The role of the central limit theorem in discovering sharp rates of convergence to equilibrium for the solution of the Kac equation

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Abstract

In Dolera, Gabetta and Regazzini (2008) it is proved that the total variation distance between the solution $f(\cdot, t)$ of Kac's equation and the Gaussian density $(0, \sigma^2)$ has an upper bound which goes to zero with an exponential rate equal to $-1/4$, as $t \to +\infty$. In the present paper we determine a lower bound which decreases exponentially to zero with the same said rate, provided that a suitable symmetrized form of f_0 has non-zero fourth cumulant κ_4. Moreover, we show that upper bounds like $\overline{C}_\delta e^{-\frac{1}{4}t}\rho_\delta(t)$ are valid for some ρ_δ vanishing at infinity when $\int_{\mathbb{R}} |v|^{4+\delta} f_0(v) \mathrm{d}v < +\infty$ for some δ in $[0, 2[$ and $\kappa_4 = 0$. Generalizations of this statement are presented together with some remarks about non-Gaussian initial conditions which yield the insuperable barrier of -1 for the rate of convergence.

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1 Introduction

In order to determine the rates of relaxation to equilibrium in kinetic theory, Kac derived the following Boltzmann-like equation, commonly known as Kac equation,

$$\frac{\partial f}{\partial t}(v,t) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}[\ f(v\ \cos\theta - w\ \sin\theta, t)\cdot f(v\ \sin\theta + w\ \cos\theta, t)\ -$$

$$- f(v,t)\cdot f(w,t)\ ]\ \mathrm{d}w\mathrm{d}\theta \qquad\qquad (v\in\mathbb{R},\ t>0) \tag{1}$$

with some specific probability density function f_0 as initial datum. The resulting Cauchy problem admits a unique solution within the class of all probability density functions on $\mathbb{R}$. Such a solution provides the probability distribution at any time of the velocity of a single particle in a chaotic bath of like molecules *moving on the line*. See Kac (1956, 1959) and McKean (1966). It is well-known that the probability measure $\mu(\cdot,t)$ determined by $f(\cdot,t)$ converges to a distinguished Gaussian law in the *variational metric*, namely

$$\mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma) := \sup_{B\in\mathscr{B}(\mathbb{R})}|\mu(B,t)-\gamma_\sigma(B)|\ \to\ 0 \qquad (t\to+\infty)\ , \tag{2}$$

where γ_σ denotes the Gaussian distribution with zero mean and variance σ^2 and, for any metric space S, $\mathscr{B}(S)$ stands for the Borel class on S. It should be recalled that (2) holds true if and only if the initial datum has finite second moment and σ^2 is just the value of such a moment. The proof of the "if" part of this assertion is given in Dolera (2007) by adapting arguments explained in Carlen and Lu (2003), whereas the proof of the "only if" part is contained in Gabetta and Regazzini (2006b).

Apropos of the speed of approach to equilibrium it has been proved that

$$\mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma)\ \le\ C_* e^{-\frac{1}{4}t} \qquad (t\ge 0) \tag{3}$$

holds with some suitable constant C_* depending only on the behaviour of f_0, when f_0 has finite fourth moment and

$$\varphi_0(\xi) \ := \int_{\mathbb{R}} e^{i\xi x} f_0(x)\mathrm{d}x \ = \ o(|\,\xi\,|^{-p}) \qquad (|\,\xi\,| \to +\infty) \tag{4}$$

is valid for some $p > 0$. See Dolera, Gabetta and Regazzini (2008). This work will be mentioned as DGR throughout the rest of the present paper. Inequality (3) is known as McKean's conjecture and the above statement represents the first satisfactory support of this very same conjecture.

At the end of Subsection 2.2 of DGR the question whether the upper bound in (3) can be improved is posed. To the best of the authors' knowledge, this problem has not been tackled yet, except a hint on page 370 of Carlen, Carvalho and Gabetta (2005). The main proposition in the present paper states that the answer is in the affirmative only in the rather peculiar case in which the *fourth cumulant* of the density $\tilde{f}_0(x) := \{f_0(x)+f_0(-x)\}/2$ is zero. The term fourth cumulant of a probability distribution Q on $\mathscr{B}(\mathbb{R})$ designates the quantity

$$\kappa_4(\mathsf{Q}) := \int_{\mathbb{R}} (x - \overline{Q})^4 \mathsf{Q}(\mathrm{d}x) - 3 \left(\int_{\mathbb{R}} (x - \overline{Q})^2 \mathsf{Q}(\mathrm{d}x) \right)^2$$

with $\overline{Q} := \int_{\mathbb{R}} x\mathsf{Q}(\mathrm{d}x)$, under the assumption that the fourth moment is finite. This cumulant is zero, for example, when Q is Gaussian.

In view of this fact one could comment on the main proposition by noting that improvements of the rate expressed by (3) turn out to be impossible when f_0 is definitely dissimilar from the class of all Gaussian distributions. For the sake of completeness, we recall that, given the Fourier-Stiltjes transform q of Q, the r-th cumulant of Q is defined to be the coefficient of $(i\xi)^r/r!$ in Taylor's expansion of $\log(q(\xi))$. See, e.g., Sections 3.14-3.15 of Stuart and Ord (1987).

As a further remark on the above-mentioned proposition, it is worth noting its resemblance between well-known facts related to the approximation of the distribution function F_n of the "standardized" sum of n independent and identically distributed random variables with finite variance, by the standard Gaussian distribution Φ. Indeed, in general, F_n is approximated by Φ except for terms of orders $1/\sqrt{n}$. But higher orders of approximation hold when the skewness and the kurtosis of the common distribution of each summand are zero. Liapounov (1901) was the pioneer of this kind of problems, followed by Cramér (1937), Esseen (1945) and others.

The structure of the paper is as follows: Section 2 contains the presentation of the main results. Section 3 deals with the basic preliminary facts which pave the way for proofs of the main results. It is split into two subsections. The former consists in a brief description of the probabilistic interpretation according to which $\mu(\cdot, t)$ can be seen as distribution of a random weighted sum of random variables. The latter is devoted to the analysis of the error made by the approximation of the law of certain weighted sums of independent random variables to the Gaussian distribution. Section 4 contains the proofs of the main results stated in Section 2. Finally, some purely technical aspects are deferred to an appendix, together with the proof of two lemmata formulated in Section 3. The various parts of the appendix are designated by A.1, ..., A.4, respectively.

2 Presentation of the new results

In order to present the main results we intend to prove in this paper it is worth mentioning the following weak version of Kac's problem (1) proposed in Bobylev (1984). Taking the Fourier transform of both sides of (1) yields

$$\frac{\partial \varphi}{\partial t}(\xi, t) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\xi \cos\theta, t) \cdot \varphi(\xi \sin\theta, t)\, \mathrm{d}\theta \; - \varphi(\xi, t) \tag{5}$$

with initial datum $\varphi_0(\xi) := \int_{\mathbb{R}} e^{i\xi x} f_0(x) \mathrm{d}x$. It should be noted that, if φ_0

is the Fourier-Stieltjes transform of any (not necessarily absolutely continuous) probability distribution μ_0 on $\mathscr{B}(\mathbb{R})$, (5) can be thought of as a new problem which generalizes (1). In any case, (5) admits a unique solution $\varphi(\cdot, t)$ which characterizes - in the form of a Fourier-Stieltjes transform - a probability distribution $\mu(\cdot, t)$ that, throughout the paper, will be said to be a solution for (5). Obviously, in problem (1) one has $\mu_0(B) := \int_B f_0(v)\mathrm{d}v$ and $\mu(B, t) := \int_B f(v, t)\mathrm{d}v$, for every B in $\mathscr{B}(\mathbb{R})$.

In order to formulate the new results exhaustively, let $\mathfrak{m}_r$ and $\overline{\mathfrak{m}}_r$ denote the r-th moment and the absolute r-th moment of μ_0 respectively, and $\tilde{\mu}_0$ be the *symmetrized form of* μ_0 defined by

$$\tilde{\mu}_0(B) \ := \ \{\mu_0(B) + \mu_0(-B)\}/2 \qquad (B \in \mathscr{B}(\mathbb{R})) \tag{6}$$

where $-B$ denotes the set $\{x \mid -x \in B\}$.

A precise statement of the fact that rate $-1/4$ may be the best possible one is contained in

Theorem 2.1. *Suppose that μ_0 possesses finite fourth moment $\mathfrak{m}_4$ and that $\kappa_4(\tilde{\mu}_0) \neq 0$. Moreover, let σ^2 be the value of $\mathfrak{m}_2$. Then, there exists a strictly positive constant C, depending only on the behaviour of μ_0, for which*

$$\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \ \geq \ Ce^{-\frac{1}{4}t} \tag{7}$$

holds true for every $t \geq 0$.

The proof of this theorem, deferred to Section 4, contains also a precise quantification for C. Since

$$\sup_{B \in \mathscr{B}(\mathbb{R})} |\mathsf{P}(B) - \mathsf{Q}(B)| = \frac{1}{2} \int_{\mathbb{R}} |p(x) - q(x)|\mathrm{d}x$$

is valid whenever P and Q are absolutely continuous probability distributions with densities p and q respectively, as an immediate consequence of Theorem 2.1, one has that

$$\frac{1}{2}\int_{\mathbb{R}} \mid f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} \mid \mathrm{d}v \ \geq \ Ce^{-\frac{1}{4}t} \qquad (t \geq 0) \tag{8}$$

turns out to be true for the solution $f(\cdot, t)$ of (1) provided that the initial datum f_0 yields a probability measure μ_0 with the same properties as in Theorem 2.1. From (8) it plainly follows that any inequality such as

$$\int_{\mathbb{R}} \mid f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} \mid \mathrm{d}v \ \leq \ C_* e^{-\frac{1}{4}t}\rho(t) \qquad (t \geq 0)$$

is *not* valid when ρ vanishes at infinity. This clarifies why inequality (3) can be viewed as sharp.

Now, we analyse the effect of assuming $\kappa_4(\tilde{\mu}_0) = 0$.

Theorem 2.2. *Consider Kac's equation* (1) *with initial datum* f_0 *such that* $\overline{\mathfrak{m}}_{4+\delta} < +\infty$ *for some* δ *in* $[0, 2[$ *and* $\kappa_4(\tilde{\mu}_0) = 0$. *Moreover, let the Fourier transform of* f_0, φ_0, *satisfy the usual tail condition* (4) *for some strictly positive* p. *Then, there are a strictly positive constant* $\overline{C}_\delta = \overline{C}_\delta(f_0; p)$ *and a function* $\rho_\delta : [0, +\infty[\to [0, +\infty[$ *which vanishes at infinity for which*

$$\int_{\mathbb{R}} \mid f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} \mid \mathrm{d}v \ \leq \ \overline{C}_\delta e^{-\frac{1}{4}t}\rho_\delta(t) \qquad (t \geq 0)\ . \tag{9}$$

In particular, if δ *belongs to* $]0, 2[$, *one can take*

$$\rho_\delta(t) = \exp\{(-3/4 + 2\alpha_{4+\delta})t\} \tag{10}$$

with $\alpha_s := \frac{1}{2\pi}\int_0^{2\pi} |\sin\theta|^s \mathrm{d}\theta$.

Useful information to quantify $\overline{C}_\delta$ can be gathered in Subsections 4.3, 4.4 and points A.2, A.4 of the Appendix.

Since even cumulants κ_{2m} of the Gaussian distribution $(0, \sigma^2)$ vanish for $m \geq 2$ and $\sup_{\xi\in\mathbb{R}} |\varphi(\xi, t) - \mathrm{Re}\varphi(\xi, t)| \leq 2e^{-t}$, one is led to think that the approach to equilibrium of $\mu(\cdot,t)$ might become faster when the symmetrized form of the initial datum presents an increasing number of zero even cumulants.

Theorem 2.3. *Consider problem* (1) *and maintain the same notation as before for* f_0, μ_0, $\tilde{\mu}_0$, φ_0 *and* α_s. *Moreover, assume that there are an integer* χ *greater than* 2 *and a number* δ *in* $[0,2[$ *for which*

i) $\int_{\mathbb{R}} |v|^{2\chi+\delta} f_0(v)\mathrm{d}v < +\infty$;

ii) *the cumulants* κ_{2m} *of* $\tilde{f}_0$ *vanish for* $m = 2,\ldots,\chi$;

iii) φ_0 *meets* (4) *for some strictly positive* p.

Then, there is a strictly positive constant $\overline{C}_{\chi,\delta} = \overline{C}_{\chi,\delta}(f_0;p)$ *for which*

$$\int_{\mathbb{R}} \mid f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} \mid \mathrm{d}v \ \leq\ \overline{C}_{\chi,\delta}\ e^{-(1-2\alpha_{2\chi+\delta})t} \qquad (t \geq 0) \qquad (11)$$

holds true.

Useful information to quantify $\overline{C}_{\chi,\delta}$ can be gathered in Subsection 4.4 and point A.2 of the Appendix.

It should be noted that, except for the centered Gaussian law, the most common distributions do not share condition ii), at least for large values of χ. So, one can legitimately believe that Theorem 2.1 covers usual applications.

In any case, $1 - 2\alpha_{2\chi+\delta}$ is strictly less than one, whichever $\chi \geq 2$ and $\delta > 0$ might be. It would be interesting to check when, under suitable conditions for the initial distribution, the value -1 for the rate of relaxation to equilibrium is actually obtained. The following propositions answer the issue, under the extra-condition that all moments of μ_0 are finite. So, it remains to check whether this moment assumption can be actually recovered from this high order of relaxation to equilibrium. This problem is going to be tackled in a separate later work.

Proposition 2.4. *If* μ_0 *possesses moments of every order and the solution* $\mu(\cdot, t)$ *of* (5) *satisfies*

$$\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \leq Ce^{-t}$$

for some strictly positive constant C, *then*

$$\mu_0(\cdot) = \gamma_\sigma(\cdot) + o_\sigma(\cdot) \tag{12}$$

where o_σ *is a finite signed measure satisfying* $o_\sigma(A) = -o_\sigma(-A)$ *and* $\gamma_\sigma(A) + o_\sigma(A) \geq 0$ *for every Borel subset* A *of* $\mathbb{R}$.

Observe that the Wild formula (cf. (13) in Subsection 3.1) implies that $\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) = |o_\sigma|e^{-t}$ when the initial datum is of the type of (12). So, if one assumes there is some $\rho : [0, +\infty[\to [0, +\infty[$ vanishing at infinity so that $\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \leq Ce^{-t}\rho(t)$, then the total variation $|o_\sigma|$ of o_σ satisfies $|o_\sigma| \leq C\rho(t)$ for all positive t, which is tantamount to asserting that o_σ is the null measure, and this provides a proof for

Corollary 2.5. *If* μ_0 *has moments of every order and the solution* $\mu(\cdot, t)$ *of* (5) *satisfies*

$$\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \leq Ce^{-t}\rho(t)$$

for some ρ *vanishing at infinity and for some positive constant* C, *then* $\mu(\cdot, t) = \gamma_\sigma(\cdot)$ *for every* $t \geq 0$.

Thus, if all the moments of μ_0 are finite, the value for the rate of convergence to equilibrium that one cannot sharpen is just -1 unless μ_0 is Gaussian.

3 Preliminaries

To pave the way for the proofs of the main statements, this section presents some necessary preliminary facts and results. Firstly, it mentions the probabilistic meaning of *Wild's series*, originally pointed out in McKean (1966).

Secondly, it gives new asymptotic expansions for the characteristic function of weighted sums of independent and identically distributed random variables, which complement analogous statements formulated, e.g., in Chapter 8 of Gnedenko and Kolmogorov (1954) or in Chapter 6 of Petrov (1975) and, moreover, in Subsection 3.2 of DGR.

3.1 McKean's interpretation of Wild's sums

Following Wild (1951) one can express the solution $\varphi(\cdot, t)$ of (5) as a time-dependent mixture of characteristic functions, i.e.

$$\varphi(\xi, t) = \sum_{n \geq 1} e^{-t}(1 - e^{-t})^{n-1} \hat{q}_n(\xi; \varphi_0) \tag{13}$$

where

$$\begin{cases} \hat{q}_1(\xi; \varphi_0) := \varphi_0(\xi) \\ \hat{q}_n(\xi; \varphi_0) = \frac{1}{n-1} \sum_{k=1}^{n-1} \hat{q}_k(\xi; \varphi_0) \star \hat{q}_{n-k}(\xi; \varphi_0) \qquad (n \geq 2) \end{cases}$$

and $\star$ denotes the so-called *Wild product* defined by

$$g_1(\xi) \star g_2(\xi) := \frac{1}{2\pi} \int_0^{2\pi} g_1(\xi \cos\theta) \cdot g_2(\xi \sin\theta) \; d\theta \; .$$

The Wild series, thanks to a symmetry property of the Wild product, yields a useful decomposition of $\mu(\cdot, t)$ we will use later on. Such a decomposition involves the symmetrized form $\tilde{\mu}$ of a probability measure μ defined by $\tilde{\mu}(B) := [\mu(B) + \mu(-B)]/2$, for any B in $\mathscr{B}(\mathbb{R})$. It is well-known that, if $\mu^{(s)}(\cdot, t)$ denotes the solution of (5) with initial datum $\tilde{\mu}_0$ (see (6)), one can write

$$\mu(\cdot, t) - \mu^{(s)}(\cdot, t) = o_0(\cdot) e^{-t} \tag{14}$$

with $o_0(\cdot) := \mu_0(\cdot) - \tilde{\mu}(\cdot)$.

The next description of the probabilistic re-interpretation of (13) follows Subsection 3.1 in DGR closely. Accordingly, introduce exactly in the same notation adopted therein the measurable space $(\Omega, \mathscr{F})$ as a product, together with its coordinate random elements ν, τ, $\theta := (\theta_n)_{n\geq 1}$, $\upsilon := (\upsilon_n)_{n\geq 1}$. Then, remember the definition of the random elements δ_j, π_j given in terms of *McKean trees* and put $\beta = (\nu, \tau, \theta)$. About the random variables π_j recall the fundamental equality

$$\sum_{j=1}^{\nu} \pi_j^2 \equiv 1 \tag{15}$$

which holds true whenever τ belongs to $\mathbb{G}(\nu)$.

Now, for some fixed initial datum μ_0 for problem (5) define a family $(\mathsf{P}_t)_{t\geq 0}$ of probability measures on $(\Omega, \mathscr{F})$ according to (12) in DGR. Next, consider the random variable

$$V = \sum_{j=1}^{\nu} \pi_j \upsilon_j \tag{16}$$

and note, through the Wild formula, that

$$\mu(B, t) = \mathsf{P}_t\{V \in B\}\ , \qquad B \in \mathscr{B}(\mathbb{R}),\ t \geq 0$$

$\mu(\cdot, t)$ being the solution of (5) with μ_0 as initial datum.

Consequently, the random variables υ_n turn out to be *conditionally independent* given β with respect to each P_t. Moreover, since β and υ are independent, one can think of the conditional probability distribution of V given β as the distribution of a weighted sum of independent random variables. Indeed, for any fixed elementary case $\overline{\omega}$ in Ω, one can define the random variable

$$\overline{V}(\cdot) := \sum_{j=1}^{\nu(\overline{\omega})} \pi_j(\overline{\omega}) \upsilon_j(\cdot) \tag{17}$$

on $(\Omega, \mathscr{F})$ for which

$$\mathsf{P}_t\{V \leq x \mid \beta\}(\overline{\omega}) = \mathsf{P}_t\{\overline{V} \leq x\} \qquad (x \in \mathbb{R}, t \geq 0) \tag{18}$$

holds P_t-almost surely in $\overline{\omega}$. This last equality plays a central role in the rest of the paper, since it allows to work on a finite sum of independent random variables using typical tools of the *central limit problem.* In this context it is important to examine the behaviour of the moments of the random variable V. Their evaluation essentially depends on sums of powers of the π_j through the following identity proved in Gabetta and Regazzini (2006a):

$$\mathsf{E}_t\left[\sum_{j=1}^{\nu} |\pi_j|^m\right] = e^{-(1-2\alpha_m)t} \tag{19}$$

α_m being the same as is in Section 2.

3.2 Some asymptotic expansions for the characteristic function of weighted sums of independent random variables

As in Subsection 3.2 of DGR, the subject to be investigated here is the behaviour of the characteristic function of weighted sums of independent and identically distributed random variables. The expansions given here turn out to be more careful than the analogous ones contained in the above-mentioned work, since now it is assumed that the common probability law of the summands possesses moments of arbitrarily high order. Cumulants will play a central role in the analysis of the remainder terms. Finally, the study of the convergence of weighted sums will provide right conditions to improve the rate of approach to equilibrium for solutions of equation (1).

In the rest of this subsection $(X_j)_{j\geq 1}$ stands for a sequence of independent and identically distributed real-valued random variables on some probability space $(E, \mathscr{E}, \mathsf{Q})$ with common non-degenerate distribution ζ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. It is assumed that ζ is symmetric (i.e. $\zeta(B) = \zeta(-B)$ for every Borel set B of $\mathbb{R}$) and possesses (finite) moments up to the order $k+\delta$, where $k = 2\chi$, χ being some integer greater than 1 and δ being an element of the interval $[0, 2[$. Denote the r-th moment and the absolute r-th moment of ζ by $\mathfrak{m}_r$ and $\overline{\mathfrak{m}}_r$, respectively, and notice that existing moments of odd order vanish. In particular, the variance σ^2 of ζ coincides with $\mathfrak{m}_2$. Set

$\psi(\xi) := \int_{\mathbb{R}} e^{i\xi x}\zeta(\mathrm{d}x)$, which turns out to be an even real-valued function, and for every positive integer n define $\{c_{1,n}, \dots, c_{n,n}\}$ to be an array of real constants such that

$$\sum_{j=1}^{n} c_{j,n}^2 = 1 \tag{20}$$

holds for every n. Then, let V_n be the sum of $Y_{1,n}, \dots, Y_{n,n}$ when

$$Y_{j,n} := \frac{1}{\sigma} c_{j,n} X_{j,n} \qquad (j = 1, \dots, n)$$

and let ψ_n be the characteristic function of V_n. Consider the r-th cumulant κ_r and recall that, in general, it can be defined by

$$\kappa_r \;=\; r! \sum_{(*)} (-1)^{s-1} \cdot (s-1)! \cdot \prod_{l=1}^{r} \frac{1}{k_l!} \left(\frac{\mathfrak{m}_l}{l!}\right)^{k_l} \qquad (r = 1, \dots, k) \tag{21}$$

where the symbol $(*)$ means that the sum is carried out over all non-negative integer solutions $(k_1, \dots, k_r)$ of equations

$$k_1 + 2k_2 + \cdots + rk_r = r$$
$$k_1 + k_2 + \cdots + k_r = s$$

with the proviso that $0^0 = 1$. Symmetry of ζ implies that existing cumulants of odd order are equal to zero. Moreover, from a technical fact proved in Appendix A.1, after defining $y_0 := \{[-6\sigma^2 + (36\sigma^4 + 12\mathfrak{m}_4)^{1/2}]/\mathfrak{m}_4\}^{1/2}$ one has $\psi(\xi) \geq 1/2$ if $|\xi| \leq y_0$ and

$$\log \psi(\xi) = \sum_{r=1}^{\chi} (-1)^r \frac{\kappa_{2r}}{(2r)!} \xi^{2r} + \xi^k \cdot \epsilon_k(\xi) \ , \tag{22}$$

where ϵ_k is continuous on $[-y_0, y_0]$, differentiable on $[-y_0, y_0] \setminus \{0\}$ and satisfies $\epsilon_k(0) = 0$, $\lim_{\xi \to 0} \varrho_k(\xi) = 0$ with $\varrho_k(\xi) := \xi \cdot \epsilon_k^{'}(\xi)$. Consequently, $M_0^{(k)} := \sup_{\xi \in [-y_0, y_0]} |\epsilon_k(\xi)|$ and $M_1^{(k)} := \sup_{\xi \in [-y_0, y_0]} |\varrho_k(\xi)|$ are two finite constants which depend only on the behaviour of the common probability law ζ.

Now, following the same line of reasoning as in Chapter 6 of Petrov (1975) we introduce the quantities

$$\tilde{\lambda}_{r,n} := \frac{\kappa_{2r}}{\sigma^{2r}} \sum_{j=1}^{n} c_{j,n}^{2r} \qquad (r = 1, \ldots, \chi) \tag{23}$$

and define the polynomials

$$\tilde{P}_{r,n}(\xi) \ := \ \sum_{(*)} \left(\prod_{m=1}^{r} \frac{1}{k_m!} \left(\frac{\tilde{\lambda}_{m+1,n}}{(2m+2)!} \right)^{k_m} \right) (-1)^{r+s} \xi^{2(r+s)} \tag{24}$$

for $r = 1, \ldots, \chi - 1$. In addition we introduce another family of functions $\eta_{k,n}$, which will be used to approximate the characteristic functions ψ_n, defined by

$$\eta_{k,n}(\xi) = e^{-\xi^2/2} + \sum_{r=1}^{\chi-1} \tilde{P}_{r,n}(\xi) \cdot e^{-\xi^2/2} \qquad (\xi \in \mathbb{R}) \ . \tag{25}$$

At this stage we are in a position to state a couple of preliminary results, which play an important role in the rest of the paper.

Lemma 3.1. *Assume* $\chi = 2$ (*i.e.* $k = 4$) *and* $\delta = 0$. *Then, there exists a positive constant* C_4^*, *depending only on the behaviour of* ζ, *such that* (*in the same notations as in* (22))

$$|\psi_n(\xi) - \eta_{4,n}(\xi)| \leq C_4^* \ \xi^4 e^{-\xi^2/2} \left[\xi^4 \sum_{j=1}^{n} c_{j,n}^4 + \sum_{j=1}^{n} c_{j,n}^4 \left| \epsilon_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \right] \tag{26}$$

$$|\psi_n(\xi) - \eta_{4,n}(\xi)| \leq C_4^* \ \xi^4 (1 + \xi^4) e^{-\xi^2/2} \left[\sum_{j=1}^{n} c_{j,n}^6 + \sum_{j=1}^{n} c_{j,n}^4 \left| \epsilon_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \right] \tag{27}$$

and

$$|\psi_n'(\xi) - \eta_{4,n}'(\xi)|$$
$$\leq C_4^* \ |\xi|^3 (1 + \xi^6) e^{-\xi^2/2} \left[\sum_{j=1}^{n} c_{j,n}^6 + \sum_{j=1}^{n} c_{j,n}^4 \left(\left| \epsilon_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \ + \ \left| \varrho_4 \left(\frac{c_{j,n}\xi}{\sigma} \right) \right| \right) \right] \tag{28}$$

hold true for every $|\xi| \leq A_{4,n} := \sigma y_0 \left(\sum_{j=1}^n c_{j,n}^4\right)^{-1/4}$.

For general $k = 2\chi$ and δ one has

Lemma 3.2. *If* $|\xi| \leq A_{k,\delta,n} := \sigma y_0 \left(\sum_{j=1}^n c_{j,n}^4\right)^{-1/(k+\delta)}$ *then*

$$|\psi_n(\xi) - \eta_{k,n}(\xi)| \leq C_{k,\delta}^* p_{0,k}(\xi) e^{-\xi^2/2} \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta}\right) \tag{29}$$

and

$$|\psi_n'(\xi) - \eta_{k,n}'(\xi)| \leq C_{k,\delta}^* p_{1,k}(\xi) e^{-\xi^2/2} \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta}\right) \tag{30}$$

where $C_{k,\delta}^*$ *is a constant depending only on the behaviour of* ζ *and* $p_{0,k}(\xi)$, $p_{1,k}(\xi)$ *are polynomials whose coefficients depend only on* k.

The proofs of these lemmata are deferred to Appendix A.2, in which one can also find directions for the evaluation of $C_{k,\delta}^*$, $p_{0,k}(\xi)$ and $p_{1,k}(\xi)$. Inequalities (29) and (30) immediately entail

$$\left(\int_{-A_{k,\delta,n}}^{A_{k,\delta,n}} |\psi_n(\xi) - \eta_{k,n}(\xi)|^2 \mathrm{d}\xi\right)^{1/2} \leq C_{k,\delta}^* a_k \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta}\right) \tag{31}$$

and

$$\left(\int_{-A_{k,\delta,n}}^{A_{k,\delta,n}} |\psi_n'(\xi) - \eta_{k,n}'(\xi)|^2 \mathrm{d}\xi\right)^{1/2} \leq C_{k,\delta}^* a_k \left(\sum_{j=1}^n |c_{j,n}|^{k+\delta}\right) \tag{32}$$

with $a_k := \max\left\{\left(\int_{\mathbb{R}} p_{0,k}^2(\xi) e^{-\xi^2} \mathrm{d}\xi\right)^{1/2}; \left(\int_{\mathbb{R}} p_{1,k}^2(\xi) e^{-\xi^2} \mathrm{d}\xi\right)^{1/2}\right\}$.

4 Proofs of the main results

First we prove Theorem 2.1 and then we focus on Proposition 2.4. In fact they rest on similar arguments. Then, we will provide proofs for Theorems 2.2 and 2.3 by adapting methods used in Section 4 of DGR.

Before starting, it is worth setting some new symbols which will be adopted hereafter. First of all, choose a version for the conditional distribution function $\mathsf{P}_t\{V \le x \mid \beta\}$ and call it $\mathsf{F}^*(x)$. In view of (18), it does not depend on t. $\mathsf{F}^*(x)[\overline{\omega}]$ will indicate dependence of $\mathsf{F}^*(x)$ on a specific sample point $\overline{\omega}$ in Ω. The Fourier-Stieltjes transform of $\mathsf{F}^*(\cdot)[\overline{\omega}]$ will be designated by $\varphi^*(\cdot)[\overline{\omega}]$. Moreover, integral over a measurable subset S of Ω will be often denoted by $\mathsf{E}[\cdot\ ; S]$. Symbols $\mathfrak{m}_r$ and $\overline{\mathfrak{m}}_r$ for $\int x^r \mu_0(\mathrm{d}v)$ and $\int |x|^r \mu_0(\mathrm{d}x)$, respectively, will continue to be used and σ^2 will designate the value of $\mathfrak{m}_2$, while y_0 will stand for the quantity $\{[-6\sigma^2 + (36\sigma^4 + 12\mathfrak{m}_4)^{1/2}]/\mathfrak{m}_4\}^{1/2}$.

4.1 Proof of Theorem 2.1

Assume initially that μ_0 is symmetric. For simplicity, introduce the *re-scaled solution* $\mu_\sigma(\cdot, t)$ defined by $\mu_\sigma(B, t) := \mu(\sigma B, t)$, where $\sigma B := \{y = \sigma x \mid x \in B\}$ for every B in the Borel class of $\mathbb{R}$. By the homogeneity of the total variation distance we have $\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) = \mathrm{d}_{TV}(\mu_\sigma(\cdot, t); \gamma)$ where γ is a shorthand for the standard normal law γ_1. Now, thanks to the elementary inequality

$$\mathrm{d}_{TV}(\mu_\sigma(\cdot, t); \gamma) \ \ge\ \frac{1}{2} \sup_{\xi \in \mathbb{R}} |\varphi(\xi/\sigma, t) - e^{-\xi^2/2}|\ , \tag{33}$$

one can employ the expansions given in Subsection 3.2. First, observe that for any small ε in $]0, \sigma y_0]$ one has

$$\sup_{\xi \in \mathbb{R}} |\varphi(\xi/\sigma, t) - e^{-\xi^2/2}| \ \ge\ |\varphi(\varepsilon/\sigma, t) - e^{-\varepsilon^2/2}|$$
$$= \left|\mathsf{E}_t\left\{\mathsf{E}_t\left[e^{i\varepsilon V/\sigma} \mid \beta\right] - e^{-\varepsilon^2/2}\right\}\right| \ = \ \left|\int_\Omega \left\{\varphi^*(\varepsilon/\sigma)[\overline{\omega}] - e^{-\varepsilon^2/2}\right\} \mathsf{P}_t(\mathrm{d}\overline{\omega})\right|\ . \tag{34}$$

Next, after fixing any $\overline{\omega}$ in Ω, substitute $\nu(\overline{\omega})$ for n and $\pi_j(\overline{\omega})$ for $c_{j,n}$ ($j = 1, 2, \dots, n$) in Lemma 3.1. This way, $\psi_n(\xi)$ changes into $\varphi^*(\xi/\sigma)$ and the restriction, that Lemma 3.1 imposes on ε, becomes $|\varepsilon| \le \sigma y_0 \left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega})\right)^{-1/4}$. Then, if ε belongs to $]0, \sigma y_0]$, these restrictions are met with P_t-probability

1 for each t. Hence, (26) can be applied with

$$\eta_4(\xi)[\overline{\omega}] := e^{-\xi^2/2} + \frac{\kappa_4}{4!\sigma^4}\left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega})\right) \xi^4 e^{-\xi^2/2}$$

in place of $\eta_{4,n}(\xi)$. If $R_4^*(\xi)[\overline{\omega}]$ stands for $\varphi^*(\varepsilon/\sigma)[\overline{\omega}] - \eta_4(\xi)[\overline{\omega}]$, the last member in (34) can be written as

$$\begin{aligned}
&\Big|\int_\Omega R_4^*(\varepsilon)[\overline{\omega}]\mathsf{P}_t(\mathrm{d}\overline{\omega}) \ + \ \frac{\kappa_4}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}\int_\Omega \left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^4(\overline{\omega})\right)\mathsf{P}_t(\mathrm{d}\overline{\omega})\Big| \\
&\qquad = \Big|\int_\Omega R_4^*(\varepsilon)\mathrm{d}\mathsf{P}_t \ + \ \frac{\kappa_4}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-\frac{1}{4}t}\Big| \\
&\qquad \geq \Big|\frac{|\kappa_4|}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-\frac{1}{4}t} \ - \ \Big|\int_\Omega R_4^*(\varepsilon)\mathrm{d}\mathsf{P}_t\Big|\Big| \qquad (35)
\end{aligned}$$

where the equality follows from (19) and the inequality from $\big|a+b\big| \geq \big||a| - |b|\big|$. Now, the claim is that there is ε independent of t and small enough to have

$$\Big|\int_\Omega R_4^*(\varepsilon)\mathrm{d}\mathsf{P}_t\Big| \ \leq \ \frac{|\kappa_4|}{2\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-\frac{1}{4}t} \qquad (36)$$

for every non-negative t. To this end recall: that ϵ_4 (see (22)) is a continuous function depending only on the initial datum μ_0 so that $\epsilon_4(0) = 0$; that $|\kappa_4|$ is strictly positive; that the constant $C_4^* = C_4^*(\mu_0)$ can never be chosen equal to zero. The inequality

$$|\epsilon_4(x)| \ \leq \ \frac{|\kappa_4|}{4\cdot 4!\sigma^4 C_4^*}$$

is surely satisfied for every x belonging to a suitable non-degenerate interval $[-\overline{x}, \overline{x}]$ included in $[-y_0, y_0]$. Thus, taking (26) into account, one can write

$$\int_\Omega \left[C_4^* \ \varepsilon^4 e^{-\varepsilon^2/2}\sum_{j=1}^{\nu(\overline{\omega})}\pi_j^4(\overline{\omega})\Big|\epsilon_4\left(\frac{\pi_j^4(\overline{\omega})\varepsilon}{\sigma}\right)\Big|\right]\mathsf{P}_t(\mathrm{d}\overline{\omega}) \ \leq \ \frac{|\kappa_4|}{4\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-\frac{1}{4}t} \qquad (37)$$

for every ε in $]0, \sigma\overline{x}]$ and $t \geq 0$. Moreover,

$$C_4^* \ \varepsilon^8 e^{-\varepsilon^2/2}e^{-\frac{1}{4}t} \ \leq \ \frac{|\kappa_4|}{4\cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}e^{-\frac{1}{4}t}$$

is valid for every non-negative t provided that ε is chosen not greater than $\overline{\overline{x}} := \left(\frac{|\kappa_4|}{4 \cdot 4! C_4^* \sigma^4}\right)^{1/4}$. Thus, in view of (26), (36) is satisfied for ε in $]0, \min\{\sigma\overline{x}; \overline{\overline{x}}\}]$. To conclude the proof in the symmetric case, fix ε as above in order to have (36) and use the following elementary fact: If $|b| \leq |a|/2$, then $\big||a| - |b|\big| = |a| - |b| \geq |a|/2$. Apply this to (35) to get

$$\Big|\frac{|\kappa_4|}{4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2} e^{-\frac{1}{4}t} \ - \ \Big|\int_{\Omega} R_4^*(\varepsilon) \mathrm{dP}_t\Big|\Big| \ \geq \ \frac{|\kappa_4|}{2 \cdot 4!\sigma^4}\varepsilon^4 e^{-\varepsilon^2/2} e^{-\frac{1}{4}t}$$

which, in view of (34), provides a lower bound for $\mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma)$. When μ_0 is symmetric, the constant $\tilde{C}$, which appears in Theorem 2.1, can be taken equal to $\frac{|\kappa_4|}{4 \cdot 4! \sigma^4}\varepsilon^4 e^{-\varepsilon^2/2}$ with ε in $]0, \min\{\sigma\overline{x}; \overline{\overline{x}}\}]$.

When μ_0 is not symmetric, employ its symmetrized form $\tilde{\mu}_0$ and recall (14) to obtain

$$\begin{aligned} |\mu^{(s)}(B, t) - \gamma_\sigma(B)| \ &= \ |\mu(B, t) - o_0(B)e^{-t} - \gamma_\sigma(B)| \\ \leq \ |\mu(B, t) - \gamma_\sigma(B)| \ + \ 2e^{-t} \ &\leq \ \mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \ + \ 2e^{-t} \qquad (B \in \mathscr{B}(\mathbb{R})) \end{aligned}$$

which plainly entails

$$\mathrm{d}_{TV}(\mu^{(s)}(\cdot, t); \gamma_\sigma) \ \leq \ \mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \ + \ 2e^{-t} \ . \tag{38}$$

From the first part of the proof one can find a constant $\tilde{C}(\tilde{\mu}_0) \leq 2$ for which

$$\mathrm{d}_{TV}(\mu^{(s)}(\cdot, t); \gamma_\sigma) \ \geq \ \tilde{C}(\tilde{\mu}_0) e^{-\frac{1}{4}t} \ .$$

Hence,

$$\begin{aligned} \mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma) \ &\geq \ \mathrm{d}_{TV}(\mu^{(s)}(\cdot, t); \gamma_\sigma) \ - \ 2e^{-t} \\ &\geq \ \tilde{C}(\tilde{\mu}_0) e^{-\frac{1}{4}t} \ - \ 2e^{-t} \ \geq \ \frac{1}{2}\tilde{C}(\tilde{\mu}_0) e^{-\frac{1}{4}t} \end{aligned}$$

holds provided that $t \geq \hat{t} := -\log[(\tilde{C}(\tilde{\mu}_0)/4)^{4/3}]$ where $\hat{t}$ is strictly positive. To conclude the proof observe that (7) is valid taking, for example,

$$\tilde{C} = \tilde{C}(\mu_0) := \min\left\{\frac{1}{2}\tilde{C}(\tilde{\mu}_0); \inf_{t \in [0, \hat{t}]} \mathrm{d}_{TV}(\mu(\cdot, t); \gamma_\sigma)\right\} \ .$$

Finally, $\inf_{t\in[0,\hat{t}]} \mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma)$ is strictly positive in view of the existence of the minimum combined with the uniqueness of the solution of Kac's equation. This point is clarified in Subsection A.3 of the Appendix.

4.2 Proof of Proposition 2.4

To prove this proposition under the assumption that all the moments of μ_0 are finite, it will suffice to prove that all the cumulants $\tilde{\kappa}_{2m}$ of even order of $\tilde{\mu}_0$ are zero for $m = 2, 3, \ldots$. Thanks to (38), the inequality, which appears in the statement of Proposition 2.4, can be re-written as

$$\mathrm{d}_{TV}(\mu^{(s)}(\cdot,t);\gamma_\sigma) \ \leq \ (C+2)e^{-t} \ . \tag{39}$$

In view of this fact we can assume, without real loss of generality, that μ_0 is symmetric. Then, supposing $\kappa_{2m} = 0$ for $m = 2, \ldots, s-1$ and $\kappa_{2s} \neq 0$ for some integer s greater than 2, one contradicts (39).

As in the previous subsection, write

$$\begin{aligned} 2\ \mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma) \ &\geq \ \sup_{\xi\in\mathbb{R}} |\varphi(\xi/\sigma,t) - e^{-\xi^2/2}| \\ &\geq \ \Big| \int_\Omega \Big\{ \varphi^*(\varepsilon/\sigma)[\overline{\omega}] - e^{-\varepsilon^2/2} \Big\} \, \mathsf{P}_t(\mathrm{d}\overline{\omega}) \Big| \end{aligned} \tag{40}$$

where ε is any positive constant not greater than σy_0. Following the general lines of Subsection 3.2, define

$$\eta_{2s}(\xi)[\overline{\omega}] := e^{-\xi^2/2} + (-1)^s \frac{\kappa_{2s}}{(2s)!\sigma^{2s}} \left(\sum_{j=1}^{\nu(\overline{\omega})} \pi_j^{2s}(\overline{\omega}) \right) \xi^{2s} e^{-\xi^2/2} \ .$$

After setting $R^*_{2s}(\xi)[\overline{\omega}] := \varphi^*(\varepsilon/\sigma)[\overline{\omega}] - \eta_{2s}(\xi)[\overline{\omega}]$, the last member in (40) becomes

$$\begin{aligned}
&\Big|\int_\Omega R^*_{2s}(\varepsilon)[\overline{\omega}]\mathsf{P}_t(\mathrm{d}\overline{\omega}) \;+\; (-1)^s\frac{\kappa_{2s}}{(2s)!\sigma^{2s}}\varepsilon^{2s}e^{-\varepsilon^2/2}\int_\Omega\left(\sum_{j=1}^{\nu(\overline{\omega})}\pi_j^{2s}(\overline{\omega})\right)\mathsf{P}_t(\mathrm{d}\overline{\omega})\Big| \\
&\qquad = \;\Big|\int_\Omega R^*_{2s}(\varepsilon)\mathrm{d}\mathsf{P}_t \;+\; (-1)^s\frac{\kappa_{2s}}{(2s)!\sigma^{2s}}\varepsilon^{2s}e^{-\varepsilon^2/2}e^{-(1-2\alpha_{2s})t}\Big| \\
&\qquad \geq \;\Big|\frac{|\kappa_{2s}|}{(2s)!\sigma^{2s}}\varepsilon^{2s}e^{-\varepsilon^2/2}e^{-(1-2\alpha_{2s})t} \;-\; \Big|\int_\Omega R^*_{2s}(\varepsilon)\mathrm{d}\mathsf{P}_t\Big|\Big| \,. \qquad (41)
\end{aligned}$$

Now, if $|\varepsilon| \leq \sigma y_0$, an application of (29) with $k = 2s$ and $\delta = 1$ combined with (19) yields

$$\begin{aligned}
&\Big|\int_\Omega R^*_{2s}(\varepsilon)\mathrm{d}\mathsf{P}_t\Big| \;\leq\; \int_\Omega \big|R^*_{2s}(\varepsilon)\big|\mathrm{d}\mathsf{P}_t \\
&\leq\; C^*_{2s,1}|\varepsilon|^{2s+1}[1+|\varepsilon|^{h(2s)}]\; e^{-\varepsilon^2/2}e^{-(1-2\alpha_{2s+1})t} \\
&\leq\; C^*_{2s,1}|\varepsilon|^{2s+1}[1+(\sigma y_0)^{h(2s)}]\; e^{-\varepsilon^2/2}e^{-(1-2\alpha_{2s+1})t} \qquad (42)
\end{aligned}$$

for every non-negative t. If ε satisfies the further restriction

$$|\varepsilon| \leq \frac{1}{2C^*_{2s,1}}\cdot\frac{1}{1+(\sigma y_0)^{h(2s)}}\cdot\frac{|\kappa_{2s}|}{(2s)!\sigma^{2s}}$$

then one can re-write (42) as

$$\Big|\int_\Omega R^*_{2s}(\varepsilon)\mathrm{d}\mathsf{P}_t\Big| \;\leq\; \frac{|\kappa_{2s}|}{2\cdot(2s)!\sigma^{2s}}\varepsilon^{2s}e^{-\varepsilon^2/2}e^{-(1-2\alpha_{2s})t}\,. \qquad (43)$$

Hence, inequalities (41) and (43) entail

$$\frac{|\kappa_{2s}|}{2\cdot(2s)!\sigma^{2s}}\varepsilon^{2s}e^{-\varepsilon^2/2}e^{-(1-2\alpha_{2s})t} \;\leq\; 2\;\mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma) \;\leq\; 2(C+2)e^{-t}$$

for every non-negative t, which contradicts the fact that $(1-2\alpha_{2s})$ is strictly smaller than 1. Thus, κ_{2s} must vanish implying $\mu_0 = \gamma_\sigma$ since γ_σ is uniquely determined by its moments. Finally, if μ_0 is not symmetric, then $\tilde{\mu}_0 = \gamma_\sigma$.

4.3 Proof of Theorem 2.2 when $k+\delta=4$

We shall closely follow the proof of Theorem 2.1 in DGR. First of all, let us assume that condition

(H) $\qquad f_0$ *and, consequently,* $f(\cdot,t)$ *are even functions*

is in force. This does not limit the generality of subsequent reasoning thanks to (9)-(10) of DGR. Since $\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(v)$ represents a version of the conditional probability density function of V given β, in view of basic properties of conditional expectation one has

$$\int_{\mathbb{R}} \mid f(v,t) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} \mid \mathrm{d}v \;=:\; ||\, f(v,t) - g_\sigma(v)\,||_1$$

$$\leq\; \mathsf{E}_t\left[||\,\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(v) - g_\sigma(v)\,||_1\right] \;=\; \mathsf{E}_t\left[||\,\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)\,||_1\right] \tag{44}$$

where $g_\sigma(v)\mathrm{d}v = \gamma_\sigma(\mathrm{d}v)$. Moreover, from Proposition 2.2 of DGR, which can be applied to f_0 thanks to the hypotheses in Theorem 2.2 and (H), there are α and λ for which

$$\mid \varphi_0(\xi) \mid \;\leq\; \left(\frac{\lambda^2}{\lambda^2+\xi^2}\right)^\alpha \tag{45}$$

holds true for every real ξ. In particular one can set $\alpha = (2\cdot\lceil 2/p\rceil)^{-1}$, p being the same as in (4) and $\lceil s\rceil$ standing for the least integer not less than s. Define $U \subset \Omega$ by

$$U \;:=\; \{\nu \leq \overline{n}\} \;\cup\; \{\prod_{j=1}^{\nu}\pi_j = 0\} \;\cup\; \{\sum_{j=1}^{\nu}\pi_j^4 \geq \overline{\delta}\} \tag{46}$$

with $\overline{n} = 17\cdot\lceil 2/p\rceil$ and

$$\overline{\delta} = \min\left\{\frac{1}{2^{\overline{n}}\;\overline{n}!};\, \frac{\sigma^8}{16\; y_0^4\overline{\mathfrak{m}}_3^4}\right\} \leq \frac{1}{2^{\overline{n}}\;\overline{n}!}\;.$$

Next, check that U belongs to $\mathscr{F}$ and rewrite the last term in (44) as

$$\mathsf{E}_t\left[||\,\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)\,||_1; U\right] \;+\; \mathsf{E}_t\left[||\,\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)\,||_1; U^c\right]\;. \tag{47}$$

By the same arguments as the ones used to prove (22) in DGR one obtains

$$\mathsf{P}_t\{\nu \leq \overline{n}\} \ \leq \ \overline{n}\ e^{-t} \qquad \text{and} \qquad \mathsf{P}_t\{\prod_{j=1}^{\nu} \pi_j = 0\} = 0 \ .$$

As to the third component of the union in the definition of U, one can combine Markov's (with power 2) and Liapounov's inequalities to get

$$\mathsf{P}_t\{\sum_{j=1}^{\nu} \pi_j^4 \geq \overline{\delta}\} \ \leq \ \frac{1}{\overline{\delta}^2}\mathsf{E}_t\left[\left(\sum_{j=1}^{\nu} \pi_j^4\right)^2\right] \ \leq \ \frac{1}{\overline{\delta}^2}\mathsf{E}_t\left[\sum_{j=1}^{\nu} \pi_j^6\right] \ \leq \ \frac{1}{\overline{\delta}^2}e^{-\frac{3}{8}t} \ .$$

Exponent 3/8 follows from application of (19) with $m = 6$. Now, combination of all the above computations provides an estimate for the probability of U under P_t, that is

$$\mathsf{P}_t(U) \ \leq \ [\overline{n} + 1/\overline{\delta}^2]e^{-\frac{3}{8}t} \qquad (t \geq 0) \ . \tag{48}$$

Inequality (48) leads immediately to the upper bound

$$\mathsf{E}_t\left[||\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)||_1 ; U\right] \ \leq \ 2\mathsf{P}_t(U) \ \leq \ 2\ [\overline{n} + 1/\overline{\delta}^2]e^{-\frac{3}{8}t} \ . \tag{49}$$

To control the integral over U^c appearing in (47), invoke the *Beurling inequality* formulated in Proposition 4.1 of DGR to obtain

$$\mathsf{E}_t\left[||\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)||_1 ; U^c\right] \ \leq \ \frac{1}{\sqrt{2}}\,\mathsf{E}_t\left[\left\{\int_{\mathbb{R}} |\Delta|^2 \mathrm{d}\xi + \int_{\mathbb{R}} |\Delta'|^2 \mathrm{d}\xi\right\}^{1/2} ; \ U^c\right] \tag{50}$$

where $\Delta := \varphi^*(\xi/\sigma) - e^{-\xi^2/2}$ and $\Delta' := \frac{\mathrm{d}}{\mathrm{d}\xi}\Delta$. Applicability of this result is justified by the fact that the restriction to U^c of the conditional characteristic function $\xi \mapsto \varphi^*(\xi) := \int_{\mathbb{R}} e^{i\xi x}\mathrm{d}\mathsf{F}^*(x)$ belongs to $H^1(\mathbb{R})$. To see this, note that $\varphi^*(\xi)[\overline{\omega}] = o(|\xi|^{-34})$ is valid for $|\xi| \to +\infty$ and for $\overline{\omega}$ in U^c. Indeed, thanks to conditional independence and (45), one has

$$|\varphi^*(\xi)| \ \leq \ \prod_{j=1}^{\overline{n}} \left(\frac{\lambda^2}{\lambda^2 + \pi_j^2 \xi^2}\right)^{\alpha}$$

and the claimed "tail behaviour" of φ^* follows from the definitions of $\overline{n}$ and α together with the fact that the random numbers π_j do not vanish on U^c. To complete the argument for $H^1(\mathbb{R})$ regularity use Remark A.3.2 in DGR. Now, the expectation in the right-hand side of (50) is dominated by

$$\mathsf{E}_t\left[\left(\int_{\{|\xi|\le \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi\right)^{1/2} ;\ U^c\right] + \mathsf{E}_t\left[\left(\int_{\{|\xi|\ge \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi\right)^{1/2} ;\ U^c\right]$$
$$+\ \mathsf{E}_t\left[\left(\int_{\{|\xi|\le \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi\right)^{1/2} ;\ U^c\right] + \mathsf{E}_t\left[\left(\int_{\{|\xi|\ge \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi\right)^{1/2} ;\ U^c\right] \quad (51)$$

with

$$\mathrm{A} = \mathrm{A}(\beta) := \frac{\sigma y_0}{\left(\sum_{j=1}^{\nu} \pi_j^4\right)^{1/4}}\ .$$

At this stage apply (27) to the evaluation of the first integral in (51) after noticing that here the function $\eta_{4,n}(\xi)$ equals $e^{-\xi^2/2}$ almost surely since $\kappa_4 = 0$. This leads to

$$\left(\int_{\{|\xi|\le \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi\right)^{1/2} \le\ 2\ \sqrt{2\ \Gamma(17/2)} C_4^* \left(\sum_{j=1}^{\nu} \pi_j^6\right)$$
$$+\ \sqrt{2}\ C_4^* \left[\int_{\mathbb{R}} \xi^8 (1+\xi^4)^2 e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left|\tilde{\epsilon}_4\left(\frac{\pi_j \xi}{\sigma}\right)\right|\right)^2 \mathrm{d}\xi\right]^{1/2} \quad (52)$$

with

$$\tilde{\epsilon}_4(x) := \begin{cases} \frac{\log\varphi_0(x)\ +\ (\sigma^2/2)\ x^2\ -\ (\kappa_4/4!)\ x^4}{x^4} & \text{if } 0 < |x| \le \sigma y_0 \\ \tilde{\epsilon}_4(\sigma y_0) & \text{if } |x| > \sigma y_0 \\ 0 & \text{if } x = 0\ . \end{cases}$$

Note that $\tilde{\epsilon}_4$ is a bounded continuous function. Take expectation of both sides of (52) and recall (19) to obtain

$$\mathsf{E}_t \left(\int\limits_{\{|\xi|\le \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi \right)^{1/2} \le 2\ \sqrt{2\ \Gamma(17/2)} C_4^*\ e^{-\frac{3}{8}t}$$

$$+ \sqrt{2}\ C_4^*\ \mathsf{E}_t \left[\int\limits_{\mathbb{R}} \xi^8 (1+\xi^4)^2 e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 \mathrm{d}\xi \right]^{1/2} . \tag{53}$$

In view of A.4 in the Appendix,

$$\lim_{t\to+\infty} \rho_0^{(1)}(t) = 0 \tag{54}$$

where

$$\rho_0^{(1)}(t) := e^{\frac{1}{4}t}\ \mathsf{E}_t \left[\int\limits_{\mathbb{R}} \xi^8 (1+\xi^4)^2 e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 \mathrm{d}\xi \right]^{1/2} .$$

Similarly, apply (28) to evaluate the second integral in (51) as follows:

$$\left(\int\limits_{\{|\xi|\le \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi \right)^{1/2} \le 4\ \sqrt{\Gamma(19/2)} C_4^* \left(\sum_{j=1}^{\nu} \pi_j^6 \right)$$

$$+ 2\sqrt{2}\ C_4^* \left[\int\limits_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 \mathrm{d}\xi \right]^{1/2}$$

$$+ 2\sqrt{2}\ C_4^* \left[\int\limits_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \left| \tilde{\varrho}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \right| \right)^2 \mathrm{d}\xi \right]^{1/2} \tag{55}$$

with

$$\tilde{\varrho}_4(x) := \begin{cases} x \frac{\mathrm{d}}{\mathrm{d}x} \tilde{\epsilon}_4(x) & \text{if } 0 < |x| < \sigma y_0 \\ l := \lim_{u \uparrow \sigma y_0} \tilde{\varrho}_4(u) & \text{if } |x| \ge \sigma y_0 \\ 0 & \text{if } x = 0\ . \end{cases}$$

Take once again expectation of both sides of (55) and use (19) to get

$$\mathsf{E}_t \left(\int\limits_{\{|\xi| \le \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi \right)^{1/2} \le 4 \sqrt{\Gamma(19/2)} C_4^* e^{-\frac{3}{8}t}$$

$$+ 2\sqrt{2}\, C_4^* \mathsf{E}_t \left[\int\limits_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \Big| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \Big| \right)^2 \mathrm{d}\xi \right]^{1/2}$$

$$+ 2\sqrt{2}\, C_4^* \mathsf{E}_t \left[\int\limits_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \Big| \tilde{\varrho}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \Big| \right)^2 \mathrm{d}\xi \right]^{1/2} . \qquad (56)$$

Another application of A.4 leads to state the following important facts:

$$\lim_{t \to +\infty} \rho_0^{(2)}(t) = \lim_{t \to +\infty} \rho_0^{(3)}(t) = 0 \qquad (57)$$

where

$$\rho_0^{(2)}(t) := e^{\frac{1}{4}t}\, \mathsf{E}_t \left[\int\limits_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \Big| \tilde{\epsilon}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \Big| \right)^2 \mathrm{d}\xi \right]^{1/2}$$

and

$$\rho_0^{(3)}(t) := e^{\frac{1}{4}t}\, \mathsf{E}_t \left[\int\limits_{\mathbb{R}} \xi^6 (1+\xi^{12}) e^{-\xi^2} \left(\sum_{j=1}^{\nu} \pi_j^4 \Big| \tilde{\varrho}_4 \left(\frac{\pi_j \xi}{\sigma} \right) \Big| \right)^2 \mathrm{d}\xi \right]^{1/2} .$$

After determining upper bounds for integrals of the type of $\int_{\{|\xi| \le \mathrm{A}\}}$ one gets down to examining the remaining summands in (51). Minkowski's inequality yields

$$\left(\int\limits_{\{|\xi| \ge \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi \right)^{1/2} \le \left(\int\limits_{\{|\xi| \ge \mathrm{A}\}} |\varphi^*(\xi/\sigma)|^2 \mathrm{d}\xi \right)^{1/2} + \left(\int\limits_{\{|\xi| \ge \mathrm{A}\}} |e^{-\frac{\xi^2}{2}}|^2 \mathrm{d}\xi \right)^{1/2}$$

and

$$\left(\int\limits_{\{|\xi| \ge \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi \right)^{1/2} \le \left(\int\limits_{\{|\xi| \ge \mathrm{A}\}} |\frac{\mathrm{d}}{\mathrm{d}\xi} \varphi^*(\xi/\sigma)|^2 \mathrm{d}\xi \right)^{1/2} + \left(\int\limits_{\{|\xi| \ge \mathrm{A}\}} |\xi e^{-\frac{\xi^2}{2}}|^2 \mathrm{d}\xi \right)^{1/2} .$$

From a well-known inequality, proved e.g. in Lemma 2 of VII.1 in Feller (1968) and since $\max_{x\geq 0} x^k e^{-\alpha x^2} = [k/(2e\alpha)]^{k/2}$, one obtains

$$\left(\int\limits_{\{|\xi|\geq \mathrm{A}\}} e^{-\xi^2}\mathrm{d}\xi\right)^{1/2} \leq \left(\frac{15}{2}\right)^{15/4} e^{-15/4}(\sigma y_0)^{-8}\sum_{j=1}^{\nu}\pi_j^6$$

and

$$\left(\int\limits_{\{|\xi|\geq \mathrm{A}\}} \xi^2 e^{-\xi^2}\mathrm{d}\xi\right)^{1/2} \leq \frac{2+\sqrt{2}}{2}\left(\frac{17}{2}\right)^{17/4} e^{-15/4}(\sigma y_0)^{-8}\sum_{j=1}^{\nu}\pi_j^6 \ .$$

Then, (19) can be applied to have

$$\mathsf{E}_t\left(\int\limits_{\{|\xi|\geq \mathrm{A}\}} e^{-\xi^2}\mathrm{d}\xi\right)^{1/2} \leq \left(\frac{15}{2}\right)^{15/4} e^{-15/4}(\sigma y_0)^{-8}e^{-\frac{3}{8}t} \qquad (58)$$

and

$$\mathsf{E}_t\left(\int\limits_{\{|\xi|\geq \mathrm{A}\}} \xi^2 e^{-\xi^2}\mathrm{d}\xi\right)^{1/2} \leq \frac{2+\sqrt{2}}{2}\left(\frac{17}{2}\right)^{17/4} e^{-15/4}(\sigma y_0)^{-8}e^{-\frac{3}{8}t} \ . \qquad (59)$$

At this point, to control the remaining integrals over $\{|\xi|\geq \mathrm{A}\}$ proceed as in formula (30) in DGR to write

$$\left[\left(\int\limits_{\{|\xi|\geq \mathrm{A}\}} |\varphi^*(\xi/\sigma)|^2\mathrm{d}\xi\right)^{1/2} + \left(\int\limits_{\{|\xi|\geq \mathrm{A}\}} |\frac{\mathrm{d}}{\mathrm{d}\xi}\varphi^*(\xi/\sigma)|^2\mathrm{d}\xi\right)^{1/2}\right]\cdot 1\!\!1_{U^c}$$

$$\leq\ 2\sqrt{2}\left(\int\limits_{\mathrm{A}}^{+\infty} |\varphi^*(\xi/\sigma)|\mathrm{d}\xi\right)^{1/2}\cdot 1\!\!1_{U^c} + \sqrt{2|\varphi^*(\mathrm{A}/\sigma)|}\cdot 1\!\!1_{U^c}\ . \qquad (60)$$

For $\overline{\omega}$ in U^c the bound

$$\mathrm{A}(\overline{\omega})\ \leq\ \frac{\sigma^3}{2\overline{\mathfrak{m}}_3\sum_{j=1}^{\nu(\overline{\omega})}|\pi_j(\overline{\omega})|^3}$$

holds true thanks to the definition of $\overline{\delta}$ and the Liapounov inequality. Thus, Lemma 12 in Chapter 6 of Petrov (1975) can be applied to the characteristic function $\varphi^*(\xi/\sigma)$ with $b = 1/2$ to deduce

$$\sqrt{2\ |\varphi^*(\mathrm{A}/\sigma)|}\ \leq\ \sqrt{2}\, e^{-\frac{1}{12}\mathrm{A}^2}\ \leq\ \sqrt{2}(48/e)^4\mathrm{A}^{-8}\ =\ \sqrt{2}\,(48/e)^4(\sigma y_0)^{-8}\sum_{j=1}^{\nu}\pi_j^6$$

which entails

$$\mathsf{E}_t\sqrt{2\ |\varphi^*(\mathrm{A}/\sigma)|}\ \leq\ \sqrt{2}\ (48/e)^4(\sigma y_0)^{-8}e^{-\frac{3}{8}t}\ . \tag{61}$$

It remains to analyse

$$\left(\int_{\mathrm{A}}^{+\infty}|\varphi^*(\xi/\sigma)|\mathrm{d}\xi\right)^{1/2}\cdot 1\!\!1_{U^c} = \left(\int_{\mathrm{A}}^{+\infty}\prod_{j=1}^{\nu}|\varphi_0(\frac{\pi_j\xi}{\sigma})|\mathrm{d}\xi\right)^{1/2}\cdot 1\!\!1_{U^c}\ .$$

An estimate of this term is made using Proposition 2.2 in DGR together with (33), (34) and (35) therein with $\overline{\varepsilon} = 1/(2\overline{n}!)$. Then,

$$\left(\int_{\mathrm{A}}^{+\infty}|\varphi^*(\xi/\sigma)|\mathrm{d}\xi\right)^{1/2}\cdot 1\!\!1_{U^c} \leq \left[\lambda\sigma\int_{\mathrm{A}/\lambda\sigma}^{+\infty}\left(\frac{1}{\overline{\varepsilon}\eta^{2\overline{n}}}\right)^{\alpha}\mathrm{d}\eta\right]^{1/2} \leq D\left(\sum_{j=1}^{\nu}\pi_j^4\right)^{\frac{2\overline{n}\alpha-1}{8}}\ . \tag{62}$$

The definition of $\overline{n}$ in (46) yields $(2\alpha\overline{n}-1)/8 = 2$. Moreover,

$$\begin{aligned} D &:= \frac{1}{4\overline{\varepsilon}^{\alpha/2}}\ \frac{(\lambda\sigma)^{17/2}}{(\sigma y_0)^8} \\ &\leq\ 2^{13/4}\left[\left(\frac{3}{2\sigma^2}\right)^{17/4} + \left(\frac{2}{1-M}\right)^{17/4}(L_p)^{17/2p}\right] \end{aligned} \tag{63}$$

with

$$L_p :=\ \sup_{\xi\in\mathbb{R}}[|\xi|^p\cdot|\varphi_0(\xi)|]$$

and

$$M\ =\ \exp\left\{-\frac{3\pi^2}{64\left(3+(L_p)^{4/p}\right)^2}\left(\frac{\sqrt{2}\sigma}{8\lceil 2/p\rceil\sigma^3 + 40\pi\sqrt{\lceil 2/p\rceil\ \mathfrak{m}_4}}\right)^2\right\}\ .$$

Taking expectation in (62) entails

$$\mathsf{E}_t\left[\left(\int_{\mathrm{A}}^{+\infty}|\varphi^*(\xi/\sigma)|\mathrm{d}\xi\right)^{1/2}\cdot 1\!\!1_{U^C}\right] \ \leq\ De^{-\frac{3}{8}t}\ . \tag{64}$$

The claimed upper bound (9) follows from (49), (53), (56), (58), (59), (61) and (64).

4.4 Proof of Theorems 2.2 and 2.3 when $2\chi+\delta>4$

This proof differs from the previous one only in the choice of the constants. One can start from (44) under hypothesis (H). Thanks to (H) and to the hypotheses of the theorems to be proved, one can apply Proposition 2.2 of DGR to get (45) with $\alpha=(2\cdot\lceil 2/p\rceil)^{-1}$.

Now, define U exactly as in (46) with $\overline{n}=[k(k+2)+1]\cdot\lceil 2/p\rceil$ and

$$\overline{\delta}=\min\left\{\frac{1}{2^{\overline{n}}\ \overline{n}!};\frac{\sigma^8}{16\ y_0^4\mathfrak{m}_3^4}\right\}\leq\frac{1}{2^{\overline{n}}\ \overline{n}!}\ .$$

The probability of U is then estimated, under each P_t, using the fact that

$$\mathsf{P}_t\{\nu\leq\overline{n}\}\ \leq\ \overline{n}\ e^{-t}\qquad\text{and}\qquad\mathsf{P}_t\{\prod_{j=1}^{\nu}\pi_j=0\}=0$$

whereas, for the third component of the union in the definition of U, one can combine Markov's (with exponent $k/2$) and Liapounov's inequalities to get

$$\mathsf{P}_t\{\sum_{j=1}^{\nu}\pi_j^4\geq\overline{\delta}\}\ \leq\ \frac{1}{\overline{\delta}^{k/2}}\mathsf{E}_t\left[\left(\sum_{j=1}^{\nu}\pi_j^4\right)^{k/2}\right]$$

$$\leq\ \frac{1}{\overline{\delta}^{k/2}}\mathsf{E}_t\left[\sum_{j=1}^{\nu}\pi_j^{k+2}\right]\ \leq\ \frac{1}{\overline{\delta}^{k/2}}e^{-(1-2\alpha_{k+2})t}\ .$$

Thus,

$$\mathsf{P}_t(U)\ \leq\ [\overline{n}+1/\overline{\delta}^{k/2}]\ e^{-(1-2\alpha_{k+2})t}\qquad(t\geq 0)\ . \tag{65}$$

Now, split the term $\mathsf{E}_t\left[||\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)||_1\right]$ into the sum of two contributes, exactly as in (47), and note that (65) entails

$$\mathsf{E}_t\left[||\frac{\mathrm{d}}{\mathrm{d}v}\mathsf{F}^*(\sigma v) - g_1(v)||_1; U\right] \leq 2\mathsf{P}_t(U) \leq 2\left[\overline{n} + 1/\overline{\delta}^{k/2}\right] e^{-(1-2\alpha_{k+2})t} . \tag{66}$$

To control the integral over U^c invoke once again *Beurling's inequality* (see Proposition 4.1 in DGR) to write (50). Applicability of this result rests on the same arguments as those provided in Subsection 4.3. The right-hand side of (50) is split into a sum of four terms, exactly as in (51), with

$$\mathrm{A} = \mathrm{A}(\beta) := \frac{\sigma y_0}{\left(\sum_{j=1}^{\nu} \pi_j^4\right)^{1/(k+\delta)}} .$$

Now apply (31) to the evaluation of the first integral in (51) noticing that the function $\eta_{k,n}(\xi)$ equals $e^{-\xi^2/2}$ almost surely since $\kappa_{2r} = 0$ for $r = 2, \dots, \chi$. This leads to

$$\mathsf{E}_t\left[\left(\int_{\{|\xi|\leq \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi\right)^{1/2}\right] \leq C^*_{k,\delta} a_k \cdot e^{-(1-2\alpha_{2\chi+\delta})t} \tag{67}$$

and

$$\mathsf{E}_t\left[\left(\int_{\{|\xi|\leq \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi\right)^{1/2}\right] \leq C^*_{k,\delta} a_k \cdot e^{-(1-2\alpha_{2\chi+\delta})t}. \tag{68}$$

After determining upper bounds for integrals of the type of $\int_{\{|\xi|\leq \mathrm{A}\}}$ one gets down to examining the remaining summands in (51). Minkowski's inequality gives

$$\left(\int_{\{|\xi|\geq \mathrm{A}\}} |\Delta|^2 \mathrm{d}\xi\right)^{1/2} \leq \left(\int_{\{|\xi|\geq \mathrm{A}\}} |\varphi^*(\xi/\sigma)|^2 \mathrm{d}\xi\right)^{1/2} + \left(\int_{\{|\xi|\geq \mathrm{A}\}} |e^{-\frac{\xi^2}{2}}|^2 \mathrm{d}\xi\right)^{1/2}$$

and

$$\left(\int_{\{|\xi|\geq \mathrm{A}\}} |\Delta'|^2 \mathrm{d}\xi\right)^{1/2} \leq \left(\int_{\{|\xi|\geq \mathrm{A}\}} |\frac{\mathrm{d}}{\mathrm{d}\xi}\varphi^*(\xi/\sigma)|^2 \mathrm{d}\xi\right)^{1/2} + \left(\int_{\{|\xi|\geq \mathrm{A}\}} |\xi e^{-\frac{\xi^2}{2}}|^2 \mathrm{d}\xi\right)^{1/2} .$$

Integrals involving the Gaussian density are controlled as in the previous subsection to give

$$\mathsf{E}_t \left(\int_{\{|\xi| \geq \mathrm{A}\}} e^{-\xi^2} \mathrm{d}\xi \right)^{1/2} \leq \left(\frac{k(k+2)-1}{2e} \right)^{\frac{k(k+2)-1}{4}} (\sigma y_0)^{-k(k+2)/2} \, e^{-(1-2\alpha_{k+2})t} \tag{69}$$

and

$$\mathsf{E}_t \left(\int_{\{|\xi| \geq \mathrm{A}\}} \xi^2 e^{-\xi^2} \mathrm{d}\xi \right)^{1/2}$$

$$\leq \frac{2+\sqrt{2}}{2} \left(\frac{k(k+2)+1}{2e} \right)^{\frac{k(k+2)+1}{4}} (\sigma y_0)^{-k(k+2)/2} \, e^{-(1-2\alpha_{k+2})t} \quad . \tag{70}$$

To control the remaining integrals over the region $\{|\xi| \geq \mathrm{A}\}$ proceed as before writing equation (60). For $\overline{\omega}$ in U^c the bound

$$\mathrm{A}(\overline{\omega}) \ \leq \ \frac{\sigma^3}{2\overline{\mathfrak{m}}_3 \sum_{j=1}^{\nu(\overline{\omega})} |\pi_j(\overline{\omega})|^3}$$

holds true thanks to the definition of $\overline{\delta}$ and the Liapounov inequality. Then, set $b = 1/2$ in Lemma 12 in Chapter 6 of Petrov (1975) to deduce

$$\sqrt{2\,|\varphi^*(\mathrm{A}/\sigma)|} \ \leq \ \sqrt{2}\; e^{-\frac{1}{12}\mathrm{A}^2}$$

$$\leq \ \sqrt{2} \left(\frac{3k(k+2)}{e} \right)^{\frac{k(k+2)}{4}} (\sigma y_0)^{-\frac{k(k+2)}{2}} \cdot \left(\sum_{j=1}^{\nu} \pi_j^4 \right)^{\frac{k(k+2)}{2(k+\delta)}}$$

$$\leq \ \sqrt{2} \left(\frac{3k(k+2)}{e} \right)^{\frac{k(k+2)}{4}} (\sigma y_0)^{-\frac{k(k+2)}{2}} \cdot \left(\sum_{j=1}^{\nu} \pi_j^{k+2} \right)$$

and, therefore,

$$\mathsf{E}_t \sqrt{2\,|\varphi^*(\mathrm{A}/\sigma)|} \ \leq \ \sqrt{2} \left(\frac{3k(k+2)}{e} \right)^{\frac{k(k+2)}{4}} (\sigma y_0)^{-\frac{k(k+2)}{2}} \cdot e^{-(1-2\alpha_{k+2})t} \ . \tag{71}$$

Finally, apropos of $\left(\int_{\mathrm{A}}^{+\infty} |\varphi^*(\xi/\sigma)| \mathrm{d}\xi\right)^{1/2} \cdot 1\!\!1_{U^c}$, one can write

$$\left(\int_{\mathrm{A}}^{+\infty} |\varphi^*(\xi/\sigma)| \mathrm{d}\xi \right)^{1/2} \cdot 1\!\!1_{U^c} \leq D \left(\sum_{j=1}^{\nu} \pi_j^4 \right)^{\frac{2\overline{n}\alpha - 1}{8}} \tag{72}$$

with the same constant D as in (63). The definition of $\overline{n}$, given at the beginning of this subsection, yields $(2\alpha\overline{n}-1)/8 > k/2$. Now, taking expectation in (72) entails

$$\mathsf{E}_t\left[\left(\int_A^{+\infty}|\varphi^*(\xi/\sigma)|\mathrm{d}\xi\right)^{1/2}\cdot 1\!\!1_{U^C}\right] \leq De^{-(1-2\alpha_{k+2})t} . \tag{73}$$

To obtain (11) it will suffice to combine the previous inequalities.

5 Appendix

This appendix contains all the elements which are necessary to complete the proofs given in Section 4. It is split into four parts: The first focuses on a quantification of numbers y_0 such that the Fourier-Stieltjes transform of a symmetric probability law turns out to be greater than 1/2 on $[-y_0, y_0]$. The second presents the proofs of Lemma 3.1 and Lemma 3.2. The third aims at clarifying the conclusion of the proof of Proposition 2.4. Finally, the fourth provides a proof for (54) and (57).

A.1

Let ψ be the Fourier-Stieltjes transform of a symmetric probability law ζ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, namely $\psi(\xi) := \int_{\mathbb{R}} e^{i\xi x}\zeta(\mathrm{d}x)$ for every real ξ. Assume that $\mathfrak{m}_4 := \int_{\mathbb{R}} x^4\zeta(\mathrm{d}x)$ is finite and put $\sigma^2 := \int_{\mathbb{R}} x^2\zeta(\mathrm{d}x)$, $y_0 := \{[-6\sigma^2 + (36\sigma^4 + 12\mathfrak{m}_4)^{1/2}]/\mathfrak{m}_4\}^{1/2}$. If $|\xi| \leq y_0$, then $\psi(\xi) \geq 1/2$.

Proof. From the Taylor expansion for characteristic functions, one can write $\psi(\xi) = 1-(\sigma^2/2)\xi^2 + R(\xi)$ with $|R(\xi)| \leq (\mathfrak{m}_4/24)\xi^4$. See, for example, Section 8.4 in Chow-Teicher (1997). The desired bound is obtained if

$$1-\frac{\sigma^2}{2}\xi^2-\frac{\mathfrak{m}_4}{24}\xi^4 \geq \frac{1}{2}$$

holds true for every ξ belonging to some interval. Now, one can note that the biquadratic equation $\mathfrak{m}_4\xi^4 + 12\sigma^2\xi^2 - 12 = 0$ possesses exactly two real solutions, namely $\pm y_0$, and the previous inequality is satisfied for every ξ in $[-y_0, y_0]$. $\square$

A.2

Proof of Lemma 3.1. Set $\psi_{j,n}$ for the characteristic function of $Y_{j,n}$ ($j = 1, 2, \ldots, n$) and use the definition of V_n combined with independence to write

$$\psi_n(\xi) = \prod_{j=1}^{n} \psi_{j,n}(\xi) = \prod_{j=1}^{n} \psi\left(\frac{c_{j,n}\xi}{\sigma}\right) .$$

If $|\xi| \leq A_{4,n}$, it easily follows that

$$\left|\frac{c_{j,n}\xi}{\sigma}\right| \leq \left|\frac{c_{j,n}\sigma y_0}{\sigma}\left(\sum_{r=1}^{n} c_{r,n}^4\right)^{-1/4}\right| \leq y_0 .$$

Now, using elementary properties of the logarithm, one can combine expansion (22) with property (20) of each array $\{c_{1,n}, \ldots, c_{n,n}\}$ to obtain

$$\log\psi_n(\xi) = \sum_{j=1}^{n} \log\psi_{j,n}(\xi)$$
$$= \sum_{j=1}^{n}\left[-\frac{1}{2}\sigma^2\frac{c_{j,n}^2\xi^2}{\sigma^2} + \frac{1}{4!}\kappa_4\frac{c_{j,n}^4\xi^4}{\sigma^4} + \frac{c_{j,n}^4\xi^4}{\sigma^4}\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right]$$
$$= -\frac{1}{2}\xi^2 + \frac{\tilde{\lambda}_{2,n}}{4!}\xi^4 + R_4(\xi)$$

where

$$R_4(\xi) := \sum_{j=1}^{n} \frac{c_{j,n}^4\xi^4}{\sigma^4}\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right) .$$

Inverting the logarithm, one gets

$$\psi_n(\xi) = e^{-\xi^2/2}\cdot\exp\{\frac{\tilde{\lambda}_{2,n}}{4!}\xi^4\}\cdot\exp\{R_4(\xi)\} . \tag{74}$$

It is easily verified that the restrictions $|u| := |\tilde{\lambda}_{2,n}\xi^4|/4! \leq \kappa_4 y_0/4!$ and $|R_4(\xi)| \leq M_0^{(4)} y_0^4$ hold true when $|\xi| \leq A_{4,n}$, and that $\tilde{\lambda}_{2,n}\xi^4/4! = \tilde{P}_{1,n}(\xi)$.

Finally, set $F(x) := e^x - 1 - x$. At this point we have all the tools to prove (26) and (27). Indeed,

$$
\begin{aligned}
|\psi_n(\xi) - \eta_{4,n}(\xi)| &= e^{-\xi^2/2}\left|e^u \exp\{R_4(\xi)\} - 1 - u\right| \\
&= e^{-\xi^2/2}\left|e^u \exp\{R_4(\xi)\} - e^u + F(u)\right| \\
&\leq e^{-\xi^2/2}\, e^u\left|\exp\{R_4(\xi)\} - 1\right| \;+\; e^{-\xi^2/2}|F(u)|\;.
\end{aligned}
$$

By elementary arguments, if x is any real number satisfying $|x| \leq c$, one has

$$
|e^x - 1| \;\leq\; e^{|x|} - 1 \;\leq\; \left(\frac{e^c - 1}{c}\right)|x| \quad .
$$

This fact can be applied to $R_4(\xi)$ to get

$$
\left|\exp\{R_4(\xi)\} - 1\right| \;\leq\; \xi^4 \cdot \left(\frac{e^{M_0^{(4)} y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \left(\sum_{j=1}^n c_{j,n}^4 \left|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|\right) \quad .
$$

Moreover, since inequality

$$
|F(u)| \;\leq\; \max_{|x| \leq \kappa_4 y_0/4!}\left[\left|\frac{F(x)}{x^2}\right|\right] \xi^8 \left(\sum_{j=1}^n c_{j,n}^4\right)^2
$$

is in force, one can conclude that

$$
\begin{aligned}
&|\psi_n(\xi) - \eta_{4,n}(\xi)| \\
&\leq\; e^{-\xi^2/2}\xi^4 \cdot \exp\{\frac{\kappa_4 y_0^4}{4!}\}\;\left(\frac{e^{M_0^{(4)} y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \left(\sum_{j=1}^n c_{j,n}^4 \left|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\right|\right) \\
&\quad + e^{-\xi^2/2} \max_{|x| \leq \kappa_4 y_0^4/4!}\left[\left|\frac{F(x)}{x^2}\right|\right] \xi^8 \left(\sum_{j=1}^n c_{j,n}^4\right)^2 \quad . \qquad (75)
\end{aligned}
$$

After setting

$$
C_4^{**} := \exp\{\frac{\kappa_4 y_0^4}{4!}\}\;\left(\frac{e^{M_0^{(4)} y_0^4} - 1}{\sigma^4 y_0^4}\right) + \max_{|x| \leq \kappa_4 y_0^4/4!}\left[\left|\frac{F(x)}{x^2}\right|\right]
$$

derivation of (26) and (27) follows by rewriting (75) conveniently. To get (26) it is enough to notice that $\sum_{j=1}^n c_{j,n}^4 \leq 1$, while to deduce (27) one can combine inequality $\left(\sum_{j=1}^n c_{j,n}^4\right)^2 \le \sum_{j=1}^n c_{j,n}^6$ with $\max\{1;\xi^4\} \le (1+\xi^4)$.

To prove (28), start from (74) and take the derivative with respect to ξ. Thus, one obtains

$$
\begin{aligned}
&|\psi_n^{'}(\xi) - \eta_{4,n}^{'}(\xi)| \\
\le\ & \exp\{R_4(\xi)\} \cdot |R_4^{'}(\xi)| \cdot |\eta_{4,n}(\xi) + F(u)e^{-\xi^2/2}| \ +\ |\eta_{4,n}^{'}(\xi)| \cdot \Big|\exp\{R_4(\xi)\} - 1\Big| \\
&+\ \exp\{R_4(\xi)\} \cdot \Big|\frac{\mathrm{d}}{\mathrm{d}\xi}F(u)\Big| \cdot e^{-\xi^2/2} \ +\ \exp\{R_4(\xi)\} \cdot |F(u)| \cdot |\xi| e^{-\xi^2/2}\ .
\end{aligned}
$$

Arguing as in the first part of this proof,

$$
\begin{aligned}
&|\eta_{4,n}^{'}(\xi)| \cdot \Big|\exp\{R_4(\xi)\} - 1\Big| \\
\le\ & \left(\frac{e^{M_0^{(4)}y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \left(1 + \frac{\kappa_4}{4!\sigma^4}\right) |\xi|^5 (1+\xi^4) e^{-\xi^2/2} \cdot \left(\sum_{j=1}^n c_{j,n}^4 \Big|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\Big|\right)
\end{aligned}
\tag{76}
$$

and

$$
\begin{aligned}
&\exp\{R_4(\xi)\} \cdot |F(u)| \cdot |\xi| e^{-\xi^2/2} \\
\le\ & \max_{|x|\le \kappa_4 y_0/4!} \left[\Big|\frac{F(x)}{x^2}\Big|\right] \exp\{M_0^{(4)}y_0^4\} |\xi|^9 e^{-\xi^2/2} \left(\sum_{j=1}^n c_{j,n}^4\right)^2 .
\end{aligned}
\tag{77}
$$

Moreover,

$$
\begin{aligned}
\exp\{R_4(\xi)\} \cdot |R_4^{'}(\xi)| \cdot |\eta_{4,n}(\xi) + F(u)e^{-\xi^2/2}| \ &=\ \exp\{R_4(\xi)\} \cdot |R_4^{'}(\xi)| \cdot e^{-\xi^2/2} e^u \\
&\le\ \exp\{M_0^{(4)} y_0^4\} \cdot \exp\{\frac{\kappa_4 y_0^4}{4!}\}\ 4\sigma^{-4} |\xi|^3 e^{-\xi^2/2} \cdot \\
&\cdot \left[\sum_{j=1}^n c_{j,n}^4 \Big|\epsilon_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\Big| + \sum_{j=1}^n c_{j,n}^4 \Big|\varrho_4\left(\frac{c_{j,n}\xi}{\sigma}\right)\Big|\right]
\end{aligned}
\tag{78}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}\xi}F(u) = \frac{\tilde{\lambda}_{2,n}}{3!}\ \xi^3\ (e^u - 1)\ .
$$

Whence,

$$\exp\{R_4(\xi)\} \cdot \Big|\frac{\mathrm{d}}{\mathrm{d}\xi}F(u)\Big| \cdot e^{-\xi^2/2}$$
$$\leq \exp\{M_0^{(4)} y_0^4\} \ \frac{\kappa_4^2}{3!\ 4!\ \sigma^8} \left(\frac{\exp\{\frac{\kappa_4 y_0^4}{4!}\} - 1}{\frac{\kappa_4 y_0^4}{4!}}\right) |\xi|^7 e^{-\xi^2/2} \ \left(\sum_{j=1}^n c_{j,n}^4\right)^2 \ . \quad (79)$$

Now, set

$$C_4^{***} := \left(\frac{e^{M_0^{(4)} y_0^4} - 1}{\sigma^4 y_0^4}\right) \cdot \Big(1 + \frac{\kappa_4}{4!\sigma^4}\Big) + \max_{|x| \leq \kappa_4 y_0/4!} \Big[\Big|\frac{F(x)}{x^2}\Big|\Big] \exp\{M_0^{(4)} y_0^4\}$$
$$+ \exp\{M_0^{(4)} y_0^4\} \cdot \exp\{\frac{\kappa_4 y_0^4}{4!}\} \ 4\sigma^{-4} + \exp\{M_0^{(4)} y_0^4\} \ \frac{\kappa_4^2}{3!\ 4!\ \sigma^8} \left(\frac{\exp\{\frac{\kappa_4 y_0^4}{4!}\} - 1}{\frac{\kappa_4 y_0^4}{4!}}\right)$$

and combine (76), (77), (78) and (79), after noticing that $|\xi|^5(1+\xi^4)+|\xi|^9+|\xi|^3+|\xi|^7 \leq |\xi|^3(1+\xi^6)$ holds for every ξ. Finally, in order to have the same multiplicative constant in the right-hand sides of (26), (27) and (28), replace C_4^{**} and C_4^{***} with $C_4^* := \max\{C_4^{**}; C_4^{***}\}$.

Proof of Lemma 3.2. In view of the independence of the random variables $X_{j,n}$ and (22) one gets

$$\log \psi_n(\xi) = -\frac{1}{2}\xi^2 + \sum_{r=2}^{\chi} (-1)^r \frac{\tilde{\lambda}_{r,n}}{(2r)!}\xi^{2r} + R_{k+\delta}(\xi)$$

where

$$R_{k+\delta}(\xi) := \sum_{j=1}^n \frac{c_{j,n}^k \xi^k}{\sigma^k} \epsilon_{k+\delta}\left(\frac{c_{j,n}\xi}{\sigma}\right) \ .$$

Whence,

$$\psi_n(\xi) = e^{-\xi^2/2} \cdot \exp\left\{\sum_{r=2}^{\chi}(-1)^r \frac{\tilde{\lambda}_{r,n}}{(2r)!}\xi^{2r}\right\} \cdot \exp\{R_{k+\delta}(\xi)\} \ . \qquad (80)$$

Now, consider the function $z \mapsto f_\xi(z) = \exp\{g_\xi(z)\}$ with

$$g_\xi(z) := \sum_{r=1}^{\chi-1} (-1)^{r+1} \frac{\tilde{\lambda}_{r+1,n}}{(2r+2)!}\xi^{2(r+1)} z^r$$

and its Taylor polynomial of order $(\chi - 1)$ at $z = 0$, say $p_{\chi-1}(z)$. Then, recall the Faà di Bruno formula, i.e.

$$\frac{\mathrm{d}^{(\chi)}}{\mathrm{d}t^{(\chi)}} \exp\{(y(t))\}$$
$$= \sum_{(*)} \frac{\chi!}{k_1!k_2!\dots k_\chi!} \exp\{(y(t))\} \left(\frac{y^{(1)}(t)}{1!}\right)^{k_1} \left(\frac{y^{(2)}(t)}{2!}\right)^{k_2} \dots \left(\frac{y^{(\chi)}(t)}{\chi!}\right)^{k_\chi}$$

with $(*)$ meaning that the sum is carried out over all non-negative integer solutions $(k_1, \dots, k_\chi)$ of the equation $k_1+2k_2+\dots+\chi k_\chi = \chi$. An application of this formula entails

$$p_{\chi-1}(z) = 1 + \sum_{r=1}^{\chi-1} \tilde{P}_{r,n}(\xi) z^r$$

the functions $\tilde{P}_{r,n}(\xi)$ having been defined in (24). Thus, when $z = 1$, the Lagrange remainder can be written with a suitable $u \in [0, 1]$ as

$$\frac{1}{\chi!} f_\xi^{(\chi)}(u)$$
$$= f_\xi(u) \sum_{(*)} \frac{1}{k_1!k_2!\dots k_\chi!} \left(\frac{g_\xi^{(1)}(u)}{1!}\right)^{k_1} \left(\frac{g_\xi^{(2)}(u)}{2!}\right)^{k_2} \dots \left(\frac{g_\xi^{(\chi)}(u)}{\chi!}\right)^{k_\chi},$$

which, after repeated applications of the multinomial formula, leads to

$$\left|\frac{1}{\chi!} f_\xi^{(\chi)}(u)\right| \leq f_\xi(u) \sum_{(*)} \prod_{m=1}^{\chi-1} \sum_{\{l_1+\dots+l_{\chi-m}=k_m\}} |A_{1,m}^{l_1}(\xi) \dots A_{\chi-m,m}^{l_{\chi-m}}(\xi)|$$

with $A_{h,m}(\xi) := (-1)^{h+m}\binom{h+m-1}{m}\frac{\tilde{\lambda}_{h+m,n}}{(2(h+m))!}\xi^{2(h+m)}$. Then, one can introduce the quantity

$$W_\chi := \left[\prod_{s=2}^{\chi} \max\{\frac{\kappa_{2s}}{\sigma^{2s}}; 1\}\right]^\chi$$

to obtain, after an application of the Liapounov inequality,

$$\sum_{\{l_1+\dots+l_{\chi-m}=k_m\}} |A_{1,m}^{l_1}(\xi) \dots A_{\chi-m,m}^{l_{\chi-m}}(\xi)|$$
$$\leq \chi^\chi W_\chi \, \xi^{2mk_m}(\xi^2 + \xi^{k-2})^{k_m} \cdot \left(\sum_{j=1}^{n} c_{j,n}^{k+2}\right)^{\frac{2mk_m}{k}}.$$

Whence,

$$\left|\frac{1}{\chi!}f_\xi^{(\chi)}(u)\right| \le f_\xi(u)\cdot\chi^{\chi^2}W_\chi^{\chi-1}\xi^k[(\xi^2+\xi^{k-2})^2+(\xi^2+\xi^{k-2})^\chi]\cdot\left(\sum_{j=1}^n c_{j,n}^{k+2}\right)$$

and, using the bound $|\xi|\le A_{k,\delta,n}$,

$$|g_\xi(u)|\le\sum_{s=2}^{\chi}\kappa_{2s}y_0^{2s}:=B_\chi\ .$$

Then,

$$\begin{aligned}|\psi_n(\xi)-\eta_{k,n}(\xi)| &\le e^{-\xi^2/2}\ \left\{[f_\xi(1)-p_{\chi-1}(1)]+[e^{R_{k+\delta}(\xi)}-1]\right\}\\ \le\ e^{-\xi^2/2}&\ \Big[e^{B_\chi}\chi^{\chi^2}W_\chi^{\chi-1}\xi^k[(\xi^2+\xi^{k-2})^2+(\xi^2+\xi^{k-2})^\chi]\cdot\left(\sum_{j=1}^n c_{j,n}^{k+2}\right)\\ &+\left(\frac{\exp\{M_0^{(k+\delta)}y_0^k\}-1}{M_0^{(k+\delta)}y_0^k}\right)\ \frac{2\overline{\mathfrak{m}}_{k+\delta}}{k!\sigma^{k+\delta}}\left(\sum_{j=1}^n|c_{j,n}|^{k+\delta}\right)\ |\xi|^{k+\delta}\Big]\ .\end{aligned}\tag{81}$$

As to $|\psi_n^{'}-\eta_{k,n}^{'}|$, note that inequality

$$\begin{aligned}&|\psi_n^{'}(\xi)-\eta_{k,n}^{'}(\xi)|\\ &\le|\xi|\cdot|\psi_n(\xi)-\eta_{k,n}(\xi)|+e^{-\xi^2/2}\left|\frac{\mathrm{d}}{\mathrm{d}\xi}f_\xi(1)\right|\cdot\left|\exp\{R_{k+\delta}(\xi)\}-1\right|\\ &+e^{-\xi^2/2}|f_\xi(1)|\exp\{R_{k+\delta}(\xi)\}\left|R_{k+\delta}^{'}(\xi)\right|+e^{-\xi^2/2}\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\left(f_\xi(1)-p_{\chi-1}(1)\right)\right|\end{aligned}\tag{82}$$

obtains. As regards the first summand, it will suffice to multiply the upper bound stated in (81) for $|\psi_n-\eta_{k,n}|$ by $|\xi|$. The latter factor in the second addend of (82) can be dominated by the last addend in (81), while for the former factor one has

$$\left|\frac{\mathrm{d}}{\mathrm{d}\xi}f_\xi(1)\right|\le\exp\{B_\chi\}\sum_{r=1}^{\chi-1}\frac{\kappa_{2r+2}y_0^{2r+1}}{(2r+1)!\sigma}\ .$$

As to the third addend, recall that $|f_\xi(1)|\le\exp\{B_\chi\}$ and that $|R_{k+\delta}(\xi)|\le y_0^kM_0^{(k+\delta)}$. Moreover, $|R_{k+\delta}^{'}(\xi)|\le\sum_{j=1}^n\sigma^{-k}\xi^{k-1}|c_{j,n}|^k\{k|\epsilon_{k+\delta}(c_{j,n}\xi/\sigma)|+|\xi\sigma^{-1}c_{j,n}\epsilon_{k+\delta}^{'}(c_{j,n}\xi/\sigma)|\}$ and, in view of Theorem 1 in Section 8.4 of Chow and Teicher (1997), $(|\epsilon_{k+\delta}(x)|+|x\epsilon'_{k+\delta}(x)|) \leq 4\overline{\mathfrak{m}}_{k+\delta}|x|^{\delta}/(k-1)!$. It remains to deal with the last summand in (82). Since $\frac{\partial}{\partial\xi}p_{\chi-1}$ is a Taylor polynomial for $\frac{\partial}{\partial\xi}f_\xi$, one can use the Bernstein integral form of the remainder to obtain

$$
\begin{aligned}
\Big|\frac{\partial}{\partial\xi}\left(f_\xi(1)-p_{\chi-1}(1)\right)\Big| \;&\leq\; \frac{1}{(\chi-1)!}\int_0^1 (1-u)^{\chi-1}\sum_{l=0}^{\chi}\binom{\chi}{l}\Big|\frac{\partial^l}{\partial u^l}f_\xi(u)\Big|\mathrm{d}u \\
&\quad\cdot\sum_{j=1}^{n}c_{j,n}^{2(\chi-l+1)}\sum_{r=\chi-l}^{\chi-1}|\xi|^{2r+1}\frac{\kappa_{2(r+1)}}{\sigma^{2(r+1)}} \\
&\leq\; \sum_{l=0}^{\chi}\frac{1}{(\chi-l)!}e^{B_\chi}\sum_{(*)_l}\prod_{m=1}^{l}\frac{1}{k_m!}\left(\frac{1}{m!}\sum_{r=m}^{\chi-1}\frac{\kappa_{2(r+1)}}{\sigma^{2(r+1)}}|\xi|^{2r+2}\right)^{k_m} \\
&\quad\cdot\sum_{r=\chi-l}^{\chi-1}|\xi|^{2r+1}\frac{\kappa_{2(r+1)}}{\sigma^{2(r+1)}}\left(\sum_{j=1}^{n}c_{j,n}^{2l+2}\right)\cdot\left(\sum_{j=1}^{n}c_{j,n}^{2(\chi-l+1)}\right) .
\end{aligned}
$$

To conclude, think of the last two sums of the $c_{j,n}$s as moments of order $2l$ and $2(\chi-l)$ respectively and apply the Liapounov inequality to each sum to write

$$
\left(\sum_{j=1}^{n}c_{j,n}^{2l+2}\right)\cdot\left(\sum_{j=1}^{n}c_{j,n}^{2(\chi-l+1)}\right)\leq\sum_{j=1}^{n}c_{j,n}^{k+2} .
$$

A.3

As to the proof of Theorem 2.1, we clarify why $\inf_{t\in[0,\hat{t}]}\mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma)$ must be strictly positive. Suppose that $\inf_{t\in[0,\hat{t}]}\mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma)=0$. Then, as $t\mapsto\mathrm{d}_{TV}(\mu(\cdot,t);\gamma_\sigma)$ is continuous on $[0,+\infty[$ - see, e.g., the Wild expansion - there is t^* in $[0,\hat{t}]$ such that $\mathrm{d}_{TV}(\mu(\cdot,t^*);\gamma_\sigma)=0$. On the one hand, if $t^*=0$, then μ_0 coincides with γ_σ and this contradicts the hypothesis that $\kappa_4(\tilde{\mu}_0)\neq 0$. On the other hand, if $t^*>0$, in view of the Wild expansion one can conclude that μ_0 possesses moments of every order and is symmetric. Then, it is easy to check that these moments, i.e.

$$
\int_{\mathbb{R}} x^n\mu(\mathrm{d}x,t) \;=\; \mathsf{E}_t\left[\left(\sum_{j=1}^{\nu}\pi_j\upsilon_j\right)^n\right] \qquad (n=1,2,\dots)
$$

are, in this case, independent of t and are the same as the moments of γ_σ. Hence, $\mu_0 = \gamma_\sigma$, which contradicts the hypothesis $\kappa_4(\tilde{\mu}_0) \neq 0$ again.

A.4

The proofs of (54) and (57) follow from this proposition: *Let $g : \mathbb{R} \to [0, +\infty[$ be an integrable function and $\epsilon : \mathbb{R} \to \mathbb{R}$ be a continuous, bounded function with $\epsilon(0) = 0$. Then*

$$\lim_{t\to+\infty} H(t) := e^{\frac{1}{4}t}\, \mathsf{E}_t \left\{ \left(\int_{\mathbb{R}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \left|\epsilon(\pi_j \xi)\right| \right]^2 \mathrm{d}\xi \right)^{1/2} \right\} = 0 \,.$$

Proof. Fix an arbitrary small positive δ and show that there exists a value t_δ for which $|H(t)| < \delta$, for every $t > t_\delta$. First of all, by the continuity of ϵ and the fact that $\epsilon(0) = 0$, note that the set of real numbers x for which

$$|\epsilon(x)| \le \frac{\delta}{3\sqrt{||\, g\, ||_1}}$$

includes an interval $[-\overline{x}, \overline{x}]$ with $\overline{x} > 0$ and $||\, g\, ||_1 := \int_{\mathbb{R}} g(\xi) \mathrm{d}\xi$. Set $\overline{\pi} := \max_{1\le j\le \nu} \pi_j$ and $B := \overline{x}/|\overline{\pi}|$. B is well-defined since, due to (15), $\overline{\pi} \neq 0$. Now,

$$\left\{ \int_{\mathbb{R}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \left|\epsilon(\pi_j \xi)\right| \right]^2 \mathrm{d}\xi \right\}^{1/2}$$

$$\le \left\{ \int_{\{|\xi|\le B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \left|\epsilon(\pi_j \xi)\right| \right]^2 \mathrm{d}\xi \right\}^{1/2} + \left\{ \int_{\{|\xi|\ge B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \left|\epsilon(\pi_j \xi)\right| \right]^2 \mathrm{d}\xi \right\}^{1/2} .$$

For the integral over the internal region one can write

$$\left\{ \int_{\{|\xi|\le B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \left|\epsilon(\pi_j \xi)\right| \right]^2 \mathrm{d}\xi \right\}^{1/2} \le \frac{\delta}{3\sqrt{||\, g\, ||_1}} \left(\sum_{j=1}^{\nu} \pi_j^4 \right) \sqrt{||\, g\, ||_1}$$

and, taking expectation,

$$e^{\frac{1}{4}t}\ \mathsf{E}_t \left\{ \left(\int\limits_{\{|\xi|\le B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \Big| \epsilon(\pi_j \xi) \Big| \right]^2 \mathrm{d}\xi \right)^{1/2} \right\} \ \le\ \delta/3$$

after a standard application of (19). At this point, define M to be the maximum of $|\epsilon|$ and determine a positive value $\overline{s}$ such that

$$\int\limits_{\{|\xi|\ge\overline{s}\}} g(\xi)\mathrm{d}\xi \ \le\ \left(\frac{\delta}{3M}\right)^2 .$$

Given $S := \{\omega \mid |\overline{\pi}(\omega)| < \overline{x}/\overline{s}\}$, write

$$\begin{aligned} & e^{\frac{1}{4}t}\ \mathsf{E}_t \left\{ \left(\int\limits_{\{|\xi|\ge B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \Big| \epsilon(\pi_j \xi) \Big| \right]^2 \mathrm{d}\xi \right)^{1/2} \right\} \\ =\ & e^{\frac{1}{4}t}\ \mathsf{E}_t \left\{ \left[\int\limits_{\{|\xi|\ge B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \Big| \epsilon(\pi_j \xi) \Big| \right]^2 \mathrm{d}\xi \right]^{1/2} ;\ S \right\} \\ +\ & e^{\frac{1}{4}t}\ \mathsf{E}_t \left\{ \left[\int\limits_{\{|\xi|\ge B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \Big| \epsilon(\pi_j \xi) \Big| \right]^2 \mathrm{d}\xi \right]^{1/2} ;\ S^c \right\} . \end{aligned}$$

One can notice that $B(\omega) > \overline{s}$ for ω in S. Then,

$$\begin{aligned} & e^{\frac{1}{4}t}\ \mathsf{E}_t \left\{ \left[\int\limits_{\{|\xi|\ge B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \Big| \epsilon(\pi_j \xi) \Big| \right]^2 \mathrm{d}\xi \right]^{1/2} ;\ S \right\} \\ \le\ & e^{\frac{1}{4}t} \left\{ \int\limits_{\{|\xi|\ge\overline{s}\}} g(\xi)\mathrm{d}\xi \right\}^{1/2} M\ \mathsf{E}_t \left[\sum_{j=1}^{\nu} \pi_j^4 \right] \ \le\ \delta/3 . \end{aligned}$$

For the remaining term,

$$e^{\frac{1}{4}t}\ \mathsf{E}_t \left\{ \left[\int\limits_{\{|\xi|\ge B\}} g(\xi) \left[\sum_{j=1}^{\nu} \pi_j^4 \Big| \epsilon(\pi_j \xi) \Big| \right]^2 \mathrm{d}\xi \right]^{1/2} ;\ S^c \right\} \ \le\ e^{\frac{1}{4}t} M \sqrt{||g||_1}\, \mathsf{P}_t(S^c) .$$

An application of Markov's inequality with exponent 6 yields an upper bound for the probability of S^c, i.e.

$$\mathsf{P}_t(S^c) \ \leq \ \mathsf{E}_t[|\overline{\pi}|^6] \cdot \left(\frac{\overline{s}}{\overline{x}}\right)^6 \ \leq \ \mathsf{E}_t\left[\sum_{j=1}^{\nu} \pi_j^6\right] \cdot \left(\frac{\overline{s}}{\overline{x}}\right)^6 \ \leq \ e^{-\frac{3}{8}t} \cdot \left(\frac{\overline{s}}{\overline{x}}\right)^6 \ .$$

Hence,

$$e^{\frac{1}{4}t} M \sqrt{||g||_1} \mathsf{P}_t(S^c) \ \leq \ e^{-\frac{1}{8}t} M \sqrt{||g||_1} \cdot \left(\frac{\overline{s}}{\overline{x}}\right)^6 .$$

Taking $t_\delta = \max\left\{-8 \log\left[(\delta/3) \cdot (\overline{x}/\overline{s})^6 \cdot M^{-1} \, ||g||_1^{-1/2}\right]; 1\right\}$ makes the right-hand side of the last inequality smaller than $\delta/3$ for every $t > t_\delta$. This completes the proof.

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