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NUMERICAL METHOD FOR OPTIMAL STOPPING OF PIECEWISE DETERMINISTIC MARKOV PROCESSES*

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Abstract We propose a numerical method to approximate the value function for the optimal stopping problem of a piecewise deterministic Markov process (PDMP). Our approach is based on quantization of the post jump location – inter-arrival time Markov chain naturally embedded in the PDMP, and path-adapted time discretization grids. It allows us to derive bounds for the convergence rate of the algorithm and to provide a computable ϵ-optimal stopping time. The paper is illustrated by a numerical example.

1. Introduction. The aim of this paper is to propose a computational method for optimal stopping of a piecewise deterministic Markov process $\{X(t)\}$ by using a quantization technique for an underlying discrete-time Markov chain related to the continuous-time process $\{X(t)\}$ and path-adapted time discretization grids.

Piecewise-deterministic Markov processes (PDMP's) have been introduced in the literature by M.H.A. Davis [6] as a general class of stochastic models. PDMP's are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP

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$\{X(t)\}$ depends on three local characteristics, namely the flow ϕ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from x the motion of the process follows the flow $\phi(x,t)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\phi(x,t))$ or when the flow $\phi(x,t)$ hits the boundary of the state-space. In either case the location of the process at the jump time T_1: $X(T_1) = Z_1$ is selected by the transition measure $Q(\phi(x,T_1),\cdot)$. Starting from Z_1, we now select the next interjump time $T_2 - T_1$ and postjump location $X(T_2) = Z_2$. This gives a piecewise deterministic trajectory for $\{X(t)\}$ with jump times $\{T_k\}$ and post jump locations $\{Z_k\}$ which follows the flow ϕ between two jumps. A suitable choice of the state space and the local characteristics ϕ, λ, and Q provides stochastic models covering a great number of problems of operations research [6].

Optimal stopping problems have been studied for PDMP's in [3, 5, 6, 9, 11, 13]. In [11] the author defines an operator related to the first jump time of the process, and shows that the value function of the optimal stopping problem is a fixed point for this operator. The basic assumption in this case is that the final cost function is continuous along trajectories, and it is shown that the value function will also have this property. In [9, 13] the authors adopt some stronger continuity assumptions and boundary conditions to show that the value function of the optimal stopping problem satisfies some variational inequalities, related to integro-differential equations. In [6], M.H.A. Davis assumes that the value function is bounded and locally Lipschitz along trajectories to show that the variational inequalities are necessary and sufficient to characterize the value function of the optimal stopping problem. In [5], the authors weakened the continuity assuptions of [6, 9, 13]. A paper related to our work is [3] by O.L.V. Costa and M.H.A. Davis. It is the only one presenting a computational technique for solving the optimal stopping problem for a PDMP based on a discretization of the state space similar to the one proposed by H. J. Kushner in [12]. In particular, the authors in [3] derive a convergence result for the approximation scheme but no estimation of the rate of convergence is derived.

Quantization methods have been developed recently in numerical probability, nonlinear filtering or optimal stochastic control with applications in finance [1, 2, 14, 15, 16, 17]. More specifically, powerful and interesting methods have been developed in [1, 2, 17] for computing the Snell-envelope associated to discrete-time Markov chains and diffusion processes. Roughly speaking, the approach developed in [1, 2, 17] for studying the optimal stopping problem for a continuous-time diffusion process $\{Y(t)\}$ is based on a time-discretization scheme to obtain a discrete-time Markov chain $\{\overline{Y}_k\}$. It is shown that the original continuous-time optimization problem can be converted to an auxiliary optimal stopping problem associated with the discrete-time Markov chain $\{\overline{Y}_k\}$. Under some suitable assumptions, a rate of converge of the auxiliary value function to the original one can be derived. Then, in order to address the optimal stopping problem of the discrete-time Markov chain, a twofold computational method is proposed. The first step consists in approximating the Markov chain by a quantized process. There exists an extensive literature on quantization methods for random variables and processes. We do not pretend to present here an exhaustive panorama of these methods. However, the interested reader may for instance, consult the following works [10, 14, 17] and references therein. The second step is to approximate the conditional expectations which are used to compute the backward dynamic programming formula by the conditional expectation related to the quantized process. This procedure leads to a tractable formula called a *quantization tree algorithm* (see Proposition 4 in [1] or section 4.1 in [17]). Providing the cost function and the Markov kernel are Lipschitz, some bounds and rates of convergence are obtained (see for example section 2.2.2 in [1]).

As regards PDMP's, it was shown in [11] that the value function of the optimal stopping problem can be calculated by iterating a functional operator, labeled L (see equation (3.5) for its definition), which is related to a continuous-time maximization and a discrete-time dynamic programming formula. Thus, in order to approximate the value function of the optimal stopping problem of a PDMP $\{X(t)\}$, a natural approach would have been to follow the same lines as in [1, 2, 17]. However their method cannot be directly applied to our problem for two main reasons related to the specificities of PDMP's.

First, PDMP's are in essence discontinuous at random times. Therefore, as pointed out in [11], it will be problematic to convert the original optimization problem into an optimal stopping problem associated to a time discretization of $\{X(t)\}$ with nice convergence properties. In particular, it appears ill-advised to propose as in [1] a fixed-step time-discretization scheme $\{X(k\Delta)\}$ of the original process $\{X(t)\}$. Besides, another important intricacy concerns the transition semigroup $\{P_t\}_{t\in\mathbb{R}_+}$ of $\{X(t)\}$. On the one hand, it cannot be explicitly calculated from the local characteristics (ϕ, λ, Q) of the PDMP (see [4, 7]). Consequently, it will be complicated to express the Markov kernel P_Δ associated to the Markov chain $\{X(k\Delta)\}$. On the other hand, the markov chain $\{X(k\Delta)\}$ is in general not even a Feller chain (see [6, pages 76-77]), therefore it will be hard to ensure it is K-Lipschitz (see Definition 1 in [1]).

Second, the other main difference stems from the fact that the function appearing in the backward dynamic programming formula associated to L and the reward function g is not continuous even if some strong regularity assumptions are made on g. Consequently, the approach developed in [1, 2, 17] has to be refined since it can only handle conditional expectations of Lipschitz-continuous functions.

However, by using the special structure of PDMP's, we are able to overcome both these obstacles. Indeed, associated to the PDMP $\{X(t)\}$, there exists a natural embedded discrete-time Markov chain $\{\Theta_k\}$ with $\Theta_k = (Z_k, S_k)$ where S_k is given by the inter-arrival time $T_k - T_{k-1}$. The main operator L can be expressed using the chain $\{\Theta_k\}$ and a continuous-time maximization. We first convert the continuous-time maximization of operator L into a discrete-time maximization by using a path-dependent time-discretization scheme. This enables us to approximate the value function by the solution of a backward dynamic programming equation in discrete-time involving conditional expectation of the Markov chain $\{\Theta_k\}$. Then, a natural approximation of this optimization problem is obtained by replacing $\{\Theta_k\}$ by its quantized approximation. It must be pointed out that this optimization problem is related to the calculation of conditional expectations of indicator functions of the Markov chain $\{\Theta_k\}$. As said above, it is not straightforward to obtain convergence results as in [1, 2, 17]. We deal successfully with indicator functions by showing that the event on which the discontinuity actually occurs is of small enough probability. This enables us to provide rate of convergence for the approximation scheme.

In addition and more importantly, this numerical approximation scheme enables us to propose a computable stopping rule which also is an ϵ-optimal stopping time of the original stopping problem. Indeed, for any $\epsilon > 0$ one can construct a stopping time, labeled τ, such that

$$V(x) - \epsilon \leq \mathbf{E}_x\Big[g(X(\tau))\Big] \leq V(x)$$

where $V(x)$ is the optimal value function associated to the original stopping problem. Our computational approach is attractive in the sense that it does not require any additional calculations. Moreover, we can characterize how far it is from optimal in terms of the value function. In [1, section 2.2.3, Proposition 6], another criteria for the approximation of the optimal stopping time has been proposed. In the context of PDMP's, it must be noticed that an optimal stopping time does not generally exists as shown in [11, section 2].

An additional result extends Theorem 1 of U.S. Gugerli [11] by showing that the iteration of operator L provides a sequence of random variables which corresponds to a *quasi*-Snell envelope associated to the reward process $\big\{g(X(t))\big\}_{t\in\mathbb{R}_+}$ where the horizon time is random and given by the jump times $(T_n)_{n\in\{0,\ldots,N\}}$ of the process $\big\{X(t)\big\}_{t\in\mathbb{R}_+}$.

The paper is organized as follows. In Section 2 we give a precise definition of PDMP's and state our notation and assumptions. In Section 3, we state the optimal stopping problem, recall and refine some results from [11]. In Section 4, we build an approximation of the value function. In Section 5, we compute the error between the approximate value function and the real one. In Section 6 we propose a computable ϵ-optimal stopping time and evaluate its sharpness. Finally in Section 7 we present a numerical example. Technical results are postponed to the Appendix.

2. Definitions and assumptions. We first give a precise definition of a piecewise deterministic Markov process. Some general assumptions are presented in the second part of this section. Let us introduce first some standard notation. Let M be a metric space. $\mathbf{B}(M)$ is the set of real-valued, bounded, measurable functions defined on M. The Borel σ-field of M is denoted by $\mathcal{B}(M)$. Let Q be a Markov kernel on $(M, \mathcal{B}(M))$ and $w \in \mathbf{B}(M)$, $Qw(x) = \int_M w(y)Q(x, dy)$ for $x \in M$. For $(a, b) \in \mathbb{R}^2$, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

2.1. *Definition of a PDMP.* Let E be an open subset of $\mathbb{R}^n$, ∂E its boundary and $\overline{E}$ its closure. A PDMP is determined by its local characteristics (ϕ, λ, Q) where:
• The flow $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a one-parameter group of homeomorphisms: ϕ is continuous, $\phi(\cdot, t)$ is an homeomorphism for each $t \in \mathbb{R}$ satisfying $\phi(\cdot, t+s) = \phi(\phi(\cdot, s), t))$.
For all x in E, let us denote

$$t^*(x) \doteq \inf\{t > 0 : \phi(x,t) \in \partial E\},$$

with the convention $\inf \emptyset = \infty$.
• The jump rate $\lambda : \overline{E} \to \mathbb{R}_+$ is assumed to be a measurable function satisfying:

$$(\forall x \in E), \quad (\exists \varepsilon > 0) \text{ such that } \int_0^{\varepsilon} \lambda(\phi(x,s))ds < \infty.$$

• Q is a Markov kernel on $(\overline{E}, \mathcal{B}(\overline{E}))$ satisfying the following property:

$$(\forall x \in \overline{E}), \quad Q(x, E - \{x\}) = 1.$$

From these characteristics, it can be shown [6, p. 62-66] that there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{\mathbf{P}_x\}_{x\in E})$ such that the motion of the process $\{X(t)\}$ starting from a point $x \in E$ may be constructed as follows. Take a random variable T_1 such that

$$\mathbf{P}_x(T_1 > t) \doteq \begin{cases} e^{-\Lambda(x,t)} & \text{for } t < t^*(x), \\ 0 & \text{for } t \geq t^*(x), \end{cases}$$

where for $x \in E$ and $t \in [0, t^*(x)]$

$$\Lambda(x,t) \doteq \int_0^t \lambda(\phi(x,s))ds.$$

If T_1 generated according to the above probability is equal to infinity, then for $t \in \mathbb{R}_+$, $X(t) = \phi(x,t)$. Otherwise select independently an E-valued random variable (labelled Z_1) having distribution $Q(\phi(x,T_1), \cdot)$, namely $\mathbf{P}_x(Z_1 \in A) = Q(\phi(x,T_1), A)$ for any $A \in \mathcal{B}(\overline{E})$. The trajectory of $\{X(t)\}$ starting at x, for $t \leq T_1$, is given by

$$X(t) \doteq \begin{cases} \phi(x,t) & \text{for } t < T_1, \\ Z_1 & \text{for } t = T_1. \end{cases}$$

Starting from $X(T_1) = Z_1$, we now select the next inter-jump time $T_2 - T_1$ and post-jump location $X(T_2) = Z_2$ is a similar way.

This gives a strong Markov process $\{X(t)\}$ with jump times $\{T_k\}_{k\in\mathbb{N}}$ (where $T_0 = 0$). Associated to $\{X(t)\}$, there exists a discrete time process $(\Theta_n)_{n\in\mathbb{N}}$ defined by $\Theta_n = (Z_n, S_n)$ with

$Z_n = X(T_n)$ and $S_n = T_n - T_{n-1}$ for $n \geq 1$ and $S_0 = 0$. Clearly, the process $(\Theta_n)_{n \in \mathbb{N}}$ is a Markov chain.

We introduce a standard assumption, see for example equations (24.4) or (24.8) in [6].

Assumption 2.1 *For all* $(x,t) \in E \times \mathbb{R}_+$, $\mathbf{E}_x \Big[\sum_k \mathbf{1}_{\{T_k \leq t\}} \Big] < \infty$.

In particular, it implies that $T_k \to \infty$ as $k \to \infty$.

For $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the family of all $\{\mathcal{F}_t\}$-stopping times which are dominated by T_n and for $n < p$, let $\mathcal{M}_{n,p}$ be the family of all $\{\mathcal{F}_t\}$-stopping times ν satisfying $T_n \leq \nu \leq T_p$. Let $\mathbf{B}^c$ denote the set of all real-valued, bounded, measurable functions, w defined on $\overline{E}$ and continuous along trajectories up to the jump time horizon: for any $x \in E$, $w(\phi(x,.))$ is continuous on $[0, t^*(x)]$. Let $\mathbf{L}^c$ be the set of all real-valued, bounded, measurable functions, w defined on $\overline{E}$ and Lipschitz along trajectories:

1. there exists $[w]_1 \in \mathbb{R}_+$ such that for any $(x,y) \in E^2$, $u \in [0, t^*(x) \wedge t^*(y)]$, one has
$$\big|w(\phi(x,u)) - w(\phi(y,u))\big| \leq [w]_1 |x-y|,$$
2. there exists $[w]_2 \in \mathbb{R}_+$ such that for any $x \in E$, and $(t,s) \in [0, t^*(x)]^2$, one has
$$\big|w(\phi(x,t)) - w(\phi(x,s))\big| \leq [w]_2 |t-s|,$$
3. there exists $[w]_* \in \mathbb{R}_+$ such that for any $(x,y) \in E^2$, one has
$$\big|w(\phi(x,t^*(x))) - w(\phi(y,t^*(y)))\big| \leq [w]_* |x-y|.$$

In the sequel, for any function f in $\mathbf{B}^c$, we denote by C_f its bound:
$$C_f = \sup_{x \in E} |f(x)|,$$
and for any Lipschitz-continuous function f in $\mathbf{B}(E)$ or $\mathbf{B}(\overline{E})$, we denote by $[f]$ its Lipschitz constant:
$$[f] = \sup_{x \neq y \in E} \frac{\big|f(x) - f(y)\big|}{|x-y|}.$$

Remark 2.2 $\mathbf{L}^c$ *is a subset of* $\mathbf{B}^c$ *and any function in* $\mathbf{L}^c$ *is Lipschitz on* $\overline{E}$ *with* $[w] \leq [w]_1$.

Finally, as a convenient abbreviation, we set for any $x \in \overline{E}$, $\lambda Q w(x) = \lambda(x) Q w(x)$.

2.2. *Assumptions.* The following assumptions will be in force throughtout.

Assumption 2.3 *The jump rate λ is bounded and there exists $[\lambda]_1 \in \mathbb{R}_+$ such for any $(x,y) \in E^2$, $u \in [0, t^*(x) \wedge t^*(y)[$,*

$$\big|\lambda(\phi(x,u)) - \lambda(\phi(y,u))\big| \leq [\lambda]_1 |x-y|.$$

Assumption 2.4 *The exit time t^* is bounded and Lipschitz-continuous on E.*

Assumption 2.5 *The Markov kernel Q is Lipschitz in the following sense : there exists $[Q] \in \mathbb{R}_+$ such that for any function $w \in \mathbf{L}^c$ the following two conditions are satisfied:*

1. *for any $(x,y) \in E^2$, $u \in [0, t^*(x) \wedge t^*(y)]$, one has*

$$\big|Qw(\phi(x,u)) - Qw(\phi(y,u))\big| \leq [Q]\,[w]_1 |x-y|,$$

2. *for any $(x,y) \in E^2$, one has*

$$\big|Qw(\phi(x,t^*(x))) - Qw(\phi(y,t^*(y)))\big| \leq [Q]\,[w]_* |x-y|.$$

The reward function g associated to the optimal stopping problem satisfies the following hypothesis.

Assumption 2.6 *g is in $\mathbf{L}^c$.*

3. Optimal stopping problem. From now on, assume that the distribution of $X(0)$ is given by δ_{x_0} for a fixed state $x_0 \in E$. Let us consider the following optimal stopping problem for a fixed integer N:

$$\sup_{\tau \in \mathcal{M}_N} \mathbb{E}_{x_0}[g(X(\tau))] \tag{3.1}$$

This problem has been studied by U.S. Gugerli in [11].

Note that Assumption 2.3 yields $\Lambda(x,t) < \infty$ for all x, t. Hence, for all x in E, the jump time horizon $s^*(x)$ defined in [11] by $t^*(x) \wedge \inf\{t \geq 0,\ \mathrm{e}^{-\Lambda(x,t)} = 0\}$ is equal to the exit time $t^*(x)$. Therefore, operators $H : \mathbf{B}(\overline{E}) \to \mathbf{B}(E \times \mathbb{R}_+)$, $I : \mathbf{B}(E) \to \mathbf{B}(E \times \mathbb{R}_+)$, $J : \mathbf{B}(E) \times \mathbf{B}(\overline{E}) \to \mathbf{B}(E \times \mathbb{R}_+)$, $K : \mathbf{B}(E) \to \mathbf{B}(E)$ and $L : \mathbf{B}(E) \times \mathbf{B}^c \to \mathbf{B}^c$ introduced by Gugerli in [11, section 2] reduce to

$$Hf(x,t) = f(\phi(x, t \wedge t^*(x)))\mathrm{e}^{-\Lambda(x, t\wedge t^*(x))},$$

$$Iw(x,t) = \int_0^{t \wedge t^*(x)} \lambda Qw(\phi(x,s))\mathrm{e}^{-\Lambda(x,s)} ds,$$

$$J(w,f)(x,t) = Iw(x,t) + Hf(x,t), \tag{3.2}$$

$$Kw(x) = \int_0^{t^*(x)} \lambda Qw(\phi(x,s))\mathrm{e}^{-\Lambda(x,s)} ds + Qw(\phi(x,t^*(x)))\mathrm{e}^{-\Lambda(x,t^*(x))}, \tag{3.3}$$

$$L(w,h)(x) = \sup_{t \geq 0} J(w,h)(x,t) \vee Kw(x).$$

It is easy to derive a probabilistic interpretation of operators H, I, K and L in terms of the embedded Markov chain $(Z_n, S_n)_{n\in\mathbb{N}}$.

Lemma 3.1 *For all $x \in E$, $w \in \mathbf{B}(E)$, $f \in \mathbf{B}(\overline{E})$, $h \in \mathbf{B}^c$ and $t \geq 0$, one has*

$$\begin{aligned}
Hf(x,t) &= f(\phi(x, t\wedge t^*(x)))\mathbf{P}_x(S_1 \geq t \wedge t^*(x)),\\
Iw(x,t) &= \mathbf{E}_x\Big[w(Z_1)\mathbf{1}_{\{S_1<t\wedge t^*(x)\}}\Big],\\
Kw(x) &= \mathbf{E}_x\big[w(Z_1)\big], \qquad (3.4)\\
L(w,h)(x) &= \sup_{u\leq t^*(x)}\Big\{\mathbf{E}_x\big[w(Z_1)\mathbf{1}_{\{S_1<u\}}\big] + h(\phi(x,u))\mathbf{P}_x(S_1\geq u)\Big\} \vee \mathbf{E}_x\big[w(Z_1)\big]. \qquad (3.5)
\end{aligned}$$

For a reward function $g \in \mathbf{B}^c$, it has been shown in [11] that the value function can be recursively constructed by the following procedure:

$$\sup_{\tau\in\mathcal{M}_N} \mathbb{E}_{x_0}[g(X(\tau))] = v_0(x_0)$$

with

$$\begin{cases} v_N = g, \\ v_k = L(v_{k+1}, g) \quad \text{for } k \leq N-1. \end{cases}$$

Definition 3.2 *Introduce the random variables $(V_n)_{n\in\{0,\ldots,N\}}$ by*

$$V_n = v_n(Z_n),$$

or equivalently

$$\begin{aligned}
V_n &= \sup_{u\leq t^*(Z_n)}\Big\{\mathbf{E}\Big[v_{n+1}(Z_{n+1})\mathbf{1}_{\{S_{n+1}<u\}} + g(\phi(Z_n,u))\mathbf{1}_{\{S_{n+1}\geq u\}}\big|Z_n\Big]\Big\}\\
&\quad \vee \mathbf{E}\Big[v_{n+1}(Z_{n+1})\big|Z_n\Big]. \qquad (3.6)
\end{aligned}$$

The following result shows that the sequence $(V_n)_{n\in\{0,\ldots,N\}}$ corresponds to a *quasi*-Snell envelope associated to the reward process $\{g(X(t))\}_{t\in\mathbb{R}_+}$ where the horizon time is random and given by the jump times $(T_n)_{n\in\{0,\ldots,N\}}$ of the process $\{X(t)\}_{t\in\mathbb{R}_+}$:

Theorem 3.3 *Consider an integer $n < N$. Then*

$$V_n = \sup_{\nu\in\mathcal{M}_{n,N}} \mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}].$$

Proof: Let $\nu \in \mathcal{M}_{n,N}$. According to Proposition B.4 and Corollary B.6 in Appendix B, there exists $\widehat{\nu}\colon E \times (\mathbb{R}_+ \times E)^n \times \Omega \to \mathbb{R}_+$ such that for all $(z_0, \gamma) \in E \times (\mathbb{R}_+ \times E)^n$ the mapping $\widehat{\nu}(z_0, \gamma)\colon \Omega \to \mathbb{R}_+$ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time satisfying $\widehat{\nu}(z_0, \gamma) \leq T_{N-n}$, and $\nu = T_n + \widehat{\nu}(Z_0, \Gamma_n, \theta_{T_n})$, where $\Gamma_n = \big(S_1, Z_1, \ldots, S_n, Z_n\big)$ and θ is the shift operator. For $(z_0, \gamma) \in E \times (\mathbb{R}_+ \times E)^n$ define $\mathcal{W}\colon E \times (\mathbb{R}_+ \times E)^n \to \mathbb{R}$ by

$$\mathcal{W}(z_0, \gamma) = \mathbf{E}_{z_n}[g(X(\widehat{\nu}(z_0, \gamma)))] \leq \sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_{Z_n}[g(X(\tau))],$$

where $\gamma = (s_1, z_1, \ldots, s_n, z_n)$. Hence, the strong Markov property of the process $\{X(t)\}$ yields

$$\mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}] = \mathbf{E}_{x_0}[g(X(T_n + \widehat{\nu}(Z_0, \Gamma_n, \theta_{T_n})))|\mathcal{F}_{T_n}] = \mathcal{W}(Z_0, \Gamma_n).$$

Consequently, one has

$$\mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}] \leq \sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_{Z_n}[g(X(\tau))],$$

and therefore, one has

$$\sup_{\nu\in\mathcal{M}_{n,N}} \mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}] \leq \sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_{Z_n}[g(X(\tau))], \tag{3.7}$$

Conversely, consider $\tau \in \mathcal{M}_{N-n}$. It is easy to show that $T_n + \tau \circ \theta_{T_n} \in \mathcal{M}_{n,N}$. The strong Markov property of the process $\{X(t)\}$ again yields

$$\mathbf{E}_{Z_n}[g(X(\tau))] = \mathbf{E}_{x_0}[g(X(T_n + \tau\circ\theta_{T_n}))|\mathcal{F}_{T_n}] \leq \sup_{\nu\in\mathcal{M}_{n,N}} \mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}],$$

hence we obtain

$$\sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_{Z_n}[g(X(\tau))] \leq \sup_{\nu\in\mathcal{M}_{n,N}} \mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}]. \tag{3.8}$$

Conbining equations (3.7) and (3.8), one has

$$\sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_{Z_n}[g(X(\tau))] = \sup_{\nu\in\mathcal{M}_{n,N}} \mathbf{E}_{x_0}[g(X(\nu))|\mathcal{F}_{T_n}].$$

Finally, it is proved in [11, Theorem 1] that $v_n(x) = \sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_x[g(X(\tau))]$, whence

$$V_n = \sup_{\tau\in\mathcal{M}_{N-n}} \mathbf{E}_{Z_n}[g(X(\tau))],$$

showing the result. □

4. Approximation of the value function. To approximate the sequence of value functions (V_n), we proceed in two steps. First, the continuous-time maximization of operator L is converted into a discrete-time maximization by using a path-dependent time-discretization scheme to give a new operator L^d. In particular, it is important to remark that these time-discretization grids depend on the the post-jump locations $\{Z_k\}$ of the PDMP (see Definition 4.1 and Remark 4.2). Second, the conditional expectations of the Markov chain (Θ_k) in the definition of L^d are replaced by the conditional expectations of its quantized approximation $(\widehat{\Theta}_k)$ to define an operator $\widehat{L}^d$.

First, we define the path-adapted discretization grids as follows.

Definition 4.1 *For $z \in E$, set $\Delta(z) \in]0, t^*(z)[$. Define $n(z) = int(\frac{t^*(z)}{\Delta(z)}) - 1$, where $int(x)$ denotes the greatest integer smaller than or equal to x. The set of points $(t_i)_{i\in\{0,\dots,n(z)\}}$ with $t_i = i\Delta(z)$ is denoted by $G(z)$. This is the grid associated to the time interval $[0, t^*(z)]$.*

Remark 4.2 *It is important to note that, for all $z \in E$, not only one has $t^*(z) \notin G(z)$, but also $\max G(z) = t_{n(z)} \leq t^*(z) - \Delta(z)$. This property is crucial for the sequel.*

Definition 4.3 *Consider for $w \in \mathbf{B}(E)$ and $z \in E$,*

$$\begin{aligned} L^d(w,g)(z) &= \max_{s\in G(z)} \Big\{\mathbf{E}\Big[w(Z_1)\mathbf{1}_{\{S_1<s\}} + g(\phi(z,s))\mathbf{1}_{\{S_1\geq s\}} \big| Z_0 = z\Big]\Big\} \\ &\quad \vee \mathbf{E}\Big[w(Z_1)\big|Z_0 = z\Big]. \end{aligned}$$

Now let us turn to the quantization of (Θ_n). The quantization algorithm will provide us with a finite grid $\Gamma_n^{\Theta} \subset E \times \mathbb{R}_+$ at each time $0 \leq n \leq N$ as well as weights for each point of the grid, see e.g. [1, 14, 17]. Set $p \geq 1$ such that Θ_n has finite moments at least up to the order p and let p_n be the closest-neighbour projection from $E \times \mathbb{R}_+$ onto Γ_n^{Θ} (for the distance of norm p; if there are several equally close neighbours, pick the one with the smallest index). Then the quantization of Θ_n is defined by

$$\widehat{\Theta}_n = (\widehat{Z}_n, \widehat{S}_n) = p_n(Z_n, S_n).$$

We will also denote Γ_n^Z the projection of Γ_n^{Θ} on E and Γ_n^S the projection of Γ_n^{Θ} on $\mathbb{R}_+$.

In practice, one will first compute the quantization grids and weights, and then compute a path-adapted time-grid for each $z \in \Gamma_n^Z$, for all $0 \leq n \leq N-1$. Hence, there is only a finite number time grids to compute, and like the quantization grids, they can be computed and stored off-line.

The definition of the discretized operators now naturally follows the characterization given in Lemma 3.1.

Definition 4.4 *For $k \in \{1, \dots, N\}$, $w \in \mathbf{B}(\Gamma_k^Z)$, $z \in \Gamma_{k-1}^Z$, and $s \in \mathbb{R}_+$*

$$\begin{aligned} \widehat{J}_k(w,g)(z,s) &= \mathbf{E}\Big[w(\widehat{Z}_k)\mathbf{1}_{\{\widehat{S}_k<s\}} + g(\phi(z,s))\mathbf{1}_{\{\widehat{S}_k\geq s\}}|\widehat{Z}_{k-1}=z\Big], \\ \widehat{K}_k(w)(z) &= \mathbf{E}\Big[w(\widehat{Z}_k)|\widehat{Z}_{k-1}=z\Big], \\ \widehat{L}_k^d(w,g)(z) &= \max_{s\in G(z)}\Big\{\widehat{J}_k(w,g)(z,s)\Big\} \vee \widehat{K}_k(w)(z). \end{aligned}$$

Note that $\widehat{\Theta}_n$ is a random variable taking finitely many values, hence the expectations above actually are finite sums, the probability of each atom being given by its weight on the quantization grid. We can now give the complete contruction of the sequence approximating (V_n).

Definition 4.5 *Consider $\widehat{v}_N(z) = g(z)$ where $z \in \Gamma_N^Z$ and for $k \in \{1, \dots, N\}$*

$$\widehat{v}_{k-1}(z) = \widehat{L}_k^d(\widehat{v}_k, g)(z), \tag{4.1}$$

where $z \in \Gamma_{k-1}^Z$.

Definition 4.6 *The approximation of V_k is denoted by*

$$\widehat{V}_k = \widehat{v}_k(\widehat{Z}_k), \tag{4.2}$$

for $k \in \{0, \dots, N\}$.

5. Error estimation for the value function. We are now able to state our main result, namely the convergence of our approximation scheme with an upper bound for the rate of convergence.

Theorem 5.1 *Set $n \in \{0, \dots, N-1\}$, and suppose that $\Delta(z)$, for $z \in \Gamma_n^z$, are chosen such that*

$$\min_{z\in\Gamma_n^z}\{\Delta(z)\} > (2C_\lambda)^{-1/2}\big([t^*]\|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p\big)^{1/2}.$$

Then the discretization error for V_n is no greater than the following:

$$\begin{aligned} \|V_n - \widehat{V}_n\|_p \leq\ & \big\|V_{n+1} - \widehat{V}_{n+1}\big\|_p + \alpha\|\Delta(\widehat{Z}_n)\|_p + \beta_n\|\widehat{Z}_n - Z_n\|_p + 2\big[v_{n+1}\big]\|\widehat{Z}_{n+1} - Z_{n+1}\|_p \\ & + \gamma\big([t^*]\|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p\big)^{1/2}, \end{aligned}$$

where $\alpha = [g]_2 + 2C_gC_\lambda$, $\beta_n = [v_n] + [v_{n+1}]_1E_2 + C_gE_4 + \big([g]_1 + [g]_2[t^*]\big) \vee \big([v_{n+1}]_*[Q]\big)$, $\gamma = 4C_g(2C_\lambda)^{1/2}$, and E_2 and E_4 are defined in Appendix A.

Recall that $V_N = g(Z_N)$ and $\widehat{V}_N = g(\widehat{Z}_N)$, hence $\|V_N - \widehat{V}_N\|_p \leq [g]\big\|\widehat{Z}_N - Z_N\big\|_p$. In addition, the quantization error $\|\Theta_n - \widehat{\Theta}_n\|_p$ goes to zero as the number of points in the grids goes to infinity, see e.g. [14]. Hence $|V_0 - \widehat{V}_0|$ can be made arbitrarily small by an adequate choice of the discretizations parameters.

Remark that the square root in the last error term is the price to pay for integrating non-continuous functions, see the definition of operator J with the indicator functions, and the introduction of section 5.2.

To prove Theorem 5.1, we split the left-hand-side difference into four terms:

$$\|V_n - \widehat{V}_n\|_p \quad \leq \quad \sum_{i=1}^{4} \Xi_i,$$

where

$$\begin{aligned}
\Xi_1 &= \|v_n(Z_n) - v_n(\widehat{Z}_n)\|_p, \\
\Xi_2 &= \|L(v_{n+1}, g)(\widehat{Z}_n) - L^d(v_{n+1}, g)(\widehat{Z}_n)\|_p, \\
\Xi_3 &= \|L^d(v_{n+1}, g)(\widehat{Z}_n) - \widehat{L}^d_{n+1}(v_{n+1}, g)(\widehat{Z}_n)\|_p, \\
\Xi_4 &= \|\widehat{L}^d_{n+1}(v_{n+1}, g)(\widehat{Z}_n) - \widehat{L}^d_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n)\|_p.
\end{aligned}$$

The first term is easy enough to handle thanks to Proposition A.7 in Appendix A.2.

Lemma 5.2 *A upper bound for* Ξ_1 *is*

$$\|v_n(Z_n) - v_n(\widehat{Z}_n)\|_p \leq [v_n]\|Z_n - \widehat{Z}_n\|_p.$$

We are going to study the other terms one by one in the following sections.

5.1. *Second term.* In this part we study the error induced by the replacement of the supremum over all non-negative t smaller than or equal to $t^*(z)$ by the maximum over the finite grid $G(z)$ in the definition of operator L.

Lemma 5.3 *Let* $w \in \mathbf{L}^c$*. Then for all* $z \in E$*,*

$$\Big| \sup_{t \leq t^*(z)} J(w,g)(z,t) - \max_{s \in G(z)} J(w,g)(z,s)\Big| \leq (C_w C_\lambda + [g]_2 + C_g C_\lambda)\Delta(z).$$

Proof: Clearly, there exists $\bar{t} \in [0, t^*(z)]$ such that $\sup_{t \leq t^*(z)} J(w,g)(z,t) = J(w,g)(z,\bar{t})$, and there exists $0 \leq i \leq n(z)$ such that $\bar{t} \in [t_i, t_{i+1}]$ (with $t_{n(z)+1} = t^*(z)$). Consequently, Lemma A.2 yields

$$\begin{aligned}
0 \leq \sup_{t \leq t^*(z)} J(w,g)(z,t) - \max_{s \in G(z)} J(w,g)(z,s) &\leq J(w,g)(z,\bar{t}) - J(w,g)(z,t_i) \\
&\leq (C_w C_\lambda + [g]_2 + C_g C_\lambda)|\bar{t} - t_i| \\
&\leq (C_w C_\lambda + [g]_2 + C_g C_\lambda)|t_{i+1} - t_i|,
\end{aligned}$$

implying the result. □

Turning back to the second error term, one gets the following bound.

Lemma 5.4 *A upper bound for Ξ_2 is*

$$\|L(v_{n+1},g)(\widehat{Z}_n) - L^d(v_{n+1},g)(\widehat{Z}_n)\|_p \leq \left([g]_2 + 2C_g C_\lambda\right)\|\Delta(\widehat{Z}_n)\|_p.$$

Proof: From the definition of L and L^d we readily obtain

$$\begin{aligned}&\|L(v_{n+1},g)(\widehat{Z}_n) - L^d(v_{n+1},g)(\widehat{Z}_n)\|_p\\ &\qquad\leq \Big\| \sup_{t\leq t^*(\widehat{Z}_n)} J(v_{n+1},g)(\widehat{Z}_n,t) - \max_{s\in G(\widehat{Z}_n)} J(v_{n+1},g)(\widehat{Z}_n,s)\Big\|_p.\end{aligned}$$

Now in view of the previous lemma, one has

$$\begin{aligned}&\|L(v_{n+1},g)(\widehat{Z}_n) - L^d(v_{n+1},g)(\widehat{Z}_n)\|_p\\ &\qquad\leq \left(C_{v_{n+1}}C_\lambda + [g]_2 + C_g C_\lambda\right)\|\Delta(\widehat{Z}_n)\|_p.\end{aligned}$$

Finaly, note that $C_{v_{n+1}} = C_g$ (see Appendix A.2), completing the proof. □

5.2. *Third term.* This is the crucial part of our derivation, where we need to compare conditional expectations relative to the real Markov chain (Z_n, S_n) and its quantized approximation $(\widehat{Z}_n, \widehat{S}_n)$. The main difficulty stems from the fact that some functions inside the expectations are indicator functions and in particular they are not Lipschitz-continuous. We manage to overcome this difficulty by proving that the event on which the discontinuity actually occurs is of small enough probability, this is the aim of the following two lemmas.

Lemma 5.5 *For all $n \in \{0,\ldots,N-1\}$ and $0 < \eta < \min_{z\in\Gamma_n^Z}\{\Delta(z)\}$,*

$$\Big\| \max_{s\in G(\widehat{Z}_n)} \mathbf{E}\big[|\mathbf{1}_{\{S_{n+1}<s\}} - \mathbf{1}_{\{\widehat{S}_{n+1}<s\}}|\,|\widehat{Z}_n\big]\Big\|_p \quad\leq\quad \frac{1}{\eta}\|S_{n+1} - \widehat{S}_{n+1}\|_p + 2C_\lambda\eta.$$

Proof: Set $0 < \eta < \min_{z\in\Gamma_n^Z}\{\Delta(z)\}$. Remark that the difference of indicator functions is non-zero if and only if S_{n+1} and $\widehat{S}_{n+1}$ are on either side of s. Hence, one has

$$|\mathbf{1}_{\{S_{n+1}<s\}} - \mathbf{1}_{\{\widehat{S}_{n+1}<s\}}| \leq \mathbf{1}_{\{|S_{n+1}-\widehat{S}_{n+1}|>\eta\}} + \mathbf{1}_{\{|S_{n+1}-s|\leq\eta\}}.$$

This yields

$$\begin{aligned}&\Big\| \max_{s\in G(\widehat{Z}_n)} \mathbf{E}\big[|\mathbf{1}_{\{S_{n+1}<s\}} - \mathbf{1}_{\{\widehat{S}_{n+1}<s\}}|\,|\widehat{Z}_n\big]\Big\|_p\\ &\quad\leq\quad \big\|\mathbf{1}_{\{|S_{n+1}-\widehat{S}_{n+1}|>\eta\}}\big\|_p + \Big\| \max_{s\in G(\widehat{Z}_n)} \mathbf{E}\big[\mathbf{1}_{\{s-\eta\leq S_{n+1}\leq s+\eta\}}|\widehat{Z}_n\big]\Big\|_p.\end{aligned}\tag{5.1}$$

On the one hand, Chebychev's inequality yields

$$\big\|\mathbf{1}_{\{|S_{n+1}-\widehat{S}_{n+1}|>\eta\}}\big\|_p^p = \mathbf{P}(|S_{n+1} - \widehat{S}_{n+1}| > \eta) \leq \frac{\big\|S_{n+1} - \widehat{S}_{n+1}\big\|_p^p}{\eta^p}.\tag{5.2}$$

On the other hand, as $s \in G(\widehat{Z}_n)$ and by definition of η, one has $s+\eta < t^*(\widehat{Z}_n)$, see Remark 4.2. Thus, one has

$$\begin{aligned}
\mathbf{E}\big[\mathbf{1}_{\{s-\eta\leq S_{n+1}\leq s+\eta\}}\big|\widehat{Z}_n\big] &= \mathbf{E}\Big[\mathbf{E}\big[\mathbf{1}_{\{s-\eta\leq S_{n+1}\leq s+\eta\}}\big|Z_n=\widehat{Z}_n\big]\Big|\widehat{Z}_n\Big] \\
&= \mathbf{E}\Big[\int_{s-\eta}^{s+\eta}\lambda(\phi(\widehat{Z}_n,u))du\Big|\widehat{Z}_n\Big] \\
&\leq 2\eta C_\lambda. \qquad (5.3)
\end{aligned}$$

Combining equations (5.1)-(5.3), the result follows. □

Lemma 5.6 *For all $n \in \{0,\ldots,N\}$ and $0<\eta<\min_{z\in\Gamma_n^Z}\{\Delta(z)\}$,*

$$\big\|\mathbf{1}_{t^*(Z_n)<t^*(\widehat{Z}_n)-\eta}\big\|_p \leq \frac{[t^*]\|Z_n-\widehat{Z}_n\|_p}{\eta}.$$

Proof: We use Chebychev's inequality again. One clearly has

$$\begin{aligned}
\mathbf{E}\Big[|\mathbf{1}_{t^*(Z_n)<t^*(\widehat{Z}_n)-\eta}|^p\Big] &= \mathbf{P}\Big(t^*(Z_n)<t^*(\widehat{Z}_n)-\eta\Big) \\
&\leq \mathbf{P}\Big(\big|t^*(Z_k)-t^*(\widehat{Z}_k)\big|>\eta\Big) \leq \frac{[t^*]^p\|Z_k-\widehat{Z}_k\|_p^p}{\eta^p},
\end{aligned}$$

showing the result. □

Now we turn to the consequences of replacing the Markov chain (Z_n, S_n) by its quantized approximation $(\widehat{Z}_n, \widehat{S}_n)$ in the conditional expectations.

Lemma 5.7 *Let $w \in \mathbf{L}^c$, then one has*

$$\begin{aligned}
&\Big|\mathbf{E}\big[w(Z_{n+1})\big|Z_n=\widehat{Z}_n\big]-\mathbf{E}\big[w(\widehat{Z}_{n+1})\big|\widehat{Z}_n\big]\Big| \\
&\qquad\leq \big(C_wE_4+[w]_1E_2+[w]_*[Q]\big)\mathbf{E}\big[|Z_n-\widehat{Z}_n|\ \big|\ \widehat{Z}_n\big]+[w]\mathbf{E}\big[|Z_{n+1}-\widehat{Z}_{n+1}|\big|\widehat{Z}_n\big].
\end{aligned}$$

Proof: First, note that

$$\begin{aligned}
\mathbf{E}\big[w(Z_{n+1})\big|Z_n=\widehat{Z}_n\big]-\mathbf{E}\big[w(\widehat{Z}_{n+1})\big|\widehat{Z}_n\big] &= \mathbf{E}\big[w(Z_{n+1})\big|Z_n=\widehat{Z}_n\big]-\mathbf{E}\big[w(Z_{n+1})\big|\widehat{Z}_n\big] \\
&\quad+\mathbf{E}\big[w(Z_{n+1})\big|\widehat{Z}_n\big]-\mathbf{E}\big[w(\widehat{Z}_{n+1})\big|\widehat{Z}_n\big].
\end{aligned}$$

On the one hand, Remark 2.2 yields

$$\Big|\mathbf{E}\big[w(Z_{n+1})\big|\widehat{Z}_n\big]-\mathbf{E}\big[w(\widehat{Z}_{n+1})\big|\widehat{Z}_n\big]\Big| \leq [w]\mathbf{E}\big[|Z_{n+1}-\widehat{Z}_{n+1}|\big|\widehat{Z}_n\big].$$

On the other hand, recall that by construction of the quantized process, one has $(\widehat{Z}_n, \widehat{S}_n) = p_n(Z_n, S_n)$. Hence we have the following property: $\sigma\{\widehat{Z}_n\} \subset \sigma\{Z_n, S_n\}$. By using the special structure of the PDMP $\{X(t)\}$, we have $\sigma\{Z_n, S_n\} \subset \mathcal{F}_{T_n}$. Now, by using the Markov property of the process $\{X(t)\}$, it follows

$$\mathbf{E}\big[w(Z_{n+1})|\widehat{Z}_n\big] = \mathbf{E}\Big[\mathbf{E}\big[w(Z_{n+1})|\mathcal{F}_{T_n}\big]\Big|\widehat{Z}_n\Big] = \mathbf{E}\Big[\mathbf{E}\big[w(Z_{n+1})|Z_n\big]\Big|\widehat{Z}_n\Big].$$

Equation (3.4) thus yields

$$\begin{aligned}
&\mathbf{E}\big[w(Z_{n+1})|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[w(Z_{n+1})|\widehat{Z}_n\big] \\
&= \mathbf{E}\Big[\mathbf{E}\big[w(Z_{n+1})|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[w(Z_{n+1})|Z_n\big]\Big|\widehat{Z}_n\Big] \\
&= \mathbf{E}\Big[Kw(\widehat{Z}_n) - Kw(Z_n)\Big|\widehat{Z}_n\Big].
\end{aligned}$$

Now we use Lemma A.4 to conclude. □

Now we combine the preceding lemmas to derive the third error term.

Lemma 5.8 *For all* $0 < \eta < \min_{z \in \Gamma_n^Z}\{\Delta(z)\}$*, an upper bound for* Ξ_3 *is*

$$\begin{aligned}
&\big\|L^d(v_{n+1}, g)(\widehat{Z}_n) - \widehat{L}^d_{n+1}(v_{n+1}, g)(\widehat{Z}_n)\big\|_p \\
&\quad \leq \Big\{\big[v_{n+1}\big]_1 E_2 + C_g E_4 + 2C_g \frac{[t^*]}{\eta} + \big([g]_1 + [g]_2[t^*]\big) \vee \big([v_{n+1}]_*[Q]\big)\Big\}\|\widehat{Z}_n - Z_n\|_p \\
&\qquad + \big[v_{n+1}\big]\|\widehat{Z}_{n+1} - Z_{n+1}\|_p + 2C_g\Big(2C_\lambda\eta + \frac{\|S_{n+1} - \widehat{S}_{n+1}\|_p}{\eta}\Big).
\end{aligned}$$

Proof: To simplify notation, set $\Psi(x, y, t) = v_{n+1}(y)\mathbf{1}_{\{t<s\}} + g(\phi(x, t))\mathbf{1}_{\{t\geq s\}}$. From the definition of L^d and $\widehat{L}^d_{n+1}$, one readily obtains

$$\begin{aligned}
&\big|L^d(v_{n+1}, g)(\widehat{Z}_n) - \widehat{L}^d_{n+1}(v_{n+1}, g)(\widehat{Z}_n)\big| \\
&\quad \leq \max_{s \in G(\widehat{Z}_n)} \Big|\mathbf{E}\big[\Psi(Z_n, Z_{n+1}, S_{n+1})|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[\Psi(\widehat{Z}_n, \widehat{Z}_{n+1}, \widehat{S}_{n+1})|\widehat{Z}_n\big]\Big| \\
&\quad \vee \Big|\mathbf{E}\big[v_{n+1}(Z_{n+1})|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[v_{n+1}(\widehat{Z}_{n+1})|\widehat{Z}_n\big]\Big|.
\end{aligned} \tag{5.4}$$

On the one hand, combining Lemma 5.7 and the fact that v_{n+1} is in $\mathbf{L}^c$ (see Proposition A.7), we obtain

$$\begin{aligned}
&\Big|\mathbf{E}\big[v_{n+1}(Z_{n+1})|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[v_{n+1}(\widehat{Z}_{n+1})|\widehat{Z}_n\big]\Big| \\
&\leq \big[v_{n+1}\big]\mathbf{E}\big[|Z_{n+1} - \widehat{Z}_{n+1} \mid \widehat{Z}_n\big] \\
&\quad + (C_g E_4 + \big[v_{n+1}\big]_1 E_2 + \big[v_{n+1}\big]_*[Q])\mathbf{E}\big[|Z_n - \widehat{Z}_n| \mid \widehat{Z}_n\big].
\end{aligned} \tag{5.5}$$

On the other hand, similar arguments as in the proof of Lemma 5.7 yield

$$\begin{aligned}
&\mathbf{E}\big[\Psi(Z_n, Z_{n+1}, S_{n+1})\big|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[\Psi(\widehat{Z}_n, \widehat{Z}_{n+1}, \widehat{S}_{n+1})\big|\widehat{Z}_n\big] \\
&= \mathbf{E}\Big[\mathbf{E}\big[\Psi(Z_n, Z_{n+1}, S_{n+1})\big|Z_n = \widehat{Z}_n\big] - \mathbf{E}\big[\Psi(Z_n, Z_{n+1}, S_{n+1})\big|Z_n = Z_n\big]\Big|\widehat{Z}_n\Big] \\
&\quad +\mathbf{E}\big[\Psi(Z_n, Z_{n+1}, S_{n+1})\big|\widehat{Z}_n\big] - \mathbf{E}\big[\Psi(\widehat{Z}_n, \widehat{Z}_{n+1}, \widehat{S}_{n+1})\big|\widehat{Z}_n\big] \qquad (5.6)\\
&= \Upsilon_1 + \Upsilon_2.
\end{aligned}$$

The second difference of the right hand side of equation (5.6), labeled Υ_2, clearly satisfies

$$\begin{aligned}
|\Upsilon_2| \leq\ & [v_{n+1}]\mathbf{E}\big[|\widehat{Z}_{n+1} - Z_{n+1}|\big|\widehat{Z}_n\big] + [g]_1\mathbf{E}\big[|\widehat{Z}_n - Z_n|\big|\widehat{Z}_n\big] \\
& +2C_g\mathbf{E}\big[|\mathbf{1}_{\{S_{n+1}<s\}} - \mathbf{1}_{\{\widehat{S}_{n+1}<s\}}|\big|\widehat{Z}_n\big]. \qquad (5.7)
\end{aligned}$$

Let us turn now to the first difference of the right hand side of equation (5.6), labeled Υ_1. We meet another difficulty here. Indeed, we know by construction that $s < t^*(\widehat{Z}_n)$, but we know nothing regarding the relative positions of s and $t^*(Z_n)$. On the event where $s \leq t^*(Z_n)$ as well, we recognize operator J inside the expectations. On the opposite event $s > t^*(Z_n)$, we crudely bound Ψ by $C_{v_{n+1}} + C_g = 2C_g$. Hence, one obtains

$$|\Upsilon_1| \leq \mathbf{E}\Big[\big|J(v_{n+1}, g)(\widehat{Z}_n, s) - J(v_{n+1}, g)(Z_n, s)\big|\mathbf{1}_{\{s\leq t^*(Z_n)\}}\Big|\widehat{Z}_n\Big] + 2C_g\mathbf{E}\big[\mathbf{1}_{\{t^*(Z_n)<s\}}\big|\widehat{Z}_n\big].$$

Now Lemma A.3 gives an upper bound for the first term. As for the indicatror function, by definition of $G(\widehat{Z}_n)$ and our choice of η, we have $s < t^*(\widehat{Z}_n) - \eta$. Thus, one has

$$|\Upsilon_1| \leq (C_gE_1 + [v_{n+1}]_1E_2 + E_3)\mathbf{E}\big[|\widehat{Z}_n - Z_n|\big|\widehat{Z}_n\big] + 2C_g\mathbf{E}\big[\mathbf{1}_{\{t^*(Z_n)<t^*(\widehat{Z}_n)-\eta\}}\big|\widehat{Z}_n\big]. \qquad (5.8)$$

Now, combining (5.4), (5.5), (5.7) and (5.8), and the fact that $C_gE_1+E_3 = C_gE_4+[g]_1+[g]_2[t^*]$, one gets

$$\begin{aligned}
&\big|L^d(v_{n+1}, g)(\widehat{Z}_n) - \widehat{L}^d_{n+1}(v_{n+1}, g)(\widehat{Z}_n)\big| \\
&\leq \Big\{[v_{n+1}]_1E_2 + C_gE_4 + \big([g]_1 + [g]_2[t^*]\big) \vee \big([v_{n+1}]_*[Q]\big)\Big\}\mathbf{E}\big[|\widehat{Z}_n - Z_n|\big|\widehat{Z}_n\big] \\
&\quad + [v_{n+1}]\mathbf{E}\big[|\widehat{Z}_{n+1} - Z_{n+1}|\big|\widehat{Z}_n\big] \\
&\quad + 2C_g\mathbf{E}\big[\mathbf{1}_{t^*(Z_n)<t^*(\widehat{Z}_n)-\eta}\big|\widehat{Z}_n\big] + 2C_g \max_{s\in G(\widehat{Z}_n)} \mathbf{E}\big[|\mathbf{1}_{\{S_{n+1}<s\}} - \mathbf{1}_{\{\widehat{S}_{n+1}<s\}}|\big|\widehat{Z}_n\big].
\end{aligned}$$

Finally, we conclude by taking the L^p norm on both sides and using Lemmas 5.5 and 5.6. □

5.3. *Fourth term.* The last error term is a mere comparison of two finite sums.

Lemma 5.9 *An upper bound for Ξ_4 is*

$$\|\widehat{L}^d_{n+1}(v_{n+1}, g)(\widehat{Z}_n) - \widehat{L}^d_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n)\|_p \leq [v_{n+1}]\big\|\widehat{Z}_{n+1} - Z_{n+1}\big\|_p + \big\|V_{n+1} - \widehat{V}_{n+1}\big\|_p.$$

Proof: By definition of operator $\widehat{L}_n^d$, one has

$$\begin{aligned}
&\|\widehat{L}_{n+1}^d(v_{n+1},g)(\widehat{Z}_n) - \widehat{L}_{n+1}^d(\widehat{v}_{n+1},g)(\widehat{Z}_n)\|_p \\
&= \Big\| \max_{s\in G(\widehat{Z}_n)} \Big\{ \mathbf{E}\Big[v_{n+1}(\widehat{Z}_{n+1})\mathbf{1}_{\{\widehat{S}_{n+1}<s\}} + g\big(\phi(\widehat{Z}_n,s)\big)\mathbf{1}_{\{\widehat{S}_{n+1}\geq s\}}|\widehat{Z}_n\Big]\Big\} \vee \mathbf{E}\Big[v_{n+1}(\widehat{Z}_{n+1})|\widehat{Z}_n\Big] \\
&\quad - \max_{s\in G(\widehat{Z}_n)} \Big\{ \mathbf{E}\Big[\widehat{v}_{n+1}(\widehat{Z}_{n+1})\mathbf{1}_{\{\widehat{S}_{n+1}<s\}} + g\big(\phi(\widehat{Z}_n,s)\big)\mathbf{1}_{\{\widehat{S}_{n+1}\geq s\}}|\widehat{Z}_n\Big]\Big\} \vee \mathbf{E}\Big[\widehat{v}_{n+1}(\widehat{Z}_{n+1})|\widehat{Z}_n\Big]\Big\|_p \\
&\leq \big\|\mathbf{E}[v_{n+1}(\widehat{Z}_{n+1}) - \widehat{v}_{n+1}(\widehat{Z}_{n+1})|\widehat{Z}_n]\big\|_p \\
&\leq \big\|v_{n+1}(\widehat{Z}_{n+1}) - v_{n+1}(Z_{n+1})\big\|_p + \big\|v_{n+1}(Z_{n+1}) - \widehat{v}_{n+1}(\widehat{Z}_{n+1})\big\|_p.
\end{aligned}$$

We conclude using the fact that $v_{n+1} \in \mathbf{L}^c$ (see Proposition A.7) and the definitions of V_n and $\widehat{V}_n$. □

5.4. *Proof of Theorem 5.1.* We can finally turn to the proof of Theorem 5.1. Lemmas 5.2, 5.4, 5.8 and 5.9 from the preceding sections directly yield, for all $0 < \eta < \min_{z\in\Gamma_n^z}\{\Delta(z)\}$,

$$\begin{aligned}
\|V_n - \widehat{V}_n\|_p \leq\ & [v_n]\|\widehat{Z}_n - Z_n\|_p + \big([g]_2 + 2C_gC_\lambda\big)\|\Delta(\widehat{Z}_n)\|_p \\
&+ \Big\{[v_{n+1}]_1E_2 + C_gE_4 + 2C_g\frac{[t^*]}{\eta} + \big([g]_1 + [g]_2[t^*]\big) \vee \big([v_{n+1}]_*[Q]\big)\Big\}\|\widehat{Z}_n - Z_n\|_p \\
&+ [v_{n+1}]\|\widehat{Z}_{n+1} - Z_{n+1}\|_p + 2C_g\Big(2C_\lambda\eta + \frac{\|S_{n+1} - \widehat{S}_{n+1}\|_p}{\eta}\Big) \\
&+ [v_{n+1}]\big\|\widehat{Z}_{n+1} - Z_{n+1}\big\|_p + \big\|V_{n+1} - \widehat{V}_{n+1}\big\|_p.
\end{aligned}$$

The optimal choice for η clearly satisfies

$$2C_\lambda\eta = \frac{1}{\eta}\big([t^*]\|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p\big),$$

providing it also satisfies the condition $0 < \eta < \min_{z\in\Gamma_n^z}\{\Delta(z)\}$. Hence, rearranging the terms above, one gets the expected result:

$$\begin{aligned}
\|V_n - \widehat{V}_n\|_p \leq & \big\|V_{n+1} - \widehat{V}_{n+1}\big\|_p + \big([g]_2 + 2C_gC_\lambda\big)\|\Delta(\widehat{Z}_n)\|_p \\
&+ \Big\{[v_n] + [v_{n+1}]_1E_2 + C_gE_4 + \big([g]_1 + [g]_2[t^*]\big) \vee \big([v_{n+1}]_*[Q]\big)\Big\}\|\widehat{Z}_n - Z_n\|_p \\
&+ 2[v_{n+1}]\|\widehat{Z}_{n+1} - Z_{n+1}\|_p + 4C_g(2C_\lambda)^{1/2}\big([t^*]\|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p\big)^{1/2}.
\end{aligned}$$

6. Numerical construction of an ϵ-optimal stopping time. In [11, Theorem 1], U.S. Gugerli defined an ϵ-optimal stopping time for the original problem. Roughly speaking, this stopping time depends on the embedded Markov chain (Θ_n), and on the optimal value function.

Therefore, a natural candidate for an ϵ-optimal stopping time should be obtained by replacing the Markov chain (Θ_n) and the optimal value function by their *quantized approximations*. However, this leads to un-tractable comparisons between some quantities involving (Θ_n) and its quantized approximation. It is then far from obvious to show that this method would provide a computable ϵ-optimal stopping rule. Nonetheless, by modifying the approach of U.S. Gugerli [11], we are able to propose a numerical construction of an ϵ-optimal stopping time of the original stopping problem.

Here is how we proceed. First, recall that p_n be the closest-neighbour projection from $E \times \mathbb{R}_+$ onto Γ_n^Θ, and for all $(z,s) \in E \times \mathbb{R}_+$ define $(\widehat{z}_n, \widehat{s}_n) = p_n(z,s)$. Note that $\widehat{z}_n$ and $\widehat{s}_n$ depend on both z and s. Now, for $n \in \{1,\dots,N\}$, define

$$s_n^*(z,s) = \min\Big\{t \in G(\widehat{z}_{n-1}) \Big| \widehat{J}_n(\widehat{v}_n, g)(\widehat{z}_{n-1}, t) = \max_{u \in G(\widehat{z}_{n-1})} \widehat{J}_n(\widehat{v}_n, g)(\widehat{z}_{n-1}, u)\Big\},$$

and

$$r_{n,\beta}(z,s) = \begin{cases} t^*(z) & \text{if } \widehat{K}_n \widehat{v}_n(\widehat{z}_{n-1}) > \max\limits_{u \in G(\widehat{z}_{n-1})} \widehat{J}_n(\widehat{v}_n, g)(\widehat{z}_{n-1}, u), \\ s_n^*(z,s)\mathbf{1}_{\{s_n^*(z,s) < t^*(z)\}} + (t^*(z) - \beta)\mathbf{1}_{\{s^*(z,s) \geq t^*(z)\}} & \text{otherwise.} \end{cases}$$

Note the use of both the real jump time horizon $t^*(z)$ and the quantized approximations of K, J and (z,s). Set

$$\tau_1 = r_{N,\beta}(Z_0, S_0) \wedge T_1,$$

and for $n \in \{1,\dots,N-1\}$, set

$$\tau_{n+1} = \begin{cases} r_{N-n,\beta}(Z_0,S_0) & \text{if } T_1 > r_{N-n,\beta}(Z_0,S_0), \\ T_1 + \tau_n \circ \theta_{T_1} & \text{otherwise.} \end{cases}$$

Our stopping rule is then defined by τ_N.

Remark 6.1 *This procedure is especially appealing because it requires no more calculation: we have already computed the values of $\widehat{K}_n$ and $\widehat{J}_n$ on the grids. One just has to store the point where the maximum of $\widehat{J}_n$ is reached.*

Lemma 6.2 *τ_N is an $\{\mathcal{F}_T\}$-stopping time.*

Proof: Set $U_1 = r_{1,\beta}(Z_0, S_0)$ and for $2 \leq k \leq N$ $U_k = r_{k,\beta}(Z_{k-1}, S_{k-1})\mathbf{1}_{\{r_{k-1,\beta}(Z_{k-2},S_{k-2}) \geq S_{k-1}\}}$. One then clearly has $\tau_N = \sum_{k=1}^{N} U_k \wedge S_k$ which is an $\{\mathcal{F}_T\}$-stopping time by Proposition B.5. □

Now let us show that this stopping time provides a good approximation of the value function V_0. Namely, for all $z \in E$ set

$$\overline{v}_n(z) \quad = \quad \mathbf{E}\big[g(X_{\tau_{N-n}})\big|Z_n = z\big],$$

and in accordance to our previous notation introduce, for $n \in \{1, \dots, N-1\}$

$$\overline{V}_n \quad = \quad \overline{v}_n(Z_n).$$

The comparison between V_0 and $\overline{V}_0$ is provided by the next two Theorems.

Theorem 6.3 *Set $n \in \{0, \dots, N-2\}$ and suppose the discretization parameters are chosen such that there exists $0 < a < 1$ satisfying*

$$\frac{\beta}{a} = (2C_\lambda)^{-1/2}\Big(\frac{[t^*]}{1-a}\|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p\Big)^{1/2} < \min_{z \in \Gamma_n^z}\{\Delta(z)\}.$$

Then one has

$$\begin{aligned}\big\|\overline{V}_n - V_n\big\|_p \quad \leq \quad & \big\|\overline{V}_{n+1} - V_{n+1}\big\|_p + \big\|\widehat{V}_{n+1} - V_{n+1}\big\|_p + \big\|\widehat{V}_n - V_n\big\|_p \\ & + 2\big[v_{n+1}\big]\big\|Z_{n+1} - \widehat{Z}_{n+1}\big\|_p + a_n\big\|Z_n - \widehat{Z}_n\big\|_p \\ & + 4C_g(2C_\lambda)^{1/2}\Big(\frac{[t^*]}{1-a}\|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p\Big)^{1/2},\end{aligned}$$

with $a_n = \Big(2\big[v_{n+1}\big]_1 E_2 + 2C_g C_{t^*}\big[\lambda\big]_1(2 + C_{t^*}C_\lambda) + \big(4C_g C_\lambda\big[t^*\big] + 2\big[v_{n+1}\big]_*\big[Q\big]\big) \vee \big(3\big[g\big]_1\big)\Big)$.

Proof: The definition of τ_n and the strong Markov property of the process $\{X(t)\}$ yield

$$\begin{aligned}&\overline{v}_n(Z_n) \\ &= \mathbf{E}\big[g(X_{r_{n+1,\beta}(Z_n,S_n)})\mathbf{1}_{\{S_{n+1} > r_{n+1,\beta}(Z_n,S_n)\}}\big|Z_n\big] + \mathbf{E}\big[\overline{v}_{n+1}(Z_{n+1})\mathbf{1}_{\{S_{n+1} \leq r_{n+1,\beta}(Z_n,S_n)\}}\big|Z_n\big] \\ &= \mathbf{1}_{\{r_{n+1,\beta}(Z_n,S_n) \geq t^*(Z_n)\}}K\overline{v}_{n+1}(Z_n) + \mathbf{1}_{\{r_{n+1,\beta}(Z_n,S_n) < t^*(Z_n)\}}J(\overline{v}_{n+1}, g)(Z_n, r_{n+1,\beta}(Z_n, S_n)).\end{aligned}$$

However, our definition of $r_{n,\beta}$ with the special use of parameter β implies

$$\big\{r_{n+1,\beta}(Z_n, S_n) \geq t^*(Z_n)\big\} = \Big\{\widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{Z}_n) > \max_{s \in G(\widehat{Z}_n)} \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n, s)\Big\}.$$

Consequently, one obtains

$$\begin{aligned}&\overline{v}_n(Z_n) = \widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{Z}_n) \vee \max_{s \in G(\widehat{Z}_n)} \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n, s) \\ &\quad + \mathbf{1}_{\{r_{n+1,\beta}(Z_n,S_n) \geq t^*(Z_n)\}}\Big[K\overline{v}_{n+1}(Z_n) - \widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{Z}_n)\Big] \\ &\quad + \mathbf{1}_{\{r_{n+1,\beta}(Z_n,S_n) < t^*(Z_n)\}}\Big[J(\overline{v}_{n+1}, g)(Z_n, r_{n+1,\beta}(Z_n, S_n)) - \max_{s \in G(\widehat{Z}_n)} \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n, s)\Big].\end{aligned} \tag{6.1}$$

Let us study the term with operator K. First, we insert V_n to be able to use our work of the previous section (we cannot directy apply it to $\overline{v}_n$ because it may not be Lipschitz-continuous). Clearly, one has

$$\begin{aligned}\left|K\overline{v}_{n+1}(Z_n)-\widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{Z}_n)\right| &\leq \mathbf{E}\big[|\overline{V}_{n+1}-V_{n+1}||Z_n\big]\\ &\quad+\left|Kv_{n+1}(Z_n)-\widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{Z}_n)\right|.\end{aligned} \tag{6.2}$$

Similar calculations to those of Lemmas A.4, 5.7 and 5.9, and Equation (5.5) yield

$$\begin{aligned}&\left|Kv_{n+1}(Z_n)-\widehat{K}_{n+1}\widehat{v}_{n+1}(\widehat{Z}_n)\right|\\ &\quad\leq \left(C_gE_4+\big[v_{n+1}\big]_1E_2+\big[v_{n+1}\big]_*[Q]\right)\Big(|Z_n-\widehat{Z}_n|+\mathbf{E}\big[|Z_n-\widehat{Z}_n||\widehat{Z}_n\big]\Big)\\ &\qquad+2\big[v_{n+1}\big]\mathbf{E}\big[|Z_{n+1}-\widehat{Z}_{n+1}||\widehat{Z}_n\big]+\mathbf{E}\big[|V_{n+1}-\widehat{V}_{n+1}||\widehat{Z}_n\big].\end{aligned} \tag{6.3}$$

Now we turn to operator J. Set $R_n=r_{n+1,\beta}(Z_n,S_n)$. We first study the case when $R_n=s^*_{n+1}(Z_n,S_n)<t^*(Z_n)$. By definition, one has

$$\widehat{J}_{n+1}(\widehat{v}_{n+1},g)(\widehat{Z}_n,R_n)=\max_{s\in G(\widehat{Z}_n)}\widehat{J}_{n+1}(\widehat{v}_{n+1},g)(\widehat{Z}_n,s).$$

As above, we insert V_n and obtain

$$\begin{aligned}&\left|\Big[J(\overline{v}_{n+1},g)(Z_n,R_n)-\max_{s\in G(\widehat{Z}_n)}\widehat{J}_{n+1}(\widehat{v}_{n+1},g)(\widehat{Z}_n,s)\Big]\mathbf{1}_{\{R_n=s^*_{n+1}(Z_n,S_n)\}}\right|\\ &\qquad\leq \mathbf{E}\big[|\overline{V}_{n+1}-V_{n+1}||Z_n\big]\mathbf{1}_{\{R_n=s^*_{n+1}(Z_n,S_n)\}}\\ &\qquad\quad+\left|J(v_{n+1},g)(Z_n,R_n)-\widehat{J}_{n+1}(\widehat{v}_{n+1},g)(\widehat{Z}_n,R_n))\right|\mathbf{1}_{\{R_n=s^*_{n+1}(Z_n,S_n)\}}.\end{aligned} \tag{6.4}$$

Again, similar arguments as those used for Lemmas A.3, 5.6 and 5.9, and Equations (5.6), (5.7) and (5.8) yield, on $\{R_n=s^*_{n+1}(Z_n,S_n)\}$

$$\begin{aligned}&\left|J(v_{n+1},g)(Z_n,R_n)-\widehat{J}_{n+1}(\widehat{v}_{n+1},g)(\widehat{Z}_n,R_n))\right|\\ &\quad\leq \left(\big[v_{n+1}\big]_1E_2+\big[g\big]_1+C_gC_{t^*}\big[\lambda\big]_1(2+C_{t^*}C_\lambda)\right)\Big(|Z_n-\widehat{Z}_n|+\mathbf{E}\big[|Z_n-\widehat{Z}_n||\widehat{Z}_n\big]\Big)\\ &\qquad+2\big[v_{n+1}\big]\mathbf{E}\big[|Z_{n+1}-\widehat{Z}_{n+1}||\widehat{Z}_n\big]+\mathbf{E}\big[|V_{n+1}-\widehat{V}_{n+1}||\widehat{Z}_n\big]\\ &\qquad+\big[g\big]_1\mathbf{E}\big[|Z_n-\widehat{Z}_n||\widehat{Z}_n\big]+2C_g\mathbf{E}\big[|\mathbf{1}_{\{S_{n+1}<R_n\}}-\mathbf{1}_{\{\widehat{S}_{n+1}<R_n\}}||\widehat{Z}_n\big].\end{aligned} \tag{6.5}$$

Note that all the constants with a factor $\big[t^*\big]$ have vanished, because we know here that both $R_n<t^*(Z_n)$ and $R_n<t^*(\widehat{Z}_n)$ hold on $\{R_n=s^*_{n+1}(Z_n,S_n)\}$.

Finally, on $\{s^*(Z_n) \geq t^*(Z_n) = R_n + \beta\}$, by construction of the grid $G(\widehat{Z}_n)$ (see Remark 4.2), one has for all $0 < \eta < \min_{z \in \Gamma_n^Z} \{\Delta(z)\}$,

$$R_n = t^*(Z_n) - \beta < s^*(Z_n) < t^*(\widehat{Z}_n) - \eta.$$

Consequently, using the crude bound

$$\Big| J(\overline{v}_{n+1}, g)(Z_n, R_n)) \Big| + \Big| \max_{s \in G(\widehat{Z}_n)} \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n, s) \Big| \leq 2C_g,$$

one obtains

$$\Big| J(\overline{v}_{n+1}, g)(Z_n, r_{n+1,\beta}(Z_n, S_n)) - \max_{s \in G(\widehat{Z}_n)} \widehat{J}_{n+1}(\widehat{v}_{n+1}, g)(\widehat{Z}_n, s) \Big| \mathbf{1}_{\{r_{n+1,\beta}(Z_n,S_n) = t^*(Z_n) - \beta\}}$$
$$\leq 2C_g \Big| \mathbf{1}_{\{t^*(Z_n) - \beta < t^*(\widehat{Z}_n) - \eta\}} \Big|. \tag{6.6}$$

Now the combination of Equations (6.1)-(6.6) and Lemmas 5.5 and 5.6 yields, for all $\beta < \eta < \min_{z \in \Gamma_n^Z} \{\Delta(z)\}$

$$\begin{aligned}
\|\overline{V}_n - \widehat{V}_n\|_p \leq \;& \|\overline{V}_{n+1} - V_{n+1}\|_p + \|V_{n+1} - \widehat{V}_{n+1}\|_p + 2[v_{n+1}]\|Z_{n+1} - \widehat{Z}_{n+1}\|_p \\
& + \|Z_n - \widehat{Z}_n\|_p \Big(2[v_{n+1}]_1 E_2 + 2C_g C_{t^*}[\lambda]_1 (2 + C_{t^*} C_\lambda) \\
& + (4C_g C_\lambda [t^*] + 2[v_{n+1}]_* [Q]) \vee (3[g]_1) \Big) \\
& + 2C_g \Big(2C_\lambda \eta + \frac{1}{\eta} \|S_{n+1} - \widehat{S}_{n+1}\|_p + \frac{[t^*]}{\eta - \beta} \|Z_n - \widehat{Z}_n\|_p \Big).
\end{aligned}$$

Now suppose there exists $0 < a < 1$ such that $\eta = a^{-1}\beta$. Then the optimal choice for η satisfies

$$2C_\lambda \eta = \frac{1}{\eta} \Big(\frac{[t^*]}{1-a} \|\widehat{Z}_n - Z_n\|_p + \|S_{n+1} - \widehat{S}_{n+1}\|_p \Big),$$

providing it also satisfies the condition $0 < \eta < \min_{z \in \Gamma_n^z} \{\Delta(z)\}$, Hence the result. □

Theroem 6.3 gives a recursive error estimation. Here is the initializing step.

Theorem 6.4 *Suppose the discretization parameters are chosen such that there exists* $0 < a < 1$ *satisfying*

$$\frac{\beta}{a} = (2C_\lambda)^{-1/2} \Big(\frac{[t^*]}{1-a} \|\widehat{Z}_{N-1} - Z_{N-1}\|_p + \|S_N - \widehat{S}_N\|_p \Big)^{1/2} < \min_{z \in \Gamma_{N-1}^z} \{\Delta(z)\}.$$

Then one has

$$\begin{aligned}\|\overline{V}_{N-1} - V_{N-1}\|_p \quad \leq \quad & \|\widehat{V}_{N-1} - V_{N-1}\|_p + 3[g]\|Z_N - \widehat{Z}_N\|_p + a_{N-1}\|Z_{N-1} - \widehat{Z}_{N-1}\|_p \\ & + 4C_g(2C_\lambda)^{1/2}\Big(\frac{[t^*]}{1-a}\|\widehat{Z}_{N-1} - Z_{N-1}\|_p + \|S_N - \widehat{S}_N\|_p\Big)^{1/2},\end{aligned}$$

with $a_{N-1} = \Big(2[g]_1 E_2 + 2C_g C_{t^*}[\lambda]_1(2 + C_{t^*}C_\lambda) + \big(4C_g C_\lambda[t^*] + 2[g]_*[Q]\big) \vee (3[g]_1)\Big)$.

Proof: As before, the strong Markov property of the process $\{X(t)\}$ yields

$$\begin{aligned}\overline{v}_{N-1}(Z_{N-1}) &= \mathbf{E}\big[g(X_{r_{N,\beta}(Z_{N-1},S_{N-1})})\mathbf{1}_{\{S_N > r_{N,\beta}(Z_{N-1},S_{N-1})\}}\big|Z_{N-1}\big] \\ &\quad + \mathbf{E}\big[g(Z_N)\mathbf{1}_{\{S_N \leq r_{N,\beta}(Z_{N-1},S_{N-1})\}}\big|Z_{N-1}\big] \\ &= \mathbf{1}_{\{r_{N,\beta}(Z_{N-1},S_{N-1}) \geq t^*(Z_{N-1})\}} Kg(Z_{N-1}) \\ &\quad + \mathbf{1}_{\{r_{N,\beta}(Z_{N-1},S_{N-1}) < t^*(Z_{N-1})\}} J(g,g)(Z_{N-1}, r_{N,\beta}(Z_{N-1},S_{N-1})).\end{aligned}$$

The rest of the proof is similar to that of the previous theorem. □

As in Section 5, it is now clear that an adequate choice of discretization parameters yields arbitrarily small errors if one uses the stopping-time τ_N.

7. Example. Now we apply the procedures decribed in Sections 4 and 6 on a simple PDMP and present numerical results.

Set $E = [0,1[$, and $\partial E = \{1\}$. The flow is defined on $[0,1]$ by $\phi(x,t) = x + vt$ for some positive v, the jump rate is defined on $[0,1]$ by $\lambda(x) = \beta x^\alpha$, with $\beta > 0$ and $\alpha \geq 1$, and for all $x \in [0,1]$, one sets $Q(x,\cdot)$ to be the uniform law on $[0,1/2]$. Thus, the process moves with constant speed v towards 1, but the closer it gets to the boundary 1, the higher the probability to jump backwards on $[0,1/2]$. Figure 1 shows two trajectories of this process for $x_0 = 0$, $v = \alpha = 1$ and $\beta = 3$ and up to the 10-th jump.

The reward function g is defined on $[0,1]$ by $g(x) = x$. Our assumptions are clearly satisfied, and we are even in the special case when the flow is Lipschitz-continuous (see Remark A.8). All the constants involved in Theorems 5.1 and 6.3 can be computed explicitely.

The real value function $V_0 = v_0(x_0)$ is unknown, but, as our stopping rule τ_N is a stopping time dominated by T_N, one clearly has

$$\overline{V}_0 = \mathbf{E}_{x_0}\big[g(X(\tau_N))\big] \leq V_0 = \sup_{\tau \in \mathcal{M}_N} \mathbb{E}_{x_0}\big[g(X(\tau))\big] \leq \mathbb{E}_{x_0}\big[\sup_{0 \leq t \leq T_N} g(X(t))\big]. \tag{7.1}$$

The first and last terms can be evaluated by Monte Carlo simulations, which provides another indicator of the sharpness of our numerical procedure. For 10^6 Monte Carlo simulations, one obtains $\mathbb{E}_{x_0}\big[\sup_{0 \leq t \leq T_N} g(X(t))\big] = 0.9878$. Simulation results (for $d = 2$, $x_0 = 0$, $v = \alpha = 1$,

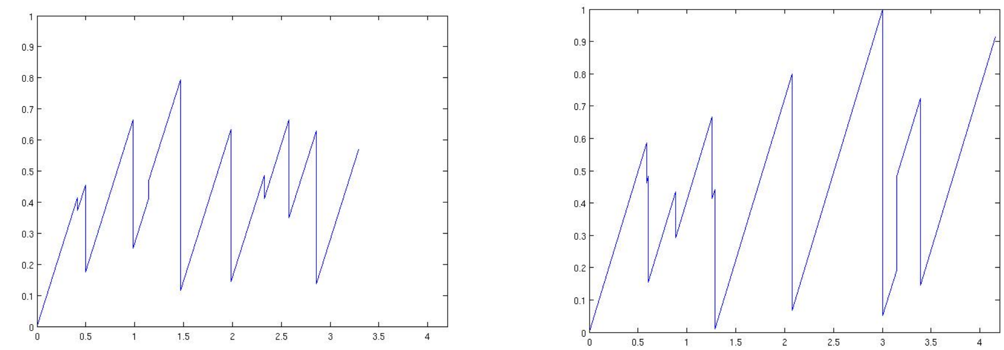

FIGURE 1. *Two trajectories of the PDMP.*

$\beta = 3$, up to the 10-th jump and for 10^5 Monte Carlo simulations) are given in Table 1. Note that, as expected, the theoretical errors decrease as the quantization error decreases. From equation (7.1), it follows

$$V_0 - \overline{V}_0 \leq \mathbb{E}_{x_0}\big[\sup_{0\leq t\leq T_N} g\big(X(t)\big)\big] - \overline{V}_0.$$

This provides an empirical upper bound for the error.

Pt	QE	Δ	$\widehat{V}_0$	$\overline{V}_0$	B_1	B_2	B_3
10	0.0943	0.151	0.7760	0.8173	0.1705	74.64	897.0
50	0.0418	0.100	0.8298	0.8785	0.1093	43.36	511.5
100	0.0289	0.083	0.8242	0.8850	0.1028	34.15	400.3
500	0.0133	0.056	0.8432	0.8899	0.0989	21.03	243.1
900	0.0102	0.049	0.8514	0.8968	0.0910	17.98	206.9

Pt	Number of points in each quantization grid
QE	Quantization error: $QE = \max_{0\leq k\leq N} \|\Theta_k - \widehat{\Theta}_k\|_2$
Δ	For all z, $\Delta(z) = \Delta$
B_1	Empirical bound $\mathbb{E}_{x_0}\big[\sup_{0\leq t\leq T_N} g\big(X(t)\big)\big] - \overline{V}_0$
B_2	Theoretical bound given by Theorem 5.1
B_3	Theoretical bound given by Theorems 6.3 and 6.4

TABLE 1
Simulation results.

APPENDIX A: AUXILIARY RESULTS

A.1. Lipschitz properties of J and K. In this section, we derive useful Lipschitz-type properties of operators J and K. The first result is straightforward.

Lemma A.1 *Let $h \in \mathbf{L}^c$. Then for all $(x,y) \in E^2$ and $(t,u) \in \mathbb{R}_+^2$, one has*

$$\Big|h\big(\phi(x,t\wedge t^*(x))\big)\mathrm{e}^{-\Lambda(x,t\wedge t^*(x))} - h\big(\phi(y,u\wedge t^*(y))\big)\mathrm{e}^{-\Lambda(y,u\wedge t^*(y))}\Big| \\ \leq D_1(h)|x-y| + D_2(h)|t-u|,$$

where

- *if $t < t^*(x)$ and $u < t^*(y)$,*

$$D_1(h) = [h]_1 + C_h C_{t^*}[\lambda]_1, \qquad D_2(h) = [h]_2 + C_h C_\lambda,$$

- *if $t = t^*(x)$ and $u = t^*(y)$,*

$$D_1(h) = [h]_* + C_h C_{t^*}[\lambda]_1 + C_h C_\lambda [t^*], \qquad D_2(h) = 0,$$

- *otherwise,*

$$D_1(h) = [h]_1 + C_h C_{t^*}[\lambda]_1 + [h]_2[t^*] + C_h C_\lambda[t^*], \qquad D_2(h) = [h]_2 + C_h C_\lambda.$$

Lemma A.2 *Let $w \in \mathbf{B}(E)$. Then for all $x \in E$, $(t,u) \in \mathbb{R}_+^2$, one has*

$$\Big|J(w,g)(x,t) - J(w,g)(x,u)\Big| \leq (C_w C_\lambda + [g]_2 + C_g C_\lambda)|t-u|.$$

Proof: By definition of J, we obtain

$$\Big|J(w,g)(x,t) - J(w,g)(x,u)\Big| \leq \Big|\int_{t\wedge t^*(x)}^{u\wedge t^*(x)} \lambda Q w(\phi(x,s)) e^{-\Lambda(x,s)} ds\Big| \\ + \Big|g(\phi(x,t\wedge t^*(x)))e^{-\Lambda(x,t\wedge t^*(x))} - g(\phi(x,u\wedge t^*(x)))e^{-\Lambda(x,u\wedge t^*(x))}\Big|.$$

Applying Lemma A.1 to $h = g$, the result follows. □

Lemma A.3 *Let $w \in \mathbf{L}^c$. Then for all $(x,y) \in E^2$, $t \in \mathbb{R}_+$,*

$$\Big|J(w,g)(x,t) - J(w,g)(y,t))\Big| \leq (C_w E_1 + [w]_1 E_2 + E_3)|x-y|,$$

where

$$\begin{aligned} E_1 &= C_\lambda[t^*] + C_{t^*}[\lambda]_1(1 + C_{t^*}C_\lambda), \\ E_2 &= C_{t^*}C_\lambda[Q], \\ E_3 &= [g]_1 + [g]_2[t^*] + C_g\{C_{t^*}[\lambda]_1 + C_\lambda[t^*]\}. \end{aligned}$$

Proof: again by definition, we obtain

$$\begin{aligned}&\Big|J(w,g)(x,t)-J(w,g)(y,t))\Big|\\&\qquad\leq\left|\int_0^{t\wedge t^*(x)}\lambda Qw(\phi(x,s))e^{-\Lambda(x,s)}ds-\int_0^{t\wedge t^*(y)}\lambda Qw(\phi(y,s))e^{-\Lambda(y,s)}ds\right|\\&\qquad+\Big|g(\phi(x,t\wedge t^*(x)))e^{-\Lambda(x,t\wedge t^*(x))}-g(\phi(y,t\wedge t^*(y)))e^{-\Lambda(y,t\wedge t^*(y))}\Big|.\end{aligned}$$

Without loss of generality it can be assumed that $t^*(x)\leq t^*(y)$. From Lemma A.1 for $h=g$ and using the fact that $\big|t\wedge t^*(x)-t\wedge t^*(y)\big|\leq\big|t^*(x)-t^*(y)\big|$, we get

$$\begin{aligned}&\Big|J(w,g)(x,t)-J(w,g)(y,t))\Big|\\&\qquad\leq\int_0^{t\wedge t^*(x)}\Big|\lambda Qw(\phi(x,s))e^{-\Lambda(x,s)}-\lambda Qw(\phi(y,s))e^{-\Lambda(y,s)}\Big|ds\\&\qquad+(C_wC_\lambda\big[t^*\big]+E_3)|x-y|.\end{aligned}$$

By using a similar results as Lemma A.1 for $h=\lambda Qw$, we obtain the result. □

Lemma A.4 *Let $w\in\mathbf{L}^c$. Then for all $(x,y)\in E^2$,*

$$\big|Kw(x)-Kw(y)\big|\leq(C_wE_4+\big[w\big]_1E_2+\big[w\big]_*\big[Q\big])|x-y|,$$

where $E_4=2C_\lambda\big[t^*\big]+C_{t^*}\big[\lambda\big]_1(2+C_{t^*}C_\lambda)$.

Proof: The proof is similar to the previous ones and therefore omitted. □

A.2. Lipschitz properties of the value functions. Now we turn to the Lipschitz continuity of the sequence of value functions (v_n). Namely, we prove that under our assumptions, v_n belongs to $\mathbf{L}^c$ for all $0\leq n\leq N$. We also compute the Lipschitz constant of v_n on $\overline{E}$ as it is much sharper in this case than $\big[v_n\big]_1$, see Remark 2.2.

We start with proving sharper results on operator J.

Lemma A.5 *Let $w\in\mathbf{L}^c$. Then for all $x\in E$ and $(s,t)\in\mathbb{R}_+^2$,*

$$\Big|\sup_{u\geq t}J(w,g)(x,u)-\sup_{u\geq s}J(w,g)(x,u)\Big|\leq(C_wC_\lambda+\big[g\big]_2+C_gC_\lambda)|t-s|.$$

Proof: Without loss of generality it can be assumed that $t\leq s$. Therefore, one has

$$\Big|\sup_{u\geq t}J(w,g)(x,u)-\sup_{u\geq s}J(w,g)(x,u)\Big|=\sup_{u\geq t}J(w,g)(x,u)-\sup_{u\geq s}J(w,g)(x,u).\tag{A.1}$$

Remark that there exists $\bar{t} \in [t \wedge t^*(x), t^*(x)]$ such that $\sup_{u \geq t} J(w,g)(x,u) = J(w,g)(x,\bar{t})$. Consequently, if $\bar{t} \geq s$ then one has $\left| \sup_{u \geq t} J(w,g)(x,u) - \sup_{u \geq s} J(w,g)(x,u) \right| = 0$.
Now if $\bar{t} \in [t \wedge t^*(x), s[$, then one has

$$\sup_{u \geq t} J(w,g)(x,u) - \sup_{u \geq s} J(w,g)(x,u) \leq J(w,g)(x,\bar{t}) - J(w,g)(x,s).$$

From Lemma A.2, we obtain the following inequality

$$\sup_{u \geq t} J(w,g)(x,u) - \sup_{u \geq s} J(w,g)(x,u) \leq (C_w C_\lambda + [g]_2 + C_g C_\lambda)|\bar{t} - s|. \tag{A.2}$$

Combining equations (A.1), (A.2) and the fact that $|\bar{t} - s| \leq |t - s|$ the result follows. □

similarly, we obtain the following result.

Lemma A.6 *Let $w \in \mathbf{L}^c$. Then for all $(x,y) \in E^2$,*

$$\Big| \sup_{t \leq t^*(x)} J(w,g)(x,t) - \sup_{t \leq t^*(y)} J(w,g)(y,t) \Big| \leq (C_w E_5 + [w]_1 E_2 + E_6)|x - y|,$$

where $E_5 = E_1 + C_\lambda [t^*]$ and $E_6 = E_3 + ([g]_2 + C_g C_\lambda)[t^*]$.

Now we turn to (v_n). Recall from [11] that for all $0 \leq n \leq N$, (v_n) is bounded with $C_{v_n} = C_g$.

Proposition A.7 *For all $0 \leq n \leq N$, $v_n \in \mathbf{L}^c$ and*

$$\begin{aligned}
[v_n]_1 &\leq e^{C_\lambda C_{t^*}} (2[v_{n+1}]_1 E_2 + C_g E_1 + C_g E_4 + C_g C_{t^*} [\lambda]_1 (1 + C_\lambda C_{t^*})) \\
&\quad + e^{C_\lambda C_{t^*}} \Big\{ ([g]_1 + [g]_2 [t^*]) \vee ([v_{n+1}]_* [Q]) \Big\}, &\text{(A.3)} \\
[v_n]_2 &\leq e^{C_\lambda C_{t^*}} \Big\{ C_g C_\lambda (4 + C_\lambda C_{t^*}) + [g]_2 \Big\}, &\text{(A.4)} \\
[v_n]_* &\leq [v_n]_1 + [v_n]_2 [t^*], \\
[v_n] &\leq [v_{n+1}]_1 E_2 + C_g E_5 + \Big\{ E_6 \vee ([v_{n+1}]_* [Q] + C_g C_{t^*} [\lambda]_1) \Big\}.
\end{aligned}$$

Proof: Clearly, $v_N = g$ is in $\mathbf{L}^c$. Assume that v_{n+1} is in $\mathbf{L}^c$, then by using the semi-group property of the drift ϕ it can be shown that for any $x \in E$, $t \in [0, t^*(x)]$, one has (see [11, Eq. (8)])

$$v_n(\phi(x,t)) = e^{\Lambda(x,t)} \Big\{ \big(\sup_{u \geq t} J(v_{n+1}, g)(x,u) \vee K v_{n+1}(x) \big) - I v_{n+1}(x,t) \Big\}. \tag{A.5}$$

Remark that for $x \in E$, $t \in \mathbb{R}_+$, one has

$$\sup_{u \geq t} J(v_{n+1}, g)(x,u) \vee K v_{n+1}(x) \leq \sup_{u} J(v_{n+1}, g)(x,u) \vee K v_{n+1}(x) = v_n(x). \tag{A.6}$$

Set $(x,y)\in E^2$ and $t\in[0,t^*(x)\wedge t^*(y)]$. It is easy to show that

$$\Big|e^{\Lambda(x,t)}-e^{\Lambda(y,t)}\Big| \;\leq\; e^{C_\lambda C_{t^*}}\left[\lambda\right]_1 C_{t^*}|x-y|, \tag{A.7}$$

$$\Big|Iv_{n+1}(x,t)-Iv_{n+1}(y,t)\Big| \;\leq\; \Big(C_{v_{n+1}}E_1+\left[v_{n+1}\right]_1 E_2\Big)|x-y|. \tag{A.8}$$

Then, equations (A.5)-(A.8) yield

$$\begin{aligned}\Big|v_n(\phi(x,t))-v_n(\phi(y,t))\Big| &\leq \Big\{|v_n(x)|+|Iv_{n+1}(x,t)|\Big\}e^{C_\lambda C_{t^*}}\left[\lambda\right]_1 C_{t^*}|x-y|\\ &+e^{\Lambda(y,t)}\Big\{\sup_{u\geq t}\Big|J(v_{n+1},g)(x,u)-J(v_{n+1},g)(y,u)\Big|\vee|Kv_{n+1}(x)-Kv_{n+1}(y)|\Big\}\\ &+e^{\Lambda(y,t)}\Big(C_{v_{n+1}}E_1+\left[v_{n+1}\right]_1 E_2\Big)|x-y|.\end{aligned} \tag{A.9}$$

For $x\in E$, $t\in[0,t^*(x)]$ and $n\in\mathbb{N}$, note that

$$e^{\Lambda(x,t)}\leq e^{C_\lambda C_{t^*}},\quad |Iv_{n+1}(x,t)|\leq C_\lambda C_{v_{n+1}}C_{t^*},\ \text{and}\ |v_{n+1}(x)|\leq C_g. \tag{A.10}$$

Therefore, we obtain inequality (A.3) by using equations (A.9), (A.10) and Lemma A.3, A.5, and the fact that $C_gE_1+E_3=C_gE_4+\left[g\right]_1+\left[g\right]_2[t^*]$.

Now, set $x\in E$ and t, $s\in[0,t^*(x)]$. Similarly, one has

$$\Big|e^{\Lambda(x,t)}-e^{\Lambda(x,s)}\Big| \;\leq\; e^{C_\lambda C_{t^*}}C_\lambda|t-s|, \tag{A.11}$$

$$\Big|Iv_{n+1}(x,t)-Iv_{n+1}(x,s)\Big| \;\leq\; C_\lambda C_{v_{n+1}}|t-s|. \tag{A.12}$$

Combining equations (A.5), (A.6), (A.11) and (A.12), it yields

$$\begin{aligned}\Big|v_n(\phi(x,t))-v_n(\phi(x,s))\Big| &\leq \Big\{|v_n(x)|+|Iv_{n+1}(x,t)|\Big\}e^{C_\lambda C_{t^*}}C_\lambda|t-s|\\ &+e^{\Lambda(x,t)}\Big\{\Big|\sup_{u\geq t}J(v_{n+1},g)(x,u)-\sup_{u\geq s}J(v_{n+1},g)(x,u)\Big|+C_\lambda C_{v_{n+1}}|t-s|\Big\}.\end{aligned} \tag{A.13}$$

Finally, inequality (A.4) follows from equations (A.10), (A.13) and Lemma A.4.

One clearly has $\left[v_n\right]_*\leq\left[v_n\right]_1+\left[v_n\right]_2[t^*]$. Finally, set $(x,y)\in\overline{E}^2$. By definition, one has

$$\begin{aligned}&|v_n(x)-v_n(y)|\\ &\quad\leq\Big|\sup_{u\leq t^*(x)}J(v_{n+1},g)(x,u)-\sup_{u\leq t^*(y)}J(v_{n+1},g)(y,u)\Big|\vee|Kv_{n+1}(x)-Kv_{n+1}(y)|,\end{aligned}$$

and we conclude using Lemmas A.6 and A.4, and the fact that $E_4=E_5+C_{t^*}\left[\lambda\right]_1$. □

Remark A.8 *Note that $\left[v_n\right]$ is much sharper than $\left[v_n\right]_1$. If in addition to our assumptions, the drift ϕ is Lipschitz-continuous in both variables, then with obvious notation, one has $\left[v_n\right]_i\leq\left[v_n\right]\left[\phi\right]_i$ for $i\in\{1,2,*\}$, which should yield better constants, see the example in Section 7.*

APPENDIX B: STRUCTURE OF THE STOPPING TIMES OF PDMP'S

Let τ be an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time. Let us recall the important result from M.H.A. Davis [6].

Theorem B.1 *There exists a sequence of nonnegative random variables $(R_n)_{n\in\mathbb{N}^*}$ such that R_n is $\mathcal{F}_{T_{n-1}}$-measurable and $\tau\wedge T_{n+1}=(T_n+R_{n+1})\wedge T_{n+1}$ on $\{\tau\geq T_n\}$.*

Lemma B.2 *Define $\overline{R}_1=R_1$, and $\overline{R}_k=R_k\mathbf{1}_{\{S_{k-1}\leq\overline{R}_{k-1}\}}$. Then one has*

$$\tau = \sum_{n=1}^{\infty}\overline{R}_n\wedge S_n.$$

Proof: Clearly, on $\{T_k\leq\tau<T_{k+1}\}$, one has $R_j\geq S_j$ and $R_{k+1}<S_{k+1}$ for all $j\leq k$. Consequently, by definition $\overline{R}_j=R_j$ for all $j\leq k+1$, whence

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{R}_n\wedge S_n &= \sum_{n=1}^{k}\overline{R}_n\wedge S_n+\{\overline{R}_{k+1}\wedge S_{k+1}\}+\sum_{n=k+2}^{\infty}\overline{R}_n\wedge S_n\\ &= T_k+R_{k+1}+\sum_{n=k+2}^{\infty}\overline{R}_n\wedge S_n.\end{aligned}$$

Since $\overline{R}_{k+1}=R_{k+1}<S_{k+1}$ we have $\overline{R}_j=0$ for all $j\geq k+2$. Therefore, $\sum_{n=1}^{\infty}\overline{R}_n\wedge S_n=T_k+R_{k+1}=\tau$, showing the result. □

There exists a sequence of measurable mappings $(r_k)_{k\in\mathbb{N}_*}$ defined on $E\times(\mathbb{R}_+\times E)^{k-1}$ with value in $\mathbb{R}_+$ satisfying

$$\begin{aligned}R_1 &= r_1(Z_0),\\ R_k &= r_k(Z_0,\Gamma_{k-1}),\end{aligned}$$

where $\Gamma_k=(S_1,Z_1,\ldots,S_k,Z_k)$.

Definition B.3 *Consider $p\in\mathbb{N}_*$. Let $(\widehat{R}_k)_{k\in\mathbb{N}_*}$ be a sequence of mappings defined on $E\times(\mathbb{R}_+\times E)^p\times\Omega$ with value in $\mathbb{R}_+$ defined by*

$$\widehat{R}_1(y,\gamma,\omega) = r_{p+1}(y,\gamma),$$

and for $k\geq 2$

$$\widehat{R}_k(y,\gamma,\omega) = r_{p+k}(y,\gamma,\Gamma_{k-1}(\omega))\mathbf{1}_{\{S_{k-1}\leq\widehat{R}_{k-1}\}}(y,\gamma,\omega).$$

Proposition B.4 *Assume that $T_p \leq \tau \leq T_N$. Then, one has*

$$\tau \quad = \quad T_p + \widehat{\tau}(Z_0, \Gamma_p, \theta_{T_p}),$$

where $\widehat{\tau}\colon E \times (\mathbb{R}_+ \times E)^p \times \Omega \to \mathbb{R}_+$ is defined by

$$\widehat{\tau}(y, \gamma, \omega) \quad = \quad \sum_{n=1}^{N-p} \widehat{R}_n(y, \gamma, \omega) \wedge S_n(\omega). \tag{B.1}$$

Proof: First, let us prove by induction that for $k \in \mathbb{N}_*$, one has

$$\widehat{R}_k(Z_0, \Gamma_p, \theta_{T_p}) \quad = \quad \overline{R}_{p+k}. \tag{B.2}$$

Indeed, one has $\widehat{R}_1(Z_0, \Gamma_p, \theta_{T_p}) = R_{p+1}$, and on the set $\{\tau \geq T_p\}$, one also has $R_{p+1} = \overline{R}_{p+1}$. Consequently, $\widehat{R}_1(Z_0, \Gamma_p) = \overline{R}_{p+1}$. Now assume that $\widehat{R}_k(Z_0, \Gamma_p, \theta_{T_p}) = \overline{R}_{p+k}$. Then, one has

$$\widehat{R}_{k+1}(Z_0(\omega), \Gamma_p(\omega), \theta_{T_p}(\omega)) = r_{p+k+1}(Z_0(\omega),\Gamma_p(\omega), \Gamma_k(\theta_{T_p}(\omega)))\mathbf{1}_{\{S_k \leq \widehat{R}_k\}}(Z_0(\omega), \Gamma_p(\omega), \theta_{T_p}(\omega)).$$

By definition, one has $\Gamma_k(\theta_{T_p}(\omega)) = \big(S_{p+1}(\omega), Z_{p+1}(\omega), \ldots, S_{p+k}(\omega), Z_{p+k}(\omega)\big)$ and the induction hypothesis easily yields $\mathbf{1}_{\{S_k \leq \widehat{R}_k\}}(Z_0(\omega), \Gamma_p(\omega), \theta_{T_p}(\omega)) = \mathbf{1}_{\{S_{p+k} \leq \overline{R}_{p+k}\}}(\omega)$. Therefore, we get $\widehat{R}_{k+1}(Z_0, \Gamma_p, \theta_{T_p}) = \overline{R}_{p+k+1}$, showing (B.2).

Combining equations (B.1) and (B.2) yields

$$\widehat{\tau}(Z_0, \Gamma_p, \theta_{T_p}) \quad = \quad \sum_{n=1}^{N-n} \overline{R}_{p+n} \wedge S_{p+n}. \tag{B.3}$$

However, we have already seen that on the set $\{T \geq T_p\}$, one has $R_k = \overline{R}_k \geq S_k$, for $k \leq p$. Consequently, using equation (B.3), we obtain

$$T_p + \widehat{\tau}(Z_0, \Gamma_p, \theta_{T_p}) = \sum_{k=1}^{p} S_k + \sum_{k=p+1}^{N} \overline{R}_k \wedge S_k = \sum_{k=1}^{N} \overline{R}_k \wedge S_k.$$

Since $\tau \leq T_N$, we obtain from Lemma B.2 and its proof that $\tau = \sum_{n=1}^{N} \overline{R}_n \wedge S_n$, showing the result. □

Proposition B.5 *Let $(U_n)_{n \in \mathbb{N}^*}$ be a sequence of nonnegative random variables such that U_n is $\mathcal{F}_{T_{n-1}}$-measurable and $U_{n+1} = 0$ on $\{S_n > U_n\}$, for all $n \in \mathbb{N}_*$. Set*

$$U \quad = \quad \sum_{n=1}^{\infty} U_n \wedge S_n.$$

Then U is an $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-stopping time.

Proof: Assumption 2.1 yields

$$\{U \leq t\} = \bigcup_{n=0}^{\infty} \Big[\Big(\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{t < T_{n+1}\} \Big) \cup \Big(\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{T_{n+1} \leq t\} \Big) \Big]. \tag{B.4}$$

From the definition of U_n, one has $\{U \geq T_n\} = \{U_n \geq S_n\}$, hence one has

$$\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{t < T_{n+1}\} = \{S_n \leq U_n\} \cap \{T_n + U_{n+1} \leq t\} \cap \{T_n \leq t\} \cap \{t < T_{n+1}\}.$$

Theorem 2.10 *ii)* in [8] now yields $\{S_n \leq U_n\} \cap \{T_n + U_{n+1} \leq t\} \cap \{T_n \leq t\} \in \mathcal{F}_t$, thus one has

$$\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{t < T_{n+1}\} \in \mathcal{F}_t. \tag{B.5}$$

On the other hand, one has

$$\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{T_{n+1} \leq t\} = \{S_n \leq U_n\} \cap \{U_{n+1} < S_{n+1}\} \cap \{T_{n+1} \leq t\}.$$

Hence Theorem 2.10 *ii)* in [8] again yields

$$\{T_n \leq U < T_{n+1}\} \cap \{U \leq t\} \cap \{T_{n+1} \leq t\} \in \mathcal{F}_t. \tag{B.6}$$

Combining equations (B.4), (B.5), and (B.6) we obtain the result. □

Corollary B.6 *For any* $(y, \gamma) \in E \times (\mathbb{R}_+ \times E)^p$, $\widehat{\tau}(y, \gamma, \cdot)$ *is an* $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$*-stopping time satisfying* $\widehat{\tau}(y, \gamma, \cdot) \leq T_{N-p}$.

Proof: It follows form the definition of $\widehat{R}_k$ that $\widehat{R}_k(y, \gamma, \omega) < S_k(\omega)$ implies $\widehat{R}_{k+1}(y, \gamma, \omega) = 0$ and the nonnegative random variable $\widehat{R}_k(y, \gamma, \cdot)$ is $\mathcal{F}_{T_{k-1}}$-measurable. Therefore, Proposition B.5 yields that $\widehat{\tau}(y, \gamma, \cdot)$ is an $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-stopping time. finally, by definition of $\widehat{\tau}$, see equation (B.1), one has $\widehat{\tau}(y, \gamma, \cdot) \leq \sum_{n=1}^{N-p} S_n = T_{N-p}$ showing the result. □

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