The random conductance model with Cauchy tails

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Abstract

We consider a random walk in an i.i.d. Cauchy-tailed conductances environment. We obtain a quenched functional CLT for the suitably rescaled random walk, and, as a key step in the arguments, we improve the local limit theorem for $p^{\omega}_{n^2t}(0, y)$ in [BD09, Theorem 5.14] to a result which gives uniform convergence for $p^{\omega}_{n^2t}(x, y)$ for all x, y in a ball.

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1 Introduction

In this paper we will establish the convergence to Brownian motion of a random walk in a symmetric random environment, in a critical case that has not been covered by the papers [BC09, BD09]. We begin by recalling the 'random conductance model' (RCM). We consider the Euclidean lattice $\mathbb{Z}^d$ with $d \geq 2$. Let E_d be the set of non-oriented nearest neighbour bonds, and, writing $e = \{x, y\} \in E_d$, let $(\mu_e, e \in E_d)$ be non-negative i.i.d.r.v on $[1, \infty)$, defined on a probability space $(\Omega, \mathbb{P})$. We write $\mu_{xy} = \mu_{\{x,y\}} = \mu_{yx}$, let $\mu_{xy} = 0$ if $x \not\sim y$, and set $\mu_x = \sum_y \mu_{xy}$.

We consider two continuous time random walks on $\mathbb{Z}^d$ which jump from x to $y \sim x$ with probability μ_{xy}/μ_x. These are called in [BD09] the *constant speed random walk* (CSRW) and *variable speed random walk* (VSRW), and have generators

$$\mathcal{L}_C(\omega)f(x) = \mu_x(\omega)^{-1} \sum_y \mu_{xy}(\omega)(f(y) - f(x)), \tag{1.1}$$

$$\mathcal{L}_V(\omega)f(x) = \sum_y \mu_{xy}(\omega)(f(y) - f(x)). \tag{1.2}$$

We write X for the CSRW, and Y for the VSRW. Thus X jumps out of a state x at rate 1, while Y jumps out at rate μ_x. We will abuse notation slightly by writing P^x_ω for

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the laws of both X and Y started at $x \in \mathbb{Z}^d$ in the random environment $(\mu_e(\omega))$. Since the generators of these processes differ by a multiple, X and Y are time changes of each other. More explicitly, as in [BC09] define the *clock process*

$$S_t = \int_0^t \mu_{Y_s} ds, \tag{1.3}$$

and let A_t be its inverse. Then the CSRW can be defined by

$$X_t = Y_{A_t}, \quad t \geq 0. \tag{1.4}$$

In the case when $\mu_e \in [0,1]$, and $\mathbb{P}(\mu_e > 0) > p_c(d)$, the critical percolation probability for $\mathbb{Z}^d$, the papers [BP07, Mat08] prove that both X and Y satisfy a quenched functional central limit theorem (QFCLT), and that the limiting process is non-degenerate. The paper [BD09] studies the case when $\mu_e \in [1,\infty)$, and proves that for $\mathbb{P}$-a.a. ω the rescaled VSRW, defined by

$$Y_t^{(n,1)} = n^{-1} Y_{n^2 t},\ t \geq 0, \tag{1.5}$$

converges to $(\sigma_V W_t, t \geq 0)$, where W is a standard Brownian motion, and $\sigma_V > 0$. It is also proved there that $S_t/t \to \mathbb{E}\mu_0 \in [1,\infty]$. It follows from (1.4) that the CSRW with the standard rescaling

$$X_t^{(n,1)} = n^{-1} X_{n^2 t},\ t \geq 0,$$

converges to $\sigma_C W$, where

$$\sigma_C = \begin{cases} \sigma_V/(2d\mathbb{E}\mu_e), & \text{if } \mathbb{E}\mu_e < \infty, \\ 0, & \text{if } \mathbb{E}\mu_e = \infty. \end{cases}$$

If $\mathbb{E}\mu_e = \infty$ it is natural to ask if a different rescaling of X will give a non-trivial limit. In the case when $d \geq 3$, $\mu_e \in [1,\infty)$ and there exists $\alpha \in (0,1)$ such that

$$\mathbb{P}(\mu_e > u) \sim \frac{c}{u^\alpha}, \tag{1.6}$$

then [BC09] proves that the process

$$X_t^{(n,\alpha)} = n^{-1} X_{n^{2/\alpha} t},\ t \geq 0,$$

converges to the 'fractional kinetic motion' with index α. (For details of this process, and its connection with aging see [BCM06, BC07, BC08].) These papers leave open the case when $\alpha = 1$. In this paper we assume that (μ_e) satisfies (1.6) with $\alpha = 1$; for simplicity we take $c = 1/(2d)$, so that μ_e satisfies

$$\mathbb{P}(\mu_e \geq 1) = 1, \tag{1.7}$$

$$\mathbb{P}(\mu_x \geq u) \sim \frac{1}{u} \quad \text{as } u \to \infty. \tag{1.8}$$

We define the process

$$X_t^{(n)} = n^{-1} X_{n^2 (\log n)\, t}, \quad t \geq 0. \tag{1.9}$$

Our main theorem is:

Theorem 1. *Let $d \geq 3$, and assume that μ_e satisfies* (1.7) *and* (1.8)*. Then for $\mathbb{P}$-a.a. ω, $(X^{(n)}, P_\omega^0)$ converges in $D([0,\infty);\mathbb{R}^d)$ to $\sigma_1 W$, where $\sigma_1 = 2^{-1/2}\sigma_V > 0$, and W is a standard d-dimensional Brownian-motion.*

As in [BC09] we prove this theorem by using (1.4) and proving convergence of a rescaled clock process. Let

$$S_t^{(n)} = \frac{1}{n^2 \log n} \int_0^{n^2 t} \mu_{Y_s} ds; \tag{1.10}$$

then it is easy to check that if $A^{(n)}$ is the inverse of $S^{(n)}$ then

$$X_t^{(n)} = Y_{A_t^{(n)}}^{(n)}, \quad t \geq 0. \tag{1.11}$$

It follows that to prove Theorem 1 it is enough to prove

Theorem 2. *Let $d \geq 3$, and assume that μ_e satisfies* (1.7) *and* (1.8)*. For $\mathbb{P}$-a.a. ω, under the law P_ω^0,*

$$(S_t^{(n)}, t \geq 0) \Rightarrow (2t, t \geq 0) \text{ on } C([0,\infty);\mathbb{R}). \tag{1.12}$$

Remark 1. If we define for any $\lambda > 0$ that $S_t^{(\lambda)} = \frac{1}{\lambda \log \sqrt{\lambda}} \int_0^{\lambda t} \mu_{Y_s} ds$, then when $n^2 \leq \lambda \leq (n+1)^2$,

$$S_t^{(n)} \sim \frac{n^2 \log n}{(n+1)^2 \log(n+1)} \cdot S_t^{(n)} \leq S_t^{(\lambda)} \leq \frac{(n+1)^2 \log(n+1)}{n^2 \log n} \cdot S_t^{(n+1)} \sim S_t^{(n+1)}.$$

It follows that the convergence (1.12) holds for $(S_t^{(\lambda)}, t \geq 0)_{\lambda > 0}$, and hence Theorem 1 extends to $(X_t^\lambda) := (\lambda^{-1/2} X_{\lambda(\log\sqrt{\lambda})\, t})$.

As in [BC09], the result is proved by estimating the growth of the clock process S_t, $0 \leq t \leq n^2 T$. Since the limit of the processes $S^{(n)}$ is deterministic overall this case is much easier than when $\alpha \in (0,1)$: after suitable truncation it is enough to use a mean-variance calculation. There is however one respect in which this case is more delicate than when $\alpha < 1$. When $\alpha < 1$ it turns out that the main contribution to $S_{n^2 T}$ is from visits by Y to x such that $\varepsilon n^{2/\alpha} \leq \mu_x \leq \varepsilon^{-1} n^{2/\alpha}$ – see Sections 5 and 7 of [BC09]. When $\alpha = 1$ one finds that each set of edges of the form $E_i = \{e : 2^{i-1} n \leq \mu_e < 2^i n\}$, $i = 1, \ldots, \log n$, has a roughly comparable contribution to $S_{n^2 T}$, so a much greater range of values of μ_e need to be considered.

To motivate the proof, consider the classical case of a sum of i.i.d.r.v. ξ_i, with $\mathbb{P}(\xi_i > t) \sim t^{-1}$. We have that if

$$U_t^{(n)} = (n \log n)^{-1} \sum_{i=1}^{[nt]} \xi_i, \tag{1.13}$$

then $\sup_{0 \leq t \leq T} |U_t^{(n)} - t| \to 0$ in probability. Let $a_n = n(\log n)^\beta$ where $\beta \in (1,2)$, and $\xi_i' = \xi_i \mathbf{1}_{(\xi_i > a_i)}$. Then $\sum P(\xi_i \neq \xi_i')$ converges, so it is enough to consider the convergence of

$$V_t^{(n)} = (n \log n)^{-1} \sum_{i=1}^{[nt]} \xi_i'. \tag{1.14}$$

A straightforward argument calculating the mean and variance of

$$M_t^{(n)} = (n \log n)^{-1} \sum_{i=1}^{[nt]} (\xi'_i - E\xi'_i) \tag{1.15}$$

then gives convergence of $U^{(n)}$. (Note however that since $P(\max_{2^{n-1} \le i \le 2^n} \xi_i > 2^n \log 2^n) \sim c/n$, one does not have a.s. convergence).

The equivalent arguments in our case rely on good control of the process Y. Define the heat kernel and Green's functions for Y by

$$p_t^\omega(x, y) = P_\omega^x(Y_t = y), \qquad g^\omega(x, y) = \int_0^\infty p_t^\omega(x, y)\, dt. \tag{1.16}$$

We extend these functions from $\mathbb{Z}^d \times \mathbb{Z}^d$ to $\mathbb{R}^d \times \mathbb{R}^d$ by linear interpolation on each cube in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. Let W be a standard Brownian motion on $\mathbb{R}^d$, and let $W_t^* = \sigma_V W_t$, so that W^* is the weak limit of the processes $Y^{(n,1)}$. Let

$$k_t(x) = (2\pi\sigma_V^2)^{-d/2} \exp(-|x|^2/2\sigma_V^2) \tag{1.17}$$

be the density of the W^*.

A key element of the arguments is the following strengthening of the local limit theorem for $p_{n^2t}^\omega(0, y)$ in [BD09, Theorem 5.14] to a result which gives uniform convergence for $p_{n^2t}^\omega(x, y)$ for all x, y in a ball.

Theorem 3. *Let $d \ge 2$, and assume μ_e satisfies* (1.7). *For any $\varepsilon > 0$, $0 < \delta < T < \infty$ and $K > 0$, we have the following $\mathbb{P}$-almost sure* uniform *convergence:*

$$\begin{aligned} \frac{1}{1+\varepsilon} &< \liminf_{n\to\infty} \inf_{\delta \le t \le T} \inf_{|x|,|y| \le K} \frac{n^d p_{n^2t}^\omega(nx, ny)}{k_t(x, y)} \\ &\le \limsup_{n\to\infty} \sup_{\delta \le t \le T} \sup_{|x|,|y| \le K} \frac{n^d p_{n^2t}^\omega(nx, ny)}{k_t(x, y)} < 1 + \varepsilon. \end{aligned} \tag{1.18}$$

This result is proved in Section 2.1.

Notation. We write

$$B(x, r) = \{y \in \mathbb{Z}^d : |x - y| \le r\}, \quad B_{\mathbb{R}}(x, r) = \{y \in \mathbb{R}^d : |x - y| \le r\}.$$

If $e = \{x_e, y_e\} \in E_d$, we write $e \in B(x, r)$ if $\{x_e, y_e\} \subset B(x, r)$. We will follow the custom of writing $f \sim g$ to mean that the ratio f/g converges to 1, and $f \asymp g$ to mean that the ratio f/g remains bounded away from 0 and ∞. For any $a, b \in \mathbb{R}$, $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. Throughout the paper, c, C, C_1, C' etc. denote generic constants whose values may change from line to line.

Remark 2. One can also consider the more general case when the tail of μ_e satisfies

$$\mathbb{P}(\mu_e \ge u) \sim c\frac{(\log u)^\beta}{u} \quad \text{as } u \to \infty,$$

where $\beta \geq -1$ (so that $\mathbb{E}\mu_e = \infty$). Define for $t \geq 0$

$$X_t^{(n)} = \begin{cases} n^{-1} X_{n^2(\log n)^{1+\beta}\, t} & \text{when } \beta > -1, \\ n^{-1} X_{n^2(\log\log n)\, t} & \text{when } \beta = -1. \end{cases}$$

Then using the same strategy as in this article one can show that for $\mathbb{P}$-a.a. ω, $(X^{(n)}, P_\omega^0)$ converges to a (multiple of a) Brownian-motion.

2 Preliminaries

2.1 Transition probabilities: Proof of Theorem 3

We collect some known estimates for $p_t^\omega(x, y)$ and $g^\omega(x, y)$ which will be used in our arguments.

Lemma 4. *Let $\eta \in (0, 1)$. There exist random variables U_x ($x \in \mathbb{Z}^d$) and constants c_i such that*

$$\mathbb{P}(U_x \geq n) \leq c_1 \exp(-c_2 n^\eta).$$

(a) [BD09, Theorem 1.2(a)] There exists $c_3 > 0$ such that for all x, y and t,

$$p_t^\omega(x, y) \leq c_3 t^{-d/2}.$$

(b) [BD09, Theorem 1.2(b)] If $|x - y| \vee \sqrt{t} \geq U_x$, then

$$p_t^\omega(x, y) \leq \begin{cases} c_4 t^{-d/2} \exp(-c_5 |x - y|^2/t) & \text{when } t \geq |x - y|, \\ c_4 \exp\big(-c_5 |x - y| (1 \vee \log(|x - y|/t))\big) & \text{when } t \leq |x - y|. \end{cases} \tag{2.1}$$

(c) [BD09, Theorem 1.2(c)] If $t \geq U_x^2 \vee |x - y|^{1+\eta}$, then

$$p_t^\omega(x, y) \geq c_6 t^{-d/2} \exp(-c_7 |x - y|^2/t).$$

(d) Let $\tau(x, R) = \inf\{t \geq 0 : |Y_t - x| > R\}$. If $R \geq U_x$ then

$$P_\omega^x(\tau(x, R) \leq t) \leq c_8 \exp(-c_9 R^2/t).$$

(e) [BC09, Lemma 3.4] When $d \geq 3$,

$$c_{10} U_x^{2-d} \leq g^\omega(x, x) \leq c_{11}. \tag{2.2}$$

(f) [BC09, Proposition 3.2(b)] When $d \geq 3$, if $|x| \geq U_0$ then

$$g^\omega(0, x) \leq \frac{c_{12}}{|x|^{d-2}}. \tag{2.3}$$

(g) [BC09, Lemma 3.3] For each $K > 0$ there exists $c_{13} > 0$ such that if

$$b_n = c_{13}(\log n)^{1/\eta}, \tag{2.4}$$

then with $\mathbb{P}$-probability no less than $1 - c_{14}K^d n^{-2}$ the following holds:

$$\max_{|x|\le Kn} U_x \le b_n. \tag{2.5}$$

In particular, (2.5) holds for all n large enough $\mathbb{P}$-a.s..

(h) [BD09, Theorem 5.14] For any $\delta > 0$, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d} \sup_{t\ge\delta} \left| n^d p^{\omega}_{n^2t}(0,x) - k_t(x/n) \right| = 0. \tag{2.6}$$

(i) There exists $\theta > 0$ such that for $x, y, y' \in \mathbb{Z}^d$

$$n^d \left| p^{\omega}_{n^2t}(x,y) - p^{\omega}_{n^2t}(x,y') \right| \le c_{15} t^{-(d+\theta)/2} \cdot \left(\frac{|y-y'| \vee U_y}{n} \right)^{\theta}. \tag{2.7}$$

Proof. (d) The tail bound on $\tau(x,R)$ in (d) follows from Proposition 2.18 and Theorem 4.3 of [BD09].
(i) This follows from [BD09, Theorem 3.7] and [BH09, Proposition 3.2]. □

We begin by improving the local limit theorem in (2.6).

Lemma 5. *For any $\varepsilon > 0$, $K > 0$ and $0 < \delta < T < \infty$, there exists $\varepsilon_b > 0$ such that $\mathbb{P}$-a.s.,*

$$\limsup_{n\to\infty} \sup_{\delta\le t\le T} \sup \left\{ \frac{p^{\omega}_{n^2t}(nx_1, ny_1)}{p^{\omega}_{n^2t}(nx_2, ny_2)} : \ |x_i|, |y_i| \le K, |x_1 - x_2| \le \varepsilon_b, |y_1 - y_2| \le \varepsilon_b \right\} < 1 + \varepsilon. \tag{2.8}$$

Proof. By Lemma 4(g), we only need to work with the event $\{\max_{|x|\le Kn} U_x \le b_n\}$, on which by Lemma 4(i) we get that for all $t \ge \delta$,

$$n^d \left| p^{\omega}_{n^2t}(nx_1, ny_1) - p^{\omega}_{n^2t}(nx_1, ny_2) \right| \le C\delta^{-(d+\theta)/2} \cdot |y_1 - y_2|^{\theta} \vee \left| \frac{b_n}{n} \right|^{\theta}.$$

On the other hand, by (c) in the same lemma, there exists $\varepsilon_1 > 0$ such that for all n large such that $n^2\delta \ge b_n^2 \vee n^{1+\eta}(2K)^{1+\eta}$, all $\delta \le t \le T$ and $|x_1|, |y_1| \le K$,

$$n^d p^{\omega}_{n^2t}(nx_1, ny_1) \ge \varepsilon_1.$$

Hence

$$\left| 1 - \frac{p^{\omega}_{n^2t}(nx_1, ny_2)}{p^{\omega}_{n^2t}(nx_1, ny_1)} \right| \le \frac{C\delta^{-(d+\theta)/2}}{\varepsilon_1} \cdot |y_1 - y_2|^{\theta} \vee \left| \frac{b_n}{n} \right|^{\theta}.$$

The conclusion follows by taking ε_b small enough so that

$$\frac{C\delta^{-(d+\theta)/2}}{\varepsilon_1} \cdot \varepsilon_b^{\theta} < \sqrt{1+\varepsilon} - 1,$$

and then interchanging the roles of x and y in the argument above. □

Proof of Theorem 3. Let $\varepsilon_0 > 0$, to be chosen later. We first show that for any fixed $|x|, |y| \le K$, $\mathbb{P}$-a.s.,

$$\frac{1}{(1+\varepsilon_0)^4} \le \liminf_{n\to\infty} \inf_{\delta\le t\le T} \frac{n^d p^\omega_{n^2t}(nx, ny)}{k_t(x,y)} \le \limsup_{n\to\infty} \sup_{\delta\le t\le T} \frac{n^d p^\omega_{n^2t}(nx, ny)}{k_t(x,y)} \le (1+\varepsilon_0)^4. \quad (2.9)$$

The proof is similar to that in Lemma 4.2 in [BC09]. First fix an ε_b so that the LHS in (2.8) in Lemma 5 is bounded by $1+\varepsilon_0$. For any path $\gamma \in D([0,\infty);\mathbb{R}^d)$, define the hitting time $\sigma(\gamma) = \inf\{t : \gamma_t \in B(x, \varepsilon_b)\}$. Then by the QFCLT for the VSRW $Y^{(n)}$ we get that $\mathbb{P}$-a.s.,

$$\lim_n E^\omega_0 \mathbf{1}\{Y^{(n)}_{\sigma(Y^{(n)})+t} \in B(y,\varepsilon_b)\} = E_0\left(\mathbf{1}\{\sigma(W^*) < \infty\} \int_{z\in B(y,\varepsilon_b)} k_t(W^*_{\sigma(W^*)}, z)\, dz\right),$$

where W^* is the limit of the VSRW $Y^{(n)}$. So, writing $\sigma = \sigma(Y^{(n)})$,

$$\begin{aligned} P^0_\omega(Y^{(n)}_{\sigma+t} \in B(y,\varepsilon_b) | Y^{(n)}_\sigma, \sigma < \infty) &= \sum_{z\in B(ny, n\varepsilon_b)} p^\omega_{n^2t}(nY^{(n)}_\sigma, z) \\ &\ge (1+\varepsilon_0)^{-1} |B(ny, n\varepsilon_b)| \cdot p^\omega_{n^2t}(nY^{(n)}_\sigma, ny) \\ &\ge (1+\varepsilon_0)^{-2} |B(ny, n\varepsilon_b)| \cdot p^\omega_{n^2t}(nx, ny). \end{aligned}$$

Note that $|B(ny, n\varepsilon_b)| \sim n^d \cdot \mathrm{Vol}(B_\mathbb{R}(y, \varepsilon_b))$; using this and the analogous result for $k_t(x,y)$ we get that

$$\limsup_n n^d p^\omega_{n^2t}(nx, ny) \cdot P^\omega_0(\sigma(Y^{(n)}) < \infty) \le (1+\varepsilon_0)^4 P_0(\sigma(W^*) < \infty) \cdot k_t(x,y).$$

But by the QFCLT for the VSRW $Y^{(n)}$ again, $\lim_n P^\omega_0(\sigma(Y^{(n)}) < \infty) = P_0(\sigma(W^*) < \infty)$, hence we get the desired upper bound. The lower bound in (2.9) can be proved similarly.

We now let x, y vary over $B_\mathbb{R}(0, K)$. Find a finite set $\{z_1, \ldots, z_\ell\}$ such that $B_\mathbb{R}(0,K)$ is covered by the balls $B_\mathbb{R}(z_i, \varepsilon_b)$. By the previous argument, $\mathbb{P}$-a.s., for all $i, j = 1, \ldots, \ell$, $n^d p^\omega_{n^2t}(nz_i, nz_j)/k_t(z_i, z_j)$ is bounded above by $(1+\varepsilon_0)^4$ for all large n. Given $x, y \in B_\mathbb{R}(0,K)$, choose z_i, z_j so that $x \in B_\mathbb{R}(z_i, \varepsilon_b)$, $y \in B_\mathbb{R}(z_j, \varepsilon_b)$. Then using (2.8)

$$\frac{n^d p^\omega_{n^2t}(nx, ny)}{k_t(x,y)} = \frac{n^d p^\omega_{n^2t}(nz_i, nz_j)}{k_t(z_i, z_j)} \cdot \frac{n^d p^\omega_{n^2t}(nx, ny)}{n^d p^\omega_{n^2t}(nz_i, nz_j)} \cdot \frac{k_t(z_i, z_j)}{k_t(x,y)} < (1+\varepsilon_0)^6$$

for all large n. Taking $(1+\varepsilon_0)^6 < 1+\varepsilon$ this gives the upper bound in (1.18), and the lower bound can be proved similarly. □

2.2 Convergences after truncation

For any given $a > 0$, we introduce the following truncation of μ_x:

$$\widetilde{\mu}_e = \widetilde{\mu}^n_e = \mu_e \cdot \mathbf{1}_{\{\mu_e \le an^2\}}, \quad \widetilde{\mu}_x = \widetilde{\mu}^n_x = \sum_{y\sim x} \widetilde{\mu}_{xy}. \quad (2.10)$$

Then we have

$$\mathbb{E}\widetilde{\mu}_x \sim \log(an^2), \qquad \mathbb{E}\widetilde{\mu}_x^2 \le Can^2, \tag{2.11}$$

where C is a constant independent of a and n. Note that $\widetilde{\mu}_x$ and $\widetilde{\mu}_y$ are independent if $|x-y|>1$.

Lemma 6. *Let $K>0$ and $d\ge 3$.*
(a) If $f: B_{\mathbb{R}}(0,K)\to\mathbb{R}$ is continuous, then,

$$\frac{1}{n^d\log n}\sum_{|x|\le Kn}\widetilde{\mu}_x f(x/n) \to 2\int_{B_{\mathbb{R}}(0,K)} f(x)\ dx, \quad \mathbb{P}\text{- a.s.} \tag{2.12}$$

(b) If $g:(B_{\mathbb{R}}(0,K))^2\to\mathbb{R}$ is continuous, then,

$$\frac{1}{n^{2d}(\log n)^2}\sum_{|x|,|y|\le Kn}\widetilde{\mu}_x\widetilde{\mu}_y g(x/n,y/n) \to 4\int_{(B_{\mathbb{R}}(0,K))^2} g(x,y)\ dxdy, \quad \mathbb{P}\text{- a.s.} \tag{2.13}$$

Proof. In both cases we use a straightforward mean-variance calculation.
(a) Write I_n for the LHS of (2.12). Then as $\mathbb{E}\widetilde{\mu}_x \sim \log(an^2)\sim 2\log n$,

$$\mathbb{E}I_n = \frac{\mathbb{E}\widetilde{\mu}_0}{\log n}\sum_{|x|\le Kn} f(x/n)n^{-d} \to 2\int_{|x|\le K} f(x)\ dx \quad \text{ as } n\to\infty. \tag{2.14}$$

If $|x-y|\le 1$ then $|\operatorname{Cov}(\widetilde{\mu}_x,\widetilde{\mu}_y)|\le \operatorname{Var}(\widetilde{\mu}_0)$, by Cauchy-Schwartz. So

$$\begin{aligned}\operatorname{Var}_{\mathbb{P}}(I_n) &\le \frac{c||f||_\infty^2}{n^{2d}(\log n)^2}\sum_{|x|\le Kn}\operatorname{Var}(\widetilde{\mu}_0)\\ &\le \frac{C}{n^d(\log n)^2}an^2 \le \frac{C'}{n^{d-2}(\log n)^2}.\end{aligned}$$

So, for any $\varepsilon>0$ we deduce

$$\mathbb{P}(|I_n-\mathbb{E}I_n|>\varepsilon)\le \frac{\operatorname{Var}_{\mathbb{P}}(I_n)}{\varepsilon^2}\le \frac{c(\varepsilon)}{n^{d-2}(\log n)^2},$$

and so by Borel-Cantelli we have that $|I_n-\mathbb{E}I_n|<\varepsilon$ for all large n.
(b) Let J_n be the left hand side of (2.13). Write $B=B(0,Kn)$ and

$$\begin{aligned}J_n' &= \frac{1}{n^{2d}(\log n)^2}\sum_{x,y\in B,|x-y|\le 3}\widetilde{\mu}_x\widetilde{\mu}_y g(x/n,y/n),\\ J_n'' &= \frac{1}{n^{2d}(\log n)^2}\sum_{x,y\in B,|x-y|>3}\widetilde{\mu}_x\widetilde{\mu}_y g(x/n,y/n),\end{aligned}$$

Then since $\widetilde{\mu}_x\widetilde{\mu}_y\le \widetilde{\mu}_x^2+\widetilde{\mu}_y^2$,

$$\mathbb{E}|J_n'|\le \frac{c}{n^{2d}(\log n)^2}\sum_{x\in B}\mathbb{E}\widetilde{\mu}_x^2\,||g||_\infty \le \frac{c||g||_\infty}{n^{d-2}(\log n)^2}.$$

As this sum converges, by Borel-Cantelli $J'_n \to 0$ $\mathbb{P}$-a.s.

For J''_n we have

$$\mathbb{E}J''_n = \frac{(\mathbb{E}\widetilde{\mu}_x)^2}{n^{2d}(\log n)^2} \sum_{x,y\in B,|x-y|>3} g(x/n,y/n) \to 4\int_{|x|,|y|\le K} g(x,y)\; dxdy.$$

Furthermore,

$$\mathrm{Var}_{\mathbb{P}}(J''_n) \le \frac{C}{n^{4d}(\log n)^4} \sum_{x,y\in B,|x-y|>3} \left(\sum_{x',y'\in B,|x'-y'|>3} |\operatorname{Cov}(\widetilde{\mu}_x\widetilde{\mu}_y,\widetilde{\mu}_{x'}\widetilde{\mu}_{y'})| \right). \tag{2.15}$$

If all of x, y, x', y' are at distance greater than 1 apart in the sum in (2.15), then $\operatorname{Cov}(\widetilde{\mu}_x\widetilde{\mu}_y,\widetilde{\mu}_{x'}\widetilde{\mu}_{y'}) = 0$. So, after relabelling, we only have to handle two cases: when $|x-x'| \le 1$ and $|y-y'| \le 1$, and when $|x-x'| \le 1$ and $|y-y'| > 1$. Write K'_n and K''_n for these two sums. Observe that in both cases, since $|x-y| > 3$ and $|x'-y'| > 3$, we have $|y'-x| > 1$ and $|y-x'| > 1$.

In the first case,

$$|\operatorname{Cov}(\widetilde{\mu}_x\widetilde{\mu}_y,\widetilde{\mu}_{x'}\widetilde{\mu}_{y'})| \le \mathbb{E}\widetilde{\mu}_x\widetilde{\mu}_{x'} \cdot \mathbb{E}\widetilde{\mu}_y\widetilde{\mu}_{y'} \le cn^4, \tag{2.16}$$

and so

$$K'_n \le \frac{cn^{2d}n^4}{n^{4d}(\log n)^4} \le \frac{c}{n^{2d-4}(\log n)^4}.$$

In the second case,

$$|\operatorname{Cov}(\widetilde{\mu}_x\widetilde{\mu}_y,\widetilde{\mu}_{x'}\widetilde{\mu}_{y'})| \le \mathbb{E}\widetilde{\mu}_x\widetilde{\mu}_{x'} \cdot \mathbb{E}\widetilde{\mu}_y\widetilde{\mu}_{y'} \le cn^2(\log n)^2,$$

and so as the sum in K''_n contains $O(n^{3d})$ terms,

$$K''_n \le \frac{cn^{3d}n^2(\log n)^2}{n^{4d}(\log n)^4} \le \frac{c}{n^{d-2}(\log n)^2}.$$

Hence $\sum_n \mathrm{Var}_{\mathbb{P}}(J''_n) < \infty$, proving (2.13). □

Finally we state a simple lemma which can be proved by direct computations.

Lemma 7. *For any $K > 0$,*

(a)

$$\sum_{1\le|x|\le Kn} |x|^{2-d} = O(n^2).$$

(b)

$$\sum_{1\le|x|\le Kn} |x|^{4-2d} = \begin{cases} O(n) & \text{when } d = 3, \\ O(\log n) & \text{when } d = 4, \\ O(1) & \text{when } d \ge 5. \end{cases}$$

3 Estimates involving Green's Functions

For the usual simple random walk on $\mathbb{Z}^d$, $d \ge 3$, the Green's function $g(x,x)$ is a positive constant for all x. In our case, the best available lower bound (see Lemma 4(e)) gives that $\mathbb{P}$-a.s., for all large n and for all $|x| \le Kn$, $g^\omega(x,x) \ge C/(\log n)^{(d-2)/\eta}$. As this is not quite strong enough for the truncation arguments in the next section, we now derive some more precise bounds on sums of Green's functions in a ball.

Recall that E_d denotes the set of edges in $\mathbb{Z}^d$, and in Lemma 4(g) we defined $b_n = c_{13}(\log n)^{1/\eta}$. For $e = \{x_e, y_e\} \in E_d$, let $B(e,r) = B(x_e,r) \cup B(y_e,r)$. For $e = \{x_e, y_e\} \in E_d$ and $z \in \mathbb{Z}^d$ let

$$\gamma_n(e) = C_{\text{eff}}[\{x_e, y_e\}, B(e,b_n)^c], \tag{3.1}$$
$$\gamma_n(z) = C_{\text{eff}}[z, B(z,b_n+1)^c], \tag{3.2}$$

where $C_{\text{eff}}[A,B]$ denotes the effective conductivity between the sets A and B – see (3.8) in [BC09] or [1, Section 9.4]. Note that both $\gamma_n(e)$ and $\gamma_n(x)$ are decreasing in n, and $\gamma_\infty(e) := \lim_n \gamma_n(e)$ is the effective conductivity between e and infinity, while $\gamma_\infty(x) := \lim_n \gamma_n(x)$ is equal to $1/g^\omega(x,x)$. By [BC09, Lemma 6.2], for any $k \ge 1$, $\lim_n \mathbb{E}\gamma_n(e)^k < \infty$. Note further that μ_e and $\gamma_n(e)$ are independent, and also that $\gamma_n(e)$ and $\gamma_n(e')$ are independent if $|e - e'| \ge 2b_n + 1$. When $d \ge 3$, by Lemma 4(e), $g^\omega(x,x) < C < \infty$, and hence

$$\gamma_n(e) \ge \gamma_n(x) \ge \gamma_\infty(x) = 1/g^\omega(x,x) \ge 1/C > 0. \tag{3.3}$$

Let a_p be large enough so that $\mathbb{P}(\mu_e > a_p) < p_c(d)$, where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$. Let $\mathcal{C}(e)$ denote the cluster containing e in the bond percolation process for which $\{e \text{ is open}\} = \{\mu_e > a_p\}$. Then we have (see Theorems 6.75 and 5.4 in [G])

$$\mathbb{P}(|\mathcal{C}(e)| > m) \le \exp(-c_1 m), \qquad \mathbb{P}(\operatorname{diam}(\mathcal{C}(e)) > m) \le \exp(-c_2 m). \tag{3.4}$$

Let

$$F_n(e) = \{\operatorname{diam}(\mathcal{C}(e)) \ge \tfrac{1}{2} b_n\},$$
$$\gamma_n'(e) = \gamma_n(e) \cdot \mathbf{1}_{F_n(e)^c}.$$

Lemma 8. *(a) For any $K > 0$, $\mathbb{P}$-a.s., for all sufficiently large n, $\gamma_n(e) = \gamma_n'(e)$ for all $e \in B(0, 2Kn)$.*
(b) There exists $\theta > 0$ and $\Gamma = \Gamma(\theta) < \infty$ such that for all n

$$\mathbb{E} e^{\theta \gamma_n'(e)} < \Gamma.$$

(c) There exists $C = C(d) > 0$ such that for any $K > 0$, $\mathbb{P}$-a.s., for all large n,

$$\inf_{|x| \le Kn} g^\omega(x,x) \ge C/\log n.$$

Proof. (a) First note that

$$\mathbb{P}(\cup_{e\in B(0,2Kn)}F_n(e)) \le cn^d \exp(-c_2 b_n) = c\exp(d\log n - c'(\log n)^{1/\eta}). \tag{3.5}$$

Since $\eta < 1$ the RHS in (3.5) is summable, so that, for all but finitely many n, $\gamma_n(e) = \gamma'_n(e)$ for all $e \in B(0, 2Kn)$.
(b) On $F_n(e)^c$ the cluster $\mathcal{C}(e)$ is contained in $B(e, b_n)$, and each bond from $\mathcal{C}(e)$ to $\mathcal{C}(e)^c$ has conductivity less than a_p. Since there are at most $2d|\mathcal{C}(e)|$ such bonds, we deduce that $\gamma_n(e) \le da_p|\mathcal{C}(e)|$. So,

$$\mathbb{P}(\gamma'_n(e) > \lambda) \le \mathbb{P}(da_p|\mathcal{C}(e)| > \lambda) \le c\exp(-c'\lambda). \tag{3.6}$$

(c) Using (3.5) it is enough to consider

$$\mathbb{P}\left(\max_{e\in B(0,Kn)} \gamma'_n(e) > \lambda \log n\right) \le cn^d e^{-c'\lambda\log n},$$

which is summable when λ is large enough. □

For any $0 < a < b \le \infty$, define the sets

$$E_n(a,b) = \{e : an^2 \le \mu_e < bn^2\}. \tag{3.7}$$

Let m_n be chosen later with $m_n \ge 3b_n$. We tile $\mathbb{Z}^d$ with cubes of the form $Q = [0, m_n - 1]^d + m_n\mathbb{Z}^d$, so that each cube contains m_n^d vertices. Let z_i, $1 \le i \le d$ be the unit vectors in $\mathbb{Z}^d$, and given a cube Q in the tiling let

$$E(Q) = \big\{\{x, x + z_i\}, x \in Q, 1 \le i \le d\big\};$$

it is clear that $E(Q)$ gives a tiling of E_d, and that $|E(Q)| = dm_n^d$ for each Q. Let $K > 0$ be fixed, and let $\mathcal{Q}_n$ be the set of Q such that $Q \cap B(0, Kn + 1) \ne \emptyset$. We have $|\mathcal{Q}_n| \asymp (Kn/m_n)^d$.

Lemma 9. *(See [BC09, Lemma 6.3].) Let $a, K, \delta > 0$ be fixed.*
(a) Suppose that $Kn/\sqrt{d} \ge m_n \ge n^{\theta_1}$ for some $\theta_1 > 2/d$. Then there exists $\lambda > 0$ such that $\mathbb{P}$-a.s. for all but finitely many n,

$$\max_{Q\in\mathcal{Q}_n} \sum_{e\in E(Q)\cap E_n(a,\infty)} \gamma_n(e) \le \lambda m_n^d (an^2)^{-1}\mathbb{E}\gamma_n(e). \tag{3.8}$$

(b) Let $\theta_2 < 1/d$. Then $B(0, n^{\theta_2}) \cap E_n(a,\infty) = \emptyset$ for all but finitely many n.

Proof. (a) By Lemma 8(a) it is enough to bound the sum (3.8) with $\gamma'_n(e)$ instead of $\gamma_n(e)$. Let $Q \in \mathcal{Q}_n$. We divide $E(Q)$ into disjoint sets $(E(Q,j), j \in J)$ such that if e and e' are distinct edges in $E(Q,j)$ then $|e - e'| \ge 3b_n - 2$, each $|E(Q,j)| = (m_n/3b_n)^d := N_n$, and $|J| \sim d(3b_n)^d$.

Let $\eta_e = \mathbf{1}_{(\mu_e > an^2)}$, $p_n = \mathbb{E}\eta_e \sim 1/(2d) \cdot 1/(an^2)$, and

$$\xi_j = \sum_{e\in E(Q,j)} \gamma'_n(e)\eta_e.$$

Then the r.v. $(\gamma_n'(e), \eta_e, e \in E(Q,j))$ are independent, and so if θ and Γ are as in Lemma 8,

$$\mathbb{E}e^{\theta\xi_j} \le (1 + p_n(\Gamma - 1))^{N_n} \le e^{N_n p_n(\Gamma-1)}.$$

Hence for any $\lambda > 0$, and writing $\mathbb{E}\xi_j = N_n p_n \mathbb{E}\gamma_n'(e)$,

$$\begin{aligned}\mathbb{P}(\xi_j > \lambda\mathbb{E}\xi_j) &\le \exp(-\lambda\theta N_n p_n \mathbb{E}\gamma_n'(e) + N_n p_n(\Gamma - 1)) \\ &= \exp\left(-N_n p_n \left(\lambda\theta\mathbb{E}\gamma_n'(e) - \Gamma + 1\right)\right).\end{aligned}$$

By (3.3),

$$\mathbb{E}\gamma_n'(e) \ge 1/C \cdot \mathbb{P}(F_n(e)^c) \to 1/C,$$

hence there exists $\lambda > 0$ such that for all n large, $\lambda\theta\mathbb{E}\gamma_n'(e) - \Gamma + 1 \ge 1$, and so

$$\mathbb{P}\left(\xi_j > \lambda\mathbb{E}\xi_j\right) \le e^{-N_n p_n}.$$

Thus

$$\mathbb{P}\left(\sum_{j\in J}\xi_j > \lambda m_n^d p_n \mathbb{E}\gamma_n'(e)\right) \le d(3b_n)^d e^{-N_n p_n},$$

and so since $|\mathcal{Q}_n| \le cn^d$ and $N_n p_n \ge n^\varepsilon$ for some $\varepsilon > 0$, (3.8) follows by Borel-Cantelli.

(b) We have

$$\mathbb{P}(B(0, n^{\theta_2}) \cap E_n(a,\infty) \ne \emptyset) \le cn^{d\theta_2}(an^2)^{-1} \le cn^{d\theta_2 - 2};$$

so again the result follows using Borel-Cantelli. □

4 Proof of Theorem 2

Lemma 10. *Let $\omega \in \Omega$. If for each $t \ge 0$,*

$$S_t^{(n)} \to 2t \quad \textit{in } P_\omega^0\textit{-probability}, \tag{4.1}$$

then (1.12) *holds.*

Proof. Note that the LHS and RHS are both increasing processes, and the RHS is continuous and deterministic. The conclusion then follows from Theorem VI.3.37 in [JS03]. □

Lemma 11. *For each $\varepsilon > 0$ and $T > 0$, there exist $K > 0$ and $a > 0$ such that for $\mathbb{P}$-a.a. ω, for all $t \le T$, the following two inequalities hold:*

$$\limsup_n P_\omega^0\left(\frac{1}{n^2\log n}\sum_{|x|\ge Kn}\int_0^{n^2 t}\mu_x \cdot \mathbf{1}_{\{Y_s = x\}}\,ds > 0\right) \le \varepsilon, \tag{4.2}$$

$$\limsup_n P_\omega^0\left(\frac{1}{n^2\log n}\sum_{|x|\le Kn}\int_0^{n^2 t}\mu_x \cdot \mathbf{1}_{\{\mu_x \ge an^2\}}\mathbf{1}_{\{Y_s = x\}}\,ds > 0\right) \le \varepsilon. \tag{4.3}$$

Proof. Write F_K for the event in (4.2). Then by Lemma 4(d),

$$P_\omega^0(F_K) \le P_\omega^0(\tau(0, Kn) < n^2 t) \le c_8 \exp(-c_9 K^2/t),$$

provided that $Kn > U_x$. So, taking K sufficiently large, (4.2) holds for all sufficiently large n.

Choose $\theta_1 = (2+\varepsilon_1)/d$, $\theta_2 = (1-\varepsilon_2)/(d-2)$, where $\varepsilon_1 > 0, \varepsilon_2 > 2/d$ (so that $\theta_2 < 1/d$) and $\varepsilon_1 + \varepsilon_2 < 1$. Let $m_n = n^{\theta_1}$, and $\mathcal{Q}_n$ be as in Lemma 9. Let n be large enough so that (3.8) holds, and also that

$$B(0, n^{\theta_2}) \cap E_n(a, \infty) = \emptyset. \tag{4.4}$$

Then

$$P_\omega^0(Y \text{ hits } E_n(a,\infty) \cap B(0, Kn)) \le \sum_{Q \in \mathcal{Q}_n} \sum_{x \in E_n(a,\infty) \cap Q} \frac{g^\omega(0,x)}{g^\omega(x,x)}. \tag{4.5}$$

For $x \in E_n(a, \infty)$, if e_x is an edge containing x, then by (3.3) $1/g^\omega(x,x) \le \gamma_n(e_x)$. By (4.4) and (2.3) we can bound $g^\omega(0,x)$ by $c|x|^{2-d}$.

Let $\mathcal{Q}'_n$ be the set of $Q \in \mathcal{Q}_n$ such that $|x| \ge m_n/2$ for all $x \in Q$. Let first $Q \in \mathcal{Q}_n - \mathcal{Q}'_n$. Then by Lemma 9 and (4.4),

$$\begin{aligned} \sum_{x \in E_n(a,\infty) \cap Q} \frac{g^\omega(0,x)}{g^\omega(x,x)} &\le \max_{x \in E_n(a,\infty) \cap Q} c|x|^{2-d} \sum_{x \in E_n(a,\infty) \cap Q} \gamma_n(e_x) \\ &\le C n^{\theta_2(2-d)} \cdot \lambda m_n^d (an^2)^{-1} \le C' n^{\varepsilon_1 + \varepsilon_2 - 1}. \end{aligned}$$

So, since there are only 2^d cubes in $\mathcal{Q}_n - \mathcal{Q}'_n$,

$$\lim_n \sum_{Q \in \mathcal{Q}_n - \mathcal{Q}'_n} \sum_{x \in E_n(0,\infty) \cap Q} \frac{g^\omega(0,x)}{g^\omega(x,x)} = 0. \tag{4.6}$$

Now let $Q \in \mathcal{Q}'_n$, and let x_Q be the point in Q closest to 0. Then if $Q \in \mathcal{Q}'_n$,

$$\begin{aligned} \sum_{x \in E_n(a,\infty) \cap Q} \frac{g^\omega(0,x)}{g^\omega(x,x)} &\le c \sum_{x \in E_n(a,\infty) \cap Q} |x|^{2-d} \gamma_n(e_x) \\ &\le c|x_Q|^{2-d} \cdot \lambda m_n^d (an^2)^{-1} \le c' \lambda a^{-1} n^{-2} \sum_{x \in Q} |x|^{2-d}. \end{aligned} \tag{4.7}$$

So, summing over $Q \in \mathcal{Q}'_n$,

$$\begin{aligned} P_\omega^0(Y \text{ hits } E_n(a,\infty) \cap (\cup_{Q \in \mathcal{Q}'_n} Q)) &\le c \lambda a^{-1} n^{-2} \sum_{x \in B(0,(K+1)n)} (1 \vee |x|)^{2-d} \\ &\le c' \lambda (K+1)^2 \, a^{-1}, \end{aligned}$$

and so taking a large enough and noting (4.6), (4.3) follows. □

By Lemma 11 to prove (1.12) it suffices to consider the convergence of

$$\widetilde{S}_t^{(n)} = \frac{1}{n^2 \log n} \sum_{|x|\le Kn} \int_0^{n^2 t} \widetilde{\mu}_x \cdot \mathbf{1}_{\{Y_s = x\}}\, ds = \frac{1}{\log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot \int_0^t \mathbf{1}_{\{Y_{n^2 s} = x\}}\, ds, \tag{4.8}$$

where $\widetilde{\mu}_x$ is as in (2.10). Taking expectations with respect to P_ω^0 we have

$$\begin{aligned} E_\omega^0 \widetilde{S}_t^{(n)} &= \frac{1}{n^2 \log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot \int_0^{n^2 t} p_s^\omega(0,x)\, ds \\ &= \frac{1}{\log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot \int_0^t p_{n^2 r}^\omega(0,x)\, dr. \end{aligned} \tag{4.9}$$

Lemma 12. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, $\mathbb{P}$-a.s. for all sufficiently large n,*

$$E_\omega^0 \widetilde{S}_\delta^{(n)} \le \varepsilon. \tag{4.10}$$

Proof. By Lemma 4(g), we can assume n is large enough so that $\{\max_{|x|\le Kn} U_x \le b_n\}$. Hence, by Lemma 4(b), if $|x| \vee \sqrt{t} \ge b_n$, then

$$p_t^\omega(0,x) \le \begin{cases} c_4 t^{-d/2} \exp(-c_5 |x|^2/t) & \text{when } t \ge |x|, \\ c_4 \exp(-c_5|x|) & \text{when } t \le |x|. \end{cases}$$

Hence, by decomposing according to whether $|x| < b_n$ or $|x| \ge b_n$ we obtain

$$\begin{aligned} E_\omega^0 \widetilde{S}_\delta^{(n)} &= \frac{1}{n^2 \log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot \int_0^{n^2 \delta} p_s^\omega(0,x)\, ds \\ &\le \frac{1}{n^2 \log n} \sum_{|x|\le b_n} \widetilde{\mu}_x \cdot \int_0^{n^2\delta} c(1 \vee s)^{-d/2}\, ds && (4.11) \\ &\quad + \frac{1}{n^2 \log n} \sum_{b_n \le |x| \le Kn} \widetilde{\mu}_x \int_0^{|x|} c_4 e^{-c_5|x|}\, ds && (4.12) \\ &\quad + \frac{1}{n^2 \log n} \sum_{b_n \le |x| \le Kn} \widetilde{\mu}_x \cdot \int_{|x|}^{n^2\delta} c_4 s^{-d/2} e^{-c_5|x|^2/s}\, ds && (4.13) \end{aligned}$$

Write $\xi_n^{(i)}$, $i = 1,2,3$ for the terms in (4.11)–(4.13). Since the integral in (4.11) is bounded by $\int_0^\infty c(1\vee s)^{-d/2}\, ds < \infty$, we have

$$\mathbb{E}\xi_n^{(1)} \le c \frac{b_n^d}{n^2 \log n} \mathbb{E}\widetilde{\mu}_x \le c n^{-2} (\log n)^{d/\eta}.$$

Similarly for (4.12) we have

$$\mathbb{E}\xi^{(2)} \le c n^{-2} \sum_{|x|\le Kn} c_4 |x| e^{-c_5|x|} \le c' n^{-2}.$$

As these sums converge it follows from Borel-Cantelli that $\xi_n^{(i)} \le \varepsilon/3$ for all large n, for $i = 1, 2$.

It remains to control (4.13). First note that when $s \ge 1$,

$$\sum_{x\in\mathbb{Z}^d} s^{-d/2} e^{-\kappa|x|^2/s} \le C(\kappa). \tag{4.14}$$

So, interchanging the order of the sum and integral in (4.13),

$$\mathbb{E}\xi_n^{(3)} \le \frac{C}{n^2 \log n} \mathbb{E}\widetilde{\mu}_0 \cdot n^2\delta \le C'\delta.$$

Setting $t = s/|x|^2$ we have

$$\int_{|x|}^{n^2\delta} c_4 s^{-d/2} e^{-c_5|x|^2/s}\, ds \le C|x|^{2-d} \int_0^\infty t^{-d/2} e^{-c_5/t} dt \le C|x|^{2-d}. \tag{4.15}$$

Hence, applying Lemma 7 we get

$$\begin{aligned} \mathrm{Var}_{\mathbb{P}}\left(\xi_n^{(3)}\right) &\le \frac{C}{n^4(\log n)^2} \cdot \sum_{b_n \le |x| \le Kn} a n^2 |x|^{4-2d} \\ &\le \begin{cases} \dfrac{C}{n(\log n)^2} & \text{when } d = 3, \\ \dfrac{C}{n^2(\log n)} & \text{when } d = 4, \\ \dfrac{C}{n^2(\log n)^2} & \text{when } d \ge 5. \end{cases} \end{aligned}$$

By Chebyshev's inequality and Borel-Cantelli we then get that for δ small enough, $\mathbb{P}$-a.s., for all sufficiently large n, $\xi_n^{(3)} \le \varepsilon/3$. □

Proposition 13. *Let*

$$A_1(K,t,\delta) = \int_{|y|\le K} \int_\delta^t k_s(x)\, dx\, ds. \tag{4.16}$$

When $d \ge 3$, for any $K > 0$, $0 < \delta < T < \infty$, and $t \in (\delta, T]$, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} E_\omega^0 \left(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)}\right) = 2A_1(K,t,\delta). \tag{4.17}$$

Proof. By Lemma 6(a), it suffices to show that $\mathbb{P}$-a.s.,

$$\frac{1}{\log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot \int_\delta^t \left(p^\omega_{n^2 s}(0,x) - n^{-d} k_s(x/n)\right)\, ds \to 0.$$

The LHS is bounded in absolute value by

$$\frac{1}{n^d \log n} \sum_{|x|\le Kn} \widetilde{\mu}_x \cdot T \sup_{x\in\mathbb{Z}^d} \sup_{s\ge\delta} |n^d p^\omega_{n^2 s}(0,x) - k_s(x/n)|.$$

This converges to 0 $\mathbb{P}$-a.s. by Lemma 6(a) and Lemma 4(h). □

Proposition 14. *When $d \geq 3$, for any $\varepsilon > 0$, $K > 0$, $0 < \delta < T < \infty$, and $t \in (\delta, T]$, $\mathbb{P}$-a.s.,*

$$\limsup_n E^0_\omega \left(\widetilde{S}^{(n)}_t - \widetilde{S}^{(n)}_\delta \right)^2 \leq \varepsilon + 8(1+\varepsilon) \int_{|x|,|y|\leq K} \int_\delta^t k_s(x) \int_0^{t-s} k_r(x,y)\ dr\, ds\, dx\, dy. \tag{4.18}$$

Proof. Using the Markov property and the symmetry of Y,

$$E^0_\omega (S^{(n)}_t - S^{(n)}_\delta)^2 = \frac{2}{(\log n)^2} \left(\sum_{|x|,|y|\leq Kn} \widetilde{\mu}_x \widetilde{\mu}_y \cdot \int_\delta^t p^\omega_{n^2 s}(0,x) \ \int_0^{t-s} p^\omega_{n^2 r}(x,y)\ dr\ ds \right).$$

We begin by proving that, given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $\mathbb{P}$-a.s., for all large n,

$$\frac{2}{(\log n)^2} \sum_{|x|,|y|\leq Kn} \widetilde{\mu}_x \widetilde{\mu}_y \cdot \int_\delta^t p^\omega_{n^2 s}(0,x) \ \int_0^{\delta_1} p^\omega_{n^2 r}(x,y)\ dr\ ds \leq \varepsilon. \tag{4.19}$$

By Lemma 4(a) we have $p^\omega_{n^2 s}(0,x) \leq cn^{-d}$ for all $s \geq \delta$ and so the left side of (4.19) is bounded by

$$\frac{C}{n^d (\log n)^2} \sum_{|x|,|y|\leq Kn} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{\delta_1} p^\omega_{n^2 r}(x,y)\ dr \tag{4.20}$$

$$= \frac{C}{n^{d+2} (\log n)^2} \sum_{|x|,|y|\leq Kn,\ |x-y|>1} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{n^2\delta_1} p^\omega_r(x,y)\, dr \tag{4.21}$$

$$+ \frac{C}{n^{d+2} (\log n)^2} \sum_{|x|\leq Kn,\ |x-y|\leq 1} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{n^2\delta_1} p^\omega_r(x,y)\, dr. \tag{4.22}$$

Write A_n and B_n for the terms in (4.21) and (4.22).

The first term can be handled in the same way as in Lemma 12. Let $B = B(0, Kn)$, and write $A_n = A^{(1)}_n + A^{(2)}_n + A^{(3)}_n$, where

$$A^{(1)}_n = \frac{C}{n^{d+2} (\log n)^2} \sum_{x,y\in B,\, 1<|x-y|\leq b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{n^2\delta_1} p^\omega_r(x,y)\, dr, \tag{4.23}$$

$$A^{(2)}_n = \frac{C}{n^{d+2} (\log n)^2} \sum_{x,y\in B,\, |x-y|\geq b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_0^{|x-y|} p^\omega_r(x,y)\, dr, \tag{4.24}$$

$$A^{(3)}_n = \frac{C}{n^{d+2} (\log n)^2} \sum_{x,y\in B,\, |x-y|\geq b_n} \widetilde{\mu}_x \widetilde{\mu}_y \int_{|x-y|}^{n^2\delta_1} p^\omega_r(x,y)\, dr. \tag{4.25}$$

For (4.23) we have

$$\begin{aligned}\mathbb{E}A^{(1)}_n &\leq \frac{C}{n^{d+2} (\log n)^2} \sum_{x,y\in B,\ 1<|x-y|<b_n} \mathbb{E}(\widetilde{\mu}_x \widetilde{\mu}_y) \int_0^\infty c_4 (1 \vee s)^{-d/2} ds \\ &\leq \frac{C}{n^{d+2}} K^d n^d b_n^d \leq c \frac{(\log n)^{d/\eta}}{n^2},\end{aligned}$$

and since this sum converges we have $A_n^{(1)} \le \varepsilon/4$ for all large n, $\mathbb{P}$-a.s. The term $\mathbb{E}A_n^{(2)}$ is bounded in the same way as was the term $\xi_n^{(2)}$ in Lemma 12.

For (4.25),

$$A_n^{(3)} \le \frac{C}{n^{d+2}(\log n)^2} \sum_{x,y\in B,\ |x-y|>b_n} \widetilde{\mu}_x\widetilde{\mu}_y \int_{|x-y|}^{n^2\delta_1} c_4 s^{-d/2} \exp(-c_5|x-y|^2/s)\, ds. \tag{4.26}$$

Using (4.14) we have

$$\mathbb{E}A_n^{(3)} \le \frac{C}{n^{d+2}(\log n)^2} \cdot n^d(\mathbb{E}\widetilde{\mu}_0)^2 \cdot n^2\delta_1 = O(\delta_1).$$

We now bound $\mathrm{Var}_{\mathbb{P}}(A_n^{(3)})$. By (4.15), the integral in (4.26) is bounded by $c|x-y|^{2-d}$, so

$$\mathrm{Var}_{\mathbb{P}}(A_n^{(3)}) \le \frac{C}{n^{2d+4}(\log n)^4} \sum_{x_1,y_1\in B,\ |x_1-y_1|>b_n} \sum_{x_2,y_2\in B,\ |x_2-y_2|>b_n} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} \cdot |\operatorname{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})|.$$

We now bound this sum in the same way as was done for the variance in Lemma 6(b). Let

$$\mathcal{C}_1 = \{|x_1|,|x_2|,|y_1|,|y_2| \in B : |x_i-y_i| > b_n, i=1,2, |x_1-x_2|\le 1, |y_1-y_2|\le 1\},$$
$$\mathcal{C}_2 = \{|x_1|,|x_2|,|y_1|,|y_2| \in B : |x_i-y_i| > b_n, i=1,2, |x_1-x_2|\le 1, |y_1-y_2| > 1\}.$$

Note that if $|x_1-x_2|\le 1$, then since $|x_i-y_i|>b_n$, neither of the y_i can be within distance 1 of x_1. If $(x_1,\dots y_2)\in\mathcal{C}_1$ then $|\operatorname{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \le cn^4$, while if $(x_1,\dots y_2)\in\mathcal{C}_2$ then $|\operatorname{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \le c(\log n)^2 n^2$. So,

$$\begin{aligned}
&\frac{C}{n^{2d+4}(\log n)^4} \sum_{(x_1,\dots y_2)\in\mathcal{C}_1} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} \cdot |\operatorname{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})| \\
&\qquad \le \frac{C}{n^{2d+4}(\log n)^4} \sum_{x_1,y_1\in B} (1\vee|x_1-y_1|)^{4-2d} cn^4 \\
&\qquad \le \frac{C}{n^{2d}(\log n)^4}\, n^d \max_{x_1\in B} \sum_{y_1\in B(x,2Kn)} (1\vee|x_1-y_1|)^{4-2d} \\
&\qquad \le \frac{Cn}{n^d(\log n)^4},
\end{aligned}$$

where in the last inequality we used Lemma 7(b).

Also,

$$\frac{C}{n^{2d+4}(\log n)^4} \sum_{(x_1,\dots y_2)\in\mathcal{C}_2} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d} \cdot |\operatorname{Cov}(\widetilde{\mu}_{x_1}\widetilde{\mu}_{y_1}, \widetilde{\mu}_{x_2}\widetilde{\mu}_{y_2})|$$
$$\le \frac{C}{n^{2d+2}(\log n)^2} \sum_{(x_1,\dots y_2)\in\mathcal{C}_2} |x_1-y_1|^{2-d}|x_2-y_2|^{2-d}$$
$$\le \frac{C}{n^{2d+2}(\log n)^2} \sum_{x_1\in B} \sum_{y_1,y_2\in B(x,2Kn)} (1\vee|x_1-y_1|)^{2-d}(1\vee|x_2-y_2|)^{2-d}$$
$$\le \frac{C}{n^{d+2}(\log n)^2} \left(\sum_{y_1\in B(0,2Kn)} (1\vee|y_1|)^{2-d} \right)^2$$
$$\le \frac{Cn^4}{n^{d+2}(\log n)^2} = \frac{C}{n^{d-2}(\log n)^2}.$$

Thus $\sum_n \operatorname{Var}_{\mathbb{P}}(A_n^{(3)}) < \infty$, and so by Chebyshev's inequality and Borel-Cantelli, when δ_1 is small enough, $\mathbb{P}$-a.s. for all sufficiently large n, $A_n^{(3)} \le \varepsilon/4$.

To finish the proof of (4.19), it remains to bound the term (4.22). By Lemma 4(a), $\int_0^{n^2\delta_1} p_r^\omega(x,y)\,dr \le C$. Therefore by Cauchy-Schwartz,

$$B_n = \frac{C}{n^{d+2}(\log n)^2} \sum_{|x|\le Kn,\ |y-x|\le 1} \widetilde{\mu}_x\widetilde{\mu}_y \int_0^{n^2\delta_1} p_{n^2r}^\omega(x,y)\,dr \le \frac{C}{n^{d+2}(\log n)^2} \sum_{|x|\le Kn} \widetilde{\mu}_x^2.$$

Hence

$$\mathbb{E}B_n \le \frac{C}{n^{d+2}(\log n)^2} \cdot n^d \cdot n^2 \to 0,$$

and since $\operatorname{Var}_{\mathbb{P}}(\widetilde{\mu}_x^2) \le cn^6$,

$$\operatorname{Var}_{\mathbb{P}}(B_n) \le \frac{C}{n^{2d+4}(\log n)^4} \cdot n^d \cdot n^6 \le \frac{C}{n^{d-2}(\log n)^4}.$$

Since this bound is summable, (4.19) follows.

It remains to show that for any $\delta_1 > 0$, $\mathbb{P}$-a.s.,

$$\limsup_n \frac{2}{(\log n)^2} \sum_{|x|,|y|\le Kn} \widetilde{\mu}_x\widetilde{\mu}_y \cdot \int_\delta^t p_{n^2s}^\omega(0,x) \int_{\delta_1}^{t-s} p_{n^2r}^\omega(x,y)\,dr\,ds$$
$$\le 8(1+\varepsilon) \int_{|x|,|y|\le K} \left(\int_\delta^t k_s(0,x) \int_0^{t-s} k_r(x,y)\,dr\,ds \right) dx\,dy.$$

This follows easily from Theorem 3 and Lemma 6. $\square$

Proof of Theorem 2. By Lemma 10, it suffices to show that for any $t > 0$ and $0 < \varepsilon < t/2$, for $\mathbb{P}$-a.a. ω,

$$\lim_n P_\omega^0(|S_t^{(n)} - 2t| \geq \epsilon) \leq \epsilon. \tag{4.27}$$

Write

$$\begin{aligned} S_t^{(n)} - 2t =& (S_t^{(n)} - \widetilde{S}_t^{(n)}) + \widetilde{S}_\delta^{(n)} + \left(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)} - E_\omega^0\left(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)}\right)\right) \\ &+ \left(E_\omega^0\left(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)}\right) - 2A_1(K,t,\delta)\right) + (2A_1(K,t,\delta) - 2t). \end{aligned} \tag{4.28}$$

By Proposition 13, $\mathbb{P}$-a.s., $\left(E_\omega^0\left(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)}\right) - 2A_1(K,t,\delta)\right) \to 0$. Let $0 < \varepsilon_0 < \varepsilon/16$, to be chosen later. Choose K large enough so that the LHS in (4.2) is bounded by ε_0, and also

$$\sup_{0<\delta\leq t/2} |A_1(K,t,\delta) - (t-\delta)| \leq \varepsilon_0 < \varepsilon/16, \tag{4.29}$$

Now choose $a > 0$ large enough so that the LHS in (4.2) is also bounded by ε_0. Hence, for all large n,

$$P_\omega^0(|S_t^{(n)} - \widetilde{S}_t^{(n)}| > 0) \leq 2\varepsilon_0 < \varepsilon/4.$$

Next choose $0 < \delta < \varepsilon/16$ so that, by Lemma 12, for all sufficiently large n, $E_\omega^0 \widetilde{S}_\delta^{(n)} < \varepsilon^2/16$, and hence $P_\omega^0(\widetilde{S}_\delta^{(n)} > \varepsilon/4) \leq \varepsilon/4$. Furthermore, by Propositions 13 and 14 and (4.29),

$$\begin{aligned} \limsup_n \mathrm{Var}_{\mathbb{P}}\left(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)}\right) &\leq \varepsilon_0 + 8(1+\varepsilon_0)\cdot(t-\delta)^2/2 - (2(t-\delta-\varepsilon_0))^2 \\ &\leq \varepsilon_0(1 + 4t^2 + 4t), \end{aligned}$$

hence by Chebyshev's inequality,

$$\limsup_n P_\omega^0(|\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)} - E_\omega(\widetilde{S}_t^{(n)} - \widetilde{S}_\delta^{(n)})| \geq \varepsilon/4) \leq 16(1+4t^2+4t)\cdot\varepsilon_0/\varepsilon^2.$$

Taking ε_0 so small that $\varepsilon_0 < \varepsilon/16$ and $16(1+4t^2+4t)\cdot\varepsilon_0/\varepsilon^2 \leq \varepsilon/4$ we obtain (4.27). □

References

[BD09] M.T. Barlow and J.-D. Deuschel. Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.*, To appear.

[BH09] M. T. Barlow and B. M. Hambly. Parabolic Harnack inequality and local limit theorem for percolation clusters. *Electron. J. Probab.*, 14:no. 1, 1–27, 2009.

[BC09] M.T. Barlow and Jiří Černý. Convergence to fractional kinetics for random walks associated with unbounded conductances. *Preprint*, 2009.

[BC07] Gérard Ben Arous and Jiří Černý. Scaling limit for trap models on $\mathbb{Z}^d$. *Ann. Probab.*, 35(6):2356–2384, 2007.

[BC08] Gérard Ben Arous and Jiří Černý. The arcsine law as a universal aging scheme for trap models. *Comm. Pure Appl. Math.*, 61(3):289–329, 2008.

[BCM06] Gérard Ben Arous, Jiří Černý, and Thomas Mountford. Aging in two-dimensional Bouchaud's model. *Probab. Theory Related Fields*, 134(1):1–43, 2006.

[BP07] Marek Biskup and Timothy M. Prescott. Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.*, 12:no. 49, 1323–1348 (electronic), 2007.

[G] G.R. Grimmett. *Percolation.* (2nd edition). Springer 1999.

[JS03] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 2003.

[1] D.A. Levin, Y. Peres, E.L. Wilmer. *Markov Chains and Mixing Times.* Amer. Math. Soc. 2009.

[Mat08] P. Mathieu. Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.*, 130(5):1025–1046, 2008.

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