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# BROWNIAN COAGULATION AND A VERSION OF SMOLUCHOWSKI'S EQUATION ON THE CIRCLE

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We introduce a 1–dimensional stochastic system where particles perform independent diffusions and interact through pairwise coagulation events, which occur at a non–trivial rate upon collision. Under appropriate conditions on the diffusion coefficients, the coagulation rates and the initial distribution of particles, we derive a spatially inhomogeneous version of the mass flow equation as the particle number tends to infinity. The mass flow equation is in one to one correspondence with Smoluchowski's coagulation equation. We prove uniqueness for this equation in a broad class of solutions, to which the weak limit of the stochastic system is shown to belong.

**1. Introduction.** Coagulation models describe the dynamics of cluster growth. Particles carrying different masses move freely through space, and every time any two of them get sufficiently close there is some chance that they coagulate into a single particle, which will be charged with the sum of the masses of the original pair.

In 1916, Smoluchowski [18] considered the model of Brownian particles moving independently in three dimensional space, such that any pair coagulates into one particle upon collision. He derived a system of equations, known as Smoluchowski's coagulation equations, that describes the time evolution of the average concentration $\mu_t(m)$ of particles carrying a given mass $m = 1, 2, \ldots$. In this original work, Smoluchowski ignored the effect of spatial fluctuations in the mass concentrations, the equations we write below are thus a natural extension allowing diffusion in the space variables:

$$\dot{\mu}_t(m) = \frac{1}{2}\, a(m)\, \Delta_x \mu_t(x,m) + \frac{1}{2} \sum_{m'+m''=m} \kappa(m',m'')\, \mu_t(x,m')\, \mu_t(x,m'') - \mu_t(x,m) \sum_{m'} \kappa(m,m')\, \mu_t(x,m')\,.$$

The differentiated term on the right describes the free motion of a particle with attached mass $m$ as a Brownian motion with diffusivity rate $a(m)$. The

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kernel $\kappa$ is determined by physical considerations, it regulates the intensity of the coagulation dynamics. The first sum corresponds to the increase in the concentration resulting from the coagulation of two particles whose masses add up to $m$. The second sum reflects the decrease caused by the coalescence of a particle carrying mass $m$ with any other particle in the system. Coagulation phenomena have been studied in many fields of the applied science, we refer to Aldous' review [1] for a comprehensive survey of the literature.

This paper is concerned with the approximation of Smoluchowski's equations by stochastic particle models. Concretely, we are interested in identifying the solution to the coagulation equations as the mass density of a system of interacting particles, when the particle number tends to infinity. This problem has been much studied in the spatially homogeneous case, both for discrete and continuous mass distributions, and with different choices of coagulation kernel $\kappa$ (cf. [8], [15], [4], [5] and references therein). The relevant stochastic process for these models is the Marcus-Lushnikov process ([14], [13]), this is the pure jump Markov process where clusters of size $m$ and $m'$ coagulate into a single cluster of size $m+m'$ at rate $\kappa(m,m')$.

In the spatially inhomogeneous case, on the other hand, the coagulation mechanism is highly dependent on the relative position of the particles, hence the space dynamics plays a predominant role in the particle interactions of the stochastic system. In the original problem proposed by Smoluchowski, for instance, pairwise collisions and the ensuing coagulation events are completely determined by the Brownian paths. The first result for this model was obtained in 1980 by Lang and Xanh [12] for the case of discrete mass and constant $a$, $\kappa$, in the limit of constant mean free time. No progress was made until the forthcoming paper [16], where Norris proves convergence for both discrete and continuous mass distributions, and variable coefficients in a class that includes the Brownian case.

Over the past few years there has been considerable interest in spatial models with stochastic dynamics of coagulation. In these models, particles coagulate at some rate while they remain at less than a prescribed distance. Deaconu and Fournier [3] consider the case when this distance is independent of the particle number, and let it go to zero after taking the weak limit. The moderate limit, where the range of interaction is long in the microscopic scale, is studied by Großkinsky et al. in [7] in a regime where the dominating particle interaction are shattering collisions. In the articles [9] [10], Hammond and Rezakhanlou work in the constant mean free time limit, for dimensions $d \geq 2$.

In this paper we introduce a diffusion model where coagulation occurs on collision as a result of a random event: $N$ mass–charged particles perform independent one dimensional diffusions, and whenever two particles are at the same location they may coagulate at a positive rate in their intersection local time. The new particle is assigned the sum of the masses of the incoming particles, and the process continues.

This model is motivated by the problem of establishing the large scale dynamics of a system of Brownian particles confined to a thin tube, interacting through pairwise coagulation when any two of them get close enough. It would be interesting to determine whether, under proper scaling of the tube and particle radii in terms of the system size $N$, the higher dimensional model can be replaced by the simpler one dimensional one.

Note that in one dimension the problem of instantaneous coagulation on collision is not interesting: due to the recurrence properties of the one–dimensional Brownian path, in the limit we would instantaneously see the distribution of the total mass in the system among clusters of macroscopic size. In fact, in order to keep the average time $T$ during which a tagged particle does not undergo a collision constant, it is necessary to set coagulation rates which are inversely proportional to the particle number $N$. The model under consideration is thus of the constant mean free time type, and in this sense it is a one dimensional version of the models studied in [12], [16] and [9] [10].

We treat the case where the mass dependent diffusivity rates blow up as the mass goes to zero, combined with our choice of rates, this leads to a large scale model favouring coagulation of large and small particles.

We describe the particle system and state the main results of the paper in Section 2. Section 3 contains a compactness result. The next step in the analysis is to prove convergence to a hydrodynamic limit. This is shown to verify a spatial version of the mass flow equation, which is closely related to Smoluchowski's coagulation equation. This is the content of Section 4. In Section 5 we derive a uniqueness result for the solutions to a broad family of such equations, thereby obtaining a law of large numbers for the empirical processes of the microscopic model.

## 2. Notation and results.

Consider a positive integer $N$. Let $\mathbb{T}$ stand for the one–dimensional torus, and $\mathbb{R}_+$ for the half line $[0,\infty)$. Let

$$P_0^N = P_0^N(\,dx^1, dm^1;\ \ldots;\ dx^N, dm^N)$$

be a sequence of measures on $(\mathbb{T}\times\mathbb{R}_+)^N$ which are symmetric on the pairs $(x^i, m^i)$ and supported on $\sum_i m^i = 1$. Denote by $\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+)$ the space of probability measures on $\mathbb{T}\times\mathbb{R}_+$ endowed with the weak topology; we will assume that there exists an initial profile $\nu\in\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+)$ such that the empirical distributions $\sum_i m^i \delta_{(x^i,Nm^i)}$ converge weakly to $\delta_\nu$ as $N \to \infty$, where by $\delta_{(x^i,Nm^i)}$ (resp., $\delta_\nu$) we denote the probability measure with a unit atom at $(x^i, Nm^i)$ (resp., $\nu$). We will use the notation $\mathcal{M}_f(\mathbb{T}\times\mathbb{R}_+)$ to refer to the space of positive and finite measures on $\mathbb{T}\times\mathbb{R}_+$.

For a complete separable metric space $\mathcal{T}$, we will denote by $D(\mathbb{R}_+,\mathcal{T})$ (or $D(I,\mathcal{T})$, $I$ an interval of the real numbers), the set of right continuous functions with left limits taking values in $\mathcal{T}$, endowed with the Skorokod topology.

2.1. *The particle model.* Let

$$\Phi(m,m') : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

be a non-negative, symmetric kernel with the property that it vanishes when either of the coordinates equals 0.

Given a point $\{(x^i,m^i)\}$ in $(\mathbb{T}\times\mathbb{R}_+)^N$, define

$$(X^i_{T_0}, M^i_{T_0}) = (x_i, m_i), \quad 1 \le i \le N,$$

and set $T_0 = 0$. Let $k \in \mathbb{N}\cup\{0\}$, and suppose that the process

$$\xi^N_\cdot = \{(X^i_\cdot, M^i_\cdot)\}_{1\le i\le N} \in C\left([0,T_k],\mathbb{T}^N\right) \times D\left([0,T_k],\mathbb{R}^N_+\right)$$

has already been defined up to the time $T_k$, a stopping time with respect to the $\sigma$-algebra $\mathcal{F}_t = \sigma\{\xi^N_s,\, 0\le s\le t\}$ generated by $\xi^N_\cdot$. Consider then a family of $N$ independent Brownian motions $\{\beta^{k,i}_\cdot\}$ on $\mathbb{T}$ with corresponding diffusion coefficients $a(NM^i_{T_k})$ and initial positions $\beta^{k,i}_0 = X^i_{T_k}$. For each pair $i<j$, denote by $L^{k,ij}$ the intersection local time of the $i$–th and $j$–th particles; that is, the local time at the origin of the difference $\beta^{k,i}-\beta^{k,j}$. Let $\{\epsilon^{k,ij}\}_{1\le i<j\le N}$ be a sequence of $\binom{N}{2}$ independent, parameter 1– exponential random variables, and define the stopping times

$$\tilde T_{k+1} = \min_{i<j}\left\{T^{ij}_{k+1}\right\},\ T^{ij}_{k+1} = \inf\left\{t\ge 0, \frac{\Phi(NM^i_{T_k}, NM^j_{T_k})}{N} L^{k,ij}_t > \epsilon^{k,ij}\right\}.$$

Let then $T_{k+1} = T_k + \tilde T_{k+1}$, and for $1\le i\le N$, set

$$X^i_t = \beta^{k,i}_{t-T_k},\ T_k < t \le T_{k+1},$$
$$M^i_t = M^i_{T_k},\ T_k < t < T_{k+1},$$

$$M^i_{T_{k+1}} = \begin{cases} M^i_{T_k} + M^j_{T_k} & \text{if } \tilde T_{k+1} = T^{ij}_{k+1}, \text{ for some } j > i; \\ 0 & \text{if } \tilde T_{k+1} = T^{ij}_{k+1}, \text{ for some } j < i; \\ M^i_{T_k} & \text{otherwise.} \end{cases}$$

We will denote by $L^{ij}$ the intersection local time of the $i$-th and $j$-th particles $X^i$ and $X^j$. Note that between two consecutive stopping times $T_k$ and $T_{k+1}$ the identity $L^{ij} = L^{k,ij}$ holds.

The dynamics is well defined except for those configurations where two or more coagulation events occur simultaneously. Let us briefly show that the set of such configurations has measure zero and can therefore be neglected. We first show that this is the case on the time interval $[0, T_1]$, when $X^i = \beta^{1,i}$, $1 \le i \le N$. The $i$–th and $j$–th masses will coagulate at a time belonging to the support of the measure $dL^{ij}$, which equals the zero set of $\beta^{1,i} - \beta^{1,j}$. Fix two pairs of indices $i < j$ and $k < l$. If the four indices are different, then $\beta^{1,i} - \beta^{1,j}$ and $\beta^{1,k} - \beta^{1,l}$ perform independent diffusions, and from the fact that point sets are polar for Brownian motion in 2 or higher dimensions it follows that these diffusions do no vanish at the same time. Let there be a repeated index: $j = k$, say. Set $\alpha = \frac{a(NM^j)+a(NM^l)}{a(NM^j)}$. Then

$$U_t = \alpha\,[\beta^{1,i}_t - \beta^{1,j}_t] + \beta^{1,j}_t - \beta^{1,l}_t \qquad \text{and} \qquad V_t = \beta^{1,j}_t - \beta^{1,l}_t$$

are independent diffusions in $\mathbb{T}^2$. The previous argument implies that with probability 1 they never vanish simultaneously, and hence the same applies to $\beta^{1,i} - \beta^{1,j}$ and $\beta^{1,j} - \beta^{1,l}$. We conclude that in any case the zero sets of $\beta^{1,i} - \beta^{1,j}$ and $\beta^{1,k} - \beta^{1,l}$ are disjoint, with probability 1. By repeating the argument on each interval $[T_k, T_{k+1}]$, $k \ge 1$, it follows that outside a set of measure zero there are no conflicting coagulation events.

2.2. *Martingales.* Let $P^N$ be the measure on $C(\mathbb{R}_+, \mathbb{T}^N) \times D(\mathbb{R}_+, \mathbb{R}^N_+)$ determined by the process $\xi^N_\cdot$. There is the representation

$$M^i_t = m^i + \int_0^t \sum_{i<j\le N} M^j\, dE^{ij} - \int_0^t \sum_{1\le k<i} M^i\, dE^{ki}$$

where $dE^{ij}$ is a counting measure, $E^{ij}([0,t]) = 0, 1$ with $P^N$–probability 1, depending on whether the $i$-th and $j$-th particles have coagulated by time $t > 0$, and

$$\mathcal{H}^{ij}_t = E^{ij}([0,t]) - \int_0^t \frac{\Phi(NM^i_s, NM^j_s)}{N}\, dL^{ij}$$

is a martingale.

In general, if $f$ is a bounded function on $(\mathbb{T} \times \mathbb{R}_+)^N$ with two continuous, bounded derivatives in the space coordinates, then

$$f(\xi_t) - \int_0^t \sum_{i=1}^N a(NM_s^i)\,\frac{\partial^2 f}{\partial {X^i}^2}(\xi_s)\,ds \\ - \int_0^t \sum_{i<j} \left[f(\xi_s^{ij}) - f(\xi_s)\right] \frac{\Phi(NM_s^i, NM_s^j)}{N}\,dL^{ij}$$

is an $(\mathcal{F}_t, P)$–martingale. Given $\xi = \{(X^i, M^i)\}$, $\xi^{ij}$ here is given by

$$(\xi^{ij})^k = \begin{cases} (X^i, M^i + M^j) & \text{if } k = i; \\ (X^j, 0) & \text{if } k = j; \\ (X^k, M^k) & \text{otherwise.} \end{cases}$$

2.3. *Scaling.* The reason for the choice of scaling of the coagulation rate is quite straighforward: if a hydrodynamic description is to hold, then it is necessary that $O(N)$ mass charged particles remain in the system at all times (note that although total mass is conserved, the number of particles carrying positive mass decreases by one after each coagulation event). Therefore a generic particle will see some fraction of the other $O(N)$ mass charged particles over any fixed time interval, while it would still be expected to coagulate with only $O(1)$ of them. This forces the rate $\Phi$ to be typically of order $1/N$.

2.4. *Assumptions.* We will consider the mapping

$$\Pi_N : C(\mathbb{R}_+, \mathbb{T}^N) \times D(\mathbb{R}_+, \mathbb{R}_+^N) \to D(\mathbb{R}_+, \mathcal{M}_1(\mathbb{T} \times \mathbb{R}_+))$$

such that

$$\Pi_N(\{X_\cdot^i, M_\cdot^i\}) = \sum_{1 \le i \le N} M_\cdot^i\ \delta_{(X_\cdot^i, NM_\cdot^i)},$$

and denote by $Q^N$ the measure on $D(\mathbb{R}_+, \mathcal{M}_1(\mathbb{T} \times \mathbb{R}_+))$ induced by $\Pi_N$,

$$Q^N = P^N \circ \Pi_N^{-1}.$$

In order to derive a hydrodynamic limit for $Q^N$, we need to specify some technical conditions on the coalescing kernel $\Phi$, the diffusion coefficients $a$, the initial measure $P_0^N$ and the profile $\nu$.

The kernel $\Phi$ satisfies a Lipschitz condition away from the origin: for each $L > 0$ there exists a positive constant $\Gamma(L)$ such that

$$\big|\Phi(m + m'', m') - \Phi(m, m')\big| \le \Gamma(L)\, m'' \quad \text{whenever} \quad m > L\,. \tag{2.1a}$$

It will also be assumed that there exists $0 \le \mathfrak{p} \le 1/2$ such that

$$\Phi(m,m') \le c\,(m^{\mathfrak{p}} + m'^{\mathfrak{p}})\ 1_{[m>0,\,m'>0]} \tag{2.1b}$$

for some positive constant $c$.

We assume that the mapping $a : (0,\infty) \to (0,\infty)$ is non–increasing, so that particles diffuse at a slower rate as they gain mass. As a consequence the kernel $\Phi$ and the diffusion coefficients do not grow simultaneously. We set $a(0) = 0$.

We are ready to introduce the coagulation rates $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ appearing in the hydrodynamic equation,

$$\kappa(m,m') = \Phi(m,m')\,\big[a(m) + a(m')\big].$$

In order to study convergence and derive the uniqueness of the limit it will be useful to consider

$$\omega(m) = [1 + c + a(1)]\,\big[m^{\mathfrak{p}} + a(m) + 1\big],$$

it verifies $\kappa(m,m') \le \omega(m)\,\omega(m')$. We will then require that

$$a(m)^{-1/2}\omega(m) \ \text{ be a subadditive function of } m. \tag{2.2}$$

Conditions (2.1a,b) and (2.2) are for instance satisfied by

$$\Phi(m,\tilde m) = C(m^\alpha + \tilde m^\alpha) \quad\text{and}\quad a(m) = \frac{1}{m^\beta} 1_{[m>0]} \ \text{ with } \alpha \le \frac{1}{2} \text{ and } \beta \le 1.$$

In this case $\kappa(m,\tilde m) = C(m^\alpha + \tilde m^\alpha)(\frac{1}{m^\beta} + \frac{1}{\tilde m^\beta})$.

The initial measures $P_0^N$ will be assumed to satisfy

$$E^{P_0^N}\big[N \textstyle\sum (m^i)^2\big] < C \ \text{ and } \ E^{P_0^N}\Big[\dfrac{1}{N}\textstyle\sum a(Nm^i)^2\Big] < C' \tag{2.3}$$

for some constants $C, C' > 0$, uniformly in $N$. In particular both $\langle m,\nu\rangle$ and $\langle a(m)^2/m, \nu\rangle$ are finite. In fact, the following assumption will hold: there exists a finite measure $\nu^*(dm)$ such that

$$\nu(dx,dm) \le \nu^*(dm)\,dx \ \text{ with } \ \Big\langle m + \frac{a(m)^2}{m}, \nu^*\Big\rangle < \infty. \tag{2.4}$$

2.5. *Results.* The first theorem of the paper is a tightness result:

THEOREM 1. *Assume that (2.1a,b), (2.2) and (2.3) hold. Then the sequence of measures $Q^N$ on $D\big(\mathbb{R}_+, \mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+)\big)$ is relatively compact, and all limit points are concentrated on continuous paths.*

The next two results concern the properties satisfied by any weak limit of the empirical distributions as we pass to the limit in the particle number. The first result provides some estimates that will ensure the hydrodynamic equation is well defined, then Theorem 2 identifies this equation, thereby establishing an existence result.

Given a kernel admitting a representation $\mu(x,dm)\,dx \in \mathcal{M}_f(\mathbb{T}\times\mathbb{R}_+)$ and a bounded test function $f(m)$, we will denote by $\ll f,\mu \gg$ the single integral $\int_{\mathbb{R}_+} f(m)\,\mu(x,dm)$. This clearly determines a signed measure on $\mathbb{T}$ by

$$\int_{\mathbb{T}} h(x) \ll f,\mu \gg dx = \int_{\mathbb{T}\times\mathbb{R}_+} h(x)\,f(m)\,\mu(x,dm)\,dx.$$

PROPOSITION 1. *Let $Q$ be a weak limit of the sequence $Q^N$ of probability measures on $C\big(\mathbb{R}_+, \mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+)\big)$. Then $Q$ is supported on the set of paths $\mu.(dx,dm)$ whose marginal $\mu_t(dx,\mathbb{R}_+) \ll dx$ on $\mathbb{T}$ for all $t$, $\mu_t(dx,dm) = \upsilon_t(x,dm)\,dx$. Moreover, the following inequalities hold with $Q$–probability 1:*

$$\sup_{t\ge 0} \Big\| \ll \frac{\omega(m)}{m}, \upsilon_t \gg \Big\|_\infty < \infty \tag{2.5}$$

*and*

$$\sup_{t\le T} \langle m, \mu_t\rangle < \infty \tag{2.6}$$

*for any fixed final time $T$.*

Denote by $C_b^2(\mathbb{T}\times\mathbb{R}_+)$ the space of continuous, bounded functions on $\mathbb{T}\times\mathbb{R}_+$ which have continuous, bounded derivatives in the space variable up to the second order.

THEOREM 2. *Let $Q$ be a weak limit of the sequence $Q^N$, as in Proposition 1, and consider $f$ in $C_b^2(\mathbb{T}\times\mathbb{R}_+)$. Then, with $Q$–probability 1, a path*

*$\mu_s(dx, dm)$ satisfies*

$$\langle f, \mu_t \rangle - \langle f, \nu \rangle = \int_0^t \langle \, \frac{1}{2} \, a(m) \frac{\partial^2 f}{\partial x^2}, \mu_s \, \rangle \, ds$$

(2.7)

$$+ \int_0^t \int_{\mathbb{T}} \int_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{\big[ f(x, m+m') - f(x,m) \big]}{m'} \; \kappa(m, m') \\ \times \, \upsilon_s(x, dm) \, \upsilon_s(x, dm') \, dx \, ds$$

*if $t \geq 0$. In this equation $\nu$ is the initial profile of the model, and the kernel $\upsilon_s(x, dm)$ is such that*

$$\mu_s(dx, dm) = \upsilon_s(x, dm) \, dx \qquad \text{for all } s \geq 0, \ Q\text{–a.e.} \, .$$

Equation (2.7) describes the evolution in time of the mass flow: if we decompose its solutions as $\upsilon_s(x, dm) \, dx = m \, \hat{\upsilon}_s(x, dm) \, dx$, an elementary computation proves that $\hat{\upsilon} \, dx$ satisfies Smoluchowski's coagulation equation with kernel $\kappa$. Theorem 1 asserts that all weak limits of the measures $Q^N$ are supported on $\mathcal{M}_1(\mathbb{T} \times \mathbb{R}_+)$; in terms of the concentration densities $\hat{\upsilon}_s(x, dm)$, this means that mass is conserved, or equivalently, that there is no gelation phenomenon.

The method applied to derive these results relies heavily on stochastic calculus computations, we try to make these quite detailed in the proof of Theorem 1 and give only an outline later on.

In [16], Norris introduces a method for proving existence and uniqueness for a general class of $d$-dimensional diffusion models with coagulation; briefly put, this consists on approximating the corresponding version of equation (2.7) in his paper by a system that depends on the coalescing kernel $\kappa$ only through its values on a given compact set. In Section 5 we develop a simplified version of his technique to obtain a uniqueness result for the solutions of a broad family of mass flow equations. Section 5 may be read independently of the rest of the paper.

Some brief consideration shows that the right side of equation (2.7) is well defined in a proper subset of $C(\mathbb{R}_+, \mathcal{M}_1(\mathbb{T} \times \mathbb{R}_+))$ consisting of those paths $\eta$ whose marginal $\eta_t(dx, \mathbb{R}_+)$ has a density with respect to Lebesgue measure satisfying some integrability conditions. It is easy to see that in fact (2.5) and (2.6) are enough, and then Proposition 1 says that all weak limits of the sequence $Q^N$ are supported on configurations where the right side of (2.7) can be evaluated. This observation motivates the following definition: we will denote by $\mathcal{D}(\omega)$ the subset of $C(\mathbb{R}_+, \mathcal{M}_f(\mathbb{T} \times \mathbb{R}_+))$ of those paths $\eta$

whose marginal $\eta_t(dx, \mathbb{R}_+) \ll dx$ , and such that (2.5) and (2.6) are satisfied. Note that the map $\omega$ depends on the diffusivity $a$ and the coagulation rate $\Phi$, and then so does $\mathcal{D}(\omega)$.

As a particular case of Theorem 3 in Section 5, we have

COROLLARY 1. *Assume conditions (2.1b), (2.2) and (2.4) on the coagulation rate $\kappa$ and the initial measure $\nu$, respectively. Then for any $T \geq 0$, (2.7) has at most one solution $\{\mu_t\}_{0 \leq t \leq T}$ in $\mathcal{D}(\omega)$.*

The four preceding results imply that the sequence of probability measures $Q^N$ has as unique weak limit point the Dirac measure concentrated on the unique solution in $\mathcal{D}(\omega)$ to equation (2.7).

**3. Existence of a weak limit.** In order to simplify notation, we will often omit the dependence of the masses and positions on the time parameters whenever we think that this would not lead to confusion. For instance, in an integral where time is parametrized by $s$, $M^i$ and $X^i$ should be read as $M^i_s$ and $X^i_s$, respectively.

Throughout the article, $\Gamma$ will denote a positive constant. Unless we are particularly interested in keeping track of its growth or dependence on the parameters, we will use the same letter $\Gamma$ to denote constants on consecutive lines which may be different, or constants appearing in totally unrelated computations.

Let us consider a fixed final time $T > 0$ for the rest of the paper. We will prove a version of Theorems 1, 2 and Proposition 1 on the compact interval $[0, T]$; the fact that the value of $T$ is arbitrary will then imply that these results hold as stated in the previous section.

The following estimates will be necessary to derive Theorem 1; we postpone their proofs until the end of this section.

LEMMA 1. *There exist nonnegative constants $C(T)$, $C'(T)$ which depend on the diffusivity $a$, the kernel $\phi$ and the bounds appearing in (2.3), such that*

$$E^{P^N}\Big[N \sum_i [M^i_T]^2\Big] < C(T) \qquad \text{and} \tag{3.1}$$

$$E^{P^N}\Big[\int_0^T \sum_{i<j} M^i M^j \Phi(NM^i, NM^j)\, dL^{ij}\Big] < C'(T) \tag{3.2}$$

*hold uniformly in $N$.*

LEMMA 2. *Given $\epsilon > 0$, there is $\delta > 0$ such that*

$$\lim_{N\to\infty} P^N \Big[ \sup_{\substack{0\le s\le t\le T \\ t-s\le\delta}} \int_s^t \sum_{i<j} M^i M^j \Phi(NM^i, NM^j)\, dL^{ij} > \epsilon \Big] < \epsilon.$$

Denote by $C_b(\mathbb{T}\times\mathbb{R}_+)$ the space of bounded, continuous functions on $\mathbb{T}\times\mathbb{R}_+$ with the topology determined by uniform convergence over compact sets. Let $\{f_k, k\in\mathbb{N}\}$ be a dense, countable family in $C_b(\mathbb{T}\times\mathbb{R}_+)$. Then the distance

$$\varrho(\mu,\nu) = \sum_{k\in\mathbb{N}} \frac{1}{2^k} \frac{|\langle f_k,\mu\rangle - \langle f_k,\nu\rangle|}{1+|\langle f_k,\mu\rangle - \langle f_k,\nu\rangle|}$$

defines a metric on $\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+)$ which is compatible with the weak topology. There is the associated modulus of continuity

$$\omega_\mu(\gamma) = \sup_{\substack{0\le s\le t\le T \\ t-s\le\gamma}} \varrho(\mu_t,\mu_s).$$

*Proof of Theorem 1.* We refer to Chapter 4 in [11] for a presentation of the Skorokod's topology as well as the characterization of the relatively compact sets in $D([0,T],\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+))$. Note that condition $(ii)$ below implies that, provided the sequence $Q^N$ has limit points, these will be supported on $C([0,T],\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+))$.

By a version of Prokhorov's Theorem applied to this setting (cf. [2], Chapter 3), the theorem will follow if we can show that

$$(i)\ \text{ For every } \varepsilon>0,\ \lim_{M\uparrow\infty} \limsup_{N\to\infty} Q^N \Big[ \sup_{0\le t\le T} \mu_t(m>M) > \varepsilon \Big] = 0$$

and

$$(ii)\ \text{ For every } \varepsilon>0,\ \lim_{\gamma\downarrow 0} \limsup_{N\to\infty} Q^N \big[ \omega_\mu(\gamma) > \varepsilon \big] = 0.$$

Note that $\langle m,\mu_t\rangle$ is non-decreasing for $0\le t\le T$, $Q^N$–a.e., a fact we will repeatedly use in the course of the article. Then $(i)$ is an easy consequence of (3.1) in Lemma 1 and Tchebyshev's inequality.

In order to conclude $(ii)$ it will be enough to prove that given $f\in C_b^2(\mathbb{T}\times\mathbb{R}_+)$, $f$ Lipschitz in $m$, we can control

$$Q^N \Big[ \sup_{\substack{0\le s\le t\le T \\ t-s\le\gamma}} |\langle f,\mu_t\rangle - \langle f,\mu_s\rangle| > \varepsilon \Big] < \varepsilon$$

provided $N$ and $\gamma$ are taken to be sufficiently large and small, respectively.

Define the stopping time

$$\tau = \inf\Big\{t \geq 0,\, \max_i(M_t^i)\,[1 + \sum_i m^i a(Nm^i)] > N^{-1/4}\Big\};$$

by Tchebyshev's inequality we compute

$$P^N[\tau \leq T] < \frac{[C(T)(1+C')]^{1/2}}{N^{1/4}}$$

where $C(T)$ and $C'$ are the constants appearing on the right of (3.1) and the second inequality in (2.3) respectively. By stopping the process as soon as $\tau$ is achieved, we may assume that

$$P^N\Big[\Big\{\max_i(M_t^i)\,[1 + \sum_i m^i a(Nm^i)] \leq N^{-1/4}\Big\}\Big] = 1. \tag{3.3}$$

Applying Itô's formula to $f$, we have

$$\begin{aligned}
&\langle f, \mu_t\rangle - \langle f, \mu_s\rangle \\
&= \int_s^t \sum_i M^i \frac{\partial f}{\partial x}(X^i, NM^i)\, dX^i + \frac{1}{2}\int_s^t \sum_i M^i \frac{\partial^2 f}{\partial x^2}(X^i, NM^i)\, a(NM^i)\, ds \\
&\qquad + \int_s^t \sum_i [F_N(X^i, M^i + M^j) - F_N(X^i, M^i) - F_N(X^i, M^j)]\, dE^{ij}
\end{aligned}$$

where we denote $F_N(x, m) = mf(x, Nm)$.

Let $\gamma > 0$. Doob's inequality, (3.3) and the monotonicity of $a$ imply

$$\begin{aligned}
&P^N\Big[\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\gamma}} \Big|\int_s^t \sum_i M^i \frac{\partial f}{\partial x}(X^i, NM^i)\, dX^i\Big| > \frac{\varepsilon}{3}\Big] \\
&\qquad \leq P^N\Big[\sup_{0\leq t\leq T} \Big|\int_0^t \sum_i M^i \frac{\partial f}{\partial x}(X^i, NM^i)\, dX^i\Big| > \frac{\varepsilon}{6}\Big] \leq \frac{\Gamma(f, \varepsilon, T)}{N^{1/4}}
\end{aligned}$$

where $\Gamma$ is a positive constant that does not depend on $N$. By taking $\gamma$ such that

$$C'\Big\|\frac{\partial^2 f}{\partial x^2}\Big\|_\infty \gamma \leq \frac{\varepsilon^2}{3} \tag{3.4}$$

we also obtain

$$P^N\Big[\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\gamma}} \Big|\int_s^t \sum_i M^i \frac{\partial^2 f}{\partial x^2}(X^i, NM^i)\, a(NM^i)\, ds\Big| > \frac{\varepsilon}{3}\Big] \leq \frac{\varepsilon}{3}.$$

It remains to estimate the Poisson integral

$$\int_s^t \sum_i \left[F_N(X^i, M^i + M^j) - F_N(X^i, M^i) - F_N(X^i, M^j)\right] dE^{ij}$$
$$= \mathcal{H}_F(0,t) - \mathcal{H}_F(0,s) + I_F(s,t)$$

if $I_F(s,t)$ denotes the integral

$$\int_s^t \sum_{i<j} \left[F_N(X^i, M^i + M^j) - F_N(X^i, M^i) - F_N(X^i, M^j)\right] \frac{\Phi(NM^i, NM^j)}{N} dL^{ij}$$

and $\mathcal{H}_F(0,t)$ is a martingale collecting the remaining terms. Its quadratic variation is given by

$$\int_0^t \sum_{i<j} \left[F_N(X^i, M^i + M^j) - F_N(X^i, M^i) - F_N(X^i, M^j)\right]^2 \frac{\Phi(NM^i, NM^j)}{N} dL^{ij}.$$

Note that

$$|F_N(x, m + m') - F_N(x, m) - F_N(x, m')| \leq \Gamma(f)\left[(m + m') \wedge (Nmm')\right].$$

In particular, due to the assumption (3.3) on the mass sizes, (3.2) and Doob's inequality, we obtain

$$P^N\Big[\sup_{\substack{0 \leq s \leq t \leq T \\ t-s \leq \gamma}} |\mathcal{H}_F(s,t)| > \frac{\varepsilon}{6}\Big] \leq \frac{\Gamma(f, \varepsilon, T)}{N^{1/4}}$$

which will decay to 0 as we pass to the limit $N \to \infty$. Finally,

$$|I_F(s,t)| \leq \Gamma(f) \int_s^t \sum_{i<j} M^i M^j \Phi(NM^i, NM^j)\, dL^{ij}.$$

The result now follows by taking

$$\gamma \leq \gamma_1 \wedge \delta$$

where $\gamma_1$ satisfies (3.4) and $\delta$ is the value given by Lemma 2 when $\epsilon$ is set equal to $\varepsilon/6[1 + \Gamma(f)]$. □

*Proof of Lemma 1.* The proof of this lemma will follow from repeated applications of Itô-Tanaka's theorem, see [6] for an exposition of this and related formulas. We write

(3.5)
$$N \sum_i [M_t^i]^2 \;=\; N \sum_i [m^i]^2 \;+\; \mathcal{H}_t \;+ 2 \int_0^t \sum_{i<j} M_s^i M_s^j\, \Phi(NM_s^i, NM_s^j)\, dL^{ij}$$

where $\mathcal{H}_t$ is the $P^N$-martingale

$$\mathcal{H}_t = \int_0^t \sum_{i<j} 2NM_s^i M_s^j \Big[dE^{ij} - \frac{\Phi(NM_s^i, NM_s^j)}{N}\, dL^{ij}\Big].$$

We focus on the last term of (3.5). Given $\zeta > 0$, let $g_\zeta \in C(\mathbb{T}) \cap C^2(\mathbb{T} - \{0\})$ be a positive, even function that equals $|x|$ in a small interval containing the origin, vanishes outside $[-1/4, 1/4]$ and satisfies $\sup_{x\in\mathbb{T}} g_\zeta(x) \le \zeta$. We then have

$$\begin{aligned}
&\int_0^t \sum_{1\le i<j\le N} M^i M^j\, \Phi(NM^i, NM^j)\, dL^{ij} \\
&\qquad = A_1(0,t) - A_2(0,t) - A_3(0,t) - A_4(0,t) - A_5(0,t)
\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}
A_1(0,t) = &\sum_{1\le i<j\le N} M_t^i M_t^j\, \Phi(NM_t^i, NM_t^j)\, g_\zeta(X_t^i - X_t^j) \\
&\qquad - \sum_{1\le i<j\le N} m^i m^j\, \Phi(Nm^i, Nm^j)\, g_\zeta(X^i - X^j), \\
A_2(0,t) = &\int_0^t \sum_{1\le i<j\le N} M^i M^j \Phi(NM^i, NM^j) g'_\zeta(X^i - X^j)\, \big[dX^i - dX^j\big]
\end{aligned}$$

and

$$\begin{aligned}
A_3(0,t) = \frac{1}{2}\int_0^t &\sum_{i<j} M^i M^j \Phi(NM^i, NM^j)\, g''_\zeta(X^i - X^j) \\
&\qquad \times \big[a(NM^i) + a(NM^j)\big]\, du\,.
\end{aligned}$$

The function $g''_\zeta$ appearing in the formula for $A_3(0,t)$ stands for what is left of the second derivative of $g_\zeta$ (in the sense of distributions) after substracting $2\delta_0$, $\delta_0$ the Dirac measure at the origin. The terms $A_4(0,t)$ corresponds to the coagulation martingale and its compensator,

$$\begin{aligned}
A_4(0,t) = \int_0^t &\sum_{\substack{i<k\\ j}} D_N(M^i, M^j, M^k)\, g_\zeta(X^i - X^j) \\
&\qquad \times \Big[dE^{ik} - \frac{\Phi(NM^i, NM^k)}{N}\, dL^{ik}\Big] \\
A_5(0,t) = \int_0^t &\sum_{\substack{i<k\\ j}} D_N(M^i, M^j, M^k)\, g_\zeta(X^i - X^j)\, \frac{\Phi(NM^i, NM^k)}{N}\, dL^{ik}
\end{aligned}$$

where $D_N(m, m', m'')$ is defined as

$$(m + m'')\, m'\Phi(Nm + Nm'', Nm') \\ - m\, m'\Phi(Nm, Nm') - m''m'\Phi(Nm'', Nm') .$$

In deriving the formula for $A_4$, we have used that at the time when the masses $M^i$ and $M^k$ coagulate, the $i$-th and $k$-th particles are occupying the same position.

We will study these terms separately. Replacing $\Phi(m, m') \le c[m^{\mathfrak{p}} + m'^{\mathfrak{p}}]$ in the definition of $A_1$ gives

$$|A_1(0, t)| \le 4\, c\, \zeta \sum_i M^i_t [N M^i_t]^{\mathfrak{p}}. \tag{3.7}$$

The bounded variation term $A_3(0, t)$ may be similarly controlled,

$$|A_3(0, t)| \le \Gamma(\zeta) \int_0^t \Big[1 + \sum_i M^i_s\, [N M^i_s]^{\mathfrak{p}}\Big] \Big[1 + \sum_i m^i a(N m^i)\Big]\, ds\, . \tag{3.8}$$

In order to bound $A_5$, we first notice that by the Lipschitz assumption (2.1a) on $\Phi$, we have

$$\begin{aligned} &|\Phi(Nm + Nm'', Nm') - \Phi(Nm, Nm')| \\ &\qquad \le \Gamma(1)\, Nm''\, 1_{\{Nm \ge 1\}} + \Big(1 + [Nm']^{\mathfrak{p}} + [Nm'']^{\mathfrak{p}}\Big)\, 1_{\{Nm < 1\}}\, . \end{aligned} \tag{3.9}$$

It then follows that

$$\begin{aligned} |A_5(0, t)| \le \Gamma\, \zeta \Bigg[ &\int_0^t \sum_{i<k} M^i M^k\, \Phi(NM^i, NM^k)\, dL^{ik} \\ &+ \Big(1 + \sum_j M^j_t [N M^j_t]^{\mathfrak{p}}\Big) \int_0^t \sum_{i<k} \frac{1}{N^2}\, \Phi(NM^i, NM^k)\, dL^{ik} \Bigg]. \end{aligned} \tag{3.10}$$

We have

$$\begin{aligned} \int_0^t \sum_{i<k} \frac{1}{N^2}\, \Phi(NM^i, NM^k)\, dL^{ik} = &\sum_i \frac{1}{N}\, 1_{\{m^i > 0\}} - \sum_i \frac{1}{N}\, 1_{\{M^i_t > 0\}} \\ &- \int_0^t \sum_{i<k} \frac{1}{N} \big[dE^{ik} - \frac{\Phi(NM^i, NM^k)}{N}\, dL^{ik}\big]\, . \end{aligned}$$

The last term above is a martingale, hence

$$E^{P^N}\left[\int_0^t \sum_{i<k} \frac{1}{N^2}\,\Phi(NM^i, NM^k)\,dL^{ik}\right] \le 1\,,$$

and

$$E^{P^N}\left[\Big(\int_0^t \sum_{i<k} \frac{1}{N^2}\,\Phi(NM^i, NM^k)\,dL^{ik}\Big)^2\right] \\ \le \left[1 + E^{P^N}\Big[\int_0^t \sum_{i<k} \frac{1}{N^3}\,\Phi(NM^i, NM^k)\,dL^{ik}\Big]\right] \le 2\,.$$

We take expectations in (3.6) and combine with (3.7), (3.8) and (3.10) to obtain

$$\begin{aligned}(1-\Gamma\zeta)E\left[\int_0^t \sum_{i<k} M^iM^k\Phi(NM^i,NM^k)dL^{ik}\right] &\le 4c\zeta E^{P^N}\Big[\sum_i M^i[NM^i_t]^{\mathfrak{p}}\Big] \\ &+ \Gamma(\zeta)\int_0^t E^{P^N}\left[\Big(1+\sum_i M^i[NM^i]^{\mathfrak{p}}\Big)\Big(1+\sum_i m^i a(Nm^i)\Big)\right] ds \\ &+ \Gamma\zeta E^{P^N}\left[(1+\sum_i M^i_t[NM^i_t]^{\mathfrak{p}})\int_0^t \sum_{i<k}\frac{1}{N^2}\,\Phi(NM^i,NM^k)\,dL^{ik}\right] \\ &\le 4c\zeta\, E^{P^N}\Big[\sum_i M^i[NM^i_t]^{\mathfrak{p}}\Big] \\ &+ \Gamma(\zeta)\, E^{P^N}\Big[1+\sum_i m^i a(Nm^i)^2\Big]\int_0^t E^{P^N}\Big[1+N\sum_i [M^i]^2\Big]\,ds \\ &+ 2\Gamma\zeta\, E^{P^N}\Big[1+N\sum_i [M^i_t]^2\Big]\,.\end{aligned}$$

In order to derive this last bound we have used Hölder's inequality and the fact that $\mathfrak{p} \le 1/2$. From (2.3), we conclude that

$$(1-\Gamma\zeta)\,E^{P^N}\left[\int_0^t \sum_{i<k} M^iM^k\,\Phi(NM^i,NM^k)\,dL^{ik}\right] \\ \le \Gamma'\,\zeta\,\Big[1+E^{P^N}\big[N\sum_i [M^i_t]^2\big]\Big] + \Gamma(\zeta)\int_0^t E^{P^N}\Big[1+N\sum_i [M^i]^2\Big]\,ds\,. \tag{3.11}$$

Choose $\zeta \leq 1/(4[\Gamma + \Gamma'])$, where $\Gamma$ and $\Gamma'$ are the constants appearing in the first line and in front of the first term on the right above, respectively. Combining (3.11) with (3.5) we get

$$E^{P^N}\Big[N\sum_i [M_t^i]^2\Big] \leq \Gamma\left[E^{P^N}\left[1 + N\sum_i [m^i]^2 + \int_0^t \Big(1 + N\sum_i [M^i]^2\Big) ds\right]\right].$$

Estimate (3.1) now follows from Gronwall's lemma and conditions (2.3) on the initial distribution of masses, and (3.2) is immediate from (3.11). □

*Proof of Lemma 2.* Choose $\zeta > 0$ and $\delta > 0$ such that

$$4\zeta\,[1+\Gamma]\,[1+C+C(T)] < \frac{\epsilon^2}{50} \qquad \text{and} \qquad 4\delta\,\Gamma(\zeta)\,[1+C(T)]\,[1+C'] \leq \frac{\epsilon^2}{50}\,,$$

where $\Gamma$ and $\Gamma(\zeta)$ are the constants appearing on the right of (3.10) and (3.8), respectively, $C$ and $C'$ are the constants from assumption (2.3), and $C(T)$ the bound established in Lemma 1. Set the parameter of $g$ equal to a value of $\zeta$ determined as above.

As in the proof of Theorem 1, we will stop the process at the finite stopping time $\tau \wedge \tau_\zeta \wedge T$, where $\tau$ is the stopping time defined in the proof of Theorem 1, and

$$\tau_\zeta = \inf\left\{t \geq 0,\ N\sum_i [M_t^i]^2 \geq \frac{1}{\zeta}\right\}.$$

By Lemma 1 and the choice of $\zeta$, we have

$$P^N[\tau_\zeta \leq T] \leq C(T)\,\zeta \leq \frac{\epsilon}{2}$$

if $\epsilon$ is small enough. We will thus assume that $P^N$ is supported on

$$\max_i\{M_T^i\}\,[1 + \sum_i m^i a(Nm^i)] \leq \frac{1}{N^{1/4}}\,, \quad N\sum_i [M_T^i]^2 \leq \frac{1}{\zeta}\,. \tag{3.12}$$

The proof will now follow by estimating the variation of the terms $\{A_i\}_{1\leq i\leq 5}$ on the right of (3.6). Let $A_i(s,t) = A_i(0,t) - A_i(0,s)$, $1 \leq i \leq 5$.

By Tchebyshev's inequality, (3.7), (3.8), and the choice of $\zeta$, $\delta$, we have

$$P^N\Big[\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} |A_1(s,t)| > \frac{\epsilon}{5}\Big] \leq \frac{\epsilon}{10}\,, \tag{3.13}$$

$$P^N\Big[\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} |A_3(s,t)| > \frac{\epsilon}{5}\Big] \leq \frac{\epsilon}{10}\,, \tag{3.14}$$

and

$$P^N\Big[\sup_{\substack{0\le s\le t\le T\\ t-s\le\delta}} |A_5(s,t)| > \frac{\epsilon}{5}\Big] \;\le\; \frac{\epsilon}{10}\,. \tag{3.15}$$

The quadratic variations $Q_2(0,t)$ and $Q_4(0,t)$ of the martingale terms $A_2$ and $A_4$ satisfy

$$\begin{aligned}
Q_2(0,t) &\le \Gamma(\zeta,T)\,\max_i\{M_T^i\}\Bigg(\Big[\sum_i m^i a(Nm^i)\Big]\Big[1+N\sum_i [M_T^i]^2\Big] \\
&\qquad\qquad + \Big[1+N\sum_i [M_T^i]^2+\sum_i m^i a(Nm^i)\Big]\Bigg) \\
&\le \frac{\Gamma(\zeta,T)}{N^{1/4}}\Big[1+\sum_i m^i a(Nm^i)+N\sum_i [M_T^i]^2\Big]\,, \\
Q_4(0,t) &\le \Gamma\,\zeta^2\Big[1+N\sum_i [M_t^i]^2\Big]\int_0^t \sum_{i<k}\,[M^iM^k+\frac{1}{N^2}]\,\Phi(NM^i,NM^k)\,dL^{ik} \\
&\le 4\,\Gamma\,\zeta\int_0^t\sum_{i<k}\,[M^iM^k+\frac{1}{N^2}]\,\Phi(NM^i,NM^k)\,dL^{ik}
\end{aligned}$$

respectively. In order to derive these inequalities we have used the assumptions on the mass sizes, (3.12), and the fact that the diffusion coefficients are decreasing, so that they can be controlled when the masses $M_t^i$ are large.

Then, by Doob's inequality,

$$\begin{aligned}
&P^N\Big[\sup_{\substack{0\le s\le t\le T\\ t-s\le\delta}} |A_2(s,t)| > \frac{\epsilon}{5}\Big] \\
&\qquad \le\; P^N\Big[2\sup_{0\le t\le T}|A_2(0,t)|>\frac{\epsilon}{5}\Big] \;\le\; \frac{\Gamma(\epsilon,\zeta,T)}{N^{1/4}}\,[1+C'+C(T)]\,,
\end{aligned} \tag{3.16}$$

and similarly,

$$P^N\Big[\sup_{\substack{0\le s\le t\le T\\ t-s\le\delta}} |A_4(s,t)| > \frac{\epsilon}{5}\Big] \le 4\,\Gamma\,\zeta\,[1+C+C(T)]\,. \tag{3.17}$$

We pass to the limit $N\to\infty$ in (3.16), and conclude the proof from the estimates obtained in (3.13), (3.14), (3.15), (3.17) and the choice of $\zeta$. □

**4. The hydrodynamic equation.** We begin with Proposition 1.

*Proof of Proposition 1.* Estimate (3.1) in Lemma 1 implies that

$$E^Q\Big[\sup_{t\le T}\langle m\wedge M,\mu_t\rangle\Big]\le C(T)$$

uniformly in $M>0$. Then (2.6) follows by letting $M\to\infty$ and monotone convergence.

In order to obtain (2.5), we will show that the probability measure $Q$ satisfies

$$E^Q\Big[\sup_{\substack{\Psi(x,s)\in L^1[\mathbb{T}\times[0,T]]\\ \|\Psi\|_1\le 1}}\int_0^T\Big\langle\,\frac{\omega(m)}{m}\,\Psi,\mu_s\,\Big\rangle\,ds\Big]<\infty. \tag{4.1}$$

Indeed, (4.1) implies that with $Q$–probability 1, $\mu_t(dx,dm)=\upsilon_t(dx,dm)\,dx$ for almost every $t\in[0,T]$, where $\upsilon_t$ satisfies estimate (2.5) in the statement of the proposition. But $Q$ is supported on $C\big([0,T],\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+)\big)$, hence the result.

We must therefore prove that

$$E^Q\Big[\sup_{k\in\mathbb{N}}\int_0^T\Big\langle\,\frac{\omega(m)}{m}\,\Psi^k,\mu_s\,\Big\rangle\,ds\Big]$$
$$=\lim_{K\to\infty}\lim_{M\to\infty}\lim_{N\to\infty}E^{Q^N}\Big[\sup_{1\le k\le K}\int_0^T\Big\langle\,\frac{\omega(m)}{m}\,\Psi^{k,M},\mu_s\,\Big\rangle\,ds\Big]<\infty$$

where we denote

$$\Psi^{k,M}(s,x)=\Psi^k(s,x)\wedge M,$$

$\{\Psi^k\}_{k\in\mathbb{N}}$ a dense family in $C\big([0,T],\mathbb{T}\big)\cap B_1\big[L^1([0,T]\times\mathbb{T})\big]$ in the supremum norm, $B_1\big[L^1([0,T]\times\mathbb{T})\big]$ the unit ball in $L^1\big[[0,T]\times\mathbb{T}\big]$. The measures $\mu_s$ are non-negative $Q$ a.e., so we may take $\Psi^k\ge 0$, $k\in\mathbb{N}$.

For each $\Psi^{k,M}$ and $m>0$, let then $u^{k,M}(s,x,m)$ be the solution to

$$\begin{cases} u^{k,M}_s+\frac{a(m)}{2}u^{k,M}_{xx}=-\Psi^{k,M}\\ u^{k,M}(T,\cdot)=0\,. \end{cases}$$

We have the representation formula

$$u^{k,M}(s,x,m)=\int_s^T\int_{\mathbb{T}}p(a(m)(u-s),x,z)\,\Psi^{k,M}(u,z)\,dz\,du \tag{4.2}$$

where $p(t,x,z)$ is the Brownian transition density on $\mathbb{T}$.

Itô's formula applied to $\frac{\omega(m)}{m} u^{k,M}$ yields

$$
\begin{aligned}
\int_0^T \sum_i M^i \, \frac{\omega(NM^i)}{NM^i} \, \Psi^{k,M}(X^i, NM^i) \, ds &= \sum_i \frac{\omega(Nm^i)}{N} \, u^{k,M}(x^i, Nm^i) \\
&+ \int_0^T \sum_i \frac{\omega(NM^i)}{N} \, \frac{\partial u^{k,M}}{\partial x} \, dX^i + \int_0^T \sum_{i<j} D_N(X^i, NM^i, NM^j)(u^{k,M}) \, dE^{ij}
\end{aligned}
\tag{4.3}
$$

if $D_N$ now denotes

$$
\begin{aligned}
&D_N(x, m, m')(f) \\
&\qquad = \frac{1}{N} \big[ \, \omega(m + m') f(x, m + m') - \omega(m) f(x, m) - \omega(m') f(x, m') \big].
\end{aligned}
$$

The last term in the expansion (4.3) is non-positive by formula (4.2), the assumption that $\Psi^k \geq 0$, and the subadditivity in $m$ of $a(m)^{-1/2}\omega(m)$. We thus get

$$
\begin{aligned}
&E^{Q^N} \Big[ \sup_{1 \leq k \leq K} \int_0^T \big\langle \, \frac{\omega(m)}{m} \, \Psi^{k,M}, \mu_s \big\rangle \, ds \Big] \\
&\qquad \leq E^{P^N} \Big[ \sup_{1 \leq k \leq K} \sum_i \frac{\omega(Nm^i)}{N} \, u^{k,M}(x^i, Nm^i) \Big] \\
&\qquad\quad + E^{P^N} \Big[ \sup_{1 \leq k \leq K} \Big| \int_0^T \sum_i \frac{\omega(NM^i)}{N} \, \frac{\partial u^{k,M}}{\partial x} \, dX^i \, \Big| \Big].
\end{aligned}
\tag{4.4}
$$

The second term on the right side above can be easily bounded by replacing the supremum by a sum over $1 \leq k \leq K$ and computing the quadratic variation of each of the resulting orthogonal martingale terms. The sum of these quadratic variations vanishes in the limit $N \to \infty$; in order to see this, it suffices to replace $u^{k,M}$ by its representation (4.2) and then apply assumptions (2.1b), (2.3) and estimate (3.1).

Finally, the hypothesis on $P_0^N$, $\nu$, $\nu^*$ and the fact that $\|\Psi^{k,M}\|_1 \leq \|\Psi^k\|_1 \leq 1$ imply that

$$
\begin{aligned}
&\lim_{N \to \infty} E^{P^N} \Big[ \sup_{1 \leq k \leq K} \sum_i \frac{\omega(Nm^i)}{N} \, u^{k,M}(x^i, Nm^i) \Big] \\
&\qquad = \sup_{1 \leq k \leq K} \big\langle \, \frac{\omega(m)}{m} \, u^{k,M}, \nu \, \big\rangle \leq \sup_{1 \leq k \leq K} \big\langle \, \frac{\omega(m)}{m}, \nu^* \, \big\rangle \, \|\Psi^{k,M}\|_1 = \Gamma(\nu^*) < \infty
\end{aligned}
$$

holds uniformly in $M$, $K$.

We pass to the limit $M \to \infty$ and then $K \to \infty$ in (4.4) to obtain (4.1). □

*Proof of Theorem 2.* It will be enough to consider $f \in C_b^2(\mathbb{T}\times\mathbb{R}_+)$ compactly supported and Lipschitz in $m$, and then use bounded convergence to obtain (2.7) for a general $f \in C_b^2(\mathbb{T}\times\mathbb{R}_+)$. We need to analyse the difference

$$Z_f(t) = \sum_i M_t^i f(X_t^i, NM_t^i) - \sum_i m^i f(x^i, Nm^i)\,.$$

We start by writing the semimartingale $Z_f$ as

$$Z_f(t) = \mathcal{H}_f(t) + A_f(t)$$

where $\mathcal{H}_f$ is the martingale obtained by adding the fluctuation terms arising from the free particle dynamics and the stochastic coagulation phenomena. These can be proved negligible by applying Doob's inequality, the integrability assumptions on $a(m)$ stated in Section 1, and Lemma 1. The term $A_f$ is given by

$$\begin{aligned}A_f(t) = &\int_0^t \sum_i M_s^i \frac{a(NM^i)}{2}\,\frac{\partial^2 f}{\partial x^2}(X_s^i, NM_s^i)\,ds\\
+ &\int_0^t \sum_{i<j}\Big[M^i\Big(f(X^i, N(M^i+M^j)) - f(X^i, NM^i)\Big)\\
&\qquad + M^j\Big(f(X^j, N(M^i+M^j) - f(X^j, NM^j)\Big)\Big]\,\frac{\Phi(NM^i, NM^j)}{N}\,dL^{ij}.\end{aligned}$$

The first term of $A_f$ will clearly have the limit

$$\int_0^t \int_{\mathbb{T}\times\mathbb{R}_+} \langle \frac{a(m)}{2}\,\frac{\partial^2 f}{\partial x^2}, \mu_s\rangle\,ds. \tag{4.5}$$

We can guess the limit of the second term from the Occupation Times Formula; we should recover the second term in the hydrodynamic equation (2.7). In order to obtain this expression, we replace $dL^{ij}$ in the second term of $A_f$ by $[a(m)+a(m')]\,V_\epsilon(X^i - X^j)\,ds$, where $V_\epsilon(x)$ approximates the Dirac $\delta$–function at the origin as $\epsilon \to 0$. The new integral will converge weakly to

$$\begin{aligned}\frac{1}{2}\int_0^t \int_{\mathbb{T}\times\mathbb{T}} \int_{\mathbb{R}_+\times\mathbb{R}_+} \Big[\frac{f(x, m+m') - f(x,m)}{m'} + \frac{f(y, m+m') - f(y,m')}{m}\Big]&\\
\times\,\kappa(m,m')\,V_\epsilon(x-y)\,\mu_s(dx,dm)\,\mu_s(dy,dm')\,ds&\end{aligned}$$

as $N \to \infty$.

We will justify this exchange by showing that there exists a sequence of measurable sets $\mathcal{C}_{N,\epsilon,T}$ with

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} P^N\left[\mathcal{C}_{N,\epsilon,T}\right] = 0,$$

such that $\Upsilon_{N,\epsilon,f}(t)$ given by

$$\int_0^t \sum_{i<j} \Big[M^i\Big(f(X^i, N(M^i+M^j)) - f(X^i, NM^i)\Big) + M^j\Big(f(X^j, N(M^i+M^j) - f(X^j, NM^j)\Big)\Big]\frac{\Phi(NM^i, NM^j)}{N} \times \Big(dL^{ij} - \left[a(NM^i) + a(NM^j)\right] V_\epsilon(X^i_s - X^j_s)\, ds\Big)$$

satisfies

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} E^{P^N}\left[\sup_{0\le t\le T}\left|\Upsilon_{N,\epsilon,f}(t)\right| 1_{\mathcal{C}^c_{N,\epsilon,T}}\right] = 0\,. \tag{4.6}$$

Here $\mathcal{C}^c$ denotes the complement of the set $\mathcal{C}$.

Suppose that (4.6) holds, and let $\delta > 0$, $l \in \mathbb{N}$. Define

$$\kappa^l(m,m') = \kappa(m,m')\, 1_{\{l^{-1}\le m\le l\}}\, 1_{\{l^{-1}\le m'\le l\}}$$
$$a^l(m) = a(m)\, 1_{\{l^{-1}\le m\le l\}}$$

and

$$\mathcal{F}_{l,\epsilon,\delta} = \Big\{ \sup_{0\le t\le T}\Big|\langle\mu_t, f\rangle - \langle\mu_0, f\rangle - \int_0^t \langle\frac{1}{2}\, a^l(m)\,\frac{\partial^2 f}{\partial x^2}, \mu_s\rangle\, ds - \int_0^t\int_{(\mathbb{T}\times\mathbb{R}_+)^2} \frac{[f(x,m+m') - f(x,m)]}{m'}\, V_\epsilon(x-y) \times \kappa^l(m,m')\,\mu_s(dx,dm)\,\mu_s(dy,dm')\, ds\Big| \le \delta\Big\}.$$

Then $\mathcal{F}_{l,\epsilon,\delta}$ is closed in $C([0,T],\mathcal{M}_1(\mathbb{T}\times\mathbb{R}_+))$ with the Skorokod topology.

By Proposition 1, $Q$ almost everywhere,

$$\lim_{l\to\infty}\int_0^t \langle\frac{1}{2}\, a^l(m)\,\frac{\partial^2 f}{\partial x^2}, \mu_s\rangle\, ds = \int_0^t \langle\frac{1}{2}\, a(m)\,\frac{\partial^2 f}{\partial x^2}, \mu_s\rangle\, ds$$

and

$$\lim_{l\to\infty}\int_0^t \Big\langle\frac{[f(x,m+m') - f(x,m)]}{m'}\, V_\epsilon(x-y)\,\kappa^l(m,m'), \mu_s\otimes\mu_s\Big\rangle\, ds = \int_0^t \Big\langle\frac{[f(x,m+m') - f(x,m)]}{m'}\, V_\epsilon(x-y)\,\kappa(m,m'), \mu_s\otimes\mu_s\Big\rangle\, ds$$

for all $t \in [0,T]$. These imply

$$\limsup_{l\uparrow\infty}\limsup_{N\uparrow\infty} Q^N\big[\mathcal{F}_{l,\epsilon,\delta}\big] \le \limsup_{l\uparrow\infty} Q\big[\mathcal{F}_{l,\epsilon,\delta}\big] \le Q\big[\mathcal{F}_{\infty,\epsilon,2\delta}\big]. \tag{4.7}$$

Now, we know that $\mu$ disintegrates as $\mu_s(dx,dm) = \upsilon_s(x,dm)\,dx$, $s \in [0,T]$, $Q$ a.e.. Letting $\epsilon \to 0$ in (4.7), by Lebesgue's Differentiation Theorem, dominated convergence and (4.6), we have

$$1 = \lim_{\epsilon\to 0}\limsup_{l\uparrow\infty}\limsup_{N\uparrow\infty} Q^N\big[\mathcal{F}_{l,\epsilon,\delta}\big] \le \lim_{\epsilon\to 0} Q\big[\mathcal{F}_{\infty,\epsilon,2\delta}\big] = Q\big[\mathcal{F}_{\infty,2\delta}\big],$$

if $\mathcal{F}_{\infty,2\delta}$ is obtained replacing $V_\epsilon(x-y)$ in the definition of $\mathcal{F}_{\infty,\epsilon,2\delta}$ by the Dirac function evaluated at $x-y$. Since $\delta > 0$ is arbitrary, this implies that (2.7) holds with $Q$–probability 1.

It remains to prove (4.6). For each $\epsilon > 0$, let the approximation of the Dirac $\delta$ function $V_\epsilon$ be such that there exists a function $u_\epsilon$ in $C^2(\mathbb{T}-\{0\}) \cap C^1(\mathbb{T})$ with support contained in $(-1/2,1/2)$, so that

$$\|u_\epsilon\|_\infty \le \gamma_1\epsilon$$

$$u_\epsilon'(0) = \frac{1}{2},\ \|u_\epsilon'\|_\infty \le \gamma_2,\ \lim_{\epsilon\to 0} u_\epsilon'(x) = 0,\ x \ne 0$$

$$\text{and }\ u_\epsilon''(x) = W_\epsilon(x),\ x \ne 0; \quad u_\epsilon''(x) < 0 \ \text{ if } \ 0 < x < \epsilon,$$

where $W_\epsilon$ is a real valued function such that $|W_\epsilon| = V_\epsilon$, and $\gamma_1$ and $\gamma_2$ are positive constants independent of $\epsilon$.

Consider the finite stopping time

$$\tau_\epsilon = \inf\Big\{t \le T,\ N\sum_i [M_t^i]^2 \ \ge\ \frac{1}{\sqrt{\epsilon}}\Big\}$$

and define

$$\mathcal{C}_{N,\epsilon,T} = \big\{\tau_\epsilon \le T\big\}\,.$$

Lemma 1 then implies

$$P^N\big[\mathcal{C}_{N,\epsilon,T}\big] \le C(T)\,\sqrt{\epsilon}$$

and clearly $\lim_{\epsilon\to 0}\limsup_{N\to\infty} P^N[\mathcal{C}_{N,\epsilon,T}] = 0$.

Let

$$G(x,y,m,m') = G_{N,f,\epsilon}(x,y,m,m') = \Big[m\big[f(x,m+m') - f(x,m)\big] + m'\big[f(y,m+m') - f(y,m')\big]\Big]\,u_\epsilon(|x-y|)\,\frac{\Phi(m,m')}{N}\,.$$

The proof follows the usual pattern after this point: we will stop the process upon achieving the stopping time $\tau_\epsilon$, so that we may assume that $P^N$ is supported on

$$N\sum_i [M_T^i]^2 \leq \frac{1}{\sqrt{\epsilon}}.$$

We will then apply Itô–Tanaka's formula to

$$\sum_{i<j} G(X_t^i, X_t^j, NM_t^i, NM_t^j)$$

in order to recover $\Upsilon_{N,\epsilon,f}$ from the non-differentiability of $u_\epsilon(|x|)$ at the origin and the particular choice of $u_\epsilon$. One then has to check that the remaining terms of the expansion converge to 0 uniformly on $[0,T]$, when taking $N \to \infty$ and $\epsilon \to 0$, in that order.

We have

$$\sum_{i<j} G(X_t^i, X_t^j, NM_t^i, NM_t^j)$$
$$= \sum_{i<j} G(x^i, x^j, NM^i, NM^j) + \frac{1}{2}\Upsilon_{N,\epsilon,f}(t) + \mathcal{H}_{N,\epsilon,f}(t) + A_{N,\epsilon,f}(t)$$

where $\mathcal{H}_{N,\epsilon,f}(t)$ is a $P^N$- martingale and $A_{N,\epsilon,f}(t)$ is a continuous, bounded variation process. We start with the former:

$$\mathcal{H}_{N,\epsilon,f} =$$
$$= \sum_i \int_0^t \sum_{j\neq i} \frac{\partial G}{\partial X^i}(X_s^i, X_s^j, NM_s^i, NM_s^j)\, dX^i$$
$$+ \sum_{i<k} \int_0^t \Big( \Big[ \sum_{j\neq i,k} G(N[M^i + M^k], NM^j)$$
$$- G(NM^i, NM^j) - G(NM^k, NM^j)\Big] - G(NM^i, NM^k)\Big)$$
$$\times \Big[ dE^{ik} - \frac{\Phi(NM^i, NM^k)}{N} dL^{ik}\Big]$$

The quadratic variation of the first term on the right above, the Brownian martingale, can be easily seen to vanish when $N \to \infty$. We proceed to show how to bound one term in the quadratic variation $Q_c$ of the coagulation martingale. Consider then

$$Q_c^1(t) =$$
$$= \int_0^t \sum_{i<k} [M^i]^2 \Big\{ \sum_{j\neq i,k} \Big[ f\big(X^i, N(M^i+M^k+M^j)\big) - f\big(X^i, N(M^i+M^k)\big) - f\big(X^i, N(M^i+M^j)\big) + f\big(X^i, NM^i\big)\Big] \times u_\epsilon(|X^i-X^j|)\, \frac{\Phi\big(N(M^i+M^k), NM^j\big)}{N} \Big\}^2 \, \frac{\Phi(NM^i, NM^k)}{N}\, dL^{ik}.$$

We now use that $f$ has compact support. Let $L \geq 0$ be such that $f(x,m) = 0$ whenever $|m| > L$. The expression between brackets in $Q_c^1$ will thus vanish whenever $NM^i > L$, so that we may bound $Q_c^1(T)$ by

$$\Gamma(f)\, \epsilon^2 \int_0^T \sum_{i<k} \frac{1}{N^2} \Big[ \big[N(M^i+M^k)\big]^{2\mathfrak{p}} + \sum_j \frac{1}{N} [NM^j]^{2\mathfrak{p}} \Big] \frac{\Phi(NM^i, NM^k)}{N} dL^{ik}$$
$$\leq \Gamma(f)\, \epsilon^2 \int_0^T \sum_{i<k} \frac{1}{N^2}\, \Phi(NM^i, Nm^k)\, dL^{ik}$$

by the assumption that $\mathfrak{p} \leq 1/2$. Now,

$$E^{P^N}\Big[ \int_0^T \sum_{i<k} \frac{1}{N^2}\, \Phi(NM^i, NM^k)\, dL^{ik} \Big]$$
$$= E^{P^N}\Big[ \sum_i \frac{1}{N} 1_{\{m^i>0\}} - \sum_i \frac{1}{N} 1_{\{M_T^i>0\}} \Big] \;\leq\; 1\,,$$

from where it follows that $\lim_{\epsilon\to 0} \lim_{N\to\infty} E^{P^N}[Q_c^1] = 0$. Similar considerations prove that the expectation of the rest of the terms in $Q_c$ vanish in the limit $N \to \infty$, $\epsilon \to 0$. Doob's inequality then implies

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} E^{P^N} \Big[ \sup_{0\leq t\leq T} |\mathcal{H}_{N,\epsilon,f}(t)| 1_{C^c_{N,\epsilon,T}} \Big] = 0\,. \tag{4.8}$$

The Lipschitz property of $f$ yields

$$E^{P^N}\Big[ \Big| \sum_{i<j} G(X_t^i, X_t^j, M_t^i, M_t^j) - \sum_{i<j} G(x^i, x^j, m^i, m^j) \Big| \Big]$$
$$\leq \Gamma(f)\, \epsilon\, E^{P^N} [1 + N \sum_i [M_t^i]^2] \leq \Gamma(f)\, \epsilon\, [1 + C(T)]\,. \tag{4.9}$$

The process $A_{N,\epsilon,f}(t)$ equals

$$A_{N,\epsilon,f}(t) = \frac{1}{2}\int_0^t \sum_{i\neq j} M^i \left[\frac{\partial^2 f}{\partial x^2}(X^i, N(M^i+M^j)) - \frac{\partial^2 f}{\partial x^2}(X^i, NM^i)\right]$$
$$\times\, u_\epsilon(|X^i - X^j|)\,\frac{\Phi(NM^i, NM^j)}{N}\,a(NM^i)\,ds$$
$$+\frac{1}{2}\int_0^t \sum_{i\neq j} M^i \left[\frac{\partial f}{\partial x}(X^i, N(M^i+M^j)) - \frac{\partial f}{\partial x}(X^i, NM^i)\right]$$
$$\times\,\mathrm{sign}(X^i - X^j)\,u'_\epsilon(|X^i - X^j|)\,\frac{\Phi(NM^i, NM^j)}{N}\,a(NM^i)\,ds$$
$$+\int_0^t \sum_{\substack{i<k\\ j\neq i,k}} \Big[G(X^i, X^j, N(M^i+M^k), NM^j) - G(X^i, X^j, NM^i, NM^j)$$
$$- G(X^k, X^j, NM^k, NM^j) - G(X^i, X^k, NM^i, NM^k)\Big]\,\frac{\Phi(NM^i, NM^k)}{N}\,dL^{ik}$$
$$= I^1_{N,\epsilon,f}(t) + I^2_{N,\epsilon,f}(t) + I^3_{N,\epsilon,f}(t)$$

where $\mathrm{sign}(x)$ takes values 1 or $-1$ according to whether $x > 0$ or $x \le 0$. It is easy to see that

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} E^{P^N}\left[\sup_{0\le t\le T} |I^1_{N,\epsilon,f}(t)|\right] = 0\,. \tag{4.10}$$

We can then bound

$$|I^2_{N,\epsilon,f}(t)| \le \Gamma(f)\int_0^t \sum_{i<j} M^i\,|u'_\epsilon|(|X^i - X^j|)\,\frac{\Phi(NM^i, NM^j)}{N}\,a(NM^i)\,ds$$
$$\le \Gamma(f)\Big[\sum_i M^i_t [NM^i_t]^{\mathfrak{p}} + \sum_i m^i a(Nm^i)\Big]\,.$$

Since $\lim_{\epsilon\to 0} u'_\epsilon(x) = 0$ for all $x \neq 0$ in $\mathbb{T}$, dominated convergence implies that

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} E^{P^N}\left[\sup_{0\le t\le T} |I^2_{N,\epsilon,f}(t)|\right] = 0 \tag{4.11}$$

as well. Finally, the fact that $f$ is Lipschitz and (2.1a) yield

$$|I^3_{N,\epsilon,f}(t)| \le \Gamma(f,\Phi)\,\epsilon\,\big[\sum_i M^i_t (NM^i_t)^{\mathfrak{p}}\big]$$
$$\times \int_0^t \sum_{i<k} \big[M^i M^k + \frac{1}{N^2}\big]\,\Phi(NM^i, NM^k)\,dL^{ik}$$

so that

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} E^{P^N}\left[\sup_{0\le t\le T}|I^3_{N,\epsilon,f}(t)|1_{C^c_{N,\epsilon,T}}\right]=0. \tag{4.12}$$

The limit (4.6) is immediate from estimates (4.8), (4.9), (4.10), (4.11) and (4.12), and the result follows.

□

**5. Uniqueness of the solution.** In this section we seek to establish the uniqueness in an appropriately defined class of the solution to the hydrodynamic equation (2.7), which in differentiated form can be written as

$$\dot\mu_t = \frac{1}{2}\,a(m)\,\frac{\partial^2}{\partial x^2}\mu_t\,+K(\mu_t),\quad \mu_0=\nu. \tag{5.1}$$

Here, $\frac{\partial^2}{\partial x^2}$ denotes the partial derivative of second order with respect to the space variable, interpreted in the weak sense, and $K$ is the coagulation kernel given by

$$\langle f,K(\mu)\rangle = \int_{\mathbb{R}_+\times\mathbb{R}_+}[f(m+m')-f(m)]\,\frac{\kappa(m,m')}{m'}\,\mu(dm)\,\mu(dm')$$
$$=\frac{1}{2}\int_{\mathbb{R}^2_+}[(m+m')f(m+m')-mf(m)-m'f(m')]\,\frac{\kappa(m,m')}{mm'}\,\mu(dm)\mu(dm')$$

if $f$ is a bounded test function, $\mu\in\mathcal{M}_f(\mathbb{R}_+)$ a finite measure such that $\kappa(m,m')/m'\in L^1(d\mu\times d\mu)$.

We will work under the assumption that there exists a pair of functions $\varpi$ and $\omega$ bounded on each compact subset of $(0,\infty)$, such that $\varpi\omega^{-1}$ is bounded, $\omega$, $a^{-1/2}\,\omega$ and $a^{-1/2}\,\omega\,\varpi$ are subadditive, and such that the following inequalities hold

$$\kappa(m,m')\le\omega(m)\,\omega(m') \tag{5.2}$$
$$\kappa(m,m')\le\omega(m)\varpi(m')+\varpi(m)\omega(m'). \tag{5.3}$$

We will also require that $a$ is non-increasing and the initial measure $\nu$ satisfies

$$\nu(dx,dm)\le\nu^*(dm)\,dx,\quad \nu^*\text{ such that }\Big\langle\frac{\omega^2}{m},\nu^*\Big\rangle<\infty. \tag{5.4}$$

Let us define the class $\mathcal{B}(\omega)$ by

$$\mathcal{B}(\omega)=\Big\{\eta\in C(\mathbb{R}_+,\mathcal{M}_f(\mathbb{T}\times\mathbb{R}_+)):\eta_t(dx,dm)=\upsilon_t(x,dm)\,dx,$$
$$\sup_{t\ge 0}\|\ll\frac{\omega(m)}{m},\upsilon_t\gg\|_\infty<\infty\ \text{ and }\ \sup_{t\le T}\langle m,\mu_t\rangle<\infty\ \forall\,T>0\Big\}.$$

The choices

$$\omega(m) = [1 + c' + a(1)]\,[m^{\mathfrak{p}} + a(m) + 1] \quad \text{and} \quad \varpi(m) = a(m) + 1$$

satisfy conditions (5.2) and (5.3) in the particular situation treated in Sections 3 and 4. In this case the coagulation kernel is given by

$$\kappa(m, m') = \Phi(m, m')\,[a(m) + a(m')]$$

with symmetric rate $\Phi$ verifying

$$\Phi(m, m') \le c\,(m^{\mathfrak{p}} + m'^{\mathfrak{p}})1_{[m>0, m'>0]}, \quad 0 \le \mathfrak{p} \le \frac{1}{2},$$

and diffusivity $a(m)$ such that $a^{-1/2}\omega$ is subadditive. Then Theorem 2 yields an existence result in $\mathcal{D}(\omega)$. The uniqueness of this solution in the larger class $\mathcal{B}(\omega)$ follows from Theorem 3 below.

Here is the main result of this section,

THEOREM 3. *Assume conditions (5.2), (5.3) and (5.4) on the coagulation rate $\kappa$ and the initial measure $\nu$. Then for any $T > 0$, (5.1) has at most one solution $\{\mu_t\}_{0\le t\le T}$ in $\mathcal{B}(\omega)$.*

We state the theorem on $\mathbb{T}$ to avoid introducing more terminology; in fact, the proof holds in $\mathbb{R}^d$ for a general diffusion model with coefficients given by $a$, undergoing coagulation at a rate determined by $\kappa$. In that case we require that (5.2), (5.3) and (5.4) are satisfied by a pair of maps $\omega$, $\varpi$ such that $\varpi^{-1}\omega$ is bounded, as before, and $a^{-d/2}\omega$ and $a^{-d/2}\omega\varpi$ are both subadditive.

The result will follow from considering an approximating system of equations to the coagulation equation, for which existence and uniqueness can be easily derived. The method is a simplification of a technique developed by Norris in [16], we have thus tried to adhere to his notation whenever possible. We are able to make a significant shortcut in the proof due to the assumption that we already have got one solution in $\mathcal{B}(\omega)$, this yields a crucial monotonicity property in the approximating scheme as a direct byproduct of its construction (compare Lemma 5.1 below with Lemmas 5.5 and 5.6 in [16]).

Before we can proceed to prove the theorem we need to introduce some definitions. Given $s > 0$ and $\mu \in \mathcal{M}_f(\mathbb{T} \times \mathbb{R}_+)$, we will denote $P_s\mu \in \mathcal{M}_f(\mathbb{T} \times \mathbb{R}_+)$ the measure given by

$$< f, P_s\mu >= \int f(z, m)\, p(a(m)s, x, z)\, \mu(dx, dm)\, dz,$$

where, as before, $p(t,x,z)$ is the Brownian transition density on $\mathbb{T}$. Hereafter, we will use the abridged notation $p_t^{x,z}(m)$ to denote $p(a(m)t,x,z)$. We also introduce the kernels $K^+$ and $K^-$ on $\mathcal{M}_f(\mathbb{R}_+)$ defined as

$$K^+(\mu)(dm) = \int_{m'+m''=m} \frac{\kappa(m',m'')}{m''}\,\mu(dm')\,\mu(dm'')$$
$$K^-(\mu)(dm) = \int_{\mathbb{R}_+} \frac{\kappa(m,m')}{m'}\,\mu(dm)\,\mu(dm').$$

With these notations, we now show that (5.1) is equivalent for $\mu \in \mathcal{B}(\omega)$ to the integral equation

$$\mu_t = P_t\nu + \int_0^t P_{t-r}K^+(\mu_r) - P_{t-r}K^-(\mu_r)\,dr. \tag{5.5}$$

By integrating against a test function and differentiating in time, it follows that any solution to (5.5) satisfies (5.1). Conversely, let $\mu_t$ be a solution to (5.1), and set $\tilde{\mu}_t$ equal to the right side of (5.5). Then $\mu - \tilde{\mu}_t$ verifies

$$\frac{d}{dt}(\tilde{\mu}_t - \mu_t) = \frac{1}{2}\,a(m)\,\frac{\partial^2}{\partial x^2}(\tilde{\mu}_t - \mu_t), \qquad \tilde{\mu}_0 - \mu_0 = 0,$$

and we conclude that $\tilde{\mu}_t = \mu_t$. In particular $\mu_t$ is a solution to (5.5).

Given a bounded map $c_s(x) : [0,T]\times\mathbb{T} \longrightarrow \mathbb{R}$, let $\tilde{p}_m(c)$ be the propagator associated with the operator $\frac{1}{2}\,a(m)\,\frac{\partial^2}{\partial x^2} - c_s(\cdot)$ on $\mathbb{T}$, and consider the kernel

$$\tilde{P}_{ts}(c)\mu(x,dm) = \int_{\mathbb{T}} \mu(dz,dm)\,\tilde{p}_m(c)(s,z;t,x)\,dz,$$

$\mu$ in $\mathcal{M}_f(\mathbb{T}\times\mathbb{R}_+)$.

Let $E_n = [1/n, n]$ for $n \in \mathbb{N}$, and define $K_n^+$, $K_n^-$ by analogy with $K^{\pm}$: $K_n^+(\mu)(dm) = K^+(\mu)(dm)\,1_{\{m\in E_n\}}$,

$$K_n^-(\mu)(dm) = \int_{\mathbb{R}_+} \frac{\kappa(m,m')}{m'}\,1_{\{m+m'\in E_n\}}\,\mu(dm)\,\mu(dm')$$

and $K_n = K_n^+ - K_n^-$. Define $\nu^n$ as the restriction of $\nu$ to $E_n$, $\nu^n(dm) = \nu(dm)1_{\{m\in E_n\}}$.

In order to keep the number of definitions to a minimum, given a test function $g$, we will use $\ll g,\mu \gg$ to denote the integral $\ll g,\nu \gg$ in the case that the decomposition $\mu(dx,dm) = \nu(x,dm)\,dx$ holds, where we recall that $\ll g,\nu \gg$ stands for the single integral $\int g(m)\,\nu(x,dm)$.

The proof of Theorem 3 will follow from the following lemma.

LEMMA 3. *Let $\{\mu_s\}_{0\le s\le T} \in \mathcal{B}(\omega)$ be a solution to (5.1) with initial value $\nu$, and assume that conditions (5.2) (5.3) and (5.4) hold. Then, for each $n \in \mathbb{N}$, there exists a unique kernel $(\tilde\mu_t^n)_{0\le t\le T}$ in $\mathcal{B}(\omega)$, such that*

$$\tilde\mu_t^n = \tilde P_t(c^n)\nu^n + \int_0^t \tilde P_{ts}(c^n)\big[K_n^+(\tilde\mu_s^n) + \delta_s^n \tilde\mu_s^n\big]\,ds, \tag{5.6}$$

*where*

$$\delta_t^n(x,m) = \ll \frac{\omega(m)\,\omega(m') - \kappa(m,m')}{m'}, \tilde\mu_t^n \gg,$$
$$c_t^n(x,m) = \ll \omega(m)\frac{\omega(m')}{m'}, P_t\nu \gg + \int_0^t \ll \omega(m)\frac{\omega(m')}{m'}, P_{t-s}\big[K_n(\tilde\mu_s^n)\big] \gg ds.$$

*Moreover, $\tilde\mu_s^n$ satisfies*

$$\tilde\mu_s^n \le \tilde\mu_s^{n+1} \le \mu_s \quad \textit{for all} \;\; s \in [0,T].$$

PROOF. The method of the proof is classical. We will take advantage of the fact that all relevant quantities are bounded in $E_n$ to define a Picard iteration procedure, and then prove by a contraction mapping argument that the scheme converges to a solution, which is finally shown to be unique.

Define

$$\tilde\mu_{t,0}^n = \mu_t 1_{\{m\in E_n\}} \le \tilde\mu_{t,0}^{n+1} \le \mu_t,$$
$$c_{t,0}^n = \ll \omega(m)\frac{\omega(m')}{m'}, P_t\nu \gg + \int_0^t \ll \omega(m)\frac{\omega(m')}{m'}, P_{t-s}\big[K_n(\tilde\mu_{s,0}^n)\big] \gg ds.$$

Condition (5.4) on $\nu$ and the fact that $\tilde\mu_{\cdot,0}^n$ is supported on $E_n$ imply that $c_{t,0}^n(x,m)$ is well defined and bounded.

Let

$$c_t(x,m) = \ll \omega(m)\frac{\omega(m')}{m'}, \mu_t \gg .$$

We claim that $0 \le c_t(x,m) \le c_{t,0}^{n+1}(x,m) \le c_{t,0}^n(x,m) \le C\,\omega(m)$, where $C$ is a positive constant that depends solely on $\mu$ . Note that $p_u^{y,x}(m)\,\omega(m)$ is $m$-subadditive, which can be easily seen by expanding $p_u^{y,x}$ as a series and using the subadditivity of $a^{-1/2}\omega$ and the monotonicity of $a$. We next write

$$\begin{aligned}
&\ll \frac{\omega(m')}{m'}, P_{t-s}\big[K_n(\tilde\mu_{s,0}^n)\big] \gg \\
&\quad = \frac{1}{2}\int_{\mathbb{R}_+^2\times\mathbb{T}} \Big[p_{t-s}^{y,x}(m'+m'')\,\omega(m'+m'') - p_{t-s}^{y,x}(m')\,\omega(m') \\
&\qquad\qquad - p_{t-s}^{y,x}(m'')\,\omega(m'')\Big] \\
&\qquad\qquad \times \frac{\kappa(m',m'')}{m'm''}\,1_{E_n}(m'+m'')\,\tilde\mu_{s,0}^n(y,dm')\,\tilde\mu_{s,0}^n(y,dm'')\,dy.
\end{aligned}$$

Together with the subadditivity of $p_u^{y,x}(m')\,\omega(m')$, the inequality $\tilde\mu^n_{s,0} \le \tilde\mu^{n+1}_{s,0} \le \mu_s$ and the obvious inclusion $E_n \subset E_{n+1}$, this implies

$$\begin{aligned}\omega(m)\,\langle\frac{\omega(m')}{m'},\nu^*\rangle &\ge \ll \omega(m)\frac{\omega(m')}{m'}, P_t\nu \gg \\ &\ge\ c^n_{t,0}(x,m) \ge c^{n+1}_{t,0}(x,m) \ge c_t(x,m) \ge 0,\end{aligned} \tag{5.7}$$

as required.

We will need the fact that the solution $\mu.$ satisfies

$$\begin{aligned}\mu_t &= \tilde P_t(c)\nu + \int_0^t \tilde P_{ts}(c)\big[K^+(\mu_s) + \delta_s\,\mu_s\big]\,ds, \\ \delta_t(x,m) &= \ll \frac{\omega(m)\,\omega(m') - \kappa(m,m')}{m'}, \mu_t \gg .\end{aligned} \tag{5.8}$$

This can be proved by an argument similar to the one used to show the equivalence of equations (5.5) and (5.1).

Fix $k \in \mathbb{N}$, and assume that a sequence of measure paths and mappings $\tilde\mu^n_{s,k} \le \tilde\mu^{n+1}_{s,k} \le \mu_s$ and $c^n_{s,k}(x,m) \ge c^{n+1}_{s,k}(x,m) \ge c_s(x,m) \ge 0$ have already been defined. Let

$$\delta^n_{t,k} = \ll \frac{\omega(m)\omega(m') - \kappa(m,m')}{m'}, \tilde\mu^n_{t,k} \gg$$

so that $0 \le \delta^n_{t,k} \le \delta^n_{t,k+1} \le \delta_t$. Finally, set

$$\begin{aligned}\tilde\mu^n_{t,k+1} &= \tilde P_t(c^n_k)\nu^n + \int_0^t \tilde P_{ts}(c^n_k)\big[K^+_n(\tilde\mu^n_{s,k}) + \delta^n_{s,k}\,\tilde\mu^n_{s,k}\big]\,ds, \\ c^n_{t,k+1}(x,m) &= \ll \omega(m)\frac{\omega(m')}{m'}, P_t\nu \gg \\ &\qquad + \int_0^t \ll \omega(m)\frac{\omega(m')}{m'}, P_{t-s}\big[K_n(\tilde\mu^n_{s,k+1})\big] \gg\, ds.\end{aligned}$$

The Feynman-Kac formula (cf. [17], Chapter 8) and identity (5.8) then imply

$$0 \le \tilde\mu^n_{s,k+1} \le \tilde\mu^{n+1}_{s,k+1} \le \mu_s$$

and thus, by the same arguments that yield (5.7),

$$\omega(m)\,\langle\frac{\omega(m')}{m'},\nu^*\rangle \ge c^n_{s,k+1} \ge c^{n+1}_{s,k+1} \ge c_s \ge 0.$$

The kernel $\tilde\mu^n_{s,k}$ is supported on $E_n$ for each $n$ and all $k$, $s$. Also, $\tilde\mu^n_{s,k} \le \mu_s$ and the fact that $\omega(m)/m$ is bounded below in $E_n$ imply that the marginal density $d\tilde{\mu}^n_{s,k}(dx,\mathbb{R}_+)/dx \in L^\infty(dx)$ and its $L^\infty-$ norm can be bounded uniformly in $k$,

$$\Big\| \frac{d\tilde{\mu}^n_{s,k}(dx,\mathbb{R}_+)}{dx}\Big\|_\infty \le \beta_n \qquad \text{for all } k\in\mathbb{N}. \tag{5.9}$$

In view of these observations, we define a norm in the vector space

$$\Big\{\eta\in\mathcal{M}_f(\mathbb{T}\times E_n),\ \frac{d\eta}{dx}(dx,\mathbb{R}_+)\in L^\infty(dx)\Big\}$$

by

$$|||\rho-\varrho||| = \Big\|\,\|\rho_x-\varrho_x\|\,\Big\|_\infty$$

if $\rho(dx,dm)=\rho_x(dm)\,dx$ and a similar formula holds for $\varrho$. Here $\|\rho_x-\varrho_x\|$ is the total variation norm of the signed measure $\rho_x(dm)-\varrho_x(dm)$, which we may compute as

$$\|\rho_x-\varrho_x\| = \int_{\mathbb{R}_+}|\rho_x-\varrho_x|(dm)$$

if $|\rho_x-\varrho_x|(dm)$ denotes the total variation of $\rho_x(dm)-\varrho_x(dm)$. Equivalently,

$$|||\rho-\varrho||| = \sup\Big\{\langle f,\rho-\varrho\rangle|\, f:\int_{\mathbb{T}}|\sup_m f(x,m)|\,dx\ \le 1\Big\}.$$

Let $f(x,m)$ be such that $\int_{\mathbb{T}}|\sup_m f(x,m)|\,dx\ \le 1$. We obtain

$$\begin{aligned}\langle f,\tilde{\mu}^n_{t,k+1}-\tilde{\mu}^n_{s,k}\rangle &= \langle f,\tilde{P}_t(c^n_k)\nu^n-\tilde{P}_t(c^n_{k-1})\nu^n\rangle\\ &\quad+\int_0^t\langle f,\tilde{P}_{ts}(c^n_k)\big[K^+_n(\tilde{\mu}^n_{s,k})\big]-\tilde{P}_{ts}(c^n_{k-1})\big[K^+_n(\tilde{\mu}^n_{s,k-1})\big]\rangle\,ds\\ &\quad+\int_0^t\langle f,\tilde{P}_{ts}(c^n_k)\big[\delta^n_{s,k}\,\tilde{\mu}^n_{s,k}\big]-\tilde{P}_{ts}(c^n_{k-1})\big[\delta^n_{s,k-1}\,\tilde{\mu}^n_{s,k-1}\big]\rangle\,ds.\end{aligned} \tag{5.10}$$

We apply Feynman-Kac formula to bound the first term on the right side of (5.10) by

$$\int_{\mathbb{T}\times\mathbb{R}_+}E^{x,m}\Big[\big(\int_0^t|c^n_{s,k}(\chi_s,m)-c^n_{s,k-1}(\chi_s,m)|\,ds\big)\,f(\chi_t,m)\Big]\,\nu^n(dx,dm),$$

where $\chi_s$ is a Brownian motion in $\mathbb{T}$ with diffusivity $a(m)$. We have

$$\begin{aligned}
&\int_0^t |c^n_{s,k}(\chi_s, m) - c^n_{s,k-1}(\chi_s, m)|\, ds \\
&\le \omega(m) \int_0^t \int_0^s \Big| \frac{\omega(m'+m'')}{m'+m''}\, p^{x',\chi_s}_{s-u}(m'+m'') - \frac{\omega(m')}{m'}\, p^{x',\chi_s}_{s-u}(m') \Big| \\
&\qquad\qquad \times \frac{\kappa(m',m'')}{m''} 1_{\{m'+m''\in E_n\}} \\
&\times \Big| \tilde\mu^n_{u,k}(x', dm')\tilde\mu^n_{u,k}(x', dm'') - \tilde\mu^n_{u,k-1}(x', dm')\tilde\mu^n_{u,k-1}(x', dm'')\Big|\, dx'\, du\, ds.
\end{aligned}$$

Note that in all these expressions the mass variable takes values $m$, $m'$, $m''$ or $m'+m''$ belonging to $E_n$, we may therefore replace all functions that have it as an argument by an upper or lower bound, as necessary, that do not depend on the mass variable.

Now, if $\rho$ and $\varrho$ are two finite, positive measures on $(X, \Omega)$, then $\rho\otimes\rho$ and $\varrho\otimes\varrho$ are finite, positive measures on $(X\times X, \Omega\times\Omega)$ and

$$|\rho\otimes\rho - \varrho\otimes\varrho| \le \big[\rho(X) + \varrho(X)\big]\, |\rho - \varrho|.$$

In particular,

$$\begin{aligned}
&\Big| \tilde\mu^n_{u,k}(x', dm')\tilde\mu^n_{u,k}(x', dm'') - \tilde\mu^n_{u,k-1}(x', dm')\tilde\mu^n_{u,k-1}(x', dm'')\Big| \\
&\qquad\qquad \le 2\,\beta_n\, |\tilde\mu^n_{u,k}(x', dm') - \tilde\mu^n_{u,k-1}(x', dm')|,
\end{aligned}$$

$\beta_n$ as in (5.9).

We thus have

$$\begin{aligned}
\langle f, \tilde P_t(c^n_k)\nu^n - \tilde P_t(c^n_{k-1})\nu^n\rangle &\le \Gamma \int_0^t \int_0^s p(s-u, x', w)\, p(s, x, w)\, p(t-s, w, y) \\
&\times \big(\sup_m |f(y,m)|\big)\, \|\tilde\mu^n_{u,k}(x') - \tilde\mu^n_{u,k-1}(x')\|\, \nu^n(dx, dm)\, dx'\, dw\, dy\, du\, ds
\end{aligned}$$

where $\Gamma$ is a positive constant that depends on $n$ and $\beta_n$ and can therefore be chosen uniformly in $k$.

Replace now $\nu^n$ by its upper bound $\nu^*(dm)\,dx$, take the $L^\infty$ norm of the total variation factor $|\tilde\mu^n_{u,k}(x') - \tilde\mu^n_{u,k-1}(x')|$, and integrate in $x$, $x'$, $w$ and $s$, to obtain

$$\langle f, \tilde P_t(c^n_k)\nu^n - \tilde P_t(c^n_{k-1})\nu^n\rangle \le \Gamma(T) \int_0^t |||\tilde\mu^n_{u,k} - \tilde\mu^n_{u,k-1}|||\; du.$$

In this last step we used that $\int_{\mathbb{T}} |\sup_m f(x,m)|\, dx \le 1$.

Similar computations yield

$$\int_0^t \langle f, \tilde{P}_{ts}(c^n_k)[K^+_n(\tilde{\mu}^n_{s,k})] - \tilde{P}_{ts}(c^n_{k-1})[K^+_n(\tilde{\mu}^n_{s,k-1})]\rangle\, ds + \int_0^t \langle f, \tilde{P}_{ts}(c^n_k)[\delta^n_{s,k}\,\tilde{\mu}^n_{s,k}] - \tilde{P}_{ts}(c^n_{k-1})[\delta^n_{s,k-1}\,\tilde{\mu}^n_{s,k-1}]\rangle\, ds \le \Gamma(T) \int_0^t |||\tilde{\mu}^n_{u,k} - \tilde{\mu}^n_{u,k-1}\,|||\; du$$

and therefore, by (5.10),

$$\langle f, \tilde{\mu}^n_{t,k+1} - \tilde{\mu}^n_{s,k}\rangle \le \Gamma(T) \int_0^t |||\tilde{\mu}^n_{u,k} - \tilde{\mu}^n_{u,k-1}\,|||\; du,$$

$\Gamma(T)$ uniform in $k$, $f$. Take the supremum over $f$ to conclude that for $t \le T$,

$$|||\tilde{\mu}^n_{t,k+1} - \tilde{\mu}^n_{t,k}||| \le \Gamma(T) \int_0^t |||\tilde{\mu}^n_{u,k} - \tilde{\mu}^n_{u,k-1}\,|||\; du. \tag{5.11}$$

Hence, by a standard contraction mapping argument, $\tilde{\mu}^{t,n}_k$ converges in $\mathcal{M}_f(\mathbb{T} \times E_n)$, uniformly in $t \le T$. The limit is a continuous map $\tilde{\mu}^n : [0,T] \to \mathcal{M}_f(\mathbb{T} \times E_n)$ that satisfies equation (5.6). Moreover, the properties that $\tilde{\mu}^n_{s,k} \le \tilde{\mu}^{n+1}_{s,k} \le \mu_s$ and $c^n_{s,k} \ge c^{n+1}_{s,k} \ge c_s$ for all $k$, $n \in \mathbb{N}^2$ imply that the same holds for $\tilde{\mu}^n$ and $c^n$:

$$\tilde{\mu}^n_s \le \tilde{\mu}^{n+1}_s \le \mu_s \qquad \text{and} \qquad c^n_s \ge c^{n+1}_s \ge c_s,$$

for all $n \in \mathbb{N}$.

Suppose now that $\tilde{\mu}^n$ and $\tilde{\eta}^n$ are two solutions in $\mathcal{D}(K)$ to (5.6) with respective values of the mapping $c$ defined as in the statement of the lemma. A careful revision of the arguments leading to (5.11) shows that the proof goes through verbatim if we replace $\tilde{\mu}^n_{k+1}$ and $\tilde{\mu}^n_k$ by fixed points of the iteration scheme $\tilde{\mu}^n$ and $\tilde{\eta}^n$. We thus have

$$|||\tilde{\mu}^n_t - \tilde{\eta}^n_t||| \le \Gamma(T) \int_0^t |||\tilde{\mu}^n_u - \tilde{\eta}^n_u\,|||\; du \qquad t \le T$$

which implies $\tilde{\mu}^n \equiv \tilde{\eta}^n$ by Gronwall's lemma. □

*Proof of Theorem 3.* Set $\lambda^n_0 = 1_{E^c_n}\nu$, and define the kernel $\tilde{K}^-_n(\mu) = K^-(\mu) - K^-_n(\mu)$. Given mappings $\tilde{\mu}^n_\cdot$ and $c^n_\cdot$ as in Lemma 3, define

$$\mu^n_t = P_t\nu^n + \int_0^t P_{t-s}\Big[K^+_n(\tilde{\mu}^n_s) - K^-(\tilde{\mu}^n_s) - \gamma^n_s\,\tilde{\mu}^n_s\Big]\, ds$$
$$\lambda^n_t = P_t\lambda^n_0 + \int_0^t P_{t-s}\Big[\tilde{K}^-_n(\tilde{\mu}^n_s) + \gamma^n_s\,\tilde{\mu}^n_s\Big]\, ds$$

where

$$\gamma_s^n(x,m) = c_s^n(x,m) - \ll \omega(m)\frac{\omega(m')}{m'}, \tilde{\mu}_s^n \gg \geq c_s^n(x,m) - c_s(x,m) \geq 0.$$

The inequality in the last line follows from $\tilde{\mu}_s^n \leq \mu_s$ and the definition of $c_s^n$; it has the consequence that $\lambda_s^n$ is a positive measure for all $s \leq T$, $n \in \mathbb{N}$.

We claim that $\mu^n = \tilde{\mu}^n$. Indeed, differentiating in the equations satisfied by $\mu$ and $\tilde{\mu}$ shows that both maps verify the equation

$$\dot{\eta}_t = \frac{1}{2}\, a(m)\, \frac{\partial^2}{\partial x^2}\eta_t + K_n^+(\tilde{\mu}_t^n) - [c_t^n - \delta_t^n]\, \tilde{\mu}_t^n$$

with initial value $\nu^n$, so that their difference is a weak solution to $\dot{\eta} = \frac{1}{2}\, a(m)\, \frac{\partial^2}{\partial x^2}\eta_t$ started from the zero measure, and therefore it must be the null measure. In particular this implies that $\mu_s^n \leq \mu_s$.

We have, by the definition of $c_t^n$,

$$\begin{aligned} c_t^n(x,m) &= \ll \omega(m)\frac{\omega(m')}{m'}, P_t\nu \gg + \int_0^t \ll \omega(m)\frac{\omega(m')}{m'}, P_{t-s}[K_n(\tilde{\mu}_s^n)] \gg\, ds \\ &= \ll \omega(m)\frac{\omega(m')}{m'}, \lambda_t^n \gg + \ll \omega(m)\frac{\omega(m')}{m'}, \mu_t^n \gg \end{aligned}$$

and on the other hand, from the definition of $\gamma_n^t$,

$$c_t^n(x,m) = \gamma_t^n + \ll \omega(m)\frac{\omega(m')}{m'}, \tilde{\mu}_t^n \gg .$$

Hence $\gamma_s^n = \ll \omega(m)\frac{\omega(m')}{m'}, \lambda_t^n \gg$.

From this point on, the proof is copied from that of Theorem 5.4 in [16]. Let us define $\alpha_t^n = \ll \frac{\omega(m)}{m}, \lambda_t^n \gg$. Then, due to the subadditivity of $\omega(m)a(m)^{-1/2}$, we get

$$\alpha_t^n + \ll \frac{\omega(m)}{m}, \mu_t^n \gg \;\geq\; \alpha_t^{n+1} + \ll \frac{\omega(m)}{m}, \mu_t^{n+1} \gg \;\geq \ll \frac{\omega(m)}{m}, \mu_t \gg$$

for all $n \in \mathbb{N}$, and it follows from Lemma 3 that $\alpha_t^n \geq \alpha_t^{n+1}$. We can hence define the monotone limits

$$\alpha_t = \lim_{n\to\infty} \alpha_t^n, \qquad \underline{\mu_t} = \lim_{n\to\infty} \mu_t^n,$$

which satisfy

$$\underline{\mu_t} \leq \mu_t, \qquad \alpha_t + \ll \frac{\omega(m)}{m}, \underline{\mu}_t \gg \;\geq\; \ll \frac{\omega(m)}{m}, \mu_t \gg .$$

Since $\omega(m)/m > 0$, in the case that $\alpha_t = 0$ a.e. we can conclude that $\underline{\mu_t} = \mu_t$ a.e., for all $t \in [0,T]$. The uniqueness of the solution to (5.1) will then follow from the uniqueness of the solution to (5.6).

We will now show that $\alpha_t$ vanishes. We start by proving that $h_n(t) = \sup_{s\geq 0} \| \ll \frac{\omega^2(m)}{m}, P_s\mu_t^n \gg \|_\infty$ can be bounded uniformly in $n$ and $t \in [0,T]$. We apply $P_s$ to the definition of $\mu_t^n$, multiply by $\omega^2(m)/m$ and integrate over $E_n$ to obtain

$$\begin{aligned} \ll \frac{\omega^2(m)}{m}, P_s\mu_t^n \gg \leq \ll \frac{\omega^2(m)}{m}, P_s\nu \gg \\ + \int_0^t \ll \frac{\omega^2(m)}{m}, P_{s+t-r}K_n(\mu_r^n) \gg dr. \end{aligned} \tag{5.12}$$

By the subadditivity of $\omega$ and $\omega\, a^{-1/2}$, for any $u \geq 0$, $x$, $z \in \mathbb{T}$, we have

$$\begin{aligned} \omega^2(m+m')\, p_u^{z,x}(m+m') - \omega^2(m)\, p_u^{z,x}(m) - \omega^2(m')\, p_u^{z,x}(m') \\ \leq \omega(m)\, p_u^{z,x}(m)\, \omega(m') + \omega(m)\, \omega(m')\, p_u^{z,x}(m'). \end{aligned}$$

Then

$$\begin{aligned} &\ll \frac{\omega^2(m)}{m}, P_{s+t-r}K_n(\mu_r^n) \gg (x) \\ &\quad\leq 2\int_{\mathbb{T}\times E_n^2} \omega(m)\, p_{s+t-r}^{z,x}(m)\, \omega(m')\, \frac{\kappa(m,m')}{mm'}\, \mu_r^n(z,dm)\mu_r^n(z,dm')\, dz \\ &\quad\leq 2\Big\| \ll \frac{\omega^2}{m}\, p_{s+t-r}^{\cdot,x}, \mu_r^n \gg \ll \frac{\omega\,\varpi}{m}, \mu_r^n \gg \\ &\qquad\qquad + \ll \frac{\omega\,\varpi}{m}\, p_{s+t-r}^{\cdot,x}, \mu_r^n \gg \ll \frac{\omega^2}{m}, \mu_r^n \gg \Big\|_1 \\ &\quad\leq 2\,\| \ll \frac{\omega\,\varpi}{m}, \mu_r^n \gg \|_\infty \ll \frac{\omega^2}{m}, P_{s+t-r}\mu_r^n \gg (x) \\ &\qquad\qquad + 2\,\| \ll \frac{\omega^2}{m}, \mu_r^n \gg \|_\infty \ll \frac{\omega\,\varpi}{m}, P_{s+t-r}\mu_r^n \gg (x) \end{aligned}$$

where we used the bound $\kappa(m,m') \leq \omega(m)\varpi(m') + \varpi(m)\omega(m')$. Now, if we replace $\omega^2$ by $\omega\,\varpi$ in (5.12), the subadditivity of $a^{-1/2}\omega\,\varpi$ implies that the time integral term is non-positive, and therefore

$$\sup_r \sup_{s\geq 0} \| \ll \frac{\omega\,\varpi}{m}, P_s\mu_r^n \gg \|_\infty \leq \langle \frac{\omega\,\varpi}{m}, \nu^* \rangle < \infty$$

by condition (5.4) on $\nu$. This assumption also implies that

$$\sup_{s\geq 0} \ll \frac{\omega^2}{m}, P_s\nu \gg \leq \Gamma < \infty$$

for some positive constant $\Gamma$. Taking the supremum over $s \geq 0$ in (5.12), we conclude that

$$h_n(t) \leq \Gamma + 2\Gamma' \int_0^t h_n(r)\, dr, \qquad \text{with } \Gamma' > 0.$$

Note that the constants $\Gamma, \Gamma'$ can be chosen independently of $n$. Then $h_n(t) \leq \Gamma e^{2\Gamma' T}$ holds uniformly in $n$ and $t \leq T$, as claimed.

We now consider the $L^1$ norm of $\alpha_t^n$. We replace $\gamma_t^n(x,m)$ by its upper bound $\omega(m)\,\alpha_t(x)$ in the definition of $\lambda^n$ and pass to the limit as $n \to \infty$. By dominated convergence we have

$$\begin{aligned}
\|\alpha_t\|_1 &\leq \int_0^t \int_{\mathbb{T}\times\mathbb{T}\times\mathbb{R}_+} \frac{\omega^2(m)}{m}\,\alpha_s(z)\, p_{t-s}^{z,x}(m)\, \underline{\mu}_s(dm,dz)\, dx\, ds \\
&\leq \int_0^t \big(\sup_n \sup_{s\geq 0} \| \ll \frac{\omega^2}{m}, \mu_s^n \gg \|_\infty \big)\, \|\alpha_s\|_1\, ds \\
&\leq \Gamma(T) \int_0^t \|\alpha_s\|_1\, ds.
\end{aligned}$$

Since $\|\alpha_t\|_1 \leq \langle \omega/m, \nu^* \rangle < \infty$, this implies $\alpha_t = 0$ a.e., as required. The theorem follows. □

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