particular, taking  $L_n = 0$ ,  $s_{n,\theta} = 1$  and  $F_{n,\theta} = \{L_n \leq s_{n,\theta}\}$ , with  $J_n = \emptyset$ , (7) of Condition 3 holds with  $\psi_{n,\theta} = \theta$  and  $V_n = Y_n$ , and (8) holds as  $1 - \mathbf{1}(F_{n,\theta}) = 0$  a.s. As  $(n - L_n, \psi_{n,\theta}) = (n,\theta)$ , Condition 4 holds with  $c_1 = c_2 = 1$ . As  $V_n = Y_n$  we may take  $B_n = 0$  in Condition 5, and as  $L_n = 0$  Condition 6a is satisfied. Hence, only Conditions 1 and 2 are in force, and Theorem 1.1 obtains with

$$r_{n,\theta}^{-1} = \frac{\mu_{n,\theta} \Psi_{n,\theta}}{\sigma_{n,\theta}^2} + \frac{\mu_{n,\theta} k_n^2}{\sigma_{n,\theta}^3},$$

yielding a Kolmogorov bound that, up to constants, agrees with the Wasserstein bound (14) in this particular case.

Bounded size bias couplings exist not only when  $Y_n$  is the sum of independent, bounded nonnegative random variables, or a sum of bounded, nonnegative locally dependent variables with bounded dependence neighborhood sizes, as studied, for instance, in Goldstein (2005). Bounded size bias couplings can also be constructed in cases of global dependence, see Goldstein and Zhang (2011) or Goldstein and Penrose (2010).

We next apply Theorem 1.1 to vertex degree counts in the Erdős-Rényi random graph. The proof of Theorem 1.1 is given in Section 3.

# 2 Vertex degree in the Erdős-Rényi random graph

We apply Theorem 1.1 to bound the error in the normal approximation to the distribution of the number of vertices of a given degree in the Erdős-Rényi (1959) random graph  $G_n$  (see also Bollobás, 1985). With  $n \in \mathbb{N}$  we take the vertex set of  $G_n$  to be  $\mathcal{I}_n = \{1, \ldots, n\}$ , and the indicators  $\xi_{u,v}$  of the presence of edges between distinct vertices u and v to be independent Bernoulli variables with a common success probability. No vertex is connected to itself, and we set  $\xi_{u,u} = 0$  for all  $u \in \mathcal{I}_n$ .

The number  $Y_n$  of vertices of degree d of  $G_n$  has been the object of much study. For a sequence of graphs with connectivity probability p depending on  $n \in \mathbb{N}$ , Karoński and Ruciński (1987) proved the asymptotic normality of  $Y_n$  when  $n^{(d+1)/d}p \to \infty$  and  $np \to 0$ , or  $np \to \infty$  and  $np - \log n - d \log \log n \to -\infty$ ; see also Palka (1984) and Bollobás (1985). Asymptotic normality of  $Y_n$  when  $np \to c > 0$ , was obtained by Barbour et al. (1989), and Kordecki (1990) for non-smooth functions of  $Y_n$  in the case d = 0. Neammanee and Suntadkarn (2009) obtain a Kolmogorov distance bound between  $Y_n$  and the normal with

rate  $n^{-1/2+\epsilon}$  for all  $\epsilon > 0$  when  $\operatorname{Var}(Y_n)$  is of order n. Other univariate results on asymptotic normality of counts on random graphs are given in Janson and Nowicki (1991), and references therein. Goldstein and Rinott (1996) obtain smooth function bounds for the vector whose k components count the number of vertices of fixed degrees  $d_1, d_2, \ldots, d_k$  when  $p = \theta/(n-1) \in (0,1)$  for fixed  $\theta$ , implying asymptotic multivariate joint normality.

We focus on the counts of vertices of some fixed degree  $d \in \mathbb{N}$ , the case d = 0 of isolated vertices having already been handled by Kordecki (1990). Set

$$\Theta_n = (0, n-1) \cap (0, b] \text{ for all } n > d+1$$
 (15)

with b some arbitrarily large constant, and let the connectivity probability between the vertices of  $G_n$  be given by  $\theta/(n-1)$  for  $n \ge d+1, \theta \in \Theta_n$ . For  $v \in \mathcal{I}_n$  let

$$D_n(v) = \sum_{w \in \mathcal{I}_n} \xi_{v,w}, \quad X_{n,v} = \mathbf{1}(D_n(v) = d) \quad \text{and} \quad Y_n = \sum_{v \in \mathcal{I}_n} X_{n,v},$$

the degree of vertex v, the indicator that vertex v has degree d, and the number of vertices of degree d of  $G_n$ , respectively.

From Goldstein and Rinott (1996), for all  $n \geq d+1$  and  $\theta \in \Theta_n$ , the mean  $\mu_{n,\theta}$  and variance  $\sigma_{n,\theta}^2$  of  $Y_n$  are given explicitly by

$$\mu_{n,\theta} = n\tau_{n,\theta}, \text{ and } \sigma_{n,\theta}^2 = n\tau_{n,\theta}^2 \left[ \frac{(d-\theta)^2}{\theta(1-\theta/(n-1))} - 1 \right] + n\tau_{n,\theta},$$
 (16)

where

$$\tau_{n,\theta} = \binom{n-1}{d} \left(\frac{\theta}{n-1}\right)^d \left(1 - \frac{\theta}{n-1}\right)^{n-1-d}.$$
 (17)

**Theorem 2.1** For any  $d \in \mathbb{N}$  and b > 0 there exists a constant C such that for all  $n \ge d+1$  and all  $\theta \in \Theta_n$  given in (15), the normalized count  $W_{n,\theta}$  given by (3), of the number  $Y_n$  of vertices with degree d in the Erdős-Rényi random graph  $G_n$  on n vertices, with edges connecting each distinct pair independently with probability  $\theta/(n-1)$ , satisfies

$$\sup_{z \in \mathbb{R}} |P_{\theta}(W_{n,\theta} \le z) - P(Z \le z)| \le C/r_{n,\theta} \quad \text{for all } n \ge d+1,$$

where Z is a standard normal variable and

$$r_{n,\theta} = \sqrt{n\tau_{\theta}} \quad with \quad \tau_{\theta} = e^{-\theta}\theta^d/d!$$
 (18)

In the metric  $d_L$  over Lipschitz functions h defined by

$$d_L(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{h} \frac{|Eh(X) - Eh(Y)|}{||h|| + ||h'||},$$
(19)

by applying Stein's method Barbour et al. (1989) obtain a bound of order  $1/\sqrt{n\tau_{n,\theta}}$ . As Lemma 2.1 shows that  $\tau_{n,\theta}/\tau_{\theta}$  converges uniformly to 1 over  $\Theta_n$ , the Kolmogorov bound of order  $1/\sqrt{n\tau_{\theta}}$  provided by Theorem 2.1 is of the same order as the  $d_L$  bound. As remarked in Barbour et al. (1989), a bound of size  $\epsilon_n$  in the  $d_L$  metric yields a bound in the Kolmogorov

metric of order  $O(\epsilon_n^{1/2})$ , which can at times be improved to  $O(\epsilon_n)$  'at the cost of much greater effort.'

Though we do not cover the case d=0 of isolated vertices, handled in Kordecki (1990), our proof can be extended to apply there by appending additional arguments that are separate, but similar to, those for the case  $d \in \mathbb{N}$ . Note, for example, the difference in the behavior of the function  $\tau_{\theta}$  at zero for these two ranges of d.

Following Lemma 1.1 for the case of vertex degrees yields a coupling where for each  $n \geq d+1$  and vertex  $v \in \mathcal{I}_n$  one constructs a graph  $G_n^v$  from  $G_n$  having the distribution of  $G_n$  conditioned on  $X_{n,v} = 1$ , or equivalently, on  $D_n(v) = d$ ; this coupling has previously been applied by Barbour et al. (1992) and Goldstein and Rinott (1996). The graph  $G_n^v$  is obtained from  $G_n$  by adding or removing edges of v as needed. Mixing over v as indicated by Lemma 1.1 yields a variable  $Y_n^s$  having the  $Y_n$ -size bias distribution.

In the course of constructing  $G_n^v$  one also obtains a set  $\mathcal{R}_n^v$  holding the collection of vertices other than v that are affected by the size bias operation. In particular, if  $D_n(v) = d$  then  $G_n^v = G_n$  and  $\mathcal{R}_n^v = \emptyset$ . If  $D_n(v) > d$  then  $G_n^v$  is formed by removing from  $G_n$  the edges between v and the vertices in the subset  $\mathcal{R}_n^v$  of neighbors  $\{u: \xi_{u,v} = 1\}$  of v, chosen with uniform conditional distribution given  $G_n$  over all subsets of the neighbors of v of size  $D_n(v) - d$ . Similarly, if  $D_n(v) < d$  then  $G_n^v$  is formed by adding edges to  $G_n$  between v and vertices in  $\mathcal{R}_n^v$ , chosen with uniform conditional distribution given  $G_n$  over all subsets of the non-neighbors  $\{u: u \neq v, \xi_{u,v} = 0\}$  of v of size  $d - D_n(v)$ .

Now let  $X_{n,w}^v$  be the indicator that vertex w has degree d in  $G_n^v$  and

$$Y_n^v = \sum_{w \in \mathcal{I}_n} X_{n,w}^v,$$

the number of degree d vertices in  $G_n^v$ . When  $I_n$  is chosen uniformly over  $\mathcal{I}_n$ , independent of all other variables, Lemma 1.1 yields that  $Y_n^s = Y_n^{I_n}$  has the  $Y_n$ -size biased distribution. Similarly setting  $\mathcal{R}_n^s = \mathcal{R}_n^{I_n}$ , all vertices not in  $\{I_n\} \cup \mathcal{R}_n^s$  have the same degree in both  $G_n$  and  $G_n^s$ , and as  $I_n \notin \mathcal{R}_n^s$ , letting

$$\mathcal{A}_n = \{I_n\} \cup \mathcal{R}_n^s \text{ we have } |\mathcal{A}_n| = 1 + |d - D_n(I_n)|.$$
 (20)

We prove Theorem 2.1 by verifying the hypotheses of Theorem 1.1 for the size bias construction just given. With  $\tau_{n,\theta}$  as in (17), and recalling (16), let

$$\delta_{n,\theta} = \tau_{n,\theta} \left[ \frac{(d-\theta)^2}{\theta(1-\theta/(n-1))} - 1 \right] + 1 \quad \text{so that} \quad \sigma_{n,\theta}^2 = n\tau_{n,\theta}\delta_{n,\theta}, \tag{21}$$

and correspondingly, with  $\tau_{\theta}$  as in (18), let

$$\delta_{\theta} = \tau_{\theta} \left[ \frac{(d-\theta)^2}{\theta} - 1 \right] + 1. \tag{22}$$

With the help of a technical lemma placed at the end of this section, we present the proof of Theorem 2.1. Throughout we let  $C_j$  denote a constant not depending on n or  $\theta$ , and not necessarily the same at each occurrence.

*Proof:* Let  $n_0 = d + 1$ . For  $n \ge n_0$  and  $\theta \in \Theta_n$  the binomial and Poisson probabilities  $\tau_{n,\theta}$  and  $\tau_{\theta}$  in (17) and (18), respectively, lie in (0,1), and hence  $\sigma_{n,\theta}^2$  of (16) and  $r_{n,\theta}$  are positive

for all such n and  $\theta$ . Let  $r_1 > 0$  be arbitrary. In place of naming  $n_1$  explicitly, we show the remaining conditions of Theorem 1.1 are satisfied for all n sufficiently large. Since  $r_{n,\theta} \leq \sqrt{n}$  inequality (4) holds for any  $n_1 \geq n_0$ .

From Chen et al. (2010, (12.17)), following Goldstein and Rinott (1996), for  $Y_n^s$  having the  $Y_n$ -size biased distribution as constructed above, we obtain

$$\Psi_{n,\theta}^2 \le C_1 n^{-1} \left( 24\theta + 48\theta^2 + 144\theta^3 + 48d^2 + 144\theta d^2 + 12 \right) \quad \text{and hence} \quad \sup_{\theta \in \Theta_n} \Psi_{n,\theta} \le \frac{C_2}{\sqrt{n}}.$$

To complete the verification of Condition 1, Lemma 2.1 gives that over  $\Theta_n$  the ratio  $\delta_{\theta}/\delta_{n,\theta} = \delta_{\theta}\mu_{n,\theta}/\sigma_{n,\theta}^2$  converges uniformly to 1, and  $\delta_{\theta}$  in (22) is bounded away from zero. Hence for all n sufficiently large and all  $\theta \in \Theta_n$  we have

$$\frac{\mu_{n,\theta}}{\sigma_{n,\theta}^2} \le \frac{2}{\delta_{\theta}} \le C_3, \text{ and so } \frac{r_{n,\theta}\mu_{n,\theta}\Psi_{n,\theta}}{\sigma_{n,\theta}^2} \le C_4\sqrt{\tau_{\theta}} \le C_4,$$
(23)

as  $\tau_{\theta} \leq 1$  for all  $\theta \in \Theta_n$ .

Turning to the Condition 2, let

$$J_n = \{I_n, D_n(I_n)\}$$
 and  $\mathcal{F}_n = \sigma\{J_n\},$ 

that is  $\mathcal{F}_n$  is the  $\sigma$ -algebra generated by the chosen vertex and its degree. Further, let

$$K_n = 1 + d + D_n(I_n).$$

Clearly  $K_n$  is  $\mathcal{F}_n$ -measurable, and recalling that vertices not in  $\mathcal{A}_n$  of (20) have the same degree in both  $G_n$  and  $G_n^s$ , taking the difference between  $Y_n^s$  and  $Y_n$  yields

$$Y_n^s - Y_n = \sum_{w \in A_n} (X_{n,w}^{I_n} - X_{n,w}), \text{ and (20) yields } |Y_n^s - Y_n| = 1 + |d - D(I_n)| \le K_n.$$

Next, for all  $m \in \mathbb{N}$  we have

$$K_n^m \le 2^{m-1} \left( (1+d)^m + D_n(I_n)^m \right).$$
 (24)

To bound the moments of  $K_n$ , using Riordan (1937) for the first equality below, with  $S_{j,m}$  the Stirling numbers of the second kind and  $(n)_j$  the falling factorial, with  $C_{5,m} = m \max_{1 \le j \le m} S_{j,m}$  and  $D \sim \text{Bin}(n-1,p)$  we obtain

$$ED^{m} = \sum_{j=1}^{m} S_{j,m}(n-1)_{j}p^{j} \le \sum_{j=1}^{m} S_{j,m}(n-1)^{j}p^{j} \le C_{5,m}((n-1)p + (n-1)^{m}p^{m}).$$

In particular  $E_{\theta}D_n(v)^m \leq C_{5,m}(b+b^m)$ , and as  $D_n(I_n)$  is the mixture of the identical distributions  $D_n(v)$  over  $v \in \mathcal{I}_n$ , it obeys the same upper bound. Taking expectation in (24), we find that there exists constants  $C_{6,m}$ ,  $m \in \mathbb{N}$  such that

$$k_{n,\theta,m} \le C_{6,m}$$
 for all  $n \in \mathbb{N}, \theta \in \Theta_n$  and  $m \in \mathbb{N}$ . (25)

Now, using (25) for the first inequality in (26), the first inequality in (23) for the second inequality, the second equality of (21) for the first equality, and Lemma 2.1 both to obtain

the third inequality, and the boundedness of  $\delta_{\theta}$  away from zero for the fourth, we obtain that for all n sufficiently large and  $\theta \in \Theta_n$ ,

$$\frac{r_{n,\theta}\mu_{n,\theta}k_{n,\theta,4}^{1/2}}{\sigma_{n,\theta}^3} \le \frac{C_{6,4}^{1/2}r_{n,\theta}\mu_{n,\theta}}{\sigma_{n,\theta}^3} \le \frac{C_7r_{n,\theta}}{\sigma_{n,\theta}} = \frac{C_7\sqrt{\tau_{\theta}}}{\sqrt{\tau_{n,\theta}\delta_{n,\theta}}} \le \frac{C_8}{\sqrt{\delta_{\theta}}} \le C_9. \tag{26}$$

Hence inequality (12), sufficient for (6), is satisfied, and Condition 2 holds.

Turning to Condition 3, for  $n \ge d + 2$  let

$$L_n = 1, \quad s_{n,\theta} = 1 \quad \psi_{n,\theta} = \left(\frac{n-2}{n-1}\right)\theta \quad \text{and} \quad F_{n,\theta} = \{L_n \le s_{n,\theta}\},$$
 (27)

and note therefore that conditions holding on  $F_{n,\theta}$  must hold on the entire probability space. Clearly  $L_n$  takes values in  $\{0, 1, ..., n\}$  as required and  $n - s_{n,\theta} \ge n_0$  for any  $n \ge d + 2$ . Being constants,  $L_n$  and  $\psi_{n,\theta}$  are  $\mathcal{F}_n$  measurable, hence  $F_{n,\theta} \in \mathcal{F}_n$ . By (27) and  $\theta \in \Theta_n$  we have that  $\psi_{n,\theta} \in (0,b] \cap (0,n-2) = \Theta_{n-1} = \Theta_{n-L_n}$ , verifying the first part of (7).

Regarding the second part of (7), let  $H_n$  be the graph  $G_n$  with the vertex  $I_n$  and its incident edges removed, relabeling the remaining vertices  $\{1, \ldots, n-1\}$  by preserving their relative order. Let  $V_n$  be the number of degree d vertices of  $H_n$ . By counting the number of degree d vertices, the distributional equality in (7) is a consequence of

$$\mathcal{L}_{\theta}(H_n|I_n, D_n(I_n)) = \mathcal{L}_{\psi_{n,\theta}}(G_{n-1}). \tag{28}$$

The graph  $H_n$  is determined by  $\{\xi_{u,v} : \{u,v\} \subset \mathcal{I}_n \setminus \{I_n\}\}$ , which is independent of the  $\sigma$ -algebra generated by  $\{I_n, \xi_{I_n,v}, v \in \mathcal{I}_n\}$ , with respect to which  $I_n$  and  $D_n(I_n)$  are measurable. Hence  $H_n$  is independent of the conditioning event in (28), and therefore its conditional and unconditional distribution agree. In particular, conditional on  $\{I_n, D_n(I_n)\}$  the edge indicators of  $H_n$  are independent with common success probability

$$\frac{\theta}{n-1} = \frac{\psi_{n,\theta}}{n-2},$$

so (28) holds. The inequality (8) holds trivially, as  $P(L_n > 1) = 0$ . Hence Condition 3 holds. By Lemma 2.1, Condition 4 holds with  $c_1 = c_2 = 2$ .

Regarding Condition 5, as only the degrees of vertex  $I_n$  and its neighbors are different in the graphs  $G_n$  and  $H_n$ , we have

$$|Y_n - V_n| \le 1 + D(I_n) \le K_n,$$

and we set  $B_n = K_n$ , so  $\mathcal{F}_n$ -measurable. We now finish the verification of Condition 5 by showing (13), sufficient for (9), is satisfied. By (25), that  $\mu_{n,\theta} = n\tau_{n,\theta}$  and the second equality in (21), for all n sufficiently large and all  $\theta \in \Theta_n$  we have

$$\frac{r_{n,\theta}^2 \mu_{n,\theta} k_{n,\theta,4}^{\frac{1}{2}} b_{n,\theta,2}^{\frac{1}{2}}}{\sigma_{n,\theta}^4} \le \frac{\tau_{\theta} (C_{6,4} C_{6,2})^{1/2}}{\tau_{n,\theta} \delta_{n,\theta}^2} \le C_{10},$$

where the final inequality follows from Lemma 2.1, yielding that  $\tau_{n,\theta}/\tau_{\theta}$  and  $\delta_{n,\theta}/\delta_{\theta}$  converge uniformly to 1 on  $\Theta_n$ , and that  $\delta_{\theta}$  is bounded away from zero on (0, b].

Lastly, Condition 6a holds with  $l_{n,0} = 1$  for all  $n \ge d + 2$ , completing the verification of all conditions of Theorem 1.1.

The proof of Lemma 2.1 is straightforward, and is therefore omitted.

**Lemma 2.1** With  $\tau_{n,\theta}$ ,  $\tau_{\theta}$ ,  $\delta_{n,\theta}$  and  $\delta_{\theta}$  given by (17), (18), (21) and (22) respectively, for all  $d \in \mathbb{N}$  and all b > 0 the function  $\delta_{\theta}$  is bounded away from zero and infinity over (0,b], and the ratios

$$\frac{\tau_{n,\theta}}{\tau_{\theta}}$$
,  $\frac{\delta_{n,\theta}}{\delta_{\theta}}$ ,  $\frac{r_{n,\theta}}{r_{n-1,\psi_{n,\theta}}}$  and  $\frac{\sigma_{n,\theta}^2}{\sigma_{n-1,\psi_{n,\theta}}^2}$ 

and their reciprocals converge uniformly to 1 on (0,b] as n tends to infinity.

### 3 Proof of Theorem 1.1

We begin the proof of Theorem 1.1 with the following lemma.

**Lemma 3.1** Suppose that for some  $n_1 \in \mathbb{N}_0$  the nonnegative numbers  $f, \{p_{n,l}\}_{n \geq n_1, 0 \leq l \leq n}$  and  $\{a_n\}_{n \geq 0}$  satisfy

$$a_n \le \sum_{l=0}^n a_{n-l} p_{n,l} + f \quad \text{for all } n \ge n_1, \text{ and } \quad \tau \in (0,1) \quad \text{where} \quad \tau = \sup_{n \ge n_1} \sum_{l=0}^n p_{n,l}.$$
 (29)

Then  $\sup_{n>0} a_n < \infty$ .

*Proof:* As for all  $n \geq n_1$  we have  $p_{n,0} \leq \tau < 1$ , letting

$$q_{n,l} = \frac{p_{n,l}}{1 - p_{n,0}}$$
 for  $1 \le l \le n$  and  $a = \frac{f}{1 - \tau}$ ,

(29) implies

$$a_n \le \sum_{l=1}^n a_{n-l} q_{n,l} + a$$
 with  $0 \le \sum_{l=1}^\infty q_{n,l} \le \frac{\tau - p_{n,0}}{1 - p_{n,0}} \le \tau$ , for all  $n \ge n_1$ .

Letting  $\alpha = \max_{0 \le n \le n_1} a_n$  and  $c = \max\{a, \alpha(1-\tau)\}$ , the sequence  $\{b_n\}_{n \ge 0}$  defined by

$$b_n = \alpha$$
 for  $0 \le n \le n_1$  and  $b_{n+1} = \tau b_n + c$  for  $n \ge n_1$ 

has, for  $n \geq n_1$ , the explicit form

$$b_n = \gamma \tau^{n-n_1} + \frac{c}{1-\tau}$$
, where  $\gamma = \alpha - \frac{c}{1-\tau}$ .

Since  $\gamma \leq 0$  and  $\tau \in (0,1)$ , the sequence  $\{b_n\}_{n\geq 0}$  is non-decreasing with limit  $c/(1-\tau)$ , and hence is bounded. We complete the proof by showing that for all  $n \in \mathbb{N}_0$  we have  $a_m \leq b_m$  for all  $0 \leq m \leq n$ . Clearly the statement holds for  $0 \leq n \leq n_1$ . Assuming it true for some  $n \geq n_1$ , using the induction hypotheses, the definition of c and that  $b_n$  is non-decreasing,

$$a_{n+1} \le \sum_{l=1}^{n+1} a_{n+1-l} q_{n+1,l} + a \le \sum_{l=1}^{n+1} b_{n+1-l} q_{n+1,l} + c \le b_n \sum_{l=1}^{n+1} q_{n+1,l} + c \le \tau b_n + c = b_{n+1}.$$

The following proof is based on the inductive argument of Bolthausen (1984).

**Proof of Theorem 1.1** With  $r \geq 0$  recall that  $\Theta_{n,r} = \{\theta \in \Theta_n : r_{n,\theta} \geq r\}$  and let

$$\delta(n,r) = \sup_{z \in \mathbb{R}, \theta \in \Theta_{n,r}} |P_{\theta}(W_{n,\theta} \le z) - P(Z \le z)| \quad \text{for } n \ge n_0.$$
 (30)

First note that (11) of Theorem 1.1 can be made to hold whenever  $r_{n,\theta} < r_1$  by taking  $C \ge r_1$ . By (4) the cases  $n_0 \le n < n_1$  and  $r_{n,\theta} \ge r_1$  can be handled in this same manner. Hence it suffices to show that

$$\delta(n,r) \le C/r \quad \text{for } n \ge n_1 \text{ and } r \ge r_1.$$
 (31)

For  $z \in \mathbb{R}$  and  $\lambda > 0$  let  $h_{z,\lambda}$  be the smoothed indicator

$$h_{z,\lambda}(x) = \begin{cases} 1 & x \le z \\ 1 + (z - x)/\lambda & z < x \le z + \lambda \\ 0 & x > z + \lambda, \end{cases}$$

and let  $Nh_{z,\lambda} = Eh_{z,\lambda}(Z)$  with Z a standard normal variable. Let f(x) be the unique bounded solution to the Stein equation for  $h_{z,\lambda}(x)$  (see, e.g. Chen et al. (2010))

$$h_{z,\lambda}(x) - Nh_{z,\lambda} = f'(x) - xf(x). \tag{32}$$

Let  $n \geq n_1, \theta \in \Theta_{n,r}$  for some  $r \geq r_1$  and  $z \in \mathbb{R}$  and  $\lambda > 0$ . Recalling  $W_{n,\theta} = (Y_n - \mu_{n,\theta})/\sigma_{n,\theta}$ , with a slight abuse of notation set

$$W_{n,\theta}^s = \frac{Y_n^s - \mu_{n,\theta}}{\sigma_{n,\theta}}.$$

Substituting  $W_{n,\theta}$  for x in (32) and taking expectation, and dropping the subscript  $\theta$  when not essential, we obtain

$$E_{\theta}h_{z,\lambda}(W_n) - Nh_{z,\lambda} = E_{\theta}[f'(W_n) - W_n f(W_n)]. \tag{33}$$

Beginning with the second term on the right hand side of (33), from the definition of  $W_{n,\theta}$  and the size bias relation (1) we have

$$E_{\theta}[W_n f(W_n)] = \frac{1}{\sigma_n} E_{\theta}[(Y_n - \mu_n) f(W_n)] = \frac{\mu_n}{\sigma_n} E_{\theta} (f(W_n^s) - f(W_n)).$$

Taking absolute value in (33) and applying the triangle inequality, we obtain

$$|E_{\theta}h_{z,\lambda}(W_{n}) - Nh_{z,\lambda}|$$

$$= |E_{\theta}[f'(W_{n}) - W_{n}f(W_{n})]|$$

$$= \left| E_{\theta}[f'(W_{n}) - \frac{\mu_{n}}{\sigma_{n}} (f(W_{n}^{s}) - f(W_{n}))] \right|$$

$$= \frac{\mu_{n}}{\sigma_{n}} \left| E_{\theta} \left[ \frac{\sigma_{n}}{\mu_{n}} f'(W_{n}) - (f(W_{n}^{s}) - f(W_{n})) \right] \right|$$

$$= \frac{\mu_{n}}{\sigma_{n}} \left| E_{\theta} \left[ \left( \frac{\sigma_{n}}{\mu_{n}} - (W_{n}^{s} - W_{n}) \right) f'(W_{n}) + (W_{n}^{s} - W_{n}) f'(W_{n}) - (f(W_{n}^{s}) - f(W_{n})) \right] \right|$$

$$\leq \frac{\mu_{n}}{\sigma_{n}} \left| E_{\theta} \left[ \left( \frac{\sigma_{n}}{\mu_{n}} - (W_{n}^{s} - W_{n}) \right) f'(W_{n}) \right] \right| + \frac{\mu_{n}}{\sigma_{n}} \left| E_{\theta} \left[ \int_{0}^{W_{n}^{s} - W_{n}} [f'(W_{n}) - f'(W_{n} + t)] dt \right] \right| .$$

From the size bias relation (1) with f(x) = x we obtain  $\mu_n E_{\theta}[Y_n^s] = E_{\theta}[Y_n^2]$ , and therefore

$$E_{\theta}[W_n^s - W_n] = E_{\theta} \left[ \frac{Y_n^s - Y_n}{\sigma_n} \right] = \frac{1}{\sigma_n} \left[ \frac{E_{\theta} Y_n^2}{\mu_n} - \mu_n \right] = \frac{1}{\sigma_n \mu_n} \sigma_n^2 = \frac{\sigma_n}{\mu_n}. \tag{35}$$

Now applying (35) and  $|f'(x)| \le 1$  from Chen and Shao (2004, Equation (4.6)), see also Chen et al. (2010, Lemma 2.5), by conditioning on  $W_n$  the first term of (34) may be bounded by

$$\frac{\mu_n}{\sigma_n} \left| E_{\theta} \left[ E_{\theta} \left( \frac{\sigma_n}{\mu_n} - (W_n^s - W_n) \middle| W_n \right) f'(W_n) \right] \right| \le \frac{\mu_n}{\sigma_n} \sqrt{\operatorname{Var} E_{\theta}(W_n^s - W_n | W_n)} = \frac{\mu_n}{\sigma_n^2} \Psi_n, (36)$$

recalling the definition of  $\Psi_n$  in (5).

Moving now to the second term of (34), Bolthausen (1984, Equation (2.4)) gives

$$|f(x)| \le 1$$
 and  $|xf(x)| \le 1$ ,

and combining these inequalities with  $|f'(x)| \leq 1$  and (32) as in Bolthausen (1984, Equation (2.5)) yields

$$|f'(x) - f'(x+t)| \le |t| \left(1 + |x| + \frac{1}{\lambda} \int_0^1 1_{[z,z+\lambda]}(x+ut) du\right).$$

Hence, applying the bound  $|Y_n^s - Y_n| \leq K_n$ , the second term in (34) may be bounded by

$$\frac{\mu_n}{\sigma_n} E_{\theta} \int_{-K_n/\sigma_n}^{K_n/\sigma_n} |t| \left( 1 + |W_n| + \frac{1}{\lambda} \int_0^1 1_{[z,z+\lambda]} (W_n + ut) du \right) dt, \tag{37}$$

yielding three terms.

For the first two terms in (37) we obtain

$$\frac{2\mu_n}{\sigma_n} E_{\theta} \left( (1 + |W_n|) \int_0^{K_n/\sigma_n} t dt \right) = \frac{\mu_n}{\sigma_n^3} E_{\theta} [(1 + |W_n|) K_n^2]. \tag{38}$$

Next, as  $|t| \leq K_n/\sigma$  in the region of integration, we may bound the expectation of the remaining term in (37) by

$$\frac{\mu_n}{\lambda \sigma_n^2} E_\theta \left( K_n \int_{-K_n/\sigma_n}^{K_n/\sigma_n} \int_0^1 \mathbf{1}_{[z,z+\lambda]} (W_n + ut) du dt \right). \tag{39}$$

Clearly,

$$\mathbf{1}_{[z,z+\lambda]}(W_n + ut) \leq (1 - \mathbf{1}_{F_{n,\theta}}) + \mathbf{1}_{[z,z+\lambda]}(W_n + ut)\mathbf{1}_{F_{n,\theta}}. \tag{40}$$

Substituting (40) into (39), the first term in (40) gives rise to the expression

$$\frac{\mu_n}{\lambda \sigma_n^2} E_{\theta} \left( K_n \int_{-K_n/\sigma_n}^{K_n/\sigma_n} \int_0^1 (1 - \mathbf{1}_{F_{n,\theta}}) du dt \right) = \frac{2\mu_n}{\lambda \sigma_n^3} E_{\theta} \left[ K_n^2 (1 - \mathbf{1}_{F_{n,\theta}}) \right]. \tag{41}$$

Substituting the second term in (40) into (39), conditioning on  $\mathcal{F}_n$  and invoking the  $\mathcal{F}_n$  measurability of  $K_n$  and  $F_{n,\theta}$  provided by Conditions 2 and 3, respectively, yields

$$\frac{\mu_n}{\lambda \sigma_n^2} E_{\theta} \left( K_n \int_{-K_n/\sigma_n}^{K_n/\sigma_n} \int_0^1 \mathbf{1}(z \le W_n + ut \le z + \lambda) \mathbf{1}_{F_{n,\theta}} du dt \right) 
= \frac{\mu_n}{\lambda \sigma_n^2} E_{\theta} \left( K_n \int_{-K_n/\sigma_n}^{K_n/\sigma_n} \int_0^1 P_{\theta}^{\mathcal{F}_n}(z \le W_n + ut \le z + \lambda) \mathbf{1}_{F_{n,\theta}} du dt \right), \tag{42}$$

where  $P_{\theta}^{\mathcal{F}_n}$  denotes conditional probability with respect to  $\mathcal{F}_n$ . To handle the indicator in (42), note that Condition 3 implies that  $n - L_n \geq n_0$  on  $F_{n,\theta}$ . Hence on  $F_{n,\theta}$  we may define

$$\underline{W_{n,\theta}} = \frac{V_n - \mu_{n-L_n,\psi_{n,\theta}}}{\sigma_{n-L_n,\psi_{n,\theta}}}$$

and write

$$W_n = \left(\frac{\sigma_{n-L_n,\psi_n}}{\sigma_n}\right) \underline{W_n} + \left(\frac{Y_n - V_n}{\sigma_n}\right) - \left(\frac{\mu_n - \mu_{n-L_n,\psi_n}}{\sigma_n}\right) := \rho_n \underline{W_n} + T_{n,1} - T_{n,2}. \tag{43}$$

By Conditions 5 and 3 we have  $|T_{n,1}| \leq B_n/\sigma_n$  and that  $\rho_n, B_n$  and  $T_{n,2}$  are  $\mathcal{F}_n$ -measurable. Using (43) we may write

$$P_{\theta}^{\mathcal{F}_{n}}(z \leq W_{n} + ut \leq z + \lambda)\mathbf{1}_{F_{n,\theta}}$$

$$= P_{\theta}^{\mathcal{F}_{n}}(\rho_{n}^{-1}(z - T_{n,1} + T_{n,2} - ut) \leq \underline{W_{n}} \leq \rho_{n}^{-1}(z - T_{n,1} + T_{n,2} - ut + \lambda))\mathbf{1}_{F_{n,\theta}}$$

$$\leq P_{\theta}^{\mathcal{F}_{n}}(\rho_{n}^{-1}(z + T_{n,2} - ut) - B_{n}/\sigma_{n-L_{n},\psi_{n}} \leq \underline{W_{n}} \leq \rho_{n}^{-1}(z + T_{n,2} - ut) + B_{n}/\sigma_{n-L_{n},\psi_{n}} + \rho_{n}^{-1}\lambda)\mathbf{1}_{F_{n,\theta}}$$

$$= P_{\theta}^{\mathcal{F}_{n}}(Q_{n} - B_{n}/\sigma_{n-L_{n},\psi_{n}} \leq W_{n} \leq Q_{n} + B_{n}/\sigma_{n-L_{n},\psi_{n}} + \rho_{n}^{-1}\lambda)\mathbf{1}_{F_{n,\theta}}, \quad (44)$$

where we have set

$$Q_n = \rho_n^{-1}(z + T_{n,2} - ut).$$

Recalling (30), we have

$$P_{\theta}(z \leq W_{n,\theta} \leq z + \lambda)$$

$$\leq |P_{\theta}(z \leq W_{n,\theta} \leq z + \lambda) - P(z \leq Z \leq z + \lambda)| + P(z \leq Z \leq z + \lambda)$$

$$\leq 2\delta(n, r_{n,\theta}) + \lambda/\sqrt{2\pi}.$$
(45)

Since the endpoints of the interval bounding  $\underline{W_n}$  in (44) are  $\mathcal{F}_n$ -measurable, using Condition 3 and (45) with the appropriate substitutions, the conditional probability in (44) is bounded by

$$\left(2\delta(n-L_n,r_{n-L_n,\psi_{n,\theta}}) + (2B_n/\sigma_{n-L_n,\psi_{n,\theta}} + \rho_n^{-1}\lambda)/\sqrt{2\pi}\right)\mathbf{1}_{F_{n,\theta}}$$

$$\leq \left(2\delta(n-L_n,r_{n,\theta}/c_2) + (2\sqrt{c_1}B_n/\sigma_{n,\theta} + \sqrt{c_1}\lambda)/\sqrt{2\pi}\right)\mathbf{1}_{F_{n,\theta}}.$$

where we have applied Condition 4, and that  $\delta(n,r)$  is non-increasing in r. As this last quantity does not depend on u or t, substitution into (42) yields the bound

$$\frac{2\mu_{n,\theta}}{\lambda\sigma_{n,\theta}^3}E_{\theta}\left[K_n^2\left(2\delta(n-L_n,r_{n,\theta}/c_2)+(2\sqrt{c_1}B_n/\sigma_{n,\theta}+\sqrt{c_1}\lambda)/\sqrt{2\pi}\right)\right]\mathbf{1}_{F_{n,\theta}}.$$
(46)

Expression (46) leads to three terms. By (6), that  $F_{n,\theta} \subset \{L_n \leq s_{n,\theta}\}$ , and since  $n - s_{n,\theta} \geq n_0$  implies  $s_n = \sup_{\theta \in \Theta_{n,r_1}} s_{n,\theta} \leq n - n_0$ , there exists a positive constant  $C_1$  such that the first term satisfies

$$\frac{4\mu_{n,\theta}}{\lambda\sigma_{n,\theta}^{3}}E_{\theta}\left[K_{n}^{2}\delta(n-L_{n},r_{n,\theta}/c_{2})\mathbf{1}_{F_{n,\theta}}\right]$$

$$=\frac{4\mu_{n,\theta}k_{n,\theta,2}}{\lambda\sigma_{n,\theta}^{3}}E_{\theta}\left[\frac{K_{n}^{2}}{E_{\theta}K_{n}^{2}}\delta(n-L_{n},r_{n,\theta}/c_{2})\mathbf{1}_{F_{n,\theta}}\right] \leq \frac{C_{1}}{\lambda r_{n,\theta}}E_{\theta}\left[\frac{K_{n}^{2}}{E_{\theta}K_{n}^{2}}\delta(n-L_{n},r_{n,\theta}/c_{2})\mathbf{1}_{F_{n,\theta}}\right]$$

$$\leq \frac{C_{1}}{\lambda r_{n,\theta}}\sum_{l=0}^{s_{n}}\delta(n-l,r_{n,\theta}/c_{2})t_{n,\theta,l}, \quad (47)$$

where  $t_{n,\theta,l}$ , given in (10), satisfy

$$\sum_{l=0}^{n} t_{n,\theta,l} = 1 \quad \text{for all } \theta \in \Theta_{n,r}. \tag{48}$$

Dropping the indicator  $\mathbf{1}_{F_{n,\theta}}$ , the sum of the second and third terms of (46) are bounded by

$$\frac{4\sqrt{c_1}\mu_n E_{\theta}[K_n^2 B_n]}{\sqrt{2\pi}\lambda \sigma_n^4} + \frac{\sqrt{2c_1}\mu_n}{\sqrt{\pi}\sigma_n^3} E_{\theta} K_n^2. \tag{49}$$

Collecting terms (36), (38), (41), (47) and (49), and letting

$$c_{n,\theta,1} = \frac{\mu_{n,\theta}}{\sigma_{n,\theta}^2} \Psi_{n,\theta} + \frac{\mu_{n,\theta}}{\sigma_{n,\theta}^3} E_{\theta} \left[ \left( \left( 1 + \frac{\sqrt{2c_1}}{\sqrt{\pi}} \right) + |W_{n,\theta}| \right) K_n^2 \right] \quad \text{and}$$

$$c_{n,\theta,2} = \frac{2\mu_{n,\theta}}{\sigma_{n,\theta}^3} E_{\theta} \left[ K_n^2 (1 - \mathbf{1}_{F_{n,\theta}}) \right] + \frac{4\sqrt{c_1}\mu_{n,\theta}}{\sqrt{2\pi}\sigma_{n,\theta}^4},$$

for all  $z \in \mathbb{R}$  we have

$$|E_{\theta}h_{z,\lambda}(W_{n,\theta}) - Nh_{z,\lambda}| \le \frac{C_1}{\lambda r_{n,\theta}} \sum_{l=0}^{s_n} \delta(n-l, r_{n,\theta}/c_2) t_{n,\theta,l} + c_{n,\theta,1} + \frac{1}{\lambda} c_{n,\theta,2}.$$
 (50)

Note that Conditions 1 and 2, and 3 and 5, respectively yield the existence of positive constants  $C_2$  and  $C_3$  that

$$c_{n,\theta,1} \le C_2/r_{n,\theta} \quad \text{and} \quad c_{n,\theta,2} \le C_3/r_{n,\theta}^2.$$
 (51)

As  $\mathbf{1}(w \leq z) \leq h_{z,\lambda}(w) \leq \mathbf{1}(w \leq z + \lambda)$  we obtain

$$P_{\theta}(W_{n,\theta} \le z) - P(Z \le z) \le |E_{\theta}h_{z,\lambda}(W_{n,\theta}) - Eh_{z,\lambda}(Z)| + Eh_{z,\lambda}(Z) - P(Z \le z)$$

with  $Eh_{z,\lambda}(Z) - P(Z \le z) \le P(z \le Z \le z + \lambda) \le \lambda/\sqrt{2\pi}$ . Along with a similar lower bound obtained by considering  $h_{z-\lambda,\lambda}(w)$ , in view of (50) and (51) we have that for every  $z \in \mathbb{R}$ 

$$|P_{\theta}(W_{n,\theta} \le z) - P(Z \le z)| \le \frac{C_1}{\lambda r_{n,\theta}} \sum_{l=0}^{s_n} \delta(n-l, r_{n,\theta}/c_2) t_{n,\theta,l} + \frac{C_2}{r_{n,\theta}} + \frac{C_3}{\lambda r_{n,\theta}^2} + \frac{\lambda}{\sqrt{2\pi}}.$$

Letting  $\lambda = 2c_2C_1/r_{n,\theta}$ , and, noting that the right hand side does not depend on z, taking supremum over  $z \in \mathbb{R}$  yields

$$\sup_{z \in \mathbb{R}} |P_{\theta}(W_{n,\theta} \le z) - P(Z \le z)| \le \sum_{l=0}^{s_n} \delta(n - l, r_{n,\theta}/c_2) t_{n,\theta,l}/2c_2 + C_4/r_{n,\theta}$$

$$\le \sum_{l=0}^{s_n} \delta(n - l, r/c_2) t_{n,\theta,l}/2c_2 + C_4/r, \quad (52)$$

for  $C_4 = C_2 + C_3/2c_2C_1 + 2c_2C_1/\sqrt{2\pi}$ , where for the last inequality we have used that  $\theta \in \Theta_{n,r}$ , and that  $\delta(n,r)$  and 1/r are non-increasing functions of r. Taking supremum over  $\Theta_{n,r_1}$  on the right hand side of (52), then over  $\Theta_{n,r} \subset \Theta_{n,r_1}$  on the left yields

$$\delta(n,r) \le \sup_{\theta \in \Theta_{n,r_1}} \sum_{l=0}^{s_n} \delta(n-l, r/c_2) t_{n,\theta,l} / 2c_2 + C_4 / r.$$
 (53)

Suppose first that Condition 6b is satisfied, so that  $L_n = l_{n,0}$  almost surely for some  $l_{0,n} \in \mathbb{N}_0$  for all  $\theta \in \Theta_{n,r_1}$ . If  $l_{0,n} > s_n$  then (10) and (53) yield  $\delta(n,r) \leq C_4/r$ , proving (31). Otherwise  $0 \leq l_{n,0} \leq s_n$ , and as  $t_{n,\theta,l} = \mathbf{1}(l = l_{n,0})$ , inequality (53) specializes to

$$\delta(n,r) \le \delta(n - l_{n,0}, r/c_2)/2c_2 + C_4/r. \tag{54}$$

When Condition 6b is satisfied, the sum in (53) is a continuous function of  $\theta$  on the compact set  $\Theta_{n,r_1}$ , and hence achieves its supremum on some  $\theta_n^* \in \Theta_{n,r_1}$ . Letting  $p_{n,l} = t_{n,\theta_n^*,l}/2$  using (48) we have

$$\delta(n,r) \le \sum_{l=0}^{s_n} \delta(n-l, r/c_2) p_{n,l}/c_2 + C_4/r \quad \text{with} \quad \sum_{l=0}^{n} p_{n,l} = 1/2.$$
 (55)

As (54) is the special case of (55) when  $p_{n,l} = \mathbf{1}(l=l_0)/2$ , it suffices to handle the latter.

When the second inequality in Condition 4 holds for some  $c_2$  it also holds for all larger values, and we may therefore assume  $c_2 \ge 1$ . Let  $a_n = 0$  for  $0 \le n < n_0$ , and  $a_n = \sup_{r > r_1} r\delta(n,r)$  for  $n \ge n_0$ . For all  $r \ge r_1$  and  $n \ge n_0$  we have

$$(r/c_2)\delta(n, r/c_2) \leq \sup_{\substack{s:s \geq r_1\\ s:s \geq r_1/c_2}} (s/c_2)\delta(n, s/c_2)$$

$$= \sup_{\substack{s:s \geq r_1/c_2\\ s:s \geq r_1/c_2 \leq s < r_1}} s\delta(n, s)$$

$$\leq \sup_{\substack{s:r_1/c_2 \leq s < r_1\\ s:s \geq r_1}} s\delta(n, s) + \sup_{\substack{s:s \geq r_1\\ s:s \geq r_1}} s\delta(n, s)$$

$$\leq r_1 + a_n.$$

Using that  $n \geq n_1$  implies  $n - s_n \geq n_0$ , multiplication by r in (55) yields, with  $f = r_1/2 + C_4$ , that for all  $n \geq n_1$ 

$$r\delta(n,r) \le \sum_{l=0}^{s_n} (r/c_2)\delta(n-l,r/c_2)p_{n,l} + C_4 \le \sum_{l=0}^{s_n} (r_1 + a_{n-l})p_{n,l} + C_4 \le \sum_{l=0}^{n} a_{n-l}p_{n,l} + f.$$

Taking supremum on the left hand side over  $r \geq r_1$  and recalling (55) now yields

$$a_n \le \sum_{l=0}^n a_{n-l} p_{n,l} + f$$
 with  $\sum_{l=0}^n p_{n,l} = 1/2$ , for all  $n \ge n_1$ .

Lemma 3.1 now implies  $\sup_{n\geq n_1} a_n < \infty$ . Hence, there exists a constant C such that  $\delta(n,r) \leq C/r$  for all  $n\geq n_1$  and all  $r\geq r_1$ , that is, (31) holds.

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