Asymptotic behavior of solutions to the fragmentation equation with shattering: an approach via self-similar Markov processes

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Abstract

The subject of this paper is a fragmentation equation with non-conservative solutions, some mass being lost to a dust of zero-mass particles as a consequence of an intensive splitting. Under some assumptions of regular variation on the fragmentation rate, we describe the large-time behavior of solutions. Our approach is based on probabilistic tools: the solutions to the fragmentation equation are constructed via non-increasing self-similar Markov processes that reach continuously 0 in finite time. Our main probabilistic result describes the asymptotic behavior of these processes conditioned on non-extinction and is then used for the solutions to the fragmentation equation.

We notice that two parameters influence significantly these large-time behaviors: the rate of formation of "nearly-1 relative masses" (this rate is related to the behavior near 0 of the Lévy measure associated with the corresponding self-similar Markov process) and the distribution of large initial particles. Correctly rescaled, the solutions then converge to a non-trivial limit which is related to the quasi-stationary solutions to the equation. Besides, these quasi-stationary solutions, or equivalently the quasi-stationary distributions of the self-similar Markov processes, are entirely described.

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1 Introduction and main results

Fragmentation processes occur in a variety of natural phenomena, among which polymer degradation, mineral grinding, droplet break-up, but also in analysis of algorithms, phylogeny, etc. The kinetic equation used in the physical literature to describe the time-evolution of masses of particles prone to fragmentation has the form

$$\partial_t n_t(x) = \int_x^{\infty} a(y)b(y,x)n_t(y)\mathrm{d}y - a(x)n_t(x), \tag{1}$$

where $n_t(x)$ is the concentration of particles of mass x at time t, $a(x)$ is the overall rate at which a particle with mass x splits and $b(x,y)$ describes the distribution of particles of mass y produced by the fragmentation of a particle of mass x. It is assumed that no mass is lost when a particle breaks up, that is $\int_0^x yb(x,y)\mathrm{d}y = x$. The integral in the right-hand side of (1) models the increase of particles of mass x due to the the fragmentation of particles of masses $y > x$, whereas the negative term $-a(x)n_t(x)$ models the loss of particles of mass x, due to their fragmention into smaller particles. This fragmentation equation has been intensively studied both by physicists and mathematicians. Among the first papers on the topic, we may cite e.g. [24, 25].

Both in the physical and mathematical literature, particular attention has been paid to models with the following self-similar dynamic:

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- $a(x) = Cx^{\alpha}$, for some fixed $C > 0$ and $\alpha \in \mathbb{R}$
- $b(x, y) = h(y/x)/x$ (with h such that $\int_0^1 uh(u)\mathrm{d}u = 1$). This means that the distribution of the ratios daughters masses/parent mass is only determined by a function of these ratios (and not by the parent mass).

The reason is twofold. These self-similar assumptions are relevant for applications, e.g. for polymer degradation ([30]), mineral crushing in the mining industry ([7] and the references therein) or to construct phylogenetic trees ([1]). But they are also more tractable mathematically. Besides, for the same reasons, there is also a significant literature on probabilistic models for the microscopic mechanism of fragmentation with a self-similar dynamic. We refer to the book of Bertoin [5] for an overview and to the papers [13] and [19] for discussions on the relations between the probabilistic models and the above equation.

The goal of this paper is to contribute to the understanding of solutions to the self-similar fragmentation equation, by describing their large time behavior. The cases when $\alpha > 0$ are treated in [13] and we will be concerned here with the negative cases $\alpha < 0$.

We will actually consider the following generalization of the weak form of the above fragmentation equation (1) with a self-similar dynamic:

$$\partial_t < \mu_t, f > \quad = \quad \int_0^{\infty} x^{\alpha} \left(\int_0^1 (f(yx) - f(x)y)\, B(\mathrm{d}y) \right) \mu_t(\mathrm{d}x), \tag{2}$$

where $(\mu_t, t \geq 0)$ denotes a family of measures on $]0, \infty[$, $\alpha \in \mathbb{R}$, B is a measure on $]0, 1[$ such that

$$\int_0^1 y(1-y)B(\mathrm{d}y) < \infty, \text{ and } B\left(]0,1[\right) > 0, \tag{3}$$

and f denotes any test function. When $B(\mathrm{d}y) = Ch(y)\mathrm{d}y$ with $\int_0^1 yh(y)\mathrm{d}y = 1$ and $\mu_t(\mathrm{d}x) = n_t(x)\mathrm{d}x$, we recover the weak form of (1) with $a(x) = Cx^{\alpha}$ and $b(x, y) = h(y/x)/x$. Informally, (2) corresponds to models in which particles with mass xy, $0 < y < 1$, are produced from the splitting of a particle with mass x at rate $x^{\alpha}B(\mathrm{d}y)$. Note that the overall rate at which a particle with mass x splits is $x^{\alpha}\int_0^1 yB(\mathrm{d}y)$, which may be infinite here. Let us add that the physical interpretation of the fragmentation equation imposes some constraints on the measure B. However other interpretations are possible and in the following we will be concerned with all measures B satisfying (3).

We focus on solutions to (2) with finite and non-zero initial total mass. The fragmentation equation being linear, we suppose, without loss of generality, that $\int_0^{\infty} x\mu_0(\mathrm{d}x) = 1$. To be precise, we call *a solution to (2) starting from* μ_0, any family of measures $(\mu_t, t \geq 0)$ on $]0, \infty[$ starting from μ_0 and such that

- $(\mu_t, t \geq 0)$ satisfies (2) for any test functions $f \in C_c^1$, the set of real-valued continuously differentiable functions on $]0, \infty[$ with compact support
- the following natural "physical properties" are respected (id denotes the identity function)

$$m(t) :=< \mu_t, \mathrm{id} > \quad \leq \quad m(0) = 1, \ \forall t \geq 0, \tag{4}$$

and

$$\mu_0([M, \infty[) = 0 \text{ for some } M > 0 \Rightarrow \mu_t([M, \infty[) = 0 \ \forall t \geq 0. \tag{5}$$

Note the *self-similarity of solutions*: if $(\mu_t, t \geq 0)$ is a solution to (2), so is $(\gamma^{-1}\mu_{t\gamma^{\alpha}} \circ (\gamma \mathrm{id})^{-1})$, for all $\gamma > 0$. Note also that if $(\mu_t, t \geq 0)$ is a solution to the equation with parameters (α, B), then for all $c > 0$, $(\mu_{ct}, t \geq 0)$ is a solution to the equation (2) with parameters (α, cB).

Many results on existence and uniqueness of solutions to (1) are available in the literature. See e.g. [2, 13, 23] and the references therein. With the definition above, we have the following result on existence and uniqueness of solutions to (2), which is a generalization of Theorem 1 of [19] (see also [17] for a similar approach). We recall that a subordinator is a non-decreasing Lévy process and that its distribution is characterized by two parameters: a non-negative drift coefficient and a so-called Lévy measure on $]0, \infty[$ that governs the jumps of the process. See Section 2 for background on this topic.

Theorem 1.1. *Let μ_0 be a measure on $]0,\infty[$ such that $\int_0^\infty x\mu_0(\mathrm{d}x) = 1$. Then*
(i) there exists a solution to (2) starting from μ_0 as soon as

$$\alpha \leq 0$$

or

$$\alpha > 0 \text{ and either } \int_1^\infty x\ln(x)\mu_0(\mathrm{d}x) < \infty \text{ or } x \in]0,1[\to x^{|\alpha|}\int_0^x yB(\mathrm{d}y) \text{ is bounded near 0.}$$

More precisely, this solution can be constructed via a subordinator ξ with zero drift and a Lévy measure given for any measurable function $g:]0,\infty[\to [0,\infty[$ by

$$\int_0^\infty g(x)\Pi(\mathrm{d}x) = \int_0^1 g(-\ln(x))xB(\mathrm{d}x), \tag{6}$$

through the formula

$$\int_0^\infty f(x)x\mu_t(\mathrm{d}x) := \int_0^\infty \mathbb{E}\left[f\left(x\exp(-\xi_{\rho(x^\alpha t)})\right)\right]x\mu_0(\mathrm{d}x), \text{ for all measurable } f:]0,\infty[\to [0,\infty[, \tag{7}$$

where ρ is defined by

$$\rho(t) := \inf\left\{u \geq 0 : \int_0^u \exp(\alpha\xi_r)\mathrm{d}r > t\right\}.$$

(ii) This solution is unique as soon as $\mu_0([M,\infty[) = 0$ for some $M > 0$.

When the family $(\mu_t, t \geq 0)$ is constructed via a subordinator by (7), some conditions on μ_0 and B for the existence of a density for μ_t, $t > 0$, can be settled explicitly. See e.g. [18, Proposition 3.10]. We also recall that there may be multiple solutions to the fragmentation equation when the Assumption (4) is dropped. We refer to [2] for some explicit examples.

The proof of Theorem 1.1, based on that of Theorem 1 in [19], is postponed to the Appendix at the end of this paper.

The main purpose of this paper is to use the construction (7) of solutions to the fragmentation equation to describe the large-time behavior of these solutions when $\alpha < 0$. From another, but equivalent, point of view, our main results describe the large-time behavior of exponentials of minus time-changed subordinators, as defined in Theorem 1.1, conditioned on non-extinction. These processes belong to the family of so-called *self-similar Markov processes.* We refer to Section 3 for a statement of our results in that context.

The study of large-time behavior of solutions to the fragmentation equation when $\alpha > 0$ is investigated in details in [13]. We point out that some results of [13] can be re-demonstrated using a probabilistic approach: it mainly consists in combining the subordinator construction of solutions to the fragmentation equation with the description of large-time behavior of time-changed subordinators when $\alpha > 0$ investigated in [6].

From now on, we consider $\alpha < 0$. It is well-known that in a such case, small particles split so quickly that they are reduced to a dust of zero-mass particles, so that the total mass of non-zero particles

$$m(t) = < \mu_t, \mathrm{id} >$$

decreases as time passes. This phenomenon, sometimes called "shattering", was studied e.g. in [2, 4, 16, 19, 24, 29]. More precisely, one can check that the total mass m is strictly decreasing and strictly positive on $[0,\infty[$ and that $m(t) \to 0$ as $t \to \infty$. See the forthcoming Proposition 3.3 for a proof in our framework.

In order to describe the behavior of $m(t)$ as $t \to \infty$ more accurately, we introduce the following function defined for all $t \geq 0$ by

$$\phi(t) := \int_0^1 \left(1 - x^t\right) xB(\mathrm{d}x). \tag{8}$$

It is not hard to check that the function $t \to t/\phi(t)$ is continuous, strictly increasing on $]0,\infty[$ and that its range is $](\int_0^1 |\ln(x)|xB(\mathrm{d}x))^{-1}, \infty[$. Note that the integral $\int_0^1 |\ln(x)|xB(\mathrm{d}x)$ may be finite or infinite. Then introduce

$$\varphi, \text{ the inverse of } t \to t/\phi(t), \tag{9}$$

which is well defined in a neighborhood of ∞. This function will play a key-role in the description of the long-time behavior of solutions to the fragmentation equation.

Most of our main results rely on the following hypothesis on the measure B:

$$\text{the function } u :]0,1[\to \int_0^{1-u} xB(\mathrm{d}x) \text{ varies regularly at 0 with an index } -\beta \in]-1,0], \tag{H}$$

which, in particular, ensures that ϕ and φ are regularly varying functions at ∞, with respective indices β and $1/(1-\beta)$. See Section 2.2 for details and background on regular variation.

Last, we mention that the large-time behavior of solutions to the fragmentation equation will depend strongly on the structure of the initial measure μ_0, mainly on the manner it distributes weight near ∞. The statements of our results are therefore split into two parts, according as to whether the initial measure has a bounded support (Subsection 1.1) or not (Subsection 1.2). Subsection 1.3 deals with the quasi-stationary solutions.

1.1 Initial measure μ_0 with bounded support

In this subsection, we adopt the following hypotheses and notations:

- $\alpha < 0$
- the measure μ_0 has a bounded support, i.e. $\mu_0([M,\infty[) = 0$ for some $M > 0$
- $(\mu_t, t \geq 0)$ denotes the unique solution to the fragmentation equation (2) starting from μ_0.

The supremum of the support of μ_0 is the real number s such that $\mu_0(]s,\infty[) = 0$ and $\mu_0(]s-\varepsilon, s]) > 0$ for all $\varepsilon < s$. Thanks to the self-similarity of solutions, we can, and will, always suppose that *this supremum is equal to* 1. In such frame, we have the following results.

Proposition 1.2. *For all $\lambda < \phi(\infty) := \lim_{x\to\infty} \phi(x)$, there exists a constant $C_\lambda < \infty$ such that*

$$m(t) \leq C_\lambda \exp(-\lambda t), \ \forall t \geq 0.$$

More precisely, under the hypothesis (H),

$$-\ln(m(t)) \underset{t\to\infty}{\sim} \frac{(1-\beta)}{|\alpha|} \varphi(|\alpha| t).$$

In particular, $t \to -\ln(m(t))$ is regularly varying at ∞ with index $1/(1-\beta)$.

Together with the following theorem, this gives a complete description of the large-time behavior of $(\mu_t, t \geq 0)$. Here, two positive functions g and h are said asymptotically equivalent if $g(x)/h(x) \to 1$ as $x \to \infty$.

Theorem 1.3. *Suppose (H) and $\int_0 |\ln(x)| xB(\mathrm{d}x) < \infty$. Then for all continuous bounded test functions $f :]0,\infty[\to \mathbb{R}$,*

$$\frac{1}{m(t)} \int_0^\infty f\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} x\right) x\mu_t(\mathrm{d}x) \underset{t\to\infty}{\to} \int_0^\infty f(x) x\mu_\infty(\mathrm{d}x),$$

where $x\mu_\infty(\mathrm{d}x)$ is a probability distribution on $]0,\infty[$ that is characterized by its moments

$$\int_0^\infty x^{|\alpha|n} x\mu_\infty(\mathrm{d}x) = \phi(|\alpha|)\phi(2|\alpha|)...\phi(n|\alpha|), \ n \geq 1. \tag{10}$$

The function $t \to \varphi(|\alpha|t)/(|\alpha|t)$ can be replaced by any asymptotically equivalent function.

It is interesting to compare this result with that obtained by Escobedo et al. [13] when the parameter α is positive. As already mentioned, part of their result can be rediscovered and completed by using Bertoin and Caballero's paper [6]. With our notations and under the assumptions $\int_0 |\ln(x)| xB(\mathrm{d}x) < \infty$ and $\alpha > 0$, the asymptotic behavior of the solution $(\mu_t, t \geq 0)$ to the fragmentation equation (α, B) starting from $\mu_0 = \delta_1$ can be described as follows:

$$\int_0^\infty f\left(t^{1/\alpha} x\right) x\mu_t(\mathrm{d}x) \underset{t\to\infty}{\to} \int_0^\infty f(x) x\eta_\infty(\mathrm{d}x),$$

for all continuous bounded functions $f :]0,\infty[\to \mathbb{R}$. The measure $x\eta_\infty(\mathrm{d}x)$ is a probability measure on $]0,\infty[$. Interestingly, the measure B is then only involved in the description of the limit measure η_∞, not on the "shape" of the speed of decrease of masses to 0.

We come back to the case $\alpha < 0$. Note that when $\int_0^{1-u} xB(\mathrm{d}x) \sim u^{-\beta}$ as $u \to 0$ for some $\beta \in [0,1[$, we have $\phi(t) \sim \Gamma(1-\beta)t^\beta$ and therefore $(\varphi(|\alpha|t)/|\alpha|t)^{1/|\alpha|} \sim C_{\alpha,\beta}t^{\beta/((1-\beta)|\alpha|)}$ as $t \to \infty$, where $C_{\alpha,\beta} = (|\alpha|^\beta\Gamma(1-\beta))^{1/((1-\beta)|\alpha|)}$. When moreover $\int_0 |\ln(x)|xB(\mathrm{d}x) < \infty$, Theorem 1.3 then reads

$$\frac{1}{m(t)}\int_0^\infty f\left(C_{\alpha,\beta}t^{\beta/((1-\beta)|\alpha|)}x\right)x\mu_t(\mathrm{d}x) \underset{t\to\infty}{\to} \int_0^\infty f(x)x\mu_\infty(\mathrm{d}x)$$

for all continuous bounded test functions $f :]0,\infty[\to \mathbb{R}$.

The existence and uniqueness of a measure μ_∞ on $]0,\infty[$ satisfying (10) actually hold without any assumption of regular variation on the measure B or on its behavior near 0. See the discussion near formula (14) in Section 3 for details. Some properties of the measure μ_∞ (tail behavior near 0, near ∞) are given in Section 5. In Subsection 1.3, we discuss its links with the *quasi-stationary solutions* to the fragmentation equation.

The proof of Theorem 1.3 consists in describing the behavior of the mass of a random typical non-dust particle, which is defined as follows: at each time t, choose a particle at random among the particles with a *strictly positive mass*, with a probability proportional to its mass. I.e. if $M(t)$ denotes the mass of this random particle, the distribution of $M(t)$ is given by

$$M(t) \overset{\mathrm{d}}{\sim} \frac{x\mu_t(\mathrm{d}x)}{m(t)}.$$

Otherwise said, in terms of the subordinator ξ related to the equation by (7), $M(t)$ is distributed as $M(0)\exp(-\xi_{\rho(M(0)^\alpha t)})$ conditioned to be strictly positive, with $M(0)$ independent of ξ. In terms of M, the statement of Theorem 1.3 rephrases as follows

$$\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} M(t) \overset{\mathrm{d}}{\to} M_\infty,$$

where M_∞ is a random variable with distribution $x\mu_\infty(\mathrm{d}x)$. Note the special case $\int_0^1 xB(\mathrm{d}x) < \infty$, where $\varphi(t)/t \to \int_0^1 xB(\mathrm{d}x) < \infty$. Then we have that $M(t)$ converges in distribution to a non-trivial limit. In the other cases satisfying the assumptions of Theorem 1.3, $\varphi(t)/t \to \infty$ and therefore $M(t) \overset{\mathbb{P}}{\to} 0$.

Using this random approach, we can also specify the behavior of masses that decrease at different speeds to 0, as follows.

Proposition 1.4. *Assume (H) and* $\kappa := \int_0^1 |\ln(x)|xB(\mathrm{d}x) < \infty$.
(i) Suppose moreover that the support of B is not included in a set of the form $\{a^n, n \in \mathbb{N}\}$ for some $a \in]0,1[$. Then for all measurable functions $g : [0,\infty[\to]0,\infty[$ converging to 0 at ∞,

$$\frac{g(t)^\alpha}{m(t)}\int_0^{g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}} x\mu_t(\mathrm{d}x) \underset{t\to\infty}{\to} \frac{1}{|\alpha|\kappa};$$

(ii) For all measurable functions $g : [0,\infty[\to]0,\infty[$ converging to ∞ at ∞,

- *if $g^{|\alpha|}(t)t/\varphi(t)$ converges to ∞ at ∞,*

$$\int_{g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}}^\infty x\mu_t(\mathrm{d}x) = 0$$

for all t sufficiently large

- *if* $g^{|\alpha|}(t)t/\varphi(t)$ *converges to* 0 *at* ∞ *and* $0<\beta<1$:

$$\limsup_{t\to\infty}\frac{1}{\phi^{-1}(g(t)^{|\alpha|})}\ln\left(\frac{\int_{g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}}^{\infty}x\mu_t(\mathrm{d}x)}{m(t)}\right)\leq-\frac{\beta}{|\alpha|},$$

where ϕ^{-1} *denotes the inverse of* ϕ.

Note that the first assertion of (ii) is obvious since $g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha}\to\infty$ (which means that for t sufficiently large, it is larger than 1, the supremum of the support of μ_t).

We finish this section with the following result on the remaining mass at time t of particles of mass 1 when $\mu_0(\{1\})>0$. The measure μ_∞ is that introduced in Theorem 1.3.

Proposition 1.5. *Suppose* $\mu_0(\{1\})>0$ *and set* $\phi(\infty):=\int_0^1 xB(\mathrm{d}x)\in]0,\infty]$. *Then for all* $t\geq 0$,

$$\mu_t(\{1\})=\exp\left(-t\phi(\infty)\right)\mu_0(\{1\}).$$

When moreover (H) is satisfied and $\int_0|\ln(x)|xB(\mathrm{d}x)<\infty$ *and* $\phi(\infty)<\infty$,

$$\frac{\mu_t(\{1\})}{m(t)}\underset{t\to\infty}{\to}\phi(\infty)^{1/|\alpha|}\mu_\infty\left(\{\phi(\infty)^{1/|\alpha|}\}\right)$$

and this limit is non-zero if and only if $\int^1\frac{B(\mathrm{d}x)}{1-x}<\infty$.

This means that under the assumptions of Proposition 1.5, for large times, the remaining total mass of 1-mass particles is proportional to the total mass of non-zero particles when $\int^1(1-x)^{-1}B(\mathrm{d}x)<\infty$, whereas it is negligible compared to the total mass of non-zero particles when $=\int^1(1-x)^{-1}B(\mathrm{d}x)=\infty$. We point out that the convergence of Proposition 1.5 is *not* necessarily true when $\mu_0(\{1\})=0$ (since then $\mu_t(\{1\})=0$ for all $t\geq 0$, whereas the term in the limit may be strictly positive).

1.2 Initial measure μ_0 with unbounded support

We still suppose that $\alpha<0$ and we denote by $(\mu_t,t\geq 0)$ the solution to the fragmentation equation (2) starting from μ_0 and constructed via a subordinator by formula (7). The asymptotic behavior of the mass $m(t)$ is then strongly modified by the presence of large masses and depends on the behavior as $t\to\infty$ both of $\phi(t)$ and $\mu_0\left([t,\infty[\right)$. We investigate two particular cases: exponential and power decreases of $\mu_0\left([t,\infty[\right)$ as $t\to\infty$.

Theorem 1.6. *Assume (H) and that* μ_0 *possesses a density, say* u_0, *in a neighborhood of* ∞ *such that*

$$\ln\left(u_0(x)\right)\underset{\infty}{\sim}-Cx^{\gamma}$$

for some $\gamma>0$.
(i) Then,

$$-\ln\left(m(t)\right)\underset{\infty}{\sim}C_{\alpha,\beta,\gamma}C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}h(t)$$

where h *is the inverse, well-defined in the neighborhood of* ∞, *of* $t\to t^{1+|\alpha|/\gamma}/\phi(t)$ *and*

$$C_{\alpha,\beta,\gamma}=\left(1+|\alpha|^{-1}\gamma(1-\beta)\right)\left(\frac{|\alpha|^{1/(1-\beta)}}{\gamma}\right)^{\frac{\gamma(1-\beta)}{\gamma(1-\beta)+|\alpha|}}.$$

In particular, $-\ln(m(t))$ *varies regularly at* ∞ *with index* $1/(1-\beta+|\alpha|/\gamma)$.
(ii) Suppose moreover that $\int_0|\ln(x)|xB(\mathrm{d}x)<\infty$, *which ensures that the function* $\ln(m)$ *is differentiable on* $]0,\infty[$. *Then, if the derivative* $(\ln(m))'$ *is regularly varying at* ∞, *one has, for all continuous bounded test functions* $f:]0,\infty[\to\mathbb{R}$,

$$\frac{1}{m(t)}\int_0^\infty f\left(\left(\frac{h(t)}{C_{\alpha,\beta,\gamma,C}t}\right)^{1/|\alpha|}x\right)x\mu_t(\mathrm{d}x)\underset{t\to\infty}{\to}\int_0^\infty f(x)x\mu_\infty(\mathrm{d}x),$$

where $\mu_\infty(\mathrm{d}x)$ *is the measure introduced in Theorem 1.3 and*

$$C_{\alpha,\beta,\gamma,C}=\frac{C_{\alpha,\beta,\gamma}C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}}{1-\beta+|\alpha|/\gamma}.$$

Assuming that the derivative $(\ln(m))'$ is regularly varying at ∞ may seem demanding. Actually, this assumption is also needed to get Theorem 1.3, but we are able to show it is always satisfied under the hypotheses of this theorem (see Lemma 3.9). Unfortunately, it seems difficult to adapt this proof to the case when the measure μ_0 has an unbounded support. However, according to a classical result on regular variation (the Monotone Density Theorem), $(\ln(m))'$ varies regularly at ∞ as soon as $\ln(m)$ varies regularly at ∞ and $(\ln(m))'$ is monotone near ∞, which can be checked in some particular cases.

There is also the following result on the decrease of the mass m when the density u_0 of μ_0 has a power decrease near ∞.

Proposition 1.7. *Assume that μ_0 possesses a density u_0 in a neighborhood of ∞ such that*

$$u_0(x) \underset{\infty}{\sim} Cx^{-\gamma}$$

for some $\gamma > 2$. Then,

$$m(t) \underset{\infty}{\sim} C' t^{\frac{\gamma-2}{\alpha}},$$

with $C' = |\alpha|^{-1} C \int_0^\infty \overline{m}(u) u^{\frac{2-\gamma}{\alpha}-1} \mathrm{d}u < \infty$, where $\overline{m}$ denotes the total mass of the solution to fragmentation equation with the same parameters α, B as that considered here, and with initial distribution δ_1, the Dirac mass at 1.

1.3 Quasi-stationary solutions

A *quasi-stationary solution* to the fragmentation equation (2) is a solution $(\mu_t, t \geq 0)$ such that

$$\mu_t = m(t)\mu_0, \forall t \geq 0,$$

with $m(t) =< \mu_t, \mathrm{id} >$. These quasi-stationary solutions are closely related to the measure μ_∞ introduced in the statement of Theorem 1.3. We have already mentioned that existence and uniqueness of such a measure μ_∞ satisfying (10) hold without any assumption of regular variation on the measure B or on its behavior near 0. The interesting fact is that, whatever the conditions on B, this measure and its self-similar counterparts

$$\mu_\infty^{(\lambda)} := \lambda^{-1} \mu_\infty \circ (\lambda \mathrm{id})^{-1},$$

$\lambda > 0$, are the only initial measures leading to quasi-stationary solutions to the fragmentation equation (2).

Theorem 1.8. *For all $\lambda > 0$, let $(\mu_{\infty,t}^{(\lambda)}, t \geq 0)$ denote the solution to the fragmentation equation (2) starting from $\mu_\infty^{(\lambda)}$ and constructed via a subordinator by (7). Then for all $t \geq 0$,*

$$\mu_{\infty,t}^{(\lambda)} = \exp(-\lambda^\alpha t)\mu_\infty^{(\lambda)} = m(t)\mu_\infty^{(\lambda)}.$$

Reciprocally, if $(\mu_t, t \geq 0)$ is a quasi-stationary solution to the fragmentation equation, then there exists a $\lambda > 0$ such that $(\mu_t, t \geq 0) = (\mu_{\infty,t}^{(\lambda)}, t \geq 0)$.

Organization of the paper. We start in Section 2 with some background on subordinators and regular variation. Section 3 is the core of this paper: our main results on large-time behavior of self-similar Markov processes conditioned on non-extinction are stated and proved there. Together with Theorem 1.1, these results imply Theorems 1.3, 1.6 and 1.8, as well as Propositions 1.2 and 1.7. Section 4 is devoted to the proof of Proposition 1.4. Some properties of the limit measure μ_∞ are given in Section 5, and used to prove Proposition 1.5. Last, some specific examples are discussed in Section 6 and the proof of Theorem 1.1 is detailed in the Appendix.

2 Background on subordinators and regular variation

2.1 Subordinators

A subordinator is a non-decreasing Lévy process, i.e. a non-decreasing càdlàg process with stationary and independent increments. We recall here the main properties we need in this paper and refer to the Chapter 3 of [3] for a more complete introduction to the subject.

The distribution of a subordinator $(\xi_t, t \geq 0)$ starting from $\xi_0 = 0$ is characterized by its so-called Laplace exponent $\phi : [0, \infty[\to [0, \infty[$ through the identity

$$\mathbb{E}\left[\exp(-\lambda \xi_t)\right] = \exp(-t\phi(\lambda)), \ \forall \lambda, t \geq 0.$$

According to the Lévy-Khintchine formula [3, Theorem 1, Chapter 1], there exists a real number $d \geq 0$ and a measure Π on $]0, \infty[$, $\int_0^\infty (1 \wedge x)\Pi(\mathrm{d}x) < \infty$, such that

$$\phi(\lambda) = d\lambda + \int_0^\infty (1 - \exp(-\lambda x))\Pi(\mathrm{d}x), \ \forall \lambda \geq 0.$$

The measure Π governs the jumps of the subordinator: the jumps process of ξ is a Poisson point process with intensity Π.

We will need the strong Markov property of subordinators ([3, Proposition 6, Chapter 1]): given a subordinator ξ and a stopping time T with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ generated by ξ, then, conditionally on $\{T < \infty\}$, the process $(\xi_{t+T} - \xi_T, t \geq 0)$ is independent of $\mathcal{F}_T$ and is distributed as ξ. Last, we recall that the semi-group of a subordinator possesses the Feller property ([3, Proposition 5, Chapter 1]).

From now on, all subordinators considered in this paper start from 0 *and have a drift* $d = 0$. Their distribution is therefore completely determined by their Lévy measure Π. Note that when Π is related to a measure B on $]0, 1[$ through the formula (6), the above expression of ϕ coincides with that given by formula (8), i.e.

$$\phi(\lambda) = \int_0^\infty (1 - \exp(-\lambda x))\Pi(\mathrm{d}x) = \int_0^1 (1 - x^\lambda) x B(\mathrm{d}x), \ \forall \lambda \geq 0.$$

2.2 Regular variation

A function $f :]0, \infty[\to]0, \infty[$ is said to vary regularly at ∞ (resp. 0) with index $\gamma \in \mathbb{R}$ if for all $a > 0$

$$\frac{f(ax)}{f(x)} \to a^\gamma \text{ as } x \to \infty \text{ (resp. 0)}.$$

We refer to [9] for background on the topic. In particular, we have already implicitely used that the inverse, when it exists, of a function regularly varying at ∞ with index $\gamma > 0$ is also regularly varying at ∞, with index $1/\gamma$ (see Section 1.5.7 of [9]).

Note that when the Lévy measure Π is related to the fragmentation measure B by the formula (6), our main assumption (H) reads "$u \in]0, \infty[\to \int_u^\infty \Pi(\mathrm{d}x)$ varies regularly at 0 with index $-\beta$". It is classical that this is equivalent to the fact that the function

$$\phi \text{ varies regularly at } \infty \text{ with index } \beta.$$

This can be easily proved from Karamata Abelian-Tauberian Theorems (see in particular Chapters 1.6 and 1.7 of [9]). We will often use this form of the assumption (H).

To prove the forthcoming Theorem 3.1, which will then imply Theorems 1.3 and 1.6 (ii), we will need the following technical lemma, which is taken from Chow and Cuzick [12].

Lemma 2.1. *(Chow and Cuzick [12, Lemma 3]) Let f be regularly varying at infinity with index $\gamma > 0$, and suppose that for all $\varepsilon > 0$ there exists some $x(\varepsilon)$ such that*

$$\lambda^{\gamma - \varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq \lambda^{\gamma + \varepsilon}, \ \forall \ \lambda \geq 1, \forall x \geq x(\varepsilon). \tag{11}$$

Then for all $\theta > -1$,

$$e^{f(t)} \left(\frac{f(t)}{t}\right)^{\theta+1} \int_t^\infty (x-t)^\theta \, e^{-f(x)} \mathrm{d}x \underset{t\to\infty}{\to} \gamma^{-1-\theta}\Gamma(1+\theta).$$

We point out that Chow and Cuzick state their result for all regularly varying functions with a positive index, but that their proof strongly relies on the key point (11), which is not true for any regularly varying function (easy counter-examples can be constructed). However, the functions we are interested in, i.e. $-\ln(m)$, and to which we will apply this result, will in general satisfy (11). In particular, see the forthcoming Lemma 3.6.

3 Asymptotic behavior of self-similar Markov processes

Given the construction (7) via subordinators of solutions to the fragmentation equation, the issue of characterizing the large-time asymptotics of these solutions is equivalent to characterizing large-time behavior of distributions of time-changed subordinators.

So, let ξ be a subordinator started from 0 with Lévy measure Π and no drift. We denote by ϕ its Laplace exponent. Then consider $\alpha < 0$ and let $X(0)$ be a strictly positive random variable, independent of ξ. Our goal is to specify the asymptotic behavior as $t \to \infty$ of the distributions of the random variables

$$X(t) := X(0) \exp\left(-\xi_{\rho(X(0)^\alpha t)}\right), \tag{12}$$

conditional on $\{X(t) > 0\}$, where ρ is given by

$$\rho(t) = \inf\left\{u \geq 0 : \int_0^u \exp(\alpha \xi_r) \mathrm{d}r > t\right\}.$$

Following Lamperti [22], the process X belongs to the so-called family of *self-similar Markov processes*. This means that it is strongly Markovian and that for all $x > 0$, if $\mathbb{P}_x$ denotes the distribution of X started from x, then for all $a > 0$,

$$\text{the distribution of } (aX(a^\alpha t), t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{ax}.$$

Moreover, X reaches 0 a.s. and it does it continuously. Conversely, Lamperti [22] also shows that any non-increasing càdlàg self-similar Markov processes on $[0, \infty[$ that reaches continuously 0 in finite time a.s. can be constructed like this via a time-changed subordinator.

Note that the moment at which X reaches 0 is $X(0)^{|\alpha|} I$ where I is the *exponential functional* defined by

$$I := \int_0^\infty \exp(\alpha \xi_r) \mathrm{d}r \tag{13}$$

which is clearly a.s. finite. The distribution of the random variable I was first studied in details in [11]. In particular, it is well known that for all integers $n \geq 1$,

$$\mathbb{E}[I^n] = \frac{n!}{\phi(|\alpha|)\phi(2|\alpha|)...\phi(n|\alpha|)}$$

and that the distribution of I is characterized by these moments ([11, Prop. 3.3]). It will also be essential for us (see [8, Propositions 1 and 2]) that there exists a unique probability measure μ_R on $]0, \infty[$ whose entire positive moments are given by

$$\int_0^\infty x^n \mu_R(\mathrm{d}x) = \phi(|\alpha|)\phi(2|\alpha|)...\phi(n|\alpha|), \ n \geq 1 \tag{14}$$

and that, moreover, if R denotes a r.v. with distribution μ_R independent of I, then

$$RI \overset{\mathrm{d}}{=} \mathbf{e}(1) \tag{15}$$

where $\mathbf{e}(1)$ has an exponential distribution with parameter 1.

We now have the material to state the main result of this section. To be consistent with the notations used for the fragmentation equation, we denote by $x\mu_0(\mathrm{d}x)$, $x > 0$, the distribution of $X(0)$. We recall also the definition of the function φ as the inverse, well-defined in a neighborhood of ∞, of $t \to t/\phi(t)$.

Theorem 3.1. *Suppose that $\int_u^\infty \Pi(\mathrm{d}x)$ varies regularly at 0 with index $-\beta, \beta \in [0,1[$ and $\int^\infty x\Pi(\mathrm{d}x) < \infty$.*
(i) If the support of μ_0 is bounded with a supremum equal to 1, then for all bounded continuous functions $f :]0,\infty[\to \mathbb{R}$,

$$\mathbb{E}\left[f\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t)\right) \Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}\left[f(R^{1/|\alpha|})\right]$$

where R is the random variable with distribution μ_R defined by (14).
(ii) If μ_0 possesses a density u_0 in a neighborhood of ∞ such that

$$\ln(u_0(x)) \underset{\infty}{\sim} -Cx^\gamma$$

for some $\gamma > 0$, then the function $t \in]0,\infty[\to \mathbb{P}(X(t) > 0)$ is continuously differentiable. If moreover the derivative of $t \to \ln(\mathbb{P}(X(t) > 0))$ is regularly varying at ∞ – which e.g. is true as soon as this derivative is monotone near ∞ – then, for all bounded continuous functions $f :]0,\infty[\to \mathbb{R}$, as $t \to \infty$,

$$\mathbb{E}\left[f\left(\left(\frac{h(t)}{C_{\alpha,\beta,\gamma,C}t}\right)^{1/|\alpha|} X(t)\right) \Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}\left[f(R^{1/|\alpha|})\right]$$

where the function h is the inverse, defined in the neighborhood of ∞, of $t \to t^{1+|\alpha|/\gamma}/\phi(t)$ and $C_{\alpha,\beta,\gamma,C}$ is the constant defined in the statement of Theorem 1.6.

We will see in the proof of this result that the function $t \to \varphi(|\alpha|t)/|\alpha|t$ in assertion (i) can be replaced by any asymptotically equivalent function. Likewise for h in the second assertion.

Now, let B be the fragmentation measure related to Π by (6). If $(\mu_t, t \geq 0)$ refers to the solution to the (α, B)-fragmentation equation contructed from ξ by the formula (7), we have

$$m(t) = \int_0^\infty x\mu_t(\mathrm{d}x) = \mathbb{P}(X(t) > 0)$$

and the distribution of $X(t)$ conditional on $X(t) > 0$ is $x\mu_t(\mathrm{d}x)/m(t)$. The above theorem then leads directly to the statements of Theorem 1.3 and Theorem 1.6 (ii) (note that $\int^\infty x\Pi(\mathrm{d}x) < \infty$ is equivalent to $\int_0 |\ln(x)|xB(\mathrm{d}x) < \infty$). The limit distribution $x\mu_\infty(\mathrm{d}x)$ mentioned in these theorems is therefore the distribution of $R^{1/|\alpha|}$. The large-time behavior of $m(t) = \mathbb{P}(X(t) > 0)$ is studied in Subsection 3.1 below, whereas Theorem 3.1 is established in Subsection 3.2.

Last, we finish with the following result on the *quasi-stationary distributions* of X, which will be proved in Subsection 3.3, and which, in terms of the fragmentation equation, will lead to Theorem 1.8. We recall that the quasi-stationary distributions of X are the distributions ς on $]0,\infty[$ such that

$$X(0) \overset{\mathrm{d}}{\sim} \varsigma \Rightarrow \mathbb{E}[f(X(t))|X(t) > 0] = \mathbb{E}[f(X(0))], \text{ for all } t \geq 0 \text{ and all test-functions } f \text{ defined on }]0,\infty[.$$

Theorem 3.2. *Let $\mu_R^{(\lambda)}$ denote the law of $\lambda R^{1/|\alpha|}$, $\lambda > 0$. Then, a probability measure ς on $]0,\infty[$ is a quasi-stationary distributions of X if and only if $\varsigma = \mu_R^{(\lambda)}$ for some $\lambda > 0$. Moreover, if $X(0) \overset{\mathrm{d}}{\sim} \mu_R^{(\lambda)}$,*

$$\mathbb{P}(X(t) > 0) = \exp(-\lambda^\alpha t), \forall t \geq 0.$$

We point out that this theorem does not directly lead to the reciprocal assertion of Theorem 1.8. However, easy manipulations of the fragmentation equation will lead to it. See Subsection 3.3 for details.

3.1 Total mass behavior

This section is devoted to the description of the behavior of the total mass

$$m(t) = \int_0^\infty x\mu_t(\mathrm{d}x) = \mathbb{P}(X(t) > 0) = \mathbb{P}(I > X(0)^\alpha t).$$

The notations are those introduced above in the introduction of Section 3. We start with the following result, which holds for all fragmentation equations with parameters $\alpha < 0, B$ and all initial measure μ_0 such that $\int_0^\infty x\mu_0(\mathrm{d}x) = 1$.

Proposition 3.3. *The total mass m is strictly positive and strictly decreasing on $[0,\infty[$. Moroever $m(t)\to 0$ as $t\to\infty$.*

Proof. Since

$$m(t)=\int_0^\infty \mathbb{P}(I>x^\alpha t)x\mu_0(\mathrm{d}x),$$

it is sufficient to show that the function $t\in[0,\infty[\to\mathbb{P}(I>t)$ is strictly positive, strictly decreasing and converges to 0 as $t\to\infty$. This last point is obvious since $I<\infty$ a.s.. Next, suppose that $\mathbb{P}(I\le t)=1$ for some $t>0$. This would imply that for all $n\ge 1$

$$\frac{n!}{\phi(|\alpha|)\phi(2|\alpha|)...\phi(n|\alpha|)}=\mathbb{E}[I^n]\le t^n.$$

But we have seen in the Introduction that $x/\phi(x)\to\infty$ as $x\to\infty$. In particular, $2t\le n/\phi(n|\alpha|)$ for large enough n, say $n>n_0$. Hence we would have

$$\frac{n_0!}{\phi(|\alpha|)\phi(2|\alpha|)...\phi(n_0|\alpha|)}(2t)^{n-n0}\le t^n$$

for all $n>n_0$, which is impossible. So, $\mathbb{P}(I>t)>0$ for all $t>0$.

Last, for all $t>0$, using the Markov property of subordinators, we get

$$\begin{aligned} I &= \int_0^t \exp(\alpha\xi_r)\mathrm{d}r+\exp(\alpha\xi_t)\int_0^\infty \exp(\alpha(\xi_{r+t}-\xi_t))\mathrm{d}r \\ &\le t+\exp(\alpha\xi_t)\tilde{I}, \end{aligned}$$

where $\tilde{I}$ is distributed as I and independent of ξ_t. Consider a such that $\mathbb{P}(I\le a)>0$ and note, using the Poisson point process construction of the subordinator, that $\mathbb{P}(\exp(\alpha\xi_t)\le t/a)>0$ for all $t>0$. Then

$$0<\mathbb{P}\left(\exp(\alpha\xi_t)\le t/a,\tilde{I}\le a\right)\le\mathbb{P}(I\le 2t),\ \forall t>0.$$

This leads to the fact that $\mathbb{P}(t\ge I>s)>0$ for all $0\le s<t$. Indeed, the event $\{I>s\}$ coincides with $\{\rho(s)<\infty\}$ and when $I>s$,

$$I=s+\exp(\alpha\xi_{\rho(s)})\int_0^\infty \exp\left(\alpha(\xi_{r+\rho(s)}-\xi_{\rho(s)}\right)\mathrm{d}r.$$

Using the strong Markov property of the subordinator at the stopping time $\rho(s)$, we get, with probability one,

$$(I-s)^+=\exp(\alpha\xi_{\rho(s)})\tilde{I},$$

with $\tilde{I}$ independent of $\xi_{\rho(s)}$ and distributed as I. Hence, for all $0\le s<t$,

$$\mathbb{P}(I>s)-\mathbb{P}(I>t)=\mathbb{P}(s<I\le t)=\mathbb{P}\left(\exp(\alpha\xi_{\rho(s)})>0,\tilde{I}\le(t-s)\exp(|\alpha|\xi_{\rho(s)})\right)$$

and this last probability is strictly positive since $\mathbb{P}(\exp(\alpha\xi_{\rho(s)})>0)=\mathbb{P}(I>s)>0$ and $\mathbb{P}(\tilde{I}\le a)>0$ for all $a>0$. □

Now we turn to the proofs of the more precise descriptions of the behavior of m stated in Proposition 1.2, Theorem 1.6 (i) and Proposition 1.7. The crucial point is the following lemma, which is basically a consequence of Rivero [28, Prop. 2] and König and Mörters [21, Lemma 2.3].

Lemma 3.4. *Assume (H), or, equivalently, that ϕ varies regularly at ∞ with index $\beta\in[0,1[$. Then*

$$-\ln\left(\mathbb{P}(I>t)\right)\underset{\infty}{\sim}\frac{(1-\beta)}{|\alpha|}\varphi(|\alpha|t)\underset{\infty}{\sim}(1-\beta)|\alpha|^{\frac{\beta}{1-\beta}}\varphi(t)$$

where φ is the inverse of $t\to t/\phi(t)$, which is well defined in the neighborhood of ∞. In particular, $-\ln\left(\mathbb{P}(I>t)\right)$ is regularly varying at ∞ with index $1/(1-\beta)$.

Proof. Note that the Laplace exponent of the subordinator $|\alpha|\xi$ is $\phi(|\alpha|\cdot)$ and that the inverse of $t \to t/\phi(|\alpha|t)$ is $\varphi(|\alpha|\cdot)/|\alpha|$. Using this remark, we can restrict our proof to the case $|\alpha| = 1$, which is supposed in the following.

When $\beta \in]0,1[$, the statement of the lemma is exactly Proposition 2 of Rivero [28]. When $\beta = 0$ and $\phi(\infty) < \infty$,

$$\frac{1}{n}\ln\left(\frac{\mathbb{E}(I^n)}{n!}\right) = -\frac{1}{n}\sum_{i=1}^{n}\ln(\phi(i)) \xrightarrow[n\to\infty]{} -\ln\phi(\infty).$$

Then by Lemma 2.3. of König and Mörters [21]

$$\lim_{t\to\infty}\frac{1}{t}\ln\left(\mathbb{P}(I>t)\right) = -\phi(\infty).$$

Last, when $\beta = 0$ and $\phi(\infty) = \infty$, we can adapt König and Mörters' proof of [21, Lemma 2.3] to get the expected result. Indeed, first note that

$$\frac{1}{n}\ln\left(\mathbb{E}\left[\frac{I^n\phi(n)^n}{n^n}\right]\right) = \frac{1}{n}\ln\left(\frac{n!}{n^n}\right) + \ln(\phi(n)) - \frac{1}{n}\sum_{i=1}^{n}\ln(\phi(i)) \underset{n\to\infty}{\to} -1, \tag{16}$$

as a consequence of Stirling's formula and of the fact that

$$\ln(\phi(n)) - \frac{1}{n}\sum_{i=1}^{n}\ln(\phi(i)) \underset{n\to\infty}{\to} 0$$

since ϕ is a slowly varying function (see Section 3.2 of Rivero [28] for a proof of this last point). Then, it is easy, using Markov's inequality, that

$$\limsup_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{P}\left(I > n/\phi(n)\right)\right) \le -1.$$

To get a lower bound for the limit inferior, set $Y_n := \ln(I\phi(n)/n)$. For every $\varepsilon > 0$ and every integer m, we have that

$$\frac{1}{\mathbb{E}[I^n]}\mathbb{E}\left[I^n\mathbf{1}_{\{Y_n\ge\varepsilon\}}\right] \le \exp(-\varepsilon m)\frac{\mathbb{E}[I^{n+m}]\phi(n)^m}{\mathbb{E}[I^n]n^m} \underset{n\to\infty}{\to} \exp(-\varepsilon m).$$

Letting $m \to \infty$, this gives

$$\frac{1}{\mathbb{E}[I^n]}\mathbb{E}\left[I^n\mathbf{1}_{\{Y_n\ge\varepsilon\}}\right] \underset{n\to\infty}{\to} 0. \tag{17}$$

Besides, for all $\varepsilon > 0$ and all $n \ge 1$,

$$\frac{1}{n}\ln\left(\mathbb{P}(I > n\exp(-\varepsilon)/\phi(n))\right) \ge \frac{1}{n}\ln\left(\mathbb{P}(|Y_n| < \varepsilon)\right).$$

But $I^{-n} > \exp(-n\varepsilon)n^{-n}\phi(n)^n$ on $\{|Y_n| < \varepsilon\}$, which gives

$$\begin{aligned}\frac{1}{n}\ln\left(\mathbb{P}(|Y_n|<\varepsilon)\right) &= \frac{1}{n}\ln\left(\frac{\mathbb{E}[I^{-n}I^n\mathbf{1}_{\{|Y_n|<\varepsilon\}}]}{\mathbb{E}[I^n]}\mathbb{E}[I^n]\right)\\ &\ge \frac{1}{n}\ln\left(\exp(-n\varepsilon)\frac{\mathbb{E}[I^n\mathbf{1}_{\{|Y_n|<\varepsilon\}}]}{\mathbb{E}[I^n]}n^{-n}\phi(n)^n\mathbb{E}[I^n]\right).\end{aligned}$$

By (16) and (17), the last line of this inequality converges to $-\varepsilon - 1$ as $n \to \infty$. Thus, since the function $t \to t/\phi(t)$ is increasing and $\varphi(t) \to \infty$ as $t \to \infty$, we have proved that

$$\limsup_{t\to\infty}\frac{1}{\varphi(t)}\ln\left(\mathbb{P}\left(I>t\right)\right) \le -1$$

$$\liminf_{t\to\infty}\frac{1}{\varphi(t\exp(\varepsilon))}\ln\left(\mathbb{P}\left(I>t\right)\right) \ge -\varepsilon - 1.$$

Using the regular variation of φ, we get the expected

$$\lim_{t\to\infty}\frac{1}{\varphi(t)}\ln\left(\mathbb{P}(I>t)\right) = -1.$$

□

3.1.1 μ_0 with bounded support: proof of Proposition 1.2

We recall that with no loss of generality, the supremum of the support of μ_0 is supposed to be equal to 1. Then,

$$m(t) = \int_0^1 \mathbb{P}\left(I > tx^\alpha\right) x\mu_0(\mathrm{d}x) \leq \mathbb{P}\left(I > t\right) \int_0^1 x\mu_0(\mathrm{d}x) = \mathbb{P}\left(I > t\right).$$

According to Proposition 3.3 in [11], $C_\lambda := \mathbb{E}[\exp(\lambda I)] < \infty$ as soon as $\lambda < \phi(\infty)$. Hence for such λ's,

$$m(t) \leq \mathbb{P}\left(I > t\right) \leq C_\lambda \exp(-\lambda t), \ \forall t \geq 0,$$

which gives the first part of the statement.

Now, assume (H). Then, on the one hand, since $m(t) \leq \mathbb{P}(I > t)$, we get by Lemma 3.4,

$$\liminf_{t\to\infty} \frac{-\ln\left(m(t)\right)}{\varphi(|\alpha|t)} \geq \frac{1-\beta}{|\alpha|}.$$

On the other hand, for all $0 < \varepsilon < 1$,

$$m(t) \geq \mathbb{P}\left(I > t(1-\varepsilon)^\alpha\right) \int_{1-\varepsilon}^1 x\mu_0(\mathrm{d}x).$$

By assumption, $\int_{1-\varepsilon}^1 x\mu_0(\mathrm{d}x) > 0$, hence

$$\limsup_{t\to\infty} \frac{-\ln\left(m(t)\right)}{\varphi(|\alpha|t)} \leq \limsup_{t\to\infty} \frac{-\ln\left(\mathbb{P}(I > t(1-\varepsilon)^\alpha)\right)}{\varphi(|\alpha|t)} = \frac{1-\beta}{|\alpha|}(1-\varepsilon)^{\frac{\alpha}{1-\beta}}.$$

Then, let $\varepsilon \downarrow 0$ to get the expected result.

3.1.2 μ_0 with unbounded support: proofs of Theorem 1.6 (i) and Proposition 1.7

Proof of Theorem 1.6 (i) 1. Suppose first that $\mu_0(\mathrm{d}x) = \exp(-Cx^\gamma)\mathrm{d}x$, $\gamma > 0$. We have

$$m(t) = \int_0^\infty \mathbb{P}\left(I > tx^\alpha\right) x\exp(-Cx^\gamma)\mathrm{d}x = \frac{t^{-2/\alpha}}{\gamma} \int_0^\infty \mathbb{P}\left(I > u^{\alpha/\gamma}\right) u^{2/\gamma-1} \exp\left(-Cut^{-\gamma/\alpha}\right) \mathrm{d}u, \tag{18}$$

using the change of variable $u = (xt^{1/\alpha})^\gamma$. Then use Lemma 3.4 and Theorem 4.12.10 (iii) of [9] to get

$$-\ln\left(\int_0^x \mathbb{P}\left(I > u^{\alpha/\gamma}\right) u^{2/\gamma-1}\mathrm{d}u\right) \underset{x\to 0}{\sim} -\ln\left(\mathbb{P}\left(I > x^{\alpha/\gamma}\right)\right) \underset{x\to 0}{\sim} (1-\beta)|\alpha|^{\beta/(1-\beta)}\varphi(x^{\alpha/\gamma})$$

which varies regularly at 0 with index $\alpha/(\gamma(1-\beta))$. Note that in a neighborhood of 0, $x \to 1/\varphi(x^{\alpha/\gamma})$ is the inverse of

$$x \to \left(x\phi\left(\frac{1}{x}\right)\right)^{-\gamma/\alpha}.$$

Hence, by de Bruijn's Tauberian Theorem [9, Th. 4.12.9], we have

$$-\ln\left(\int_0^\infty \mathbb{P}\left(I > u^{\alpha/\gamma}\right) u^{2/\gamma-1} \exp\left(-ut\right)\mathrm{d}u\right) \underset{t\to\infty}{\sim} C_{\alpha,\beta,\gamma}/h_0(t),$$

where h_0 is the inverse, well-defined in the neighborhood of ∞, of $x \to x^{-1}\left(x\phi(1/x)\right)^{\gamma/\alpha}$ and $C_{\alpha,\beta,\gamma}$ is the constant defined in the statement of Theorem 1.6 (i). Together with (18), this leads to

$$-\ln\left(m(t)\right) \underset{\infty}{\sim} C_{\alpha,\beta,\gamma} C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}/h_0(t^{\gamma/|\alpha|}).$$

Otherwise said,

$$-\ln\left(m(t)\right) \underset{\infty}{\sim} C_{\alpha,\beta,\gamma} C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} h(t)$$

where h is the inverse of $t^{1+|\alpha|/\gamma}/\phi(t)$.

2. Now, suppose that μ_0 possesses a density u_0 in a neighborhood of ∞ such that $\ln(u_0(x)) \underset{\infty}{\sim} -Cx^\gamma$, $\gamma > 0$. Fix $\varepsilon > 0$ and let C_ε be such that $u_0(x)$ exists for $x \geq C_\varepsilon$ and

$$\exp(-(1+\varepsilon)Cx^\gamma) \leq u_0(x) \leq \exp(-(1-\varepsilon)Cx^\gamma), \ \forall x \geq C_\varepsilon. \tag{19}$$

Then write

$$m(t) = \int_0^{C_\varepsilon} \mathbb{P}(I > tx^\alpha)\, x\mu_0(\mathrm{d}x) + \int_{C_\varepsilon}^\infty \mathbb{P}(I > tx^\alpha)\, xu_0(x)\mathrm{d}x.$$

On the one hand, following the argument developed in Section 3.1.1, we get

$$\limsup_{t\to\infty} \frac{\ln\left(\int_0^{C_\varepsilon} \mathbb{P}(I > tx^\alpha)\, x\mu_0(\mathrm{d}x)\right)}{\varphi(t)} \leq -(1-\beta)|\alpha|^{\beta/(1-\beta)} C_\varepsilon^{\alpha/(1-\beta)},$$

which actually holds for any initial measure μ_0. Note that $\varphi(t)/h(t) \to \infty$ as $t \to \infty$, where h is the function defined above, in the first part of this proof.

On the other hand, the inequalities (19) and the results of the first part of this proof imply that

$$\limsup_{t\to\infty} \frac{-\ln\left(\int_{C_\varepsilon}^\infty \mathbb{P}(I > tx^\alpha)\, xu_0(x)\mathrm{d}x\right)}{h(t)} \leq C_{\alpha,\beta,\gamma}((1+\varepsilon)C)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}$$

and

$$\liminf_{t\to\infty} \frac{-\ln\left(\int_{C_\varepsilon}^\infty \mathbb{P}(I > tx^\alpha)\, xu_0(x)\mathrm{d}x\right)}{h(t)} \geq C_{\alpha,\beta,\gamma}((1-\varepsilon)C)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}.$$

We have therefore proved that

$$C_{\alpha,\beta,\gamma}((1-\varepsilon)C)^{(1+(1-\beta)\gamma/|\alpha|)^{-1}} \leq \liminf_{t\to\infty} \frac{-\ln(m(t))}{h(t)} \leq \limsup_{t\to\infty} \frac{-\ln(m(t))}{h(t)} \leq C_{\alpha,\beta,\gamma}((1+\varepsilon)C^{(1+(1-\beta)\gamma/|\alpha|)^{-1}}$$

for all $\varepsilon > 0$. The result follows by letting $\varepsilon \downarrow 0$. □

Proof of Proposition 1.7. Suppose that $u_0(x) = Cx^{-\gamma}$ on $[a, \infty[$ for some $a > 0$ and $\gamma > 2$. Then,

$$m(t) = C\int_a^\infty \mathbb{P}(I > x^\alpha t)x^{1-\gamma}\mathrm{d}x + \int_0^a \mathbb{P}(I > x^\alpha t)x\mu_0(\mathrm{d}x).$$

With the change of variables $u = x^\alpha t$,

$$\int_a^\infty \mathbb{P}(I > x^\alpha t)x^{1-\gamma}\mathrm{d}x = \frac{t^{\frac{\gamma-2}{\alpha}}}{|\alpha|}\int_0^{a^\alpha t} \mathbb{P}(I > u)u^{\frac{2-\gamma}{\alpha}-1}\mathrm{d}u$$

and this last integral converges to a finite limit as $t \to \infty$ since $\mathbb{P}(I > u) \leq C_\lambda \exp(-\lambda u)$ for all $u \geq 0$ and some $\lambda > 0$ sufficiently small (see the proof of Proposition 1.2 for this last point). Using the same upper bound for $\mathbb{P}(I > x^\alpha t)$, we get that

$$\int_0^a \mathbb{P}(I > x^\alpha t)x\mu_0(\mathrm{d}x) \leq C_\lambda \exp(-\lambda a^\alpha t)\int_0^a x\mu_0(\mathrm{d}x).$$

Thus,

$$m(t) \underset{t\to\infty}{\sim} \frac{C}{|\alpha|}t^{\frac{\gamma-2}{\alpha}}\int_0^\infty \mathbb{P}(I > u)u^{\frac{2-\gamma}{\alpha}-1}\mathrm{d}u.$$

It is not hard to extend this proof to the cases when $u_0(x) \underset{\infty}{\sim} Cx^{-\gamma}$, for some $\gamma > 2$, which is left to the reader. □

3.2 Proof of Theorem 3.1

We start with the following lemma.

Lemma 3.5. *Suppose that $-\ln(m)$ varies regularly at ∞ with a positive index γ and satisfies (11). Then, for any function $g : [0,\infty[\to]0,\infty[$ such that $g(t)/(-\ln(m(t))) \to 1$ as $t \to \infty$, we have*

$$\mathbb{E}\left[f\left(\left(\frac{\gamma g(t)}{t}\right)^{1/|\alpha|} X(t)\right) \Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}\left[f(R^{1/|\alpha|})\right],$$

for all continuous bounded test functions f on $]0,\infty[$.

Proof. First note that when $X(0)^{|\alpha|}I > t$,

$$\begin{aligned} & X(0)^{|\alpha|}I \\ = \;& X(0)^{|\alpha|}\int_0^{\rho(X(0)^\alpha t)} \exp(\alpha\xi_r)\mathrm{d}r + X(0)^{|\alpha|}\exp(\alpha\xi_{\rho(X(0)^\alpha t)})\int_0^\infty \exp(\alpha(\xi_{r+\rho(X(0)^\alpha t)} - \xi_{\rho(X(0)^\alpha t)}))\mathrm{d}r \\ = \;& t + X(0)^{|\alpha|}\exp(\alpha\xi_{\rho(X(0)^\alpha t)})\int_0^\infty \exp(\alpha(\xi_{r+\rho(X(0)^\alpha t)} - \xi_{\rho(X(0)^\alpha t)}))\mathrm{d}r. \end{aligned}$$

Then use the strong Markov property of ξ at the (randomized) stopping time $\rho(X(0)^\alpha t)$ to get

$$(X(0)^{|\alpha|}I - t)^+ = X(0)^{|\alpha|}\exp(\alpha\xi_{\rho(X(0)^\alpha t)})\tilde{I} = X(t)^{|\alpha|}\tilde{I}, \tag{20}$$

where $\tilde{I}$ is distributed as I and is independent of $X(t)$. This gives, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} m(t)^{-1}\mathbb{E}\left[\left(\left(\frac{\gamma g(t)}{t}\right)^{1/|\alpha|} X(t)\right)^{|\alpha|n}\right]\mathbb{E}[I^n] &= m(t)^{-1}\left(\frac{\gamma g(t)}{t}\right)^n \mathbb{E}\left[X(t)^{|\alpha|n}\right]\mathbb{E}[I^n] \\ &= m(t)^{-1}\left(\frac{\gamma g(t)}{t}\right)^n \mathbb{E}\left[\left((X(0)^{|\alpha|}I - t)^+\right)^n\right]. \end{aligned}$$

Then recall that

$$m(t) = \mathbb{P}(X(0)^{|\alpha|}I > t),\ t \geq 0.$$

Integrating by parts,

$$m(t)^{-1}\left(\frac{\gamma g(t)}{t}\right)^n \mathbb{E}\left[\left((X(0)^{|\alpha|}I - t)^+\right)^n\right] = nm(t)^{-1}\left(\frac{\gamma g(t)}{t}\right)^n \int_t^\infty (x-t)^{n-1}m(x)\mathrm{d}x,$$

which, according to Lemma 2.1 and the assumptions we have made on $-\ln(m)$ and g, converges as $t \to \infty$ to $n!$. Next note that $\mathbb{E}[R^n]\mathbb{E}[I^n] = n!$, using the factorization property (15) of the exponential random variable with parameter 1. Putting all the pieces together, we have proved that for all integers $n \geq 1$,

$$\mathbb{E}\left[\left(\left(\frac{\gamma g(t)}{t}\right)^{1/|\alpha|} X(t)\right)^{|\alpha|n} \Big| X(t) > 0\right] \underset{t\to\infty}{\to} \mathbb{E}[R^n].$$

To sum up: let ν_t denote the distribution of $\gamma t^{-1}g(t)X(t)^{|\alpha|}$ conditional on $X(t) > 0$ (ν_t is a probability measure on $]0,\infty[$). We have shown that for all $n \geq 1$,

$$\int_0^\infty x^n \nu_t(\mathrm{d}x) \to \int_0^\infty x^n \mu_R(\mathrm{d}x),$$

where μ_R is the distribution of R. Of course this still holds for $n = 0$. But the distribution of R is characterized by its moments. It is then well-known ([15, Chapter VIII, p.269]) that this implies that ν_t converges in distribution to μ_R. □

3.2.1 Proof of Theorem 3.1 (i)

By Proposition 1.2, under the hypothesis (H), $-\ln(m)$ varies regularly at ∞ with index $1/(1-\beta)$ and more precisely

$$-\ln(m(t)) \underset{t\to\infty}{\sim} \frac{(1-\beta)}{|\alpha|}\varphi(|\alpha|t).$$

Together with Lemma 3.5, this implies the statement of Theorem 3.1, provided $-\ln(m)$ satisfies (11). The goal of this section is to proof this last point when μ_0 has a bounded support.

Lemma 3.6. *Let*

$$f(x) = -\ln(m(x)), x \geq 0,$$

and assume (H), that $\int^{\infty} x\Pi(\mathrm{d}x) < \infty$ *and that* μ_0 *has a bounded support. Then for all* $\varepsilon > 0$, *there exists some* $x(\varepsilon)$ *such that*

$$\lambda^{\frac{1}{1-\beta}-\varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq \lambda^{\frac{1}{1-\beta}+\varepsilon}, \ \forall \lambda \geq 1 \text{ and } \forall x \geq x(\varepsilon).$$

This lemma is a direct consequence of the forthcoming Lemmas 3.7 and 3.10.

Lemma 3.7. *Let* $g :]0,\infty[\to]0,\infty[$ *be a continuously differentiable function such that*

$$\frac{xg'(x)}{g(x)} \to c > 0 \text{ as } x \to \infty.$$

Then for all $\varepsilon > 0$ *there exists some* $x(\varepsilon)$ *such that*

$$\lambda^{c-\varepsilon} \leq \frac{g(\lambda x)}{g(x)} \leq \lambda^{c+\varepsilon}, \ \forall \lambda \geq 1 \text{ and } \forall x \geq x(\varepsilon).$$

Proof. For $\varepsilon > 0$, let $x(\varepsilon)$ be such that

$$c - \varepsilon \leq \frac{xg'(x)}{g(x)} \leq c + \varepsilon \text{ for all } x \geq x(\varepsilon).$$

For such xs, and all $\lambda \geq 1$

$$(c-\varepsilon)\ln(\lambda) = (c-\varepsilon)\int_x^{\lambda x} y^{-1}\mathrm{d}y \leq \int_x^{\lambda x} \frac{g'(y)}{g(y)}\mathrm{d}y \leq (c+\varepsilon)\int_x^{\lambda x} y^{-1}\mathrm{d}y = (c+\varepsilon)\ln(\lambda).$$

Since $\int_x^{\lambda x} \frac{g'(y)}{g(y)}\mathrm{d}y = \ln(g(\lambda x)) - \ln(g(x))$, the result is proved. □

Lemma 3.8. *Suppose that* ϕ *is regularly varying at* ∞ *with index* $\beta \in [0,1[$ *and that* $\phi(x) \to \infty$ *as* $x \to \infty$. *Let* $\beta' \in]\beta, 1[$. *Then there exists some* $x_1(\beta')$ *such that for* $x \geq x_1(\beta')$ *and all* $\lambda \geq 1$,

$$1 \leq \phi(x) \leq \phi(\lambda x) \leq \lambda^{\beta'}\phi(x)$$

and

$$\phi(x) \leq x^{\beta'}.$$

Proof. Note that ϕ is infinitely differentiable on $]0,\infty[$, with derivative

$$\phi'(x) = \int_0^{\infty} v\exp(-xv)\Pi(\mathrm{d}v),$$

which is non-increasing. It is then a classical result on regular variation (see the Monotone Density Theorem, [9, Th.1.7.2]) that ϕ' is regularly varying with index $\beta - 1$ and

$$\frac{x\phi'(x)}{\phi(x)} \underset{x\to\infty}{\to} \beta.$$

The first part of the lemma is then a consequence of the above Lemma 3.7 and of the fact that ϕ is increasing and converges to ∞. We also have that $\phi(x)/x^{\beta'}$ converges to 0 at ∞ (since $\beta' > \beta$), hence the second assertion holds for x large enough. □

Lemma 3.9. *Let*
$$f(x) = -\ln\left(\mathbb{P}(I > x)\right), x \geq 0,$$
which, as proved in Lemma 3.4 is regularly varying with index $1/(1-\beta)$*, under the assumption (H). Suppose moreover that* $\int^{\infty} x\Pi(\mathrm{d}x) < \infty$*. Then* f *is infinitely differentiable and*
$$\frac{xf'(x)}{f(x)} \to \frac{1}{1-\beta} \text{ as } x \to \infty.$$

Proof. According to [11, Prop. 2.1], when $\int^{\infty} x\Pi(\mathrm{d}x) < \infty$, there exists an infinitely differentiable function $k :]0,\infty[\to [0,\infty[$ such that $k(x)\mathrm{d}x$ is the distribution of I. Moreover,
$$\begin{aligned} k(x) &= \int_x^{\infty} \overline{\Pi}\left(|\alpha|^{-1}\ln(u/x)\right) k(u)\mathrm{d}u \\ &= \int_0^{\infty} \left(\int_x^{xe^{v|\alpha|}} k(u)\mathrm{d}u\right) \Pi(\mathrm{d}v). \end{aligned}$$
To simplify notations, we suppose in the following that $|\alpha| = 1$. The proof is identical for $|\alpha| \neq 1$. In particular, we have,
$$\mathbb{P}(I > x) = \int_x^{\infty} k(u)\mathrm{d}u,$$
and
$$f'(x) = \frac{k(x)}{\mathbb{P}(I > x)} = \int_0^{\infty} \left(1 - \exp\left(f(x) - f(xe^v)\right)\right)\Pi(\mathrm{d}v), x > 0. \tag{21}$$
Note that since f is regularly varying with a positive index, we have that $f(x) \to \infty$ as $x \to \infty$, and therefore, for all $v > 0$,
$$f(x) - f(xe^v) = f(x)\left(1 - \frac{f(xe^v)}{f(x)}\right) \underset{\infty}{\sim} f(x)\left(1 - e^{v/(1-\beta)}\right) \underset{x\to\infty}{\to} -\infty.$$

- When $\Pi(]0,\infty[) < \infty$, this implies the expected result, since, by dominated convergence,
$$f'(x) \underset{x\to\infty}{\to} \Pi(]0,\infty[) = \lim_{x\to\infty} \frac{f(x)}{x}.$$

- The proof is much more technical when $\Pi(]0,\infty[) = \infty$, which is supposed for the rest of this proof. We proceed in two steps.

Step 1. The goal of this step is to prove that
$$\liminf_{x\to\infty} \frac{xf'(x)}{f(x)} \geq \frac{1}{1-\beta}.$$
To start with, suppose there exists some x_0 and some *non-decreasing* positive function g such that $f'(x) \geq g(x)$ for all $x \geq x_0$. Then for $x \geq x_0$ and $v > 0$
$$f(xe^v) - f(x) = \int_x^{xe^v} f'(u)\mathrm{d}u \geq g(x)x(e^v - 1) \geq g(x)xv.$$
Using (21), this gives
$$f'(x) \geq \phi(g(x)x), \ x \geq x_0. \tag{22}$$
Now, note that $f'(x) \to \infty$ as $x \to \infty$ since for all $a > 0$
$$\liminf_{x\to\infty} f'(x) \geq \liminf_{x\to\infty} \int_a^{\infty} \left(1 - \exp\left(f(x) - f(xe^v)\right)\right)\Pi(\mathrm{d}v) = \int_a^{\infty} \Pi(\mathrm{d}v)$$
(by dominated convergence) and this right-hand side converges to ∞ as $a \to 0$. In particular, $f'(x) \geq 1$ for x sufficiently large (say $x \geq x_0$). Replacing g by 1 in (22), we get
$$f'(x) \geq \phi(x), \forall x \geq x_0.$$

Recall that ϕ is non-decreasing and then iterate the procedure to get, for all $n \geq 0$

$$f'(x) \geq h_n(x), \ \forall x \geq x_0, \tag{23}$$

where the functions $h_n :]0,\infty[\to]0,\infty[$ are defined by induction by

$$\begin{aligned} h_0(x) &= 1, \text{ for all } x \geq 0 \\ h_n(x) &= \phi(h_{n-1}(x)x), \text{ for all } x \geq 0. \end{aligned}$$

Now the interesting fact is that for x large enough, $h_n(x) \to \varphi(x)/x$ as $n \to \infty$. Indeed, let $\beta' \in]\beta, 1[$. With the notations of Lemma 3.8 we have for $x \geq x_1(\beta')$, $1 \leq \phi(x) \leq x^{\beta'}$, i.e. $h_0(x) \leq h_1(x) \leq x^{\beta'}$. Using that ϕ is non-decreasing, we easily have, by induction, that

$$1 \leq h_n(x) \leq h_{n+1}(x) \leq x^{\beta' + \dots + \beta'^{n+1}} \leq x^{\beta'/(1-\beta')} < \infty,$$

for all $n \geq 1$. Let $l(x) := \lim_{n\to\infty} h_n(x)$. We have shown that $0 < l(x) < \infty$. Then necessarily $l(x) = \phi(l(x)x)$, otherwise said $l(x)x/\phi(l(x)x) = x$ and finally, $l(x)x = \varphi(x)$, $\forall x \geq x_1(\beta')$. To conclude, for x large enough, letting $n \to \infty$ in (23), we get $f'(x) \geq \varphi(x)/x$, which, combined with Lemma 3.4, gives the expected liminf.

Step 2. The proof of the limsup is similar, but more technical. To start with, note that for all $\varepsilon > 0$ and all $a < \ln(1+\varepsilon)$, $a > 0$,

$$\begin{aligned} \liminf_{x\to\infty} \int_0^{\ln(1+\varepsilon)} (1 - \exp(f(x) - f(xe^v)))\,\Pi(\mathrm{d}v) &\geq \liminf_{x\to\infty} \int_a^{\ln(1+\varepsilon)} (1 - \exp(f(x) - f(xe^v)))\,\Pi(\mathrm{d}v) \\ &= \int_a^{\ln(1+\varepsilon)} \Pi(\mathrm{d}v) \text{ (by dominated convergence)} \\ &\underset{a\to 0}{\to} \infty, \end{aligned}$$

whereas

$$\lim_{n\to\infty} \int_{\ln(1+\varepsilon)}^{\infty} (1 - \exp(f(x) - f(xe^v)))\,\Pi(\mathrm{d}v) = \int_{\ln(1+\varepsilon)}^{\infty} \Pi(\mathrm{d}v) < \infty.$$

Hence, there exists some $x_1(\varepsilon)$ such that for $x \geq x_1(\varepsilon)$

$$f'(x) \leq (1+\varepsilon) \int_0^{\ln(1+\varepsilon)} (1 - \exp(f(x) - f(xe^v)))\,\Pi(\mathrm{d}v). \tag{24}$$

Next, fix some $\beta' \in]\beta, 1[$ and consider some $\delta > 0$ and $\varepsilon > 0$ such that $(1+\delta)(1+\varepsilon)^{1/(\beta-1)}\beta' < 1$. Since f is regularly varying with index $1/(1-\beta)$, there exists some $x_2(\delta,\varepsilon)$ such that

$$f(x(1+\varepsilon)) \leq (1+\delta)(1+\varepsilon)^{1/(1-\beta)} f(x), \ \forall x \geq x_2(\delta,\varepsilon). \tag{25}$$

We will need this later. For the moment, let $x_0 = \max(x_1(\beta'), x_1(\varepsilon), x_2(\delta,\varepsilon))$, with $x_1(\beta')$ as introduced in Lemma 3.8. Next, suppose that for all $x \geq x_0$

$$f'(x) \leq g(x)$$

for some *non-decreasing* function g s.t. $g(x) \geq 1$ for all $x \geq x_0$. Note that this implies that

$$f(xe^v) - f(x) \ = \ \int_x^{xe^v} f'(u)\mathrm{d}u \leq g(xe^v)x(e^v - 1).$$

The function $v \to v^{-1}(e^v - 1)$ is increasing on $[0,\infty[$, hence $e^v - 1 \leq v\gamma(\varepsilon)$ for all $v \leq \ln(1+\varepsilon)$, where $\gamma(\varepsilon) = \varepsilon/(\ln(1+\varepsilon))$. Together with (24) this leads to

$$\begin{aligned} f'(x) &\leq (1+\varepsilon) \int_0^{\ln(1+\varepsilon)} (1 - \exp(-g(x(1+\varepsilon))xv\gamma(\varepsilon)))\,\Pi(\mathrm{d}v) \\ &\leq (1+\varepsilon)\phi\left(g(x(1+\varepsilon))x\gamma(\varepsilon)\right) \end{aligned} \tag{26}$$

for all $x \geq x_0$.

Then we claim that for all $n \geq 1$ and all $x \geq x_0$

$$f'(x) \leq (1+\varepsilon)^{1+\beta'+2\beta'^2+\ldots+n\beta'^n} \gamma(\varepsilon)^{\beta'+\beta'^2+\ldots+\beta'^n} g\left(x(1+\varepsilon)^n\right)^{\beta'^n} h_n(x), \tag{27}$$

where the sequence of functions h_n is that introduced in the step 1 of this proof. We will prove this by induction on n. Before doing this, let us just mention that, by an easy induction, using Lemma 3.8,

$$h_n(x(1+\varepsilon)) \leq (1+\varepsilon)^{\beta'+\ldots+\beta'^n} h_n(x), \text{for all } x \geq x_0 \text{ and } n \geq 1.$$

We now turn to the proof of (27). For $n = 1$, use (26) and Lemma 3.8 to get (note that $\gamma(\varepsilon) \geq 1$, hence $\gamma(\varepsilon)g \geq 1$)

$$f'(x) \leq (1+\varepsilon)g(x(1+\varepsilon))^{\beta'} \gamma(\varepsilon)^{\beta'} \phi(x), x \geq x_0,$$

which leads to (27) for $n = 1$. Now assume (27) is true for some integer n. Note that the function in the right-hand side of this inequality, which we call g_1, is larger than 1 for all $x \geq x_0$. Note also that it is non-decreasing. Hence we get, replacing g by g_1 in (26), for $x \geq x_0$,

$$\begin{aligned} f'(x) &\leq (1+\varepsilon)\phi\left((1+\varepsilon)^{1+\beta'+2\beta'^2+\ldots+n\beta'^n}\gamma(\varepsilon)^{\beta'+\beta'^2+\ldots+\beta'^n} g\left(x(1+\varepsilon)^{n+1}\right)^{\beta'^n} h_n(x(1+\varepsilon))x\gamma(\varepsilon)\right) \\ &\leq (1+\varepsilon)\phi\left((1+\varepsilon)^{1+2\beta'+3\beta'^2+\ldots+(n+1)\beta'^n}\gamma(\varepsilon)^{1+\beta'+\beta'^2+\ldots+\beta'^n} g\left(x(1+\varepsilon)^{n+1}\right)^{\beta'^n} h_n(x)x\right) \\ &\leq (1+\varepsilon)^{1+\beta'+2\beta'^2+\ldots+(n+1)\beta'^{n+1}}\gamma(\varepsilon)^{\beta'+\beta'^2+\ldots+\beta'^{n+1}} g\left(x(1+\varepsilon)^{n+1}\right)^{\beta'^{n+1}} \phi\left(h_n(x)x\right), \end{aligned}$$

where for the last inequality, we have used Lemma 3.8. Hence we have (27) for all $n \geq 1$.

Now, thanks to the assumptions (H) and $\int^\infty x\Pi(\mathrm{d}x) < \infty$ and to Lemma 1 in [20], we know that the function k is bounded from above on $]0,\infty[$, say by some constant $C \geq 1$. Hence $f'(x) = k(x)/\mathbb{P}(I > x) \leq C\exp(f(x))$ for all $x > 0$. Since f is non-decreasing and non-negative, the function $x \to C\exp(f(x))$ is non-decreasing, greater than 1 hence we can replace g by this function in (27) to get for all $n \geq 1$ and all $x \geq x_0$

$$f'(x) \leq (1+\varepsilon)^{1+\beta'+2\beta'^2+\ldots+n\beta'^n}\gamma(\varepsilon)^{\beta'+\beta'^2+\ldots+\beta'^n} C^{\beta'^n} \exp\left(\beta'^n f(x(1+\varepsilon)^n)\right) h_n(x). \tag{28}$$

Our goal now, is to let $n \to \infty$ in this inequality. Iterating inequality (25), we get for $x \geq x_0$ and for all $n \geq 1$

$$f(x(1+\varepsilon)^n) \leq (1+\delta)^n (1+\varepsilon)^{n/(1-\beta)} f(x).$$

Since

$$(1+\delta)(1+\varepsilon)^{1/(\beta-1)}\beta' < 1,$$

this leads, for $x \geq x_0$, to

$$\exp\left(\beta'^n f(x(1+\varepsilon)^n)\right) \underset{n\to\infty}{\to} 1.$$

As $n \to \infty$, we also have

$$C^{\beta'^n} \to 1 \text{ and } (1+\varepsilon)^{1+\beta'+2\beta'^2+\ldots+n\beta'^n} \to (1+\varepsilon)^{1+\beta'/(1-\beta')^2} \text{ and } \gamma(\varepsilon)^{\beta'+\beta'^2+\ldots+\beta'^n} \to \gamma(\varepsilon)^{\beta'/(1-\beta')}.$$

Last, recall that for x large enough, $h_n(x) \to \varphi(x)/x$ as $n \to \infty$. Letting $n \to \infty$ in (28), we therefore have for x large enough

$$f'(x) \leq C_\varepsilon \varphi(x)/x,$$

where $C_\varepsilon \to 1$ as $\varepsilon \to 0$. This gives

$$\limsup_{x\to\infty} \frac{xf'(x)}{f(x)} \leq \frac{1}{1-\beta}.$$

□

Lemma 3.10. *Let*

$$f(x) := -\ln\left(m(x)\right), x \geq 0,$$

and suppose (H), that $\int^\infty x\Pi(\mathrm{d}x) < \infty$ and that μ_0 has a bounded support. Then f is differentiable on $]0,\infty[$ and

$$\frac{xf'(x)}{f(x)} = -\frac{xm'(x)}{m(x)f(x)} \to \frac{1}{1-\beta} \text{ as } x \to \infty.$$

Proof. With no loss of generality, we suppose that the supremum of the support of μ_0 is equal to 1. Under the assumptions of the lemma, we know (see the proof of the previous lemma) that $x \to \mathbb{P}(I > x)$ is differentiable on $]0, \infty[$, with derivative $-k$. By Lemma 1 in [20], we also know that the function $x \in]0, \infty[\to xk(x)$ is bounded. Let M denotes an upper bound. Recall then that

$$m(x) = \int_0^1 \mathbb{P}(I > xy^\alpha) y\mu_0(\mathrm{d}y)$$

and note that for all $x > a > 0$ and all $y \in]0, 1[$,

$$\left| \partial_x \left(\mathbb{P}(I > xy^\alpha) \right) \right| = k(xy^\alpha) y^\alpha \leq \frac{M}{a}.$$

Hence, by dominated convergence, m is continuously differentiable on $]0, \infty[$, with derivative

$$m'(x) = -\int_0^1 k(xy^\alpha) y^\alpha y\mu_0(\mathrm{d}y), x > 0.$$

Now, fix $\delta > 0$. By Lemma 3.9, there exists some $x(\delta)$ such that for $x \geq x(\delta)$,

$$\frac{1-\delta}{1-\beta} \leq \frac{-xk(x)}{\mathbb{P}(I > x) \ln\left(\mathbb{P}(I > x)\right)} \leq \frac{1+\delta}{1-\beta}.$$

Then for $x \geq x(\delta)$,

$$\frac{1-\delta}{(1-\beta)x} \int_0^1 \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y) \leq m'(x) \leq \frac{1+\delta}{(1-\beta)x} \int_0^1 \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y). \tag{29}$$

Now, let $\varepsilon > 0$. On the one hand, we claim that

$$\int_{1-\varepsilon}^1 \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y) \underset{x\to\infty}{\sim} \int_0^1 \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y). \tag{30}$$

Indeed, for all $0 < y < 1-\varepsilon$,

$$\frac{\mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right)}{\mathbb{P}(I > x(1-\varepsilon)^\alpha) \ln\left(\mathbb{P}(I > x(1-\varepsilon)^\alpha)\right)} \to 0 \text{ as } x \to \infty$$

since $x \to -\ln\left(\mathbb{P}(I > x)\right)$ is regularly varying at ∞ with a positive index and $\alpha < 0$. It is then not hard to see, using Lemmas 3.7 and 3.9 that for x large enough this function is bounded from above by

$$\exp\left(1 - \left(\frac{y}{1-\varepsilon}\right)^{\frac{\alpha(1-\varepsilon)}{1-\beta}}\right) \left(\frac{y}{1-\varepsilon}\right)^{\frac{\alpha(1+\varepsilon)}{1-\beta}}$$

which, in turn, is bounded for $y \in]0, 1-\varepsilon[$. Hence, by dominated convergence, we see that

$$\begin{aligned} \left| \int_0^{1-\varepsilon} \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y) \right| &\underset{x\to\infty}{\ll} \left| \mathbb{P}(I > x(1-\varepsilon)^\alpha) \ln\left(\mathbb{P}(I > x(1-\varepsilon)^\alpha)\right) \right| \\ &\leq \frac{1}{\varepsilon} \left| \int_{1-\varepsilon}^1 \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y) \right|, \end{aligned}$$

where we have used for the last inequality that the function $x \to -x\ln(x)$ is increasing in a neighborhood of 0. Hence (30). A similar, but simpler, argument leads to

$$\int_{1-\varepsilon}^1 \mathbb{P}(I > xy^\alpha) y\mu_0(\mathrm{d}y) \underset{x\to\infty}{\sim} \int_0^1 \mathbb{P}(I > xy^\alpha) y\mu_0(\mathrm{d}y). \tag{31}$$

On the other hand, using that $x \to \ln\left(\mathbb{P}(I > x)\right)$ is regularly varying with index $1/(1-\beta)$, we have for $1-\varepsilon \leq y \leq 1$ and x sufficiently large (say $x \geq x(\varepsilon)$)

$$(1+\varepsilon)(1-\varepsilon)^{\alpha/(1-\beta)} \ln\left(\mathbb{P}(I > x)\right) \leq \ln\left(\mathbb{P}(I > x(1-\varepsilon)^\alpha)\right) \leq \ln\left(\mathbb{P}(I > xy^\alpha)\right) \leq \ln\left(\mathbb{P}(I > x)\right).$$

Thus

$$\int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^\alpha) y\mu_0(\mathrm{d}y) \leq \frac{\int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y)}{\ln\left(\mathbb{P}(I > x)\right)} \leq (1+\varepsilon)(1-\varepsilon)^{\alpha/(1-\beta)} \int_{1-\varepsilon}^{1} \mathbb{P}(I > xy^\alpha) y\mu_0(\mathrm{d}y).$$

Which, taking $x(\varepsilon)$ larger if necessary and using (30) and (31), gives for $x \geq x(\varepsilon)$

$$(1-\delta)m(x) \leq \frac{\int_0^1 \mathbb{P}(I > xy^\alpha) \ln\left(\mathbb{P}(I > xy^\alpha)\right) y\mu_0(\mathrm{d}y)}{\ln\left(\mathbb{P}(I > x)\right)} \leq (1+\delta)(1+\varepsilon)(1-\varepsilon)^{\alpha/(1-\beta)} m(x).$$

Plugging this in (29) and letting first $\varepsilon \to 0$ and then $\delta \to 0$, we get the expected convergence, since $f(x) \sim -\ln(\mathbb{P}(I > x))$ as $x \to \infty$. □

3.2.2 Proof of Theorem 3.1 (ii)

The fact that the function

$$x \in]0, \infty[\to f(x) := -\ln(m(x)) = -\ln\left(\mathbb{P}(X(x) > 0)\right)$$

is continuously differentiable on $]0, \infty[$ can be proved exactly as when the support of μ_0 is compact. See the beginning of the proof of Lemma 3.10. Next, by Karamata's Theorem (Th. 1.5.11 of [9]), if f varies regularly at ∞ with index $\lambda > 0$ and if its derivative is also regularly varying at ∞, then

$$\frac{xf'(x)}{f(x)} \to \lambda \text{ as } x \to \infty.$$

Together with Theorem 1.6 (i), Lemma 3.7 and Lemma 3.5, this implies Theorem 3.1 (ii).

3.3 Quasi-stationary distributions

Proof of Theorem 3.2. When $X(0) \sim \mu_R^{(\lambda)}$, the distribution of $X(0)^{|\alpha|} I$ is that of $\lambda^{|\alpha|} RI$, with R independent of I, i.e. that of an exponential random variable with parameter λ^α. We then immediately have that for $n \geq 1$ and $t \geq 0$,

$$\mathbb{E}\left[\left((X(0)^{|\alpha|} I - t)^+\right)^n\right] = \lambda^{|\alpha| n} n! \exp(-\lambda^\alpha t),$$

and

$$\mathbb{P}(X(t) > 0) = \mathbb{P}(X(0)^{|\alpha|} I > t) = \exp(-\lambda^\alpha t).$$

Following the beginning of the proof of Lemma 3.5 this gives

$$\mathbb{E}\left[(X(t))^{|\alpha| n}\right] \mathbb{E}[I^n] = \mathbb{E}\left[\left((X(0)^{|\alpha|} I - t)^+\right)^n\right] = \lambda^{|\alpha| n} n! \exp(-\lambda^\alpha t)$$

and then

$$\mathbb{E}\left[(X(t))^{|\alpha| n} | X(t) > 0\right] = \mathbb{E}[\lambda^{|\alpha| n} R^n] = \mathbb{E}\left[X(0)^{|\alpha| n}\right].$$

Hence $\mu_R^{(\lambda)}$ is a quasi-stationary distribution, since the distribution of R is characterized by its entire positive moments. Note there is no other quasi-stationary distribution. Indeed, let ς be a quasi-sationary distribution and suppose $X(0) \sim \varsigma$. Then necessarily, by the Markov property of X, $\mathbb{P}(X(t+s) > 0) = \mathbb{P}(X(t) > 0)\mathbb{P}(X(s) > 0)$ which implies that $X(0)^{|\alpha|} I$ has an exponential distribution. Say with parameter ℓ, i.e. $\ell X(0)^{|\alpha|} I$ has a exponential distribution with parameter 1. Since the factorization (15) characterizes the distribution of R, we get that $\varsigma = \mu_R^{(\ell^{1/\alpha})}$. □

Proof of Theorem 1.8. The first part of this theorem is an obvious consequence of Theorem 3.2. The reverse cannot be directly deduced from Theorem 3.2, since we do not know if uniqueness holds for the fragmentation equation when the initial measure has an unbounded support.

So, consider $(\mu_t, t \geq 0)$ a quasi-stationary solution to the fragmentation equation (2). We want to prove that this solution belongs to the family of solutions $\left((\mu_{\infty,t}^{(\lambda)}, t \geq 0), \lambda > 0\right)$ as defined in Theorem 1.8. Replacing μ_t by $m(t)\mu_0$ in equation (2), we get that

$$(1 - m(t)) < \mu_0, f >= -\int_0^t m(s)\mathrm{d}s < \mu_0, G(f) >, \ \forall f \in C_c^1,$$

where $G(f)(x) = x^\alpha \int_0^1 (f(xy) - f(x)y)B(\mathrm{d}y)$. Otherwise said, there exists some constant $C > 0$ such that

$$m(t) = \exp(-Ct), \ \forall t \geq 0,$$

and

$$< \mu_0, f >= -C^{-1} < \mu_0, G(f) >, \ \forall f \in C_c^1.$$

When $f \in C_c^1$, the function $x \to xf(x)$ is also in C_c^1. Hence the above identity rewrites

$$< x\mu_0, f >= -C^{-1} < x\mu_0, \tilde{A}(f) >, \ \forall f \in C_c^1, \tag{32}$$

where $\tilde{A}(f)(x) = x^\alpha \int_0^1 (f(xy) - f(x))yB(\mathrm{d}y)$.

To show that this characterizes μ_0, we need the following fact: for all $\beta > 0$, there exists a non-decreasing sequence of functions $f_{\beta,n} :]0,\infty[\to [0,\infty[$ such that $f_{\beta,n}(x) \to x^\beta$ as $n \to \infty$, $\forall x > 0$; and $f_{\beta,n} \in C_c^1$ and $|f'_{\beta,n}(x)| \leq \beta x^{\beta-1}$ for all $x > 0$ and all $n \geq 1$. This sequence can, for example, be constructed by considering first a non-decreasing sequence of continuous functions $g_{\beta,n} :]0,\infty[\to [0,\infty[$ such that $g_{\beta,n}(x) \leq \beta x^{\beta-1}$, $\forall x > 0, n \geq 1$, $g_{\beta,n}(x) = \beta x^{\beta-1}$ for $x \in [n^{-1}, n]$ and $g_{\beta,n}(x) = 0$ for $x \in]0,(2n)^{-1}] \cup [2n, \infty[$. Then set $f_{\beta,n}(x) := \int_0^x g_{\beta,n}(u)\mathrm{d}u$ for $x \in]0, 2n]$ and extend these functions on $]2n, \infty[$ so that $f_{\beta,n} \in C_c^1$ and $|f'_{\beta,n}(x)| \leq \beta x^{\beta-1}$, for all $x > 0$ and all $n \geq 1$, and the sequence $(f_{\beta,n}, n \geq 1)$ is non-decreasing. For all $\beta > 0$, this implies that for all $x > 0$

$$\tilde{A}(f_{\beta,n})(x) \underset{n\to\infty}{\to} x^{\alpha+\beta} \int_0^1 (y^\beta - 1)yB(\mathrm{d}y) = -x^{\alpha+\beta}\phi(\beta), \tag{33}$$

together with

$$|\tilde{A}(f_{\beta,n})(x)| \leq (2+\beta)x^{\alpha+\beta} \int_0^1 (1-y)yB(\mathrm{d}y). \tag{34}$$

Indeed, this is obvious when $\beta \geq 1$: we just need that

$$|f_{\beta,n}(xy) - f_{\beta,n}(x)| \leq \sup_{z \in [xy,x]} |f'_{\beta,n}(z)| x(1-y) \leq \beta x^\beta (1-y), \text{ for } y \in]0,1[, x > 0,$$

and then use the dominated convergence theorem. The case $0 < \beta < 1$ needs more care. Using the above mentioned properties of $f_{\beta,n}$ and also that $f_{\beta,n}(x) \leq x^\beta$ we obtain, for $x > 0$ and $y \in]0,1[$

$$\begin{aligned} x|f_{\beta,n}(xy) - f_{\beta,n}(x)| &\leq xf_{\beta,n}(xy)(1-y) + |xyf_{\beta,n}(xy) - xf_{\beta,n}(x)| \\ &\leq x^{1+\beta}(1-y) + \sup_{z\in[xy,x]} |(\mathrm{id}f_{\beta,n})'(z)| x(1-y) \\ &\leq x^{1+\beta}(1-y) + (1+\beta)x^{1+\beta}(1-y), \end{aligned}$$

which leads to (33) and (34).

Now, take $\beta = |\alpha|$. Then use (33), (34) and the dominated convergence theorem in the right-hand side of (32) (recall that $x\mu_0(\mathrm{d}x)$ is a probability measure), together with the monotone convergence theorem in the left-hand side of (32) to get

$$\int_0^\infty x^{|\alpha|} x\mu_0(\mathrm{d}x) = C^{-1}\phi(|\alpha|) < \infty.$$

Then, by an obvious induction, taking successively $\beta = 2|\alpha|$, $\beta = 3|\alpha|$, etc., we get for all $n \geq 1$

$$\int_0^\infty x^{n|\alpha|} x\mu_0(\mathrm{d}x) = C^{-1}\phi(n|\alpha|) \int_0^\infty x^{(n-1)|\alpha|} x\mu_0(\mathrm{d}x) = C^{-n}\phi(n|\alpha|)...\phi(|\alpha|).$$

We recognize the moments formula (14). Hence $x\mu_0(\mathrm{d}x) = \mu_R^{(C^{1/\alpha})}(\mathrm{d}x) = x\mu_\infty^{(C^{1/\alpha})}(\mathrm{d}x)$ and for all $t \geq 0$, $\mu_t = m(t)\mu_0 = \exp(-Ct)\mu_\infty^{(C^{1/\alpha})} = \mu_{\infty,t}^{(C^{1/\alpha})}$. □

4 Different speeds of decrease: proof of Proposition 1.4

4.1 Proof of Proposition 1.4 (i)

Recall that the support of μ_0 is supposed bounded with supremum 1. The goal of this section is to prove the forthcoming Corollory 4.3, which is the statement of Proposition 1.4 (i) translated in terms of the process X defined by (12), provided the Lévy measure Π of the subordinator ξ involved in the construction of X is related to the fragmentation measure B by (6) and $X(0)$ is distributed according to $x\mu_0(\mathrm{d}x)$. We recall that the distribution of $X(t)$ conditional on $X(t)>0$ is then $x\mu_t(\mathrm{d}x)/m(t)$, $t\geq 0$, where $(\mu_t, t\geq 0)$ denotes the solution to the fragmentation equation starting from μ_0. We start with some preliminary lemmas.

Lemma 4.1. *Suppose (H) and that $\int_0 |\ln(x)|xB(\mathrm{d}x)<\infty$. Consider some r.v. I independent of X, with distribution that of $\int_0^\infty \exp(\alpha\xi_r)\mathrm{d}r$. Then*
(i) there exists some $t_0>0$ such that

$$\sup_{t\geq t_0, a>0} a^{\alpha}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|}X(t)I^{1/|\alpha|}\leq a \;\Big|\; X(t)>0\right)<\infty$$

(ii) for all positive functions $g:[0,\infty[\to]0,\infty[$ converging to 0 at ∞, we have, as $t\to\infty$,

$$g(t)^{\alpha}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|}X(t)I^{1/|\alpha|}\leq g(t) \;\Big|\; X(t)>0\right)\to 1.$$

Proof. To simplify notations, suppose $\alpha=-1$ (the proof is identical for all $\alpha<0$). Recall then the key equality in law (20), which leads to the following identities for all $a>0$

$$\begin{aligned} a^{-1}\mathbb{P}\left(\frac{\varphi(t)}{t}X(t)I\leq a \;\Big|\; X(t)>0\right) &= \frac{1}{am(t)}\mathbb{P}\left(0<X(0)I-t\leq \frac{at}{\varphi(t)}\right)\\ &= \frac{m(t)-m(t+at/\varphi(t))}{am(t)} \\ &= \frac{1}{a}\left(1-\exp\left(-\ln(m(t))\left(1-\frac{\ln\left(m(t(1+a/\varphi(t)))\right)}{\ln(m(t))}\right)\right)\right). \end{aligned} \tag{35}$$

Use then the regular variation of $-\ln(m)$ with index $1/(1-\beta)$ (Prop. 1.2) and Lemma 3.6 to see that for all $\varepsilon>0$, there exists a real number $t(\varepsilon)$ such that for all $t\geq t(\varepsilon)$ and all $a>0$,

$$1-(1+a/\varphi(t))^{\frac{1}{1-\beta}+\varepsilon}\leq 1-\frac{\ln\left(m(t(1+a/\varphi(t)))\right)}{\ln(m(t))}\leq 1-(1+a/\varphi(t))^{\frac{1}{1-\beta}-\varepsilon}. \tag{36}$$

Now, let $0<\varepsilon<1-\beta$. Since

$$1-(1+x)^{\frac{1}{1-\beta}+\varepsilon}\geq -x\left(\frac{1}{1-\beta}+2\varepsilon\right)$$

for all $x>0$ sufficiently small, since moreover $\varphi(t)\to\infty$ as $t\to\infty$ and $-\ln(m)\underset{\infty}{\sim}(1-\beta)\varphi$, we have that for all $0<a\leq 1$ and all $t\geq t'(\varepsilon)$ (for some $t'(\varepsilon)$ depending on ε but not on $0<a\leq 1$),

$$\begin{aligned} -\ln(m(t))\left(1-(1+a/\varphi(t))^{\frac{1}{1-\beta}+\varepsilon}\right) &\geq \frac{\ln(m(t))}{\varphi(t)}a\left(\frac{1}{1-\beta}+2\varepsilon\right)\\ &\geq -a(1-\beta+\varepsilon)\left(\frac{1}{1-\beta}+2\varepsilon\right). \end{aligned}$$

Together with the identities (35) and inequalities (36) this implies that for all $t\geq\max(t(\varepsilon),t'(\varepsilon))$

$$\sup_{0<a\leq 1} a^{-1}\mathbb{P}\left(\frac{\varphi(t)}{t}X(t)I\leq a \;\Big|\; X(t)>0\right)<\infty.$$

This is enough to get (i), since for $a\geq 1$, a^{-1} times a probability is bounded by 1.

The proof of (ii) relies on the same idea. Since $g(t)/\varphi(t) \to 0$ as $t \to \infty$,

$$-\ln(m(t))\left(1-(1+g(t)/\varphi(t))^{\frac{1}{1-\beta}+\varepsilon}\right) \underset{\infty}{\sim} -g(t)(1-\beta)\left(\frac{1}{1-\beta}+\varepsilon\right),$$

and a similar result holds by replacing ε by $-\varepsilon$. Together with the inequalities (36) and the identities (35) (replacing there a by $g(t)$), using also that $g(t) \to 0$ as $t \to \infty$, we get (ii). □

Lemma 4.2. *Suppose* $\kappa := \int_0^1 |\ln(x)| x B(\mathrm{d}x) < \infty$. *Then,*

(i) I *possesses a density* $k \in \mathcal{C}^\infty(]0,\infty[)$,

(ii) $\mathbb{E}[I^{-1}] = \kappa|\alpha| < \infty$,

(iii) if moreover the support of B *is not included in a set of the form* $\{a^n, n \in \mathbb{N}\}$ *for some* $a \in]0,1[$, *then the function*

$$x \in \mathbb{R} \to \mathbb{E}[I^{ix-1}] = \int_0^\infty y^{ix-1} k(y)\mathrm{d}y$$

is well-defined and non-zero for all real number x.

Proof. If Π is the Lévy measure associated with the fragmentation equation, the assumption $\kappa < \infty$ is equivalent to $\int_0^\infty x\Pi(\mathrm{d}x) < \infty$, which, by Propositions 3.1 and 2.1 of [11] implies (i) and (ii). Next, it was proved in the proof of Theorem 2 of [20] that $\mathbb{E}[I^{ix-1}] \neq 0$ for all $x \in \mathbb{R}$ under the additional assumption that the support of Π is not included in a set of the form $\{rn, n \geq 0\}$ for some $r > 0$. □

Corollary 4.3. *Suppose (H), that* $\kappa = \int_0^1 |\ln(x)| x B(\mathrm{d}x) < \infty$ *and that the support of* B *is not included in a set of the form* $\{a^n, n \in \mathbb{N}\}$ *for some* $a \in]0,1[$. *Then, for all measurable functions* $g : [0,\infty[\to]0,\infty[$ *converging to* $\to 0$ *at* ∞,

$$g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} \frac{X(t)}{g(t)} \leq 1 \;\Big|\; X(t) > 0\right) \to \frac{1}{|\alpha|\kappa}.$$

Proof. Set for $x \geq 0$

$$U_t(x) := g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) \leq x g(t) \;\Big|\; X(t) > 0\right)$$

and note this quantity increases in x when t is fixed. Then consider some r.v. I independent of X, with distribution that of $\int_0^\infty \exp(\alpha\xi_r)\mathrm{d}r$. Consider b such that $\mathbb{P}(I \leq b) > 0$. Then

$$U_t(x)\mathbb{P}(I \leq b) \leq g(t)^\alpha \mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) I^{1/|\alpha|} \leq b^{1/|\alpha|} x g(t) \;\Big|\; X(t) > 0\right),$$

which, according to Lemma 4.1 (i), is bounded from above by some constant (independent of t and x) times $bx^{|\alpha|}$ for all $x \geq 0$ and $t \geq t_0$. I.e., there exists some finite constant C such that for all t sufficiently large and all $x \geq 0$,

$$x^\alpha U_t(x) \leq C. \tag{37}$$

Now, consider an increasing function $l : \mathbb{N} \to \mathbb{N}$. For all $x \geq 0$, the sequence $(U_{l(n)}(x), n \geq 0)$ is bounded. Hence there exist some non-decreasing right-continuous function $U : [0,\infty[\to [0,\infty[$, with $U(0) = 0$, and a subsequence $(U_{\tilde{l}(n)}, n \geq 0)$ of $(U_{l(n)}, n \geq 0)$ such that $U_{\tilde{l}(n)}(x) \to U(x)$ for a.e. $x > 0$. See e.g. [Theorem 2, Section VIII.7][15]. Hence if we prove that the limit U is given by

$$U(x) = \frac{x^{|\alpha|}}{|\alpha|\kappa}, \; \forall x \geq 0, \tag{38}$$

for all sequences $(l(n), n \geq 0), (\tilde{l}(n), n \geq 0)$ as defined above, we will have the expected result (note that the continuity of the function involved in (38) implies that the convergence will hold for every $x \geq 0$).

To prove (38), recall that by Lemma 4.2 (ii), $\int_0^\infty x^{-1}k(x)\mathrm{d}x < \infty$. Hence by dominated convergence, for all $a > 0$, $\int_0^\infty U_{l(n)}(ax^{1/\alpha})k(x)\mathrm{d}x \to \int_0^\infty U(ax^{1/\alpha})k(x)dx$. By Lemma 4.1 (ii), we therefore have

$$\int_0^\infty U(ax^{1/\alpha})k(x)\mathrm{d}x = a^{|\alpha|}, \ \forall a > 0. \tag{39}$$

We claim that this equation characterizes U under the additional assumption that the support of B is not included in a set of the form $\{a^n, n \in \mathbb{N}\}$ for some $a \in]0,1[$. Indeed, note first that by setting $V(x) := \exp(x)U(\exp(x/\alpha))$ and $\overline{k}(x) := k(\exp(-x))$ for all $x \in \mathbb{R}$, the above equation rewrites

$$\int_{-\infty}^\infty V(x)\,\overline{k}(y-x)\,\mathrm{d}x = 1, \ \forall y \in \mathbb{R}.$$

But the function V is bounded a.e. on $\mathbb{R}$, by (37). Moreover, by Lemma 4.2 (ii), $\overline{k} \in L^1(\mathbb{R})$ and by Lemma 4.2 (iii), the Fourier transform of $\overline{k}$ is non-zero on $\mathbb{R}$. We conclude with the Wiener approximation Theorem for $L^1(\mathbb{R})$ ([9, Th.4.8.4]) that the above equation in V has a unique bounded solution (in the sense that two solutions are equal a.e.). This determines V, hence U, almost everywhere. Since U is right-continuous, it is determined for all $x \geq 0$. Last, it is not hard to check that the expression of U given by (38) indeed satisfies equation (39). □

4.2 Proof of Proposition 1.4 (ii)

We just have to prove the second part of Proposition 1.4 (ii), the first one being obvious since $\mu_t(]1,\infty[) = 0$ for all $t \geq 0$ and $g(t)(\varphi(|\alpha|t)/|\alpha|t)^{1/\alpha} \to \infty$ as $t \to \infty$. We keep the notations of the previous section and we recall that we work under the assumption (H). From the proof of Lemma 4.1, we get that

$$\ln\left(\frac{m\left(t\left(1+\frac{|\alpha|h(t)^{|\alpha|}}{\varphi(|\alpha|t)}\right)\right)}{m(t)}\right) \underset{t\to\infty}{\sim} -h(t)^{|\alpha|},$$

for all positive functions h such that $h(t)^{|\alpha|}/\varphi(t) \to 0$ as $t \to \infty$. Otherwise said, for such functions h,

$$\ln\left(m(t)^{-1}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t)I^{1/|\alpha|} \geq h(t)\right)\right) \underset{t\to\infty}{\sim} -h(t)^{|\alpha|}.$$

Note that for all $t \geq 0$ and all $c > 0$, since X is independent of I,

$$\ln\left(m(t)^{-1}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) \geq \left(c\phi(h(t)^{|\alpha|})\right)^{1/|\alpha|}\right)\right) + \ln\left(\mathbb{P}\left(I^{1/|\alpha|} \geq h(t)/\left(c\phi(h(t)^{|\alpha|})\right)^{1/|\alpha|}\right)\right)$$
$$\leq \ln\left(m(t)^{-1}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t)I^{1/|\alpha|} \geq h(t)\right)\right).$$

Suppose moreover that $h(t) \to \infty$ as $t \to \infty$ and that $\beta < 1$, which implies that $h(t)^{|\alpha|}/\phi(h(t)^{|\alpha|}) \to \infty$. By Lemma 3.4, we have, for all real number $c > 0$,

$$-\ln\left(\mathbb{P}\left(I \geq h(t)^{|\alpha|}/c\phi(h(t)^{|\alpha|})\right)\right) \underset{\infty}{\sim} \frac{1-\beta}{|\alpha|}\varphi\left(\frac{|\alpha|h(t)^{|\alpha|}}{c\phi(h(t)^{|\alpha|})}\right) \underset{\infty}{\sim} \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}h(t)^{|\alpha|},$$

using both the regular variation of φ and the fact that φ is the inverse of $t \to t/\phi(t)$ near ∞. Now let $\varepsilon \in]0,1[$ and c be such that $c^{1/(1-\beta)} > (1-\beta)|\alpha|^{\beta/(1-\beta)}$. We have proved that

$$\ln\left(m(t)^{-1}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) \geq \left(c\phi(h(t)^{|\alpha|})\right)^{1/|\alpha|}\right)\right) \leq -(1-\varepsilon)\left(1-\frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\right)h(t)^{|\alpha|}$$

for t large enough. Next, let $g_{h,c}(t) = \left(c\phi(h(t)^{|\alpha|})\right)^{1/|\alpha|}, t \geq 0$, and suppose $\beta > 0$ (hence the existence of the inverse of ϕ near ∞). We have for t large enough

$$\begin{aligned}\ln\left(m(t)^{-1}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) \geq g_{h,c}(t)\right)\right) &\leq -(1-\varepsilon)\left(1-\frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\right)\phi^{-1}(g_{h,c}(t)^{|\alpha|}/c) \\ &\underset{\infty}{\sim} -(1-\varepsilon)\left(1-\frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\right)c^{-1/\beta}\phi^{-1}(g_{h,c}(t)^{|\alpha|}).\end{aligned}$$

It is not hard to check that the maximum of

$$\left\{ \left(1 - \frac{(1-\beta)|\alpha|^{\beta/(1-\beta)}}{c^{1/(1-\beta)}}\right) c^{-1/\beta}, c > (1-\beta)^{1-\beta}|\alpha|^{\beta} \right\}$$

is equal to $\beta/|\alpha|$ and is reached at $c = |\alpha|^{\beta}$. Finally, if $g_h = g_{h,|\alpha|^{\beta}}$, letting $\varepsilon \to 0$, we have proved that

$$\limsup_{t\to\infty} \frac{1}{\phi^{-1}(g_h(t)^{|\alpha|})} \ln\left(m(t)^{-1}\mathbb{P}\left(\left(\frac{\varphi(|\alpha|t)}{|\alpha|t}\right)^{1/|\alpha|} X(t) \geq g_h(t)\right)\right) \leq -\frac{\beta}{|\alpha|}. \tag{40}$$

To conclude, to get the second part of the statement of Proposition 1.4 (ii), suppose $0 < \beta < 1$ and consider some positive function g that converges to ∞ at ∞, such that $g(t)^{|\alpha|}t/\varphi(t) \to 0$. Set $h(t) = (\phi^{-1}(g(t)^{|\alpha|}/|\alpha|^{\beta})^{1/|\alpha|}, t \geq 0$. Then, $h(t)$ converges to ∞ as $t \to \infty$ and it is easily seen that $h(t)^{|\alpha|}/\varphi(t) \to 0$ as $t \to \infty$. Since $g = g_h$ with the notations above, the result follows from (40).

5 Some properties of the limit measure μ_∞

Recall that the distribution $x\mu_\infty(\mathrm{d}x)$ on $]0,\infty[$ is that of $R^{1/|\alpha|}$, where R denotes a random variable with entire positive moments

$$\mathbb{E}[R^n] = \phi(|\alpha|)....\phi(n|\alpha|),\ n \geq 1, \tag{41}$$

that characterize its distribution. Using this particular moments' shape, we get the following description of the measure μ_∞ near 0 and ∞. Some of these properties are then used at the end of this section to prove Proposition 1.5.

Proposition 5.1. *(Behavior at ∞)*
(i) Suppose (H) for some $\beta \in]0,1[$. Then,

$$-\ln\left(\int_t^\infty x\mu_\infty(\mathrm{d}x)\right) = -\ln\left(\mathbb{P}(R > t^{|\alpha|})\right) \underset{\infty}{\sim} \frac{\beta}{|\alpha|}\phi^{-1}(t^{|\alpha|})$$

where ϕ^{-1} denotes the inverse of ϕ (and is therefore a function regularly varying at ∞ with index $1/\beta$).
(ii) Suppose $\phi(\infty) := \int_0^1 xB(\mathrm{d}x) < \infty$. Then μ_∞ has a bounded support with supremum $\phi(\infty)^{1/|\alpha|}$ and

$$\mu_\infty\left(\{\phi(\infty)^{1/|\alpha|}\}\right) > 0 \Leftrightarrow \int^1 \frac{B(\mathrm{d}x)}{1-x} < \infty.$$

Proposition 5.2. *(Behavior at 0) Suppose that $\int_0^{\exp(-u)} xB(\mathrm{d}x)$ varies regularly at ∞ with index $-\gamma, \gamma \in [0,1]$. Then, as $s \to 0$*

$$\int_s^\infty x^{1+\alpha}\mu_\infty(\mathrm{d}x) = \mathbb{E}[\mathbf{1}_{\{R>s^{|\alpha|}\}}R^{-1}] \sim \frac{1}{(|\alpha|^{\gamma}\Gamma(1+\gamma)\phi(-1/\ln(s^{|\alpha|})))}$$

and

$$\int_0^s x\mu_\infty(\mathrm{d}x) = \mathbb{P}(R < s^{|\alpha|}) = \circ\left(\frac{s^{|\alpha|}}{\phi(-1/\ln(s^{|\alpha|}))}\right).$$

Proof. This is a direct consequence of Corollary 1 of Caballero-Rivero [10], which gives these results in terms of the random variable R. □

Proof of Proposition 5.1. (i) Our proof strongly relies on the proof of Proposition 2 of Rivero [28]. Rivero shows there that if a positive random variable Y has entire moments satisfying

$$\mathbb{E}[Y^n] = \prod_{i=1}^{n} \psi(i)$$

for some function ψ regularly varying at ∞ with index $\gamma \in]0,1[$, then

$$-\ln\left(\mathbb{P}(Y>t)\right) \underset{\infty}{\sim} \gamma\psi^{\leftarrow}(t),$$

where $\psi^{\leftarrow}$ is the right-inverse of ψ. Apply then this result to the r.v. R, by taking $\psi = \phi(|\alpha|\cdot)$ and $\gamma = \beta$.

(ii) Using (41) and that ϕ is increasing, we get for all $n \geq 0$,

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] \leq 1. \tag{42}$$

Besides, writing

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] = \mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n \mathbf{1}_{\{R>\phi(\infty)\}}\right] + \mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n \mathbf{1}_{\{R<\phi(\infty)\}}\right] + \mathbb{P}\left(R=\phi(\infty)\right),$$

and using the monotone and dominated convergence theorems, we see that

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] \underset{n\to\infty}{\to} \begin{cases} \infty & \text{if } \mathbb{P}(R>\phi(\infty))>0 \\ \mathbb{P}(R=\phi(\infty)) & \text{otherwise.} \end{cases}$$

In particular, from (42), we see that $\mathbb{P}(R>\phi(\infty)) = 0$. Similarly, it is easy to show, using (41), that for all $0<\varepsilon<\phi(\infty)$,

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)-\varepsilon}\right)^n\right] \to \infty,$$

which implies that $\mathbb{P}(R>\phi(\infty)-\varepsilon)>0$. Last, to get the remaining part of the statement, note that

$$\ln\left(\frac{\phi(n|\alpha|)}{\phi(\infty)}\right) \underset{n\to\infty}{\sim} \frac{\phi(n|\alpha|)}{\phi(\infty)} - 1 = \frac{-1}{\phi(\infty)}\int_0^1 x^{n|\alpha|+1} B(\mathrm{d}x).$$

Therefore

$$\ln\left(\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right]\right) = \sum_{i=1}^n \ln\left(\frac{\phi(i|\alpha|)}{\phi(\infty)}\right)$$

converges to $-\infty$ as $n\to\infty$ if and only if

$$\int_0^1 \sum_{i=1}^\infty x^{i|\alpha|+1} B(\mathrm{d}x) = \int_0^1 \frac{x^{1+|\alpha|}}{1-x^{|\alpha|}} B(\mathrm{d}x) = \infty.$$

Since $\int_0 xB(\mathrm{d}x) < \infty$,

$$\mathbb{E}\left[\left(\frac{R}{\phi(\infty)}\right)^n\right] \underset{n\to\infty}{\to} 0 \text{ i.f.f. } \int^1 \frac{1}{1-x} B(\mathrm{d}x) = \infty,$$

which ends the proof. □

Proof of Proposition 1.5. From the construction (7) of μ_t, we see that

$$\mu_t(\{1\}) = \mu_0(\{1\})\mathbb{P}\left(\xi(\rho(t))=0\right) = \mu_0(\{1\})\mathbb{P}\left(\xi(t)=0\right),$$

and from the Poisson point process construction of a pure-jump subordinator with Lévy measure Π we have that $\mathbb{P}\left(\xi(t)=0\right) = \exp(-t\Pi(]0,\infty[)) = \exp\left(-t\phi(\infty)\right)$. Next, we get from the factorization (15), that

$$\exp\left(-t\phi(\infty)\right) = \mathbb{P}\left(RI > t\phi(\infty)\right) \geq \mathbb{P}(I>t)\mathbb{P}(R\geq\phi(\infty)).$$

On the one hand, from the proof of Proposition 5.1, we see that when $\phi(\infty)<\infty$, $\mathbb{P}(R\geq\phi(\infty)) = \mathbb{P}(R=\phi(\infty))$ and that this quantity is non-zero i.f.f. $\int^1(1-x)^{-1}B(\mathrm{d}x)<\infty$. On the other hand, under (H), we get from the regular variation of $-\ln(\mathbb{P}(I>t))$ that $\mathbb{P}(I>x^\alpha t)/\mathbb{P}(I>t) \to 0$ for all $0<x<1$, as $t\to\infty$; and then, from the dominated convergence theorem that

$$\frac{m(t)}{\mathbb{P}(I>t)} = \int_0^1 \frac{\mathbb{P}(I>x^\alpha t)}{\mathbb{P}(I>t)} x\mu_0(\mathrm{d}x) \to \mu_0(\{1\}) \text{ as } t\to\infty.$$

Otherwise said, we have proved that under the hypothesis (H), when $\mu_0(\{1\}) > 0$ and $\phi(\infty) < \infty$,

$$\liminf_{t\to\infty} \frac{\mu_t(\{1\})}{m(t)} \geq \mathbb{P}(R = \phi(\infty)) = \phi(\infty)^{1/|\alpha|}\mu_\infty(\{\phi(\infty)^{1/|\alpha|}\}).$$

Next, suppose (H), that $\int_0 |\ln(x)|xB(\mathrm{d}x) < \infty$ and that $\phi(\infty) < \infty$. According to Theorem 1.3, for all $\varepsilon \in]0,1[$ such that $(1-\varepsilon)\phi(\infty)^{1/|\alpha|}$ is not an atom of μ_∞,

$$\frac{\mu_t(\{1\})}{m(t)} \leq \frac{\int_{1-\varepsilon}^1 x\mu_t(\mathrm{d}x)}{m(t)} \underset{t\to\infty}{\to} \int_{(1-\varepsilon)\phi(\infty)^{1/|\alpha|}}^{\phi(\infty)^{1/|\alpha|}} x\mu_\infty(\mathrm{d}x).$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{t\to\infty} \frac{\mu_t(\{1\})}{m(t)} \leq \phi(\infty)^{1/|\alpha|}\mu_\infty(\{\phi(\infty)^{1/|\alpha|}\}).$$

□

6 Examples

Below is a list of standard examples where the main quantities involved in our results can be computed explicitly. More precisely, for each of these examples, we specify the distributions of I (defined in (13)) and R (defined in (14)), which leads to explicit expressions of the limit measure μ_∞ (since $R^{1/|\alpha|} \overset{\mathrm{d}}{\sim} x\mu(\mathrm{d}x)$) and of the mass

$$m_1(t) = \mathbb{P}(I > t),\ t \geq 0,$$

which is the mass of the solution to the fragmentation equation starting from $\mu_0 = \delta_1$. We also specify the behavior as $t \to \infty$ of the quantity $\varphi(|\alpha|t)/|\alpha|t$, involved in the statement of Theorem 1.3. For all these examples, we give the main tools to get the distributions of I and R, but we leave the calculation details to the reader. We recall that β denotes the index of regular variation of hypothesis (H) and that when $(\mu_t, t \geq 0)$ is a solution to the equation with parameters (α, B), $(\mu_{ct}, t \geq 0)$ is a solution to the equation with parameters (α, cB). For this reason, in the examples below, given a measure B we choose its "representative" among the measures cB, $c > 0$, which is the most convenient for the statement of the results.

The first four examples concerns absolutely continuous measures $B(\mathrm{d}u) = b(u)\mathrm{d}u$, where b is a function defined on $]0,1[$. The Lévy measure is therefore also absolutely continuous and we denote by π its density. It turns out that the limit distribution μ_∞ is also absolutely continuous. We denote by u_∞ its density.

Ex.1. $b(u) = bu^{b-2}$, $b > 0$; $\alpha < 0$.

- $\beta = 0$
- $\varphi(t) \sim t$ as $t \to \infty$
- $I \overset{\mathrm{d}}{\sim} \Gamma(b/|\alpha| + 1, 1)$
- $m_1(t) = \frac{1}{\Gamma(b/|\alpha|+1)} \int_t^\infty x^{b/|\alpha|} \exp(-x)\mathrm{d}x$, $t \geq 0$
- $R \overset{\mathrm{d}}{\sim} \beta(1, b/|\alpha|)$
- $u_\infty(x) = bx^{|\alpha|-2}\left(1 - x^{|\alpha|}\right)^{\frac{b}{|\alpha|}-1}$, $0 < x < 1$

The notations $\Gamma(x,y)$ (resp. $\beta(x,y)$) refer to the classical Gamma distribution with parameters $x, y > 0$ (resp. Beta distribution). In these examples, the density of the Lévy measure associated with B is $\pi(x) = b\exp(-bx)$, $x > 0$, hence the Lévy measure associated with the subordinator $|\alpha|\xi$ (where ξ has Lévy measure Π) has a density given by $b\exp(-bx/|\alpha|)/|\alpha|$, $x > 0$. According to the Example B, p.5 of [11], the density of I is then proportional to $x^{b/|\alpha|}\exp(-x), x > 0$. Last we refer to the formula (4), Section 3 of [8], to get the distribution of R.

We point out that the solutions to the fragmentation equation with this measure B are studied in [24]. In particular, when $\alpha = -b/2$ and $\mu_0 = \delta_1$ the solutions $(\mu_t, t \geq 0)$ has the explicit expression

$$\mu_t(\mathrm{d}x) = \exp(-t)\left(\delta_1(\mathrm{d}x) + bx^{b-2}\left(t - \frac{1}{2}t^2(1 - x^{-b/2})\right)\mathbf{1}_{\{0<x<1\}}\mathrm{d}x\right),$$

which gives

$$m_1(t) = \exp(-t)\left(1 + t + \frac{t^2}{2}\right)$$

and for all bounded test functions $f :]0, \infty[\to \mathbb{R}$

$$\frac{1}{m_1(t)} \int_0^1 f(x) x \mu_t(\mathrm{d}x) \underset{t\to\infty}{\to} \int_0^1 f(x) b x^{b-1} (x^{-b/2} - 1) \mathrm{d}x,$$

which is consistent with the above expressions of m_1 and u_∞.

Ex.2. $b(u) = |\alpha|\Gamma(1-\gamma)^{-1} u^{\frac{|\alpha|}{\gamma}-2}(1 - u^{\frac{|\alpha|}{\gamma}})^{-\gamma-1}$; $0 < \gamma < 1$; $\alpha < 0$.

- $\beta = \gamma$
- $\varphi(t) \sim \left(\frac{\gamma}{|\alpha|}\right)^{\frac{\gamma}{1-\gamma}} t^{\frac{1}{1-\gamma}}$ as $t \to \infty$
- $I \overset{\mathrm{d}}{\sim} \tau_\gamma^{-\gamma}$
- $m_1(t) = \int_t^\infty g_\gamma(x)\mathrm{d}x$, $t > 0$.
- $R \overset{\mathrm{d}}{\sim} \mathbf{e}(1)^\gamma$
- $\mu_\infty(x) = |\alpha|\gamma^{-1} x^{\frac{|\alpha|}{\gamma}-2} \exp\left(-x^{|\alpha|/\gamma}\right)$, $x > 0$

Here, $\mathbf{e}(1)$ denotes a r.v. with exponential distribution with parameter 1 and τ_γ a γ-stable random variable, i.e. with Laplace transform $t \in [0, \infty[\to \exp(-t^\gamma)$. Hence $\tau_\gamma^{-\gamma}$ has the so-called Mittag-Leffler distribution. We recall that it possesses a density given by

$$g_\gamma(x) = \frac{1}{\pi\gamma} \sum_{i=0}^{\infty} \frac{(-x)^{i-1}}{i!} \Gamma(\gamma i + 1) \sin(\pi\gamma i), \ x > 0$$

and its entire positive moments are equal to $n!/\Gamma(\gamma n + 1)$, $\forall n \geq 1$ (see e.g. [27, Section 0.3]). The Lévy measure associated with B has a density given for $x > 0$ by

$$\pi(x) = \frac{|\alpha| \exp(-|\alpha| x/\gamma)}{\Gamma(1-\gamma)(1 - \exp(-|\alpha| x/\gamma))^{\gamma+1}}.$$

Using formula (5) and the following discussion in [8], we get that $I \overset{\mathrm{d}}{\sim} \tau_\gamma^{-\gamma}$ and $R \overset{\mathrm{d}}{\sim} \mathbf{e}(1)^\gamma$.

Ex.3. $b(u) = |\alpha|\gamma^2 \left((1-\gamma)\Gamma(2-\gamma)\right)^{-1} u^{\frac{\gamma|\alpha|}{1-\gamma}-2}(1 - u^{|\alpha|/(1-\gamma)})^{-\gamma-1}$; $0 < \gamma < 1; \alpha < 0$.

- $\beta = \gamma$
- $\varphi(t) \sim (1-\gamma)^{-1}|\alpha|^{\frac{\gamma}{\gamma-1}} t^{\frac{1}{1-\gamma}}$ as $t \to \infty$
- $I \overset{\mathrm{d}}{\sim} \mathbf{e}(1)^{1-\gamma}$
- $m_1(t) = \exp(-t^{1/(1-\gamma)}), \quad t \geq 0$
- $R \overset{\mathrm{d}}{\sim} \tau_{1-\gamma}^{\gamma-1}$
- $\mu_\infty(x) = |\alpha| x^{|\alpha|-2} g_{1-\gamma}(x^{|\alpha|})$, $x > 0$,

where $g_{1-\gamma}$ is the Mittag-Leffler density given in the previous example. Note the duality with this previous example. In the present example,

$$\pi(x) = \frac{|\alpha|\gamma^2 \exp(|\alpha| x/(1-\gamma))}{(1-\gamma)\Gamma(2-\gamma)(\exp(|\alpha| x/(1-\gamma)) - 1)^{1+\gamma}}, x > 0,$$

and we again refer to the formula (5) and the following discussion in [8] to get the distributions of I and R.

Ex.4. $b(u) = |\alpha|\Gamma(2+\alpha)^{-1} u^{|\alpha|-2}(1-u)^{\alpha-1}$; $-1 < \alpha < 0$.

- $\beta = |\alpha|$
- $\varphi(t) \sim \left(\frac{t}{1+\alpha}\right)^{\frac{1}{1-|\alpha|}}$ as $t \to \infty$
- $I/(1+\alpha)$ is a size-biased version of the Mittag-Leffler distribution with parameter $|\alpha|$, i.e. for all test functions f

$$\mathbb{E}[f(I)] = \frac{\mathbb{E}\left[f\left((1+\alpha)\tau_{|\alpha|}^{-|\alpha|}\right)\tau_{|\alpha|}^{-|\alpha|}\right]}{\mathbb{E}\left[\tau_{|\alpha|}^{-|\alpha|}\right]}$$

- $m_1(t) = \Gamma(|\alpha|+1)\int_{t/(1+\alpha)}^{\infty} x g_{|\alpha|}(x)\mathrm{d}x$, $t > 0$
- $((1+\alpha)R)^{1/|\alpha|} \overset{\mathrm{d}}{\sim} \Gamma(|\alpha|, 1)$
- $\mu_\infty(x) = \frac{(1+\alpha)}{\Gamma(|\alpha|)} x^{|\alpha|-2}\exp(-(1+\alpha)^{1/|\alpha|}x)$, $x > 0$.

Indeed, here

$$\pi(x) = \frac{|\alpha|\exp(x)}{\Gamma(2+\alpha)(\exp(x)-1)^{1-\alpha}}, \ x > 0.$$

Following the end of the proof of Lemma 4 of Miermont [26], we get that I has its moment of order k equal to

$$\frac{k!(1+\alpha)^k\Gamma(|\alpha|)}{\Gamma\left((k+1)|\alpha|\right)},$$

for all $k \in \mathbb{N}$. Hence

$$\mathbb{E}[R^k] = \frac{k!}{\mathbb{E}[I^k]} = \frac{\Gamma\left((k+1)|\alpha|\right)}{(1+\alpha)^k\Gamma(|\alpha|)} = \frac{1}{(1+\alpha)^k\Gamma(|\alpha|)}\int_0^\infty x^{(k+1)|\alpha|-1}\exp(-x)\mathrm{d}x.$$

Remark. Note that the Examples 2, 3 and 4 give, for all $0 < \gamma < 1$,

- if $b(u) = u^{-1}(1-u)^{-\gamma-1}$ and $\alpha = -\gamma$,

$$x\mu_\infty(\mathrm{d}x) \overset{\mathrm{d}}{\sim} c_1(\gamma)\mathbf{e}(1)$$

- if $b(u) = u^{\gamma-2}(1-u)^{-\gamma-1}$ and $\alpha = \gamma - 1$,

$$x\mu_\infty(\mathrm{d}x) \overset{\mathrm{d}}{\sim} c_2(\gamma)\tau_{1-\gamma}^{-1},$$

- if $b(u) = u^{\gamma-2}(1-u)^{-\gamma-1}$ and $\alpha = -\gamma$,

$$x\mu_\infty(\mathrm{d}x) \overset{\mathrm{d}}{\sim} c_3(\gamma)\Gamma(\gamma, 1)$$

where $c_1(\gamma), c_2(\gamma)$ and $c_3(\gamma)$ are real numbers that depend on γ. Hence, both α and the behavior of b near 0 play a significant role on the shape of the limit measure μ_∞.

Last we turn to the case where B is a Dirac measure.

Ex.5. $B = a^{-1}\delta_a$ for some $a \in]0,1[$; $\alpha < 0$.

- $\beta = 0$
- $\varphi(t) \sim t$ as $t \to \infty$
- I has a density k on $]0,\infty[$ given by

$$k(x) = \sum_{i \geq 0}\exp\left(\alpha\ln(a)i - x\exp(\alpha\ln(a)i)\right)\prod_{p \neq i}\left(1-\exp(\alpha\ln(a)(i-p))\right)^{-1}$$

- $m_1(t) = \int_t^\infty k(x)\mathrm{d}x$, $t \geq 0$

In this case, $\Pi = \delta_{-\ln(a)}$, i.e. the associated subordinator is a Poisson process. We then refer to [11, Prop. 6.5 (ii)] for the expression of the density k. Note that $\phi(t) = (1-a^t)$ for all $t \geq 0$, hence $\mathbb{E}[R^n] = \prod_{i=1}^n(1-(|\alpha|a)^i)$ for all $n \geq 1$.

7 Appendix: Existence and uniqueness of solutions

This appendix is devoted to the proof of Theorem 1.3 on existence and uniqueness of solutions to the fragmentation equation (2). So in this section, $\alpha \in \mathbb{R}$. The proof follows the main lines of that of Theorem 1 in [19], which states existence and uniqueness of solutions to a slightly restricted form of the fragmentation equation (2), and which was concentrated on solutions starting from $\mu_0 = \delta_1$. We note that it was implicit in the statement of this theorem that a solution should satisfy assumptions (4) and (5).

Let ξ denotes a subordinator with Lévy measure Π and zero drift, such that $\xi_0 = 0$. We recall that its semigroup possesses the Feller property and that the domain of its infinitesimal generator contains at least all functions f that are continuously differentiable on $\mathbb{R}$ and such that f and f' tend to 0 at infinity. See e.g. Chapter 1 of [3]. As a consequence, the domain of the infinitesimal generator of $\exp(-\xi)$ contains continuously differentiable functions f on $]0,\infty[$ with compact support.

One can easily checked that when $f:]0,\infty[\to\mathbb{R}$ is bounded and continuous, the function

$$x \to \mathbb{E}\left[f(x\exp(-\xi_{\rho(x^\alpha t)}))\right]$$

is also bounded and continuous on $]0,\infty[$. This mainly relies on the càdlàg and quasi-left-continuity ([3, Prop.7, Chapter 1]) of subordinators.

Now, for every $0<a<b$, let $\mathcal{C}_{a,b}$ be the set of continuous functions $f:]0,b]\to\mathbb{R}$ that are null on $]0,a]$, and $\mathcal{C}^1_{a,b}$ be the set of continuously differentiable functions $f:]0,b]\to\mathbb{R}$ that are null on $]0,a]$. It is clear from the remark above that for all $0<a<b$, the linear operators T_t and $\tilde{T}_t$, $t\geq 0$, defined by

$$T_t(f)(x) = \mathbb{E}\left[f(x\exp(-\xi_t))\right]$$

and

$$\tilde{T}_t(f)(x) = \mathbb{E}\left[f(x\exp(-\xi_{\rho(x^\alpha t)}))\right]$$

send $\mathcal{C}_{a,b}$ into $\mathcal{C}_{a,b}$. Following the proof of Theorem 1 of [19] (see also [22]), we see that both family of operators define strongly continuous contraction semigroups on $\mathcal{C}_{a,b}$, and that the domains of their infinitesimal generators are identical and contain $\mathcal{C}^1_{a,b}$. These generators are respectively given, for $f\in\mathcal{C}^1_{a,b}$ and $x\in]0,b]$, by

$$A(f)(x) = \int_0^\infty \left(f(x\exp(-y)) - f(x)\right)\Pi(\mathrm{d}y)$$

and

$$\tilde{A}(f)(x) = x^\alpha A(f)(x).$$

Note that when B is a measure on $]0,1[$ defined from Π by (6), we have

$$\tilde{A}(f)(x) = x^\alpha \int_0^1 \left(f(xy) - f(x)\right) yB(\mathrm{d}y).$$

Existence of solutions to (2). With the above remarks, and Kolmogorov's backward equation (see Prop.15, p.9 of [14]), we have that

$$\tilde{T}_t(f)(x) = f(x) + \int_0^t \tilde{T}_s(\tilde{A}(f))(x)\mathrm{d}s, \tag{43}$$

$\forall x\in]0,b]$, $\forall f\in\mathcal{C}^1_{a,b}$, $\forall 0<a<b$, $\forall b>0$. Otherwise said: let $f:]0,\infty[\to\mathbb{R}$ be null near 0 and continuously differentiable. Then, considering its restriction to $]0,b]$ and $x\leq b$, we have that f and x satisfy (43).

Now, consider ν_0, a probability measure on $]0,\infty[$, and set

$$<\nu_t, g> := <\nu_0, \tilde{T}_t(g)>$$

for all bounded, measurable functions g on $]0,\infty[$. Note that for all $t\geq 0$, $\nu_t(]0,\infty[)\leq 1$ and $\nu_t(x\geq M)=0$ as soon as $\nu_0(x\geq M)=0$ for some $M>0$. Then let f be some continuously differentiable function on $]0,\infty[$ with compact support. It is clear that $\tilde{A}(f)$ is null near 0 and it is easy to see, using Fubini's theorem, that there exists some constants $b,c>0$ such that $|\tilde{A}(f)|(x)\leq cx^\alpha\overline{\Pi}(\ln(x/b))$ for large enough x (here

$\overline{\Pi}(y) = \int_y^\infty \Pi(\mathrm{d}x)$). In particular, $\tilde{A}(f)$ is bounded on $]0,\infty[$ when $x^\alpha \overline{\Pi}(\ln(x))$ is bounded near ∞ (hence when $\alpha \leq 0$). It is then clear that in such case we can apply Fubini's theorem when integrating (43) with respect to ν_0 to get

$$< \nu_t, f > = < \nu_0, f > + \int_0^t < \nu_s, \tilde{A}(f) > \mathrm{d}s.$$

This holds for all continuously differentiable functions f on $]0,\infty[$ with compact support. Therefore, defining the measures μ_t on $]0,\infty[$ by $< \mu_t, g > := < \nu_t, \tilde{g} >$, where g denotes any test-function on $]0,\infty[$ and $\tilde{g}(x) = g(x)/x$, $x > 0$, we have proved that $(\mu_t, t \geq 0)$ is a solution to the fragmentation equation, as defined in the Introduction. To sum up: provided that the function $x \to x^\alpha \overline{\Pi}(\ln(x))$ is bounded near ∞, for all measure μ_0 on $]0,\infty[$ such that $\int_0^\infty x\mu_0(\mathrm{d}x) = 1$, there exists a solution constructed via subordinators to the fragmentation equation.

When $\alpha > 0$, the function $x \to x^\alpha \overline{\Pi}(\ln(x))$ may not be bounded near ∞. Another way to tackle the problem in this case is to use the definition of ρ to get that

$$\int_0^\infty \int_0^t \tilde{T}_s(|\tilde{A}(f)|(x)\mathrm{d}s\nu_0(\mathrm{d}x) = \int_0^\infty \mathbb{E}\left[\int_0^{\rho(x^\alpha t)} |A(f)|(x\exp(-\xi_u))\mathrm{d}u\right]\nu_0(\mathrm{d}x),$$

the function f being still supposed continuously differentiable on $]0,\infty[$ with compact support. For such f, the function $A(f)$ is bounded on $]0,\infty[$. Hence the double integral involved in the identity above is bounded by a constant times $\int_0^\infty \mathbb{E}\left[\rho(x^\alpha t)\right]\nu_0(\mathrm{d}x)$, which is finite as soon as $\int^\infty \ln(x)\nu_0(\mathrm{d}x) < \infty$: indeed, according to Proposition 2 of [8], for all $x \geq 0$, $\mathbb{E}[\rho(x)] = \int_0^x \mathbb{E}[\exp(-sR)]\mathrm{d}s$, where R is a random variable with distribution μ_R defined by (14). Let then I be a random variable defined by (13), independent of R, and consider a real number a such that $\mathbb{P}(I \leq a) > 0$. Using the factorization property (15), we get

$$\mathbb{E}[\rho(x)]\mathbb{P}(I \leq a) = \int_0^x \mathbb{E}[\exp(-sR)\mathbf{1}_{\{I \leq a\}}]\mathrm{d}s \leq \int_0^x \mathbb{E}[\exp(-sa^{-1}\mathbf{e}(1))]\mathrm{d}s = a\ln(1 + a^{-1}x).$$

It this then possible to apply Fubini's theorem when integrating (43) with respect to ν_0 and we conclude as above to the existence of a solution to (2).

Uniqueness of solutions to (2). Let ν_0 be a probability measure with support included in $]0,b]$ for some $b > 0$ and suppose that $(\nu_t, t \geq 0)$ is a family of measures with support included in $]0,b]$ such that

$$< \nu_t, f > = < \nu_0, f > + \int_0^t < \nu_s, \tilde{A}(f) > \mathrm{d}s, \ \forall f \in \cup_{0<a<b}\mathcal{C}^1_{a,b}.$$

Suppose moreover that $\nu_t(]0,\infty[) \leq 1$, $\forall t \geq 0$. Our goal is to prove that $(\nu_t, t \geq 0)$ is uniquely determined. Using that the total weight of ν_t is less or equal to 1, we get that $\sup_{t\geq 0} < \nu_t, |f| > < \infty$ and $\sup_{t\geq 0} < \nu_t, |A(f)| > < \infty$ for each $f \in \cup_{0<a<b}\mathcal{C}^1_{a,b}$. It is then possible to follow the proof of Proposition 18, Sect. 4.9 of [14] to deduce that uniqueness holds provided that, for all $\lambda > 0$, $(\lambda \mathrm{id} - \tilde{A}(f))(\mathcal{C}^1_{a,b})$ is dense (for the uniform norm) in $\mathcal{C}_{a,b}$, for all $0 < a < b$. Following the proof of Theorem 1 in [19], we see that $\mathcal{C}^1_{a,b}$ is a core for the strongly contraction semi-group $\tilde{T}_t : \mathcal{C}_{a,b} \to \mathcal{C}_{a,b}$, $t \geq 0$. Hence the result.

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