Near Critical Catalyst Reactant Branching Processes with Controlled Immigration*

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Abstract: Near critical catalyst-reactant branching processes with controlled immigration are studied. The reactant population evolves according to a branching process whose branching rate is proportional to the total mass of the catalyst. The bulk catalyst evolution is that of a classical continuous time branching process; in addition there is a specific form of immigration. Immigration takes place exactly when the catalyst population falls below a certain threshold, in which case the population is instantaneously replenished to the threshold. Such models are motivated by problems in chemical kinetics where one wants to keep the level of a catalyst above a certain threshold in order to maintain a desired level of reaction activity. A diffusion limit theorem for the scaled processes is presented, in which the catalyst limit is described through a reflected diffusion, while the reactant limit is a diffusion with coefficients that are functions of both the reactant and the catalyst. Stochastic averaging principles under fast catalyst dynamics are established. In the case where the catalyst evolves "much faster" than the reactant, a scaling limit, in which the reactant is described through a one dimensional SDE with coefficients depending on the invariant distribution of the reflected diffusion, is obtained. Proofs rely on constrained martingale problem characterizations, Lyapunov function constructions, moment estimates that are uniform in time and the scaling parameter, and occupation measure techniques.

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1. Introduction

This work is concerned with catalytic branching processes that model the dynamics of catalyst-reactant populations in which the activity level of the reactant depends on the amount of

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catalyst present. Branching processes in catalytic environments have been studied extensively and are motivated, for instance, by biochemical reaction networks (see [6, 8, 15, 11] and references therein). A typical setting consists of populations of multiple types such that the rate of growth (depletion) of one population type is directly affected by population sizes of other types. The simplest such model consists of a continuous time countable state branching process describing the evolution of the catalyst population and a second branching process for which the branching rate is proportional to the total mass of the catalyst population modeling the evolution of reactant particles. Such processes were introduced in [6] in the setting of super-Brownian motions ([15]). For classical catalyst-reactant branching processes, the catalyst population dies out with positive probability and subsequent to the catalyst extinction, the reactant population stays unchanged and therefore the population dynamics are modeled until the time the catalyst becomes extinct. In this work, we consider a setting where the catalyst population is maintained above a positive threshold through a specific form of controlled immigration. Branching process models with immigration have also been well studied in literature (see [2, 15] and references therein). However, typical mechanisms that have been considered correspond to adding an independent Poisson component (see e.g. [12]). Here, instead, we consider a model where immigration takes place only when the population drops below a certain threshold. Roughly speaking, we consider a sequence $\{X^{(n)}\}_{n\in\mathbb{N}}$ of continuous time branching processes, where $X^{(n)}$ starts with n particles. When the population drops below n, it is instantaneously restored to the level n.

There are many settings where controlled immigration models of the above form arise naturally. One class of examples arise from predator-prey models in ecology, where one may be concerned with the restoration of populations that are close to extinction by reintroducing species when they fall below a certain threshold. In our work, the motivation for the study of such controlled immigration models comes from problems in chemical reaction networks where one wants to keep the levels of certain types of molecules above a threshold in order to maintain a desired level of production (or inhibition) of other chemical species in the network. Such questions are of interest in the study of control and regulation of chemical reaction networks. A control action where one minimally adjusts the levels of one chemical type to keep it above a fixed threshold is one of the simplest regulatory mechanism and the goal of this research is to study system behavior under such mechanisms with the long term objective of designing optimal control policies. The specific goal of the current work is to derive simpler approximate and reduced models, through the theory of diffusion approximations and stochastic averaging techniques, that are more tractable for simulation and mathematical treatment than the original branching process models. In order to keep the presentation simple, we consider the setting of one catalyst and one reactant. However similar limit theorems can be obtained for a more general chemical reaction network in which the levels of some of the chemical species are regulated in a suitable manner. Settings where some of the chemical species act as inhibitors rather than catalysts are also of interest and can be studied using similar techniques. These extensions will be pursued elsewhere.

Our main goal is to establishes diffusion approximations for such regulated catalyst-reactant systems under suitable scalings. We consider two different scaling regimes; in the first setting the catalyst and reactant evolve on "comparable timescales", while in the second setting the

catalyst evolves "much faster" than the reactant. In the former setting, the limit model is described through a coupled system of reflected stochastic differential equations with reflection in the space $[1, \infty) \times \mathbb{R}$. The precise result (Theorem 2.1) is stated in Section 2. Such limit theorems are of interest for various analytic and computational reasons. It is simpler to simulate (reflected) diffusions than branching processes, particularly for large network settings. Analytic properties such as hitting time probabilities and steady state behavior are more easily analyzed for the diffusion models than for their branching process counterparts. In general, such diffusion limits give parsimonious model representations and provide useful qualitative insight to the underlying stochastic phenomena.

For the second scaling regime, where the catalyst evolution is much faster, we establish a stochastic averaging limit theorem. A key ingredient here is an ergodicity result, which says that under a suitable "criticality from below" assumption on the catalyst dynamics, the limiting catalyst reflected diffusion admits a unique stationary distribution, which takes an explicit form (Proposition 3.1). Characterization of the invariant distribution is based on a variant of Echeverria's criterion for constrained Markov processes [14]. Next, by constructing suitable *uniform* Lyapunov functions, we show that the stationary distribution of the scaled catalyst branching process converges to that of the catalyst diffusion (Theorem 3.1). These results are then used to establish a stochastic averaging principle that governs the dynamics of the reactant population in the fast catalyst limit. Proofs proceed by developing suitable moment estimates that are uniform in time and the scaling parameter and by using characterization results for probability laws of reflected diffusions through certain constrained martingale problems [13]. The limit evolution of the reactant population is given through an autonomous one dimensional SDE with coefficients that depend on the stationary distribution of a reflected diffusion in $[1, \infty)$. Such model reductions are important in that they not only help in better understanding the dynamics of the system but also help in reducing computational costs in simulations. Indeed, since in the model considered here the invariant distribution is explicit, the coefficients in the one dimensional averaged diffusion model are easily computed and consequently this model is significantly easier to analyze and simulate than the original two dimensional model. We refer the reader to [11] and references therein for similar results in the setting of (non-regulated) chemical reaction networks. It will be of interest to see if similar model reductions can be obtained for general multi-dimensional regulated chemical-reaction networks. Key mathematical challenges will be to identify suitable conditions for ergodicity of multi-dimensional reflected diffusions in polyhedral domains that arise from the regulated part of the network, and to develop uniform (in time and the scaling parameter) moment estimates for such multi-dimensional constrained diffusions.

We consider two different formulations of models with multiple time scales. In Theorem 4.1 we consider the setting where both catalyst and reactant processes are described through (reflected) diffusions and the time scale parameter appears in the coefficients of the catalyst evolution equation. An important step here is to argue that the generator of the two dimensional catalyst-reactant reflected diffusion is suitably close to the generator of the one dimensional averaged diffusion, for large values of the scaling parameter. Bounds on the exponential moments of the catalyst process, obtained in Lemma 8.1, play a key role in this argument. The second formulation is considered in Theorem 4.2. Here, both catalyst and reactant populations

evolve according to near critical countable state branching processes and the branching rate in the catalyst dynamics is of higher order than that for the reactant process. In this setting one encounters the additional difficulty of showing that the steady state distributions of the scaled catalyst branching process, for large values of the scaling parameter, are suitably close to the stationary distribution of the limiting catalyst reflected diffusion. The approach taken here is based on characterizing the limit points of a certain sequence of random measures on the path space of the catalyst process and the associated reflection process, as time and the scaling parameter together approach infinity.

The model considered in this work does not incorporate any spatial dynamics of the two chemical species. As noted earlier in the introduction, in the unregulated setting, the paper [6] considered catalyst-reactant systems, with chemical species moving continuously in a spatial domain, given in terms of super-Brownian motions. It will be of interest to develop analogous continuous spatial models for the regulated catalyst-reactant systems of the form considered in the current work. This question will be explored in a future work.

The paper is organized as follows. We begin in Section 2 by presenting the basic limit theorem in the setting of "comparable time scales". Section 3 studies the time asymptotic behavior of the catalyst process under a suitable *criticality from below* assumption. Section 4 presents our main results for the multiple time scale setting. Section 5 collects some auxiliary estimates that are needed in our proofs. Section 6 proves Theorem 2.1 and Section 7 is devoted to the proofs of Proposition 3.1 and Theorem 3.1. Finally, in Section 8 we present proofs of stochastic averaging principles stated in Section 4.

1.1. Notation

The following notation will be used throughout this work. Denote by $\mathbb{N}$ the natural numbers, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denote the set of integers by $\mathbb{Z}$, and let $\mathbb{R}_+ := [0, \infty)$ be the set of nonnegative real numbers. The state spaces of the scaled catalyst, reactant, and auxiliary processes, $\hat{X}^{(n)}, \hat{Y}^{(n)}$, and $\hat{Z}^{(n)}$, respectively, introduced below in (2.1), are $\mathbb{S}_X^{(n)} := \{\frac{l}{n} | l \in \mathbb{N}_0\} \cap [1, \infty)$, $\mathbb{S}_Y^{(n)} := \{\frac{l}{n} | l \in \mathbb{N}_0\}$, and $\mathbb{S}_Z^{(n)} := \{\frac{l}{n} | l \in \mathbb{Z}\}$. Let $\mathbb{W}^{(n)} := \mathbb{S}_X^{(n)} \times \mathbb{S}_Y^{(n)} \times \mathbb{S}_Z^{(n)}$ and $\mathbb{W} := [1, \infty) \times \mathbb{R}_+ \times \mathbb{R}$. Let $C^k(\mathbb{W})$ denote the space of k-times continuously differentiable, real valued functions on $\mathbb{W}$, and denote by $C_c^k(\mathbb{W})$ the space of $C^k(\mathbb{W})$ functions with compact support. Here, by a (k-times) differentiable function f on a set $D \subset \mathbb{R}^n$ we mean a function that can be extended to a (k-times) differentiable function $\tilde{f}$ on an open domain $U \supset D$ such that $\tilde{f}$ restricted to D equals f. Given a metric space S, the space of probability measures on S will be denoted by $\mathcal{P}(S)$, the Borel σ-field on S by $\mathcal{B}(S)$, and the space of real valued, bounded, measurable functions on S by $\mathrm{BM}(S)$. Let

$$D(\mathbb{R}_+ : S) := \{f : \mathbb{R}_+ \to S | f \text{ is right continuous and has left limits}\}$$

and $D_1(\mathbb{R}_+ : \mathbb{R}) := \{f \in D(\mathbb{R}_+ : \mathbb{R}) | f(0) \geq 1\}$, where these D-spaces are endowed with the usual Skorohod topology. Let $C(\mathbb{R}_+ : \mathbb{R}_+)$ be the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$

endowed with the local uniform topology. We say a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of random variables with values in some Polish space $\mathcal{E}$ is tight if the corresponding probability laws are a tight sequence in $\mathcal{P}(\mathcal{E})$. For a function $\xi : \mathbb{R}_+ \to \mathbb{R}^n$, let the jump at time t be defined as $\Delta\xi_t := \xi_t - \xi_{t-}$, $t > 0$, and $\Delta\xi_0 := 0$. For a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $t \geq 0$, let $|f|_{*,t} := \sup_{s\leq t} |f(s)|$. For two semimartingales ξ and ζ, the quadratic covariation (or bracket process) and predictable (or conditional) quadratic covariation are denoted by $\{[\xi,\zeta]_t\}_{t\in\mathbb{R}_+}$ and $\{\langle\xi,\zeta\rangle_t\}_{t\in\mathbb{R}_+}$ respectively; their definition will be recalled in Section 5.

2. Diffusion Limit under Comparable Timescales

Consider a sequence of pairs of continuous time, countable state Markov branching processes $(X^{(n)}, Y^{(n)})$, where $X^{(n)}$ and $Y^{(n)}$ represent the number of catalyst and reactant particles, respectively. The dynamics are described as follows. Each of the $X_t^{(n)}$ particles alive at time t has an exponentially distributed life time with parameter $\lambda_1^{(n)}$ (mean life time $1/\lambda_1^{(n)}$). When it dies, each such particle gives rise to a number of offspring, according to the offspring distribution $\mu_1^{(n)}(\cdot)$. Additionally, if the catalyst population drops below n, it is instantaneously replenished back to the level n (*controlled immigration*). The branching rate of the reactant process $Y^{(n)}$ is of the order of the current total mass of the catalyst population, i.e. $X^{(n)}/n$, and we denote the offspring distribution of $Y^{(n)}$ by $\mu_2^{(n)}(\cdot)$. A precise definition of the pair $(X^{(n)}, Y^{(n)})$ will be given below. We are interested in the study of asymptotic behavior of $(X^{(n)}, Y^{(n)})$, under suitable scaling, as $n \to \infty$.

To facilitate some weak convergence arguments, we will consider an auxiliary sequence of processes $Z^{(n)}$ that "shadow" $X^{(n)}$ in the following manner. The process $Z^{(n)}$ will be a $\mathbb{Z}$ valued pure jump process whose jump instances and sizes are the same as that of $X^{(n)}$ away from the boundary $\{n\}$, whereas when $X^{(n)}$ is at the boundary, $Z^{(n)}$ has a negative jump of size 1 whenever there is immigration of a catalyst particle into the system. This description is made precise through the infinitesimal generator given in (2.2). The process $\hat{Z}^{(n)}$ will not appear in the statements of the results, nevertheless it plays an important role in our proofs.

We now give a precise description of the various processes and the scaling that is considered. Roughly speaking, time is accelerated by a factor of n and mass is scaled down by a factor of n. Define RCLL processes

$$\hat{\mathbf{W}}_t^{(n)} := \left(\hat{X}_t^{(n)}, \hat{Y}_t^{(n)}, \hat{Z}_t^{(n)}\right) := \left(\frac{X_{nt}^{(n)}}{n}, \frac{Y_{nt}^{(n)}}{n}, \frac{Z_{nt}^{(n)}}{n}\right), \quad t \in \mathbb{R}_+, \tag{2.1}$$

and let $\hat{\mathbf{W}}_0^{(n)} = (x_0^{(n)}, y_0^{(n)}, z_0^{(n)}) \in \mathbb{W}^{(n)}$, where $(nx_0^{(n)}, ny_0^{(n)})$ is the initial number of catalyst and reactant particles and $z_0^{(n)} = x_0^{(n)}$. Then $\{\hat{\mathbf{W}}_t^{(n)}\}_{t\in\mathbb{R}_+}$ is characterized as the $\mathbb{W}^{(n)}$ valued Markov process with sample paths in $D(\mathbb{R}_+ : \mathbb{W}^{(n)})$, starting at $\hat{\mathbf{W}}_0^{(n)} = (x_0^{(n)}, y_0^{(n)}, z_0^{(n)})$, and

having infinitesimal generator $\hat{\mathcal{A}}^{(n)}$ given as

$$\hat{\mathcal{A}}^{(n)}\phi(\mathbf{w}) = \lambda_1^{(n)} n^2 x \sum_{k=0}^{\infty} \Big[\phi\Big(1 \vee x + \frac{k-1}{n}, y, z + \frac{k-1}{n}\Big) - \phi(\mathbf{w})\Big]\mu_1^{(n)}(k)$$
$$+ \lambda_2^{(n)} n^2 xy \sum_{k=0}^{\infty} \Big[\phi\Big(x, y + \frac{k-1}{n}, z\Big) - \phi(\mathbf{w})\Big]\mu_2^{(n)}(k), \tag{2.2}$$

where $\mathbf{w} = (x, y, z) \in \mathbb{W}^{(n)}$ and $\phi \in \mathrm{BM}(\mathbb{W})$. From the definition of the generator we see that, for each $k \geq 0$, given $\hat{\mathbf{W}}_t^{(n)} = (x, y, z) \in \mathbb{W}^{(n)}$, the process jumps to $(x, y + \frac{k-1}{n}, z)$ with rate $\lambda_2^{(n)} n^2 xy \mu_2^{(n)}(k)$, and to $(x + \frac{k-1}{n}, y, z + \frac{k-1}{n})$ with rate $\lambda_1^{(n)} n^2 x \mu_1^{(n)}(k)$, except when $k = 0$ and $x = 1$, in which case the latter jump is to $(x, y, z + \frac{k-1}{n})$ with rate $\lambda_1^{(n)} n^2 \mu_1^{(n)}(0)$. This property of the generator at $x = 1$ accounts for the instantaneous replenishment of the (unscaled) catalyst population to level n, whenever the catalyst drops below n.

For $i = 1, 2$, let

$$m_i^{(n)} := \sum_{k=0}^{\infty} k\mu_i^{(n)}(k) \quad \text{and} \quad \alpha_i^{(n)} = \sum_{k=0}^{\infty} (k-1)^2 \mu_i^{(n)}(k).$$

We make the following basic assumption on the parameters of the branching rates and offspring distributions as well as on the initial configurations of the catalyst and reactant populations.

Condition 2.1. *(i) For $i = 1, 2$ and for $n \in \mathbb{N}$, $\alpha_i^{(n)}, \lambda_i^{(n)} \in (0, \infty)$ and $m_i^{(n)} = 1 + \frac{c_i^{(n)}}{n}$, $c_i^{(n)} \in (-n, \infty)$.*

(ii) For $i = 1, 2$, as $n \to \infty$, $c_i^{(n)} \to c_i \in \mathbb{R}$, $\alpha_i^{(n)} \to \alpha_i \in (0, \infty)$, and $\lambda_i^{(n)} \to \lambda_i \in (0, \infty)$.

(iii) For $i = 1, 2$ and for every $\epsilon \in (0, \infty)$, $\lim_{n\to\infty} \sum_{l: l > \epsilon\sqrt{n}} (l - m_i^{(n)})^2 \mu_i^{(n)}(l) = 0$.

(iv) As $n \to \infty$, $(x_0^{(n)}, y_0^{(n)}) \to (x_0, y_0) \in [1, \infty) \times \mathbb{R}_+$.

Condition 2.1 and the form of the generator in (2.2) ensure that the scaled catalyst and reactant processes transition on comparable time scales, namely $\mathcal{O}(n^2)$. In order to state the limit theorem for $(\hat{X}^{(n)}, \hat{Y}^{(n)})$, we need some notation and definitions associated with the one dimensional Skorohod map with reflection at 1. Let $\Gamma : D_1(\mathbb{R}_+ : \mathbb{R}) \to D(\mathbb{R}_+ : [1, \infty))$ be defined as

$$\Gamma(\psi)(t) := (\psi(t) + 1) - \inf_{0 \leq s \leq t} \{\psi(s) \wedge 1\}, \quad \text{for } \psi \in D(\mathbb{R}_+ : \mathbb{R}). \tag{2.3}$$

The function Γ, known as Skorohod map, can be characterized as follows (see e.g. Appendix B in [3] and references therein): If $\psi, \phi, \eta^* \in D(\mathbb{R}_+ : \mathbb{R})$ are such that (i) $\psi(0) \geq 1$, (ii) $\phi = \psi + \eta^*$, (iii) $\phi \geq 1$, (iv) η^* is non-decreasing, $\int_{[0,\infty)} 1_{\{\phi(s) \neq 1\}} d\eta^*(s) = 0$, and $\eta^*(0) = 0$, then $\phi = \Gamma(\psi)$ and $\eta^* = \phi - \psi$. The process η^* can be regarded as the reflection term that is applied to the original trajectory ψ to produce a trajectory ϕ that is constrained to $[1, \infty)$. From the definition of the Skorohod map and using the triangle inequality, we get the following Lipschitz property: For $\psi, \tilde{\psi} \in D(\mathbb{R}_+ : \mathbb{R})$,

$$\sup_{s \leq t} |\Gamma(\psi)(s) - \Gamma(\tilde{\psi})(s)| \leq 2 \sup_{s \leq t} |\psi(s) - \tilde{\psi}(s)|. \tag{2.4}$$

The diffusion limit of $(\hat{X}^{(n)}, \hat{Y}^{(n)})$ will be the process (X, Y), starting at (x_0, y_0), which is given through a system of stochastic integral equations as in the following proposition.

Proposition 2.1. *Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t\})$ be a filtered probability space on which are given independent standard $\{\bar{\mathcal{F}}_t\}$ Brownian motions B^X and B^Y. Let X_0, Y_0 be square integrable $\bar{\mathcal{F}}_0$ measurable random variables with values in $[1, \infty)$ and $\mathbb{R}_+$, respectively. Then the following system of stochastic integral equations has a unique strong solution:*

$$X_t = \Gamma\left(X_0 + \int_0^{\cdot} c_1\lambda_1 X_s ds + \int_0^{\cdot} \sqrt{\alpha_1\lambda_1 X_s} dB_s^X\right)(t), \tag{2.5}$$

$$Y_t = Y_0 + \int_0^t c_2\lambda_2 X_s Y_s ds + \int_0^t \sqrt{\alpha_2\lambda_2 X_s Y_s} dB_s^Y, \tag{2.6}$$

$$\eta_t = X_t - X_0 - \int_0^t c_1\lambda_1 X_s ds - \int_0^t \sqrt{\alpha_1\lambda_1 X_s} dB_s^X, \tag{2.7}$$

where Γ is the Skorohod map defined in (2.3).

In the above proposition, by a strong solution of (2.5) – (2.7) we mean an $\bar{\mathcal{F}}$-adapted continuous process (X, Y, η) with values in $[1, \infty) \times \mathbb{R}_+ \times \mathbb{R}_+$ that satisfies (2.5) – (2.7). The following is the main result of this section.

Theorem 2.1. *Suppose Condition 2.1 holds. Then the process $(\hat{X}^{(n)}, \hat{Y}^{(n)})$ converges weakly in $D(\mathbb{R}_+ : [1, \infty) \times \mathbb{R}_+)$ to the process (X, Y) given in Proposition 2.1 with $(X_0, Y_0) = (x_0, y_0)$.*

Proposition 2.1 follows by standard arguments, so its proof is relegated to the appendix. Theorem 2.1 will be proved in Section 6.

3. Asymptotic Behavior of the Catalyst Population

Stochastic averaging results in this work rely on understanding the time asymptotic behavior of the catalyst process. Such behavior, of course, is also of independent interest. We begin with the following result on the stationary distribution of X, where X is the reflected diffusion from Proposition 2.1, approximating the catalyst dynamics (Theorem 2.1). The proof uses an extension of the Echeverria criterion for stationary distributions of diffusions to the setting of constrained diffusions (see Section 7.1). We will make the following additional assumption. Recall the constants $c_1^{(n)} \in (-n, \infty)$ and $c_1 \in \mathbb{R}$ introduced in Condition 2.1.

Condition 3.1. *For all $n \in \mathbb{N}$, $c_1^{(n)} < 0$ and $c_1 < 0$.*

Proposition 3.1. *Suppose Condition 3.1 holds. The process X defined through* (2.5) *has a unique stationary distribution, ν_1, which has density*

$$p(x) := \begin{cases} \frac{\theta}{x}\exp(2\frac{c_1}{\alpha_1}x), & \text{if } x \geq 1 \\ 0, & \text{if } x < 1, \end{cases} \tag{3.1}$$

where $\theta := \left(\int_1^\infty (\frac{1}{x}\exp(2\frac{c_1}{\alpha_1}x))dx\right)^{-1}$.

The following result shows that the time asymptotic behavior of the catalyst population is well approximated by that of its diffusion approximation given through (2.5). We make the following additional assumption on the moment generating function of the offspring distribution, which will allow us to construct certain "uniform Lyapunov functions" (see Theorem 7.2 and the function $\hat{V}^{(n)}$ defined in (7.5)) that play a key role in the analysis.

Condition 3.2. *For some* $\bar{\delta} > 0$,

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} e^{\bar{\delta}k}\mu_1^{(n)}(k) < \infty. \tag{3.2}$$

Theorem 3.1. *Suppose Conditions 2.1, 3.1, and 3.2 hold. Then, for each* $n \in \mathbb{N}$, *the process* $\hat{X}^{(n)}$ *has a unique stationary distribution* $\nu_1^{(n)}$, *and the family* $\{\nu_1^{(n)}\}_{n\in\mathbb{N}}$ *is tight. As* $n \to \infty$, $\nu_1^{(n)}$ *converges weakly to* ν_1.

Proposition 3.1 and Theorem 3.1 will be proved in Section 7.

4. Diffusion Limit of the Reactant under Fast Catalyst Dynamics

As noted in Section 2, the catalyst and reactant populations whose scaled evolution is described through (2.2) transition on comparable time scales. In situations in which the catalyst evolves "much faster" than the reactant, one can hope to find a simplified model that captures the dynamics of the reactant population in a more economical fashion. One would expect that the reactant population can be approximated by a diffusion whose coefficients depend on the catalyst only through the catalyst's stationary distribution. Indeed, we will show that the (scaled) reactant population can be approximated by the solution of

$$\check{Y}_t = \check{Y}_0 + \int_0^t c_2\lambda_2 m_X \check{Y}_s ds + \int_0^t \sqrt{\alpha_2\lambda_2 m_X \check{Y}_s} dB_s, \quad \check{Y}_0 = y_0, \tag{4.1}$$

where $m_X = \int_1^\infty x\nu_1(dx) = -\frac{\alpha_1\theta}{2c_1}\exp(2c_1/\alpha_1)$.

Such model reductions (see [11] and references therein for the setting of chemical reaction networks) not only help in better understanding the dynamics of the system but also help in reducing computational costs in simulations. In this section we will consider such stochastic averaging results in two model settings. First, in Section 4.1, we consider the simpler setting where the population mass evolutions are described through (reflected) stochastic integral equations and a scaling parameter in the coefficients of the model distinguishes the time scales of the two processes. In Section 4.2 we will consider a setting which captures the underlying physical dynamics more accurately in the sense that the mass processes are described in terms of continuous time branching processes, rather than diffusions.

4.1. Stochastic Averaging in a Diffusion Setting

In this section we consider the setting where the catalyst and reactant populations evolve according to (reflected) diffusions similar to X and Y from Proposition 2.1, but where the evolution of the catalyst is accelerated by a factor of a_n such that $a_n \uparrow \infty$ as $n \uparrow \infty$ (i.e. drift and diffusion coefficients are scaled by a_n). More precisely, we consider a system of catalyst and reactant populations that are given as solutions of the following system of stochastic integral equations: For $t \geq 0$,

$$\begin{aligned}
\check{X}_t^{(n)} &= \Gamma\left(\check{X}_0^{(n)} + \int_0^{\cdot} a_n c_1 \lambda_1 \check{X}_s^{(n)} ds + \int_0^{\cdot} \sqrt{a_n \alpha_1 \lambda_1 \check{X}_s^{(n)}} dB_s^X\right)(t) \\
\check{Y}_t^{(n)} &= \check{Y}_0^{(n)} + \int_0^t c_2 \lambda_2 \check{X}_s^{(n)} \check{Y}_s^{(n)} ds + \int_0^t \sqrt{\alpha_2 \lambda_2 \check{X}_s^{(n)} \check{Y}_s^{(n)}} dB_s^Y,
\end{aligned}$$

where $(\check{X}_0^{(n)}, \check{Y}_0^{(n)}) = (x_0, y_0)$, $c_1, c_2 \in \mathbb{R}$, $\alpha_i, \lambda_i \in (0, \infty)$, B^X and B^Y are independent standard Brownian motions, and Γ is the Skorohod map described above Proposition 2.1.

The following result says that if $c_1 < 0$, then the reactant population process $\check{Y}^{(n)}$, which is given through a coupled two dimensional system, can be well approximated by the one dimensional diffusion $\check{Y}$ in (4.1), whose coefficients are given in terms of the stationary distribution of the catalyst process.

Theorem 4.1. *Suppose Condition 3.1 holds. The process $\check{Y}^{(n)}$ converges weakly in $C(\mathbb{R}_+ : \mathbb{R}_+)$ to the process $\check{Y}$.*

The proof of Theorem 4.1 is given in Section 8.

4.2. Stochastic Averaging for Scaled Branching Processes

We now consider stochastic averaging for the setting where the catalyst and reactant populations are described through branching processes. Consider catalyst and reactant populations evolving according to the branching processes introduced in Section 2, but where the catalyst evolution is sped up by a factor of a_n such that $a_n \uparrow \infty$ monotonically as $n \uparrow \infty$. That is, we consider a sequence of catalyst populations $\tilde{X}_t^{(n)} := X_{a_n t}^{(n)}$, $t \geq 0$, where $X^{(n)}$ are the branching processes introduced in Section 2. The reactant population evolves according to a branching process, $\tilde{Y}^{(n)}$, whose branching rate, as before, is of the order of the current total mass of the catalyst population, $\tilde{X}^{(n)}/n$. The infinitesimal generator $\check{\mathcal{G}}^{(n)}$ of the scaled process

$$\left(\check{X}_t^{(n)}, \check{Y}_t^{(n)}\right) := \left(\frac{1}{n}\tilde{X}_{nt}^{(n)}, \frac{1}{n}\tilde{Y}_{nt}^{(n)}\right), \quad t \geq 0,$$

is given as

$$\check{\mathcal{G}}^{(n)}\phi(x,y) = \lambda_1^{(n)} n^2 a_n x \sum_{k=0}^{\infty} \Big[\phi\Big(1 \vee (x + \frac{k-1}{n}), y\Big) - \phi(x,y)\Big]\mu_1^{(n)}(k)$$
$$+ \lambda_2^{(n)} n^2 xy \sum_{k=0}^{\infty} \Big[\phi\Big(x, y + \frac{k-1}{n}\Big) - \phi(x,y)\Big]\mu_2^{(n)}(k), \tag{4.2}$$

where $(x,y) \in \mathbb{S}_X^{(n)} \times \mathbb{S}_Y^{(n)}$ and $\phi \in \mathrm{BM}([1,\infty) \times \mathbb{R}_+)$.

We note that a key difference between the generators $\check{\mathcal{G}}^{(n)}$ above and $\hat{\mathcal{A}}^{(n)}$ in (2.2) is the extra factor of a_n in the first term of (4.2), which says that, for large n, the catalyst dynamics are much faster than that of the reactant.

We will show in Theorem 4.2 that the reactant population process $\check{Y}^{(n)}$ can be well approximated by the one dimensional diffusion $\check{Y}$ in (4.1). Once again, the result provides a model reduction that is potentially useful for simulations and also for a general qualitative understanding of reactant dynamics near criticality.

Theorem 4.2. *Suppose Conditions 2.1, 3.1, and 3.2 hold. Then, as $n \to \infty$, $\check{Y}^{(n)}$ converges weakly in $D(\mathbb{R}_+ : \mathbb{R}_+)$ to the process $\check{Y}$.*

We will prove the above theorem in Section 8.

5. Auxiliary Results

In this section we collect several auxiliary results, which will be used in the proofs of our main results. Recall that the quadratic covariation (or bracket process) of two semimartingales ξ and ζ is the process $\{[\xi,\zeta]_t\}_{t\in\mathbb{R}_+}$ defined by

$$[\xi,\zeta]_t := \xi_t\zeta_t - \int_0^t \xi_{s-} d\zeta_s - \int_0^t \zeta_{s-} d\xi_s, \quad t \geq 0,$$

where $\xi_{0-} := 0, \zeta_{0-} := 0$. The predictable quadratic covariation of ξ and ζ is the unique predictable process $\{\langle\xi,\zeta\rangle_t\}_{t\in\mathbb{R}_+}$ such that $\{[\xi,\zeta]_t - \langle\xi,\zeta\rangle_t\}_{t\in\mathbb{R}_+}$ is a local martingale. If $\xi = \zeta$, then $[\xi] \equiv [\xi,\xi]$ and $\langle\xi\rangle \equiv \langle\xi,\xi\rangle$ are, respectively, the quadratic and predictable quadratic variation processes of ξ.

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{W}$, let $\phi_i(\mathbf{x}) = x_i$, $i = 1,2,3$, and $h := \phi_1 - \phi_3$. Note that for a locally bounded measurable function f on $\mathbb{W}$

$$M_t^{(n)}(f) := f(\hat{\mathbf{W}}_t^{(n)}) - f(\hat{\mathbf{W}}_0^{(n)}) - \int_0^t \hat{\mathcal{A}}^{(n)} f(\hat{\mathbf{W}}_s^{(n)}) ds, \quad t \geq 0, \tag{5.1}$$

is a local martingale with respect to the filtration $\sigma(\hat{\mathbf{W}}_s^{(n)} : s \leq t)$. For the rest of the paper, we suppress the filtration, and simply refer to $M^{(n)}(f)$ as a local martingale.

Let

$$\hat{\eta}_t^{(n)} := \lambda_1^{(n)} n \mu_1^{(n)}(0) \int_0^t 1_{\{\hat{X}_s^{(n)}=1\}} ds. \tag{5.2}$$

This process will play the role of the reflection term in the dynamics of the catalyst, arising from the controlled immigration. The following tightness result will be used in the weak convergence proofs.

Proposition 5.1. *Suppose Conditions 2.1 and 3.1 hold. Then the family* $\{(\hat{X}^{(n)}, \hat{Y}^{(n)}, \hat{\eta}^{(n)})\}_{n\in\mathbb{N}}$ *is tight in* $D(\mathbb{R}_+ : [1,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+)$*. If additionally Condition 3.2 holds, then the family* $\{(\hat{X}_{s+\cdot}^{(n)}, \hat{\eta}_{s+\cdot}^{(n)} - \hat{\eta}_s^{(n)})\}_{n\in\mathbb{N}, s\in\mathbb{R}_+}$ *is tight in* $D(\mathbb{R}_+ : [1,\infty) \times \mathbb{R}_+)$*.*

The proof of Proposition 5.1 will be based on the following results. Lemma 5.1 below gives some useful representations for the catalyst and reactant processes. Lemmas 5.2 – 5.5 and Corollary 5.1 provide moment bounds that are useful for arguing tightness. Proofs of these results are given in Section 5.1.

Lemma 5.1. *Suppose Condition 2.1 (i) holds. Then the process* $(\hat{X}^{(n)}, \hat{Y}^{(n)})$ *can be represented as*

$$\begin{aligned}\hat{X}_t^{(n)} &= \hat{X}_0^{(n)} + c_1^{(n)}\lambda_1^{(n)} \int_0^t \hat{X}_s^{(n)} ds + M_t^{(n)}(\phi_1) + \hat{\eta}_t^{(n)} \\ &= \Gamma\left(\hat{X}_0^{(n)} + c_1^{(n)}\lambda_1^{(n)} \int_0^{\cdot} \hat{X}_s^{(n)} ds + M_{\cdot}^{(n)}(\phi_1)\right)(t)\end{aligned} \tag{5.3}$$

and

$$\hat{Y}_t^{(n)} = \hat{Y}_0^{(n)} + c_2^{(n)}\lambda_2^{(n)} \int_0^t \hat{X}_t^{(n)} \hat{Y}_t^{(n)} ds + M_t^{(n)}(\phi_2). \tag{5.4}$$

Moreover, for $t \geq 0$*,*

$$\langle M^{(n)}(\phi_1)\rangle_t = \lambda_1^{(n)}\alpha_1^{(n)} \int_0^t \hat{X}_s^{(n)} ds - \lambda_1^{(n)}\mu_1^{(n)}(0) \int_0^t 1_{\{\hat{X}_s^{(n)}=1\}} ds \leq \lambda_1^{(n)}\alpha_1^{(n)} \int_0^t \hat{X}_s^{(n)} ds \tag{5.5}$$

and

$$\langle M^{(n)}(\phi_2)\rangle_t = \lambda_2^{(n)}\alpha_2^{(n)} \int_0^t \hat{X}_s^{(n)} \hat{Y}_s^{(n)} ds. \tag{5.6}$$

Let

$$\hat{N}_t^{(n)} := \hat{X}_0^{(n)} + c_1^{(n)}\lambda_1^{(n)} \int_0^t \hat{X}_s^{(n)} ds + M_t^{(n)}(\phi_1). \tag{5.7}$$

Then we have the following second moment estimate.

Lemma 5.2. *Suppose Conditions 2.1 (i) and (ii) hold. Then there is a $K \in (0,\infty)$ such that for all $n \in \mathbb{N}$ and $T \geq 0$*

$$E\left(\sup_{t\leq T}\left((\hat{X}_t^{(n)})^2 + (M_t^{(n)}(\phi_1))^2 + (\hat{N}_t^{(n)})^2 + (\hat{\eta}_t^{(n)})^2\right)\right) \leq \exp(KT^2)(x_0^{(n)})^2, \tag{5.8}$$

and for each $k \in \mathbb{N}$

$$E\left(\sup_{t\leq T}\left(\hat{Y}^{(n)}_{\sigma_k^{(n)}\wedge t}\right)^2\right) \leq \exp(KT^2k^2)(y_0^{(n)})^2, \tag{5.9}$$

where $\sigma_k^{(n)} := \inf\{t > 0 : \hat{X}_t^{(n)} \geq k\}$.

In order to study properties of invariant measures of $\hat{X}^{(n)}$, it will be convenient to allow the initial random variable $\hat{X}_0^{(n)}$ to have an arbitrary distribution on $\mathbb{S}_X^{(n)}$. When $X_0^{(n)}$ has distribution μ on $\mathbb{S}_X^{(n)}$, we will denote the corresponding probability and expectation operator by P_μ and E_μ, respectively. If $\mu = \delta_x$ for some $x \in \mathbb{S}_X^{(n)}$, we will instead write P_x and E_x, respectively. When considering an initial condition x for $\hat{X}^{(n)}$, x will always be in $\mathbb{S}_X^{(n)}$, although this will frequently be suppressed in the notation. The symbols E and P (without any subscripts) will correspond to the initial distribution as in Condition 2.1.

Lemma 5.3. *Suppose Conditions 2.1 (i) and (ii), 3.1, and 3.2 hold. Then there exist $\delta, \rho \in (0,\infty)$ such that for every $M > 0$*

$$\sup_{n\in\mathbb{N}, x\leq M} E_x\left(\sup_{0\leq t\leq\rho} e^{\delta\hat{X}_t^{(n)}}\right) =: d(\delta,\rho,M) < \infty. \tag{5.10}$$

Lemma 5.4. *Suppose Conditions 2.1 (i) and (ii), 3.1, and 3.2 hold. Then there exist $\delta, \tilde{d} \in (0,\infty)$ such that for every $x \in \mathbb{S}_X^{(n)}$, $n \in \mathbb{N}$, and $t \geq 0$*

$$E_x(e^{\frac{\delta}{2}\hat{X}_t^{(n)}}) \leq \tilde{d}e^{\delta x}. \tag{5.11}$$

The following is immediate from Lemmas 5.2 and 5.4.

Corollary 5.1. *Suppose Conditions 2.1 (i) and (ii), 3.1, and 3.2 hold. Let δ be as in Lemma 5.4 and $T \in \mathbb{R}_+$. Then there exists a $d(\delta,T) \in (0,\infty)$ such that for all $x \in \mathbb{S}_X^{(n)}$ and $n \in \mathbb{N}$*

$$\sup_{s\in\mathbb{R}_+} E_x\left(\sup_{s\leq u\leq s+T}(\hat{X}_u^{(n)})^2\right) \leq d(\delta,T)e^{\delta x}. \tag{5.12}$$

The next lemma follows on combining Lemma 5.4 with arguments as in the proof of Lemma 5.2. The proof is omitted.

Lemma 5.5. *Suppose Conditions 2.1 (i) and (ii), 3.1, and 3.2 hold. Let δ be as in Lemma 5.4. Then for each $T \geq 0$ there are $L_T, \tilde{L}_T \in (0, \infty)$ such that for all $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$*

$$E\left(\sup_{t\leq T}\left((\hat{X}^{(n)}_{s+t} - \hat{X}^{(n)}_s)^2 + (M^{(n)}_{s+t}(\phi_1) - M^{(n)}_s(\phi_1))^2 + (\hat{N}^{(n)}_{s+t} - \hat{N}^{(n)}_s)^2 + (\hat{\eta}^{(n)}_{s+t} - \hat{\eta}^{(n)}_s)^2\right)\right)$$
$$\leq L_T E\left(\hat{X}^{(n)}_s\right)^2 \leq \tilde{L}_T\left(e^{\delta x^{(n)}_0}\right). \tag{5.13}$$

In order to prove weak convergence results for the scaled catalyst and reactant processes, we will need to argue that the limit processes are continuous, which will be a consequence of the following bounds on the jumps. The somewhat stronger estimate on the jumps of the catalyst population in (5.15), below, will be used in the stochastic averaging argument in the proof of Theorem 4.2. Recall that for a process $\{\xi_t\}_{t\in\mathbb{R}_+}$ the jump at instant $t > 0$ is defined as $\Delta\xi_t := \xi_t - \xi_{t-}$ and $\Delta\xi_0 := 0$.

Lemma 5.6. *Suppose Condition 2.1 holds. Fix $T, \epsilon > 0$. Then, as $n \to \infty$,*

$$P\left(\sup_{0\leq t\leq T}\left(|\Delta\hat{X}^{(n)}_t| + |\Delta\hat{Y}^{(n)}_t|\right) \geq \epsilon\right) \to 0. \tag{5.14}$$

If additionally Conditions 3.1 and 3.2 hold, then, as $n \to \infty$,

$$\sup_{s\in\mathbb{R}_+} P\left(\sup_{0\leq t\leq T}|\Delta\hat{X}^{(n)}_{s+t}| \geq \epsilon\right) \to 0. \tag{5.15}$$

5.1. Proofs of Auxiliary Results

In this section we prove the results stated in Section 5. We begin with the proofs of Lemmas 5.1 – 5.5. Using these results, we will then prove Proposition 5.1. The proof of Lemma 5.6 is given at the end.

Proof of Lemma 5.1. Recall that $\hat{\mathbf{W}}^{(n)} = (\hat{X}^{(n)}, \hat{Y}^{(n)}, \hat{Z}^{(n)})$ and that for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{W}$, $\phi_i(\mathbf{x}) = x_i$, $i = 1, 2, 3$, and $h := \phi_1 - \phi_3$. From (5.1),

$$\hat{Z}^{(n)}_t = \phi_3(\hat{\mathbf{W}}^{(n)}_t) = \hat{Z}^{(n)}_0 + \int_0^t \hat{\mathcal{A}}^{(n)}\phi_3(\hat{\mathbf{W}}^{(n)}_s)ds + M^{(n)}_t(\phi_3). \tag{5.16}$$

Using (2.2), we get

$$\hat{\mathcal{A}}^{(n)}\phi_3(\hat{\mathbf{W}}^{(n)}_t) = \lambda^{(n)}_1 n\hat{X}^{(n)}_t\sum_{k=0}^{\infty}(k-1)\mu^{(n)}_1(k) = c^{(n)}_1\lambda^{(n)}_1\hat{X}^{(n)}_t. \tag{5.17}$$

Next, since $\hat{X}_0^{(n)} = \hat{Z}_0^{(n)}$, we have

$$\hat{X}_t^{(n)} - \hat{Z}_t^{(n)} = h(\hat{\mathbf{W}}_t^{(n)}) = \int_0^t \hat{\mathcal{A}}^{(n)} h(\hat{\mathbf{W}}_s^{(n)}) ds + M_t^{(n)}(h),$$

and, once more using (2.2),

$$\hat{\mathcal{A}}^{(n)} h(\mathbf{w}) = \lambda_1^{(n)} n \mu_1^{(n)}(0) 1_{\{x=1\}}, \quad \mathbf{w} = (x, y, z).$$

Thus with $\hat{\eta}^{(n)}$ as in (5.2), we get

$$\hat{X}_t^{(n)} - \hat{Z}_t^{(n)} = h(\hat{\mathbf{W}}_t^{(n)}) = \hat{\eta}_t^{(n)} + M_t^{(n)}(h). \tag{5.18}$$

Noting that $M^{(n)}(\phi_1) = M^{(n)}(h) + M^{(n)}(\phi_3)$ and using (5.16), (5.17), and (5.18), we have

$$\hat{X}_t^{(n)} = \hat{X}_0^{(n)} + c_1^{(n)} \lambda_1^{(n)} \int_0^t \hat{X}_s^{(n)} ds + M_t^{(n)}(\phi_1) + \hat{\eta}_t^{(n)}. \tag{5.19}$$

Since $\hat{\eta}^{(n)}$ is non-decreasing and $\int_0^\infty 1_{\{\hat{X}_s^{(n)} \neq 1\}} d\hat{\eta}_s^{(n)} = 0$, we have from the characterization given above (2.4) that

$$\hat{X}_t^{(n)} = \Gamma\left(\hat{X}_0^{(n)} + c_1^{(n)} \lambda_1^{(n)} \int_0^{\cdot} \hat{X}_s^{(n)} ds + M_{\cdot}^{(n)}(\phi_1)\right)(t).$$

Next, for the reactant population, using similar calculations as for $\hat{X}^{(n)}$, we get

$$\hat{Y}_t^{(n)} = \hat{Y}_0^{(n)} + \int_0^t \hat{\mathcal{A}}^{(n)} \phi_2(\hat{\mathbf{W}}_s^{(n)}) ds + M_t^{(n)}(\phi_2) = \hat{Y}_0^{(n)} + c_2^{(n)} \lambda_2^{(n)} \int_0^t \hat{X}_s^{(n)} \hat{Y}_s^{(n)} ds + M_t^{(n)}(\phi_2)$$

Finally, routine calculations then show (see [10], Lemma 3.1.3) that (5.5) and (5.6) hold. Details are omitted. □

Proof of Lemma 5.2. Using (5.5) and Doob's inequality, we have

$$E\left(\sup_{t \le T} \left(M_t^{(n)}(\phi_1)\right)^2\right) \le 4\lambda_1^{(n)} \alpha_1^{(n)} E\left(\int_0^T \hat{X}_s^{(n)} ds\right). \tag{5.20}$$

Next, from (5.3), $\hat{X}_t^{(n)} = \Gamma(\hat{N}_{\cdot}^{(n)})(t)$. The Lipschitz continuity of the Skorohod map implies

$$\sup_{t \le T} |\hat{X}_t^{(n)} - 1| \le 2 \sup_{t \le T} |\hat{N}_t^{(n)} - 1|. \tag{5.21}$$

Letting $|\hat{X}^{(n)}|_{*,T}^2 := \sup_{t \le T} |\hat{X}_t^{(n)}|^2$, we now get

$$|\hat{X}^{(n)}|_{*,T}^2 = \left(|\hat{X}^{(n)} - 1|_{*,T} + 1\right)^2 \le 2|\hat{X}^{(n)} - 1|_{*,T}^2 + 2 \le 8|\hat{N}^{(n)} - 1|_{*,T}^2 + 2 \le 16|\hat{N}^{(n)}|_{*,T}^2 + 18.$$

Combining this with (5.7) and (5.20), we obtain

$$E\left(|\hat{X}^{(n)}|^2_{*,T}\right) \leq 18 + 16E\left(|\hat{N}^{(n)}|^2_{*,T}\right) \tag{5.22}$$
$$\leq 18 + 48\left[E(\hat{X}^{(n)}_0)^2 + \left(T\big(c^{(n)}_1\lambda^{(n)}_1\big)^2 + 4\lambda^{(n)}_1\alpha^{(n)}_1\right)\int_0^T E\big(|\hat{X}^{(n)}|^2_{*,s}\big)ds\right].$$

Using Gronwall's inequality, we get, since $E(\hat{X}^{(n)}_0)^2 = (x^{(n)}_0)^2 \geq 1$,

$$E\left(|\hat{X}^{(n)}|^2_{*,T}\right) \leq 66(x^{(n)}_0)^2 \exp\left(K^{(n)}_{1,T}\right),$$

where $K^{(n)}_{1,T} := 48T(T(c^{(n)}_1\lambda^{(n)}_1)^2 + 4\lambda^{(n)}_1\alpha^{(n)}_1)$. Since $c^{(n)}_1, \lambda^{(n)}_1$, and $\alpha^{(n)}$ converge as $n \to \infty$, we have that for some $K \in (0,\infty)$ and all $n \in \mathbb{N}$

$$E\big(\sup_{t\leq T}(\hat{X}^{(n)}_t)^2\big) \leq 66\exp(KT^2)(x^{(n)}_0)^2. \tag{5.23}$$

Using (5.23) in (5.20), (5.22), and (5.19), we have the estimate in (5.8) by choosing K sufficiently large.

We next establish (5.9). Using Doob's inequality once more and applying (5.6), we have

$$E\left(\sup_{t\leq T}\left(M^{(n)}_{\sigma^{(n)}_k\wedge t}(\phi_2)\right)^2\right) \leq 4E\left(\langle M^{(n)}(\phi_2)\rangle_{\sigma^{(n)}_k\wedge T}\right) \leq 4\lambda^{(n)}_2\alpha^{(n)}_2 E\left(\int_0^{\sigma^{(n)}_k\wedge T}\hat{X}^{(n)}_s\hat{Y}^{(n)}_s ds\right).$$

Thus, by (5.4),

$$E\big(|\hat{Y}^{(n)}|^2_{*,T\wedge\sigma^{(n)}_k}\big) \leq 3\left((y^{(n)}_0)^2 + \left[T\big(c^{(n)}_2\lambda^{(n)}_2k\big)^2 + 4\lambda^{(n)}_2\alpha^{(n)}_2k\right]\int_0^T E\big(|\hat{Y}^{(n)}|^2_{*,s\wedge\sigma^{(n)}_k}\big)ds\right).$$

The estimate in (5.9) now follows by choosing K sufficiently large and applying Gronwall's inequality. □

Proof of Lemma 5.3. First we show, using Conditions 2.1 (i) and (ii), 3.1, and 3.2, that there are $\delta_0, d_1, d_2 \in (0,\infty)$ such that for all $\delta \in [0,\delta_0]$ and $n \in \mathbb{N}$

$$-\delta d_2 \leq \sum_{k=0}^{\infty} n^2\left[e^{\frac{(k-1)\delta}{n}} - 1\right]\mu^{(n)}_1(k) \leq -\delta d_1. \tag{5.24}$$

Note that

$$\sum_{k=0}^{\infty} n^2\left[e^{\frac{(k-1)\delta}{n}} - 1\right]\mu^{(n)}_1(k) = n^2\sum_{k=0}^{\infty}\left(\sum_{l=1}^{\infty}\frac{1}{l!}\left(\frac{(k-1)\delta}{n}\right)^l\right)\mu^{(n)}_1(k)$$
$$= n\delta\left(\sum_{k=0}^{\infty}k\mu^{(n)}_1(k) - 1\right) + \frac{1}{2}\delta^2\sum_{k=0}^{\infty}(k-1)^2\mu^{(n)}_1(k) + n^2\sum_{k=0}^{\infty}\left(\sum_{l=3}^{\infty}\frac{1}{l!}\left(\frac{(k-1)\delta}{n}\right)^l\right)\mu^{(n)}_1(k).$$

Now

$$n\delta\left(\sum_{k=0}^{\infty} k\mu_1^{(n)}(k) - 1\right) = n\delta(m_1^{(n)} - 1) = \delta c_1^{(n)} \to \delta c_1 \in (-\infty, 0) \quad \text{as } n \to \infty$$

and

$$\frac{1}{2}\delta^2 \sum_{k=0}^{\infty}(k-1)^2\mu_1^{(n)}(k) = \frac{1}{2}\delta^2\alpha_1^{(n)} \to \frac{1}{2}\delta^2\alpha_1 \quad \text{as } n \to \infty.$$

Noting that $c_1^{(n)} < 0$, we can choose $\delta_0 > 0$ sufficiently small, and $d_1, d_2 \in (0, \infty)$ suitably, such that (5.24) holds.

For δ_0 as above and $\delta \le \delta_0$, let

$$\alpha_\delta^{(n)} := ne^\delta \sum_{k=1}^{\infty}\left(e^{\frac{(k-1)\delta}{n}} - 1\right)\frac{\mu_1^{(n)}(k)}{\mu_1^{(n)}(0)}$$

and

$$\beta_t^{(n),\delta} := n^2\lambda_1^{(n)}\int_0^t \hat{X}_s^{(n)} \sum_{k=0}^{\infty}\left(\left[e^{\frac{(k-1)\delta}{n}} - 1\right]\mu_1^{(n)}(k)\right)1_{\{\hat{X}_s^{(n)}>1\}}ds.$$

Note that, by (5.24), for any $t \ge u \ge 0$,

$$-\delta d_2\lambda_1^{(n)}\int_u^t \hat{X}_s^{(n)}1_{\{\hat{X}_s^{(n)}>1\}}ds \le \beta_t^{(n),\delta} - \beta_u^{(n),\delta} \le -\delta d_1\lambda_1^{(n)}\int_u^t \hat{X}_s^{(n)}1_{\{\hat{X}_s^{(n)}>1\}}ds. \tag{5.25}$$

Moreover,

$$0 \le \alpha_\delta^{(n)} \le e^\delta\delta. \tag{5.26}$$

The first inequality in the last display is immediate, the second inequality can be seen as follows:

$$\alpha_\delta^{(n)} = ne^\delta\sum_{k=1}^{\infty}\left(e^{\frac{(k-1)\delta}{n}} - 1\right)\frac{\mu_1^{(n)}(k)}{\mu_1^{(n)}(0)} = ne^\delta\sum_{k=0}^{\infty}\left(e^{\frac{(k-1)\delta}{n}} - 1\right)\frac{\mu_1^{(n)}(k)}{\mu_1^{(n)}(0)} - ne^\delta(e^{-\frac{\delta}{n}} - 1)\frac{\mu_1^{(n)}(0)}{\mu_1^{(n)}(0)}.$$

By (5.24), the first term on the right hand side of the last display is smaller or equal to 0. Thus

$$\alpha_\delta^{(n)} \le -ne^\delta(e^{-\frac{\delta}{n}} - 1) \le e^\delta\delta.$$

We now argue that

$$M_t^{(n),\delta} := \exp\left(\delta\hat{X}_t^{(n)} - \beta_t^{(n),\delta}\right) - \alpha_\delta^{(n)}\int_0^t \exp\left(-\beta_s^{(n),\delta}\right)d\hat{\eta}_s^{(n)}$$

is a local martingale. Let $f(x) = e^{\delta x}$ and

$$q(x) := \frac{\hat{\mathcal{L}}^{(n)}f(x)}{f(x)}1_{\{x>1\}}.$$

Here $\hat{\mathcal{L}}^{(n)}$ is the generator of $\hat{X}^{(n)}$, that is, for $x \in \mathbb{S}_X^{(n)}$

$$\hat{\mathcal{L}}^{(n)} f(x) = \lambda_1^{(n)} n^2 x \sum_{k=0}^{\infty} \left[f\left(1 \vee \left(x + \frac{k-1}{n}\right)\right) - f(x) \right] \mu_1^{(n)}(k). \tag{5.27}$$

Note that

$$q(x) = n^2 \lambda_1^{(n)} x \sum_{k=0}^{\infty} \left(\left[e^{\frac{(k-1)\delta}{n}} - 1 \right] \mu_1^{(n)}(k) \right) 1_{\{x>1\}},$$

and thus

$$\int_0^t q(\hat{X}_s^{(n)}) ds = \beta_t^{(n),\delta}. \tag{5.28}$$

Also,

$$\int_0^t \hat{\mathcal{L}}^{(n)} f(\hat{X}_s^{(n)}) 1_{\{\hat{X}_s^{(n)}=1\}} ds = \alpha_\delta^{(n)} \hat{\eta}_t^{(n)}. \tag{5.29}$$

Consider the Markov process $V^{(n)}$ defined by

$$V_t^{(n)} := \left(\hat{X}_t^{(n)}, \exp\left(- \int_0^t q(\hat{X}_s^{(n)}) ds \right) \right), \quad t \geq 0.$$

Denote by $\bar{\mathcal{L}}^{(n)}$ the generator of $V^{(n)}$. Then the action of the generator on the function $f(x)g(y)$ with $f(x) = e^{\delta x}$ and $g(y) = y$ is given by

$$\bar{\mathcal{L}}^{(n)}(f(x)g(y)) = y \left(\hat{\mathcal{L}}^{(n)} f(x) - q(x) f(x) \right) = y \hat{\mathcal{L}}^{(n)} f(x) 1_{\{x=1\}}.$$

Using (5.28) we now have that

$$\begin{aligned} &(fg)(V_t^{(n)}) - \int_0^t \bar{\mathcal{L}}^{(n)}(fg)(V_s^{(n)}) ds \\ &\quad = e^{\delta \hat{X}_t^{(n)} - \beta_t^{(n),\delta}} - \int_0^t e^{-\beta_s^{(n),\delta}} \hat{\mathcal{L}}^{(n)} f(\hat{X}_s^{(n)}) 1_{\{\hat{X}_s^{(n)}=1\}} ds, \quad t \geq 0, \end{aligned}$$

is a local martingale. From (5.29) we now see that the last expression equals $M_t^{(n),\delta}$, $t \geq 0$, which is thus a local martingale.

We next show that for every $M > 0$, $\delta \leq \delta_0$, and $t \geq 0$

$$d_3(\delta, t, M) := \sup_{x \leq M} \sup_{n \in \mathbb{N}} E_x \left(e^{\delta \hat{X}_t^{(n)}} \right) < \infty. \tag{5.30}$$

Note that

$$\begin{aligned} e^{\delta \hat{X}_t^{(n)}} &= \left(e^{\delta \hat{X}_t^{(n)} - \beta_t^{(n),\delta}} - \alpha_\delta^{(n)} \int_0^t e^{-\beta_s^{(n),\delta}} d\hat{\eta}_s^{(n)} + \alpha_\delta^{(n)} \int_0^t e^{-\beta_s^{(n),\delta}} d\hat{\eta}_s^{(n)} \right) e^{\beta_t^{(n),\delta}} \\ &= \left(M_t^{(n),\delta} + \alpha_\delta^{(n)} \int_0^t e^{-\beta_s^{(n),\delta}} d\hat{\eta}_s^{(n)} \right) e^{\beta_t^{(n),\delta}} \end{aligned} \tag{5.31}$$

Applying Itô's formula and using (5.24), (5.28), and (5.31), we see that

$$\begin{aligned}E_x\left(e^{\delta\hat{X}_t^{(n)}}\right) &= e^{\delta x} + \alpha_\delta^{(n)} E_x \int_0^t e^{\int_0^s q(\hat{X}_u^{(n)})du} e^{-\int_0^s q(\hat{X}_u^{(n)})du} d\hat{\eta}_s^{(n)} \\ &\quad + E_x\left(\int_0^t q(\hat{X}_s^{(n)})\left(M_s^{(n),\delta} + \alpha_\delta^{(n)}\int_0^s e^{-\beta_u^{(n),\delta}} d\hat{\eta}_u^{(n)}\right) e^{\int_0^s q(\hat{X}_u^{(n)})du} ds\right) \\ &= e^{\delta x} + \alpha_\delta^{(n)} E_x \hat{\eta}_t^{(n)} + E_x\left(\int_0^t q(\hat{X}_s^{(n)}) e^{\delta \hat{X}_s^{(n)}} ds\right) \le e^{\delta x} + \alpha_\delta^{(n)} E_x \hat{\eta}_t^{(n)}.\end{aligned}$$

The estimate in (5.30) now follows on combining the above inequality with Lemma 5.2 and (5.26).

Fix $M > 0$, $x \le M$, and $\delta \le \frac{\delta_0}{4}$. Then, since $\beta_t^{(n),\delta} \le 0$ for all $t \ge 0$, we have for $\rho > 0$,

$$E_x\left(\sup_{0\le t\le\rho} e^{\delta\hat{X}_t^{(n)}}\right)^2 \le E_x\left(\sup_{0\le t\le\rho} e^{\delta\hat{X}_t^{(n)} - \beta_t^{(n),\delta}}\right)^2 \le 4E_x\left(e^{2\delta\hat{X}_\rho^{(n)} - 2\beta_\rho^{(n),\delta}}\right),$$

where the last inequality follows on noting that $e^{\delta\hat{X}_t^{(n)} - \beta_t^{(n),\delta}}$ is a sub-martingale and applying Doob's inequality. Now from (5.25) and (5.30),

$$\begin{aligned}E_x\left(e^{2\delta\hat{X}_\rho^{(n)} - 2\beta_\rho^{(n),\delta}}\right) &\le \left(E_x\left(e^{4\delta\hat{X}_\rho^{(n)}}\right)\right)^{\frac{1}{2}} \left(E_x\left(e^{-4\beta_\rho^{(n),\delta}}\right)\right)^{\frac{1}{2}} \\ &\le (d_3(4\delta,\rho,M))^{\frac{1}{2}} E_x\left(\exp\left(4\delta d_2 \lambda_1^{(n)} \rho \sup_{0\le t\le\rho} \hat{X}_t^{(n)}\right)\right).\end{aligned}$$

Choose $\rho < \left(8d_2 \sup_{n\in\mathbb{N}} \lambda_1^{(n)}\right)^{-1}$. Then, by combining the above estimates, we can find a $d_4(\delta,\rho,M) < \infty$ such that for all $x \le M$, $n \in \mathbb{N}$ and $\delta \le \frac{\delta_0}{4}$

$$\begin{aligned}E_x\left(\sup_{0\le t\le\rho} e^{\delta\hat{X}_t^{(n)}}\right) &\le d_4(\delta,\rho,M) E_x\left(\exp\left(4\delta d_2 \lambda_1^{(n)} \rho \sup_{0\le t\le\rho} \hat{X}_t^{(n)}\right)\right) \\ &\le d_4(\delta,\rho,M) E_x\left(\exp\left(\frac{\delta}{2}\sup_{0\le t\le\rho} \hat{X}_t^{(n)}\right)\right) \le d_4(\delta,\rho,M)\left[E_x\left(\sup_{0\le t\le\rho} e^{\delta\hat{X}_t^{(n)}}\right)\right]^{\frac{1}{2}}.\end{aligned}$$

Dividing both sides by $\left[E_x\left(\sup_{0\le t\le\rho} e^{\delta\hat{X}_t^{(n)}}\right)\right]^{\frac{1}{2}}$ yields $\left[E_x\left(\sup_{0\le t\le\rho} e^{\delta\hat{X}_t^{(n)}}\right)\right]^{\frac{1}{2}} \le d_4(\delta,\rho,M)$, for any $x \le M$ and $n \in \mathbb{N}$. The result follows. □

Proof of Lemma 5.4. For $\delta \in (0,1), n \in \mathbb{N}$, define

$$\begin{aligned}b_\delta^{(n),1}(x) &:= \lambda_1^{(n)} n^2 x \sum_{k=0}^\infty \left(e^{\delta\frac{k-1}{n}} - 1\right)\mu_1^{(n)}(k), \\ b_\delta^{(n),2}(x) &:= \lambda_1^{(n)} n^2 x \sum_{k=1}^\infty \left(e^{\delta\frac{k-1}{n}} - 1\right)\mu_1^{(n)}(k), \quad \text{and} \\ b_\delta^{(n)}(x) &:= b_\delta^{(n),1}(x)1_{\{x>1\}} + b_\delta^{(n),2}(x)1_{\{x=1\}}.\end{aligned}$$

From (5.24), we have, for some $\kappa \in (0, \infty)$,

$$\sup_{n\in\mathbb{N}} b_\delta^{(n),1}(x) \le -\delta d_1 x \inf_{n\in\mathbb{N}} \lambda^{(n)} \le -\delta\kappa x \le -\delta\kappa$$

for all $\delta \le \delta_0$ (with δ_0 as above (5.24)) and $x \ge 1$. Observing that with $f(x) = e^{\delta x}$, $\frac{\hat{\mathcal{L}}^{(n)} f(x)}{f(x)} = b_\delta^{(n)}(x)$, where $\hat{\mathcal{L}}^{(n)}$ is the generator of $\hat{X}^{(n)}$ defined in (5.27), we have that

$$U_t^{(n)} := e^{\delta \hat{X}_t^{(n)} - \int_0^t b_\delta^{(n)}(\hat{X}_s^{(n)})ds}, \quad t \ge 0, \tag{5.32}$$

is a local martingale. Fix δ and ρ as in the statement of Lemma 5.3. Without loss of generality, we can assume that $\delta \le \delta_0$. Note that on the set

$$\{\omega : \hat{X}_s^{(n)}(\omega) > 1 \text{ for all } s \in [(j-1)\rho, j\rho)\},$$

we have

$$\delta[\hat{X}_{j\rho}^{(n)} - \hat{X}_{(j-1)\rho}^{(n)}] \le \delta[\hat{X}_{j\rho}^{(n)} - \hat{X}_{(j-1)\rho}^{(n)}] - \int_{(j-1)\rho}^{j\rho} b_\delta^{(n)}(\hat{X}_s^{(n)})ds - \delta\kappa\rho \equiv v_j^{(n)} - \delta\kappa\rho. \tag{5.33}$$

Fix $t > 0$ and let $N \in \mathbb{N}$ be such that $(N-1)\rho \le t < N\rho$. Then, similarly, on the set

$$\{\omega : \hat{X}_t^{(n)}(\omega) > 1 \text{ for all } s \in [(N-1)\rho, t)\},$$

$\delta[\hat{X}_t^{(n)} - \hat{X}_{(N-1)\rho}^{(n)}] \le v_N^{(n)}(t)$, where

$$v_j^{(n)}(t) := \delta[\hat{X}_t^{(n)} - \hat{X}_{(j-1)\rho}^{(n)}] - \int_{(j-1)\rho}^{t} b_\delta^{(n)}(\hat{X}_s^{(n)})ds.$$

Now, for a fixed ω, let $m \equiv m(\omega)$ be such that $[(m-1)\rho, m\rho)$ is the last interval in which $\hat{X}^{(n)}$ visits 1 before time $N\rho$. We set $m = 0$ if 1 is not visited before time $N\rho$. We distinguish between the cases $0 < m < N$, $m = N$, and $m = 0$, where the latter corresponds to the case where 1 is not visited before time $N\rho$.

Case 1: $0 < m < N$.

In this case

$$\delta\hat{X}_t^{(n)} \le \delta\hat{X}_{m\rho}^{(n)} + \sum_{j=m+1}^{N-1} (v_j^{(n)} - \delta\kappa\rho) + v_N^{(n)}(t).$$

For $j \in \mathbb{N}$, let

$$\gamma_j^{(n)} := \inf\{t \ge (j-1)\rho | \hat{X}_t^{(n)} = 1\} \wedge j\rho$$

and

$$\theta_j^{(n)} := \sup_{0\le t\le\rho} [\hat{X}_{(t+\gamma_j^{(n)})\wedge j\rho}^{(n)} - \hat{X}_{\gamma_j^{(n)}}^{(n)}]. \tag{5.34}$$

Then $\delta\hat{X}_{m\rho}^{(n)} \le \delta\theta_m^{(n)} + \delta$. Combining the above estimates, we have

$$\delta\hat{X}_t^{(n)} \le \delta\theta_m^{(n)} + \delta + \sum_{j=m+1}^{N-1} (v_j^{(n)} - \delta\kappa\rho) + v_N^{(n)}(t). \tag{5.35}$$

Thus, in this case

$$\delta\hat{X}_t^{(n)} \leq \delta\hat{X}_0^{(n)} + \max_{0\leq l\leq N}\left\{\sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + \delta\theta_l^{(n)}\right\} + v_N^{(n)}(t),$$

where by convention $\sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) = 0$ for $l = N-1, N$, and $\theta_0^{(n)} := 0$.
Case 2: $m = 0$.
In this case, 1 is not visited before time $N\rho$ and thus

$$\begin{aligned}\delta\hat{X}_t^{(n)} &\leq \delta\hat{X}_0^{(n)} + \sum_{j=1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + v_N^{(n)}(t)\\ &\leq \delta\hat{X}_0^{(n)} + \max_{0\leq l\leq N}\left\{\sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + \delta\theta_l^{(n)}\right\} + v_N^{(n)}(t).\end{aligned}$$

Case 3: $m = N$.
Suppose first that there is an $s \in [(N-1)\rho, t]$ such that $\hat{X}_s^{(n)} = 1$. It then follows that

$$\delta\hat{X}_t^{(n)} \leq \delta\theta_N^{(n)} + \delta.$$

Now suppose that there is no such $s \in [(N-1)\rho, t]$. Define $m' \in \{1, 2, \ldots, N-1\}$ to be such that $[(m'-1)\rho, m'\rho)$ is the last interval in which $\hat{X}^{(n)}$ visits 1 before $(N-1)\rho$. Once again we set $m' = 0$ if there is no such interval.

If $m' = 0$, we get exactly as in Case 2 that

$$\delta\hat{X}_t^{(n)} \leq \delta\hat{X}_0^{(n)} + \max_{0\leq l\leq N}\left\{\sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + \delta\theta_l^{(n)}\right\} + v_N^{(n)}(t).$$

If $1 \leq m' \leq N-1$, then

$$\begin{aligned}\delta\hat{X}_t^{(n)} &\leq \delta\theta_{m'}^{(n)} + \delta + \sum_{j=m'+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + v_N^{(n)}(t)\\ &\leq \delta\hat{X}_0^{(n)} + \max_{0\leq l\leq N}\left\{\sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + \delta\theta_l^{(n)}\right\} + v_N^{(n)}(t).\end{aligned}$$

Combining the three cases, we have

$$\delta\hat{X}_t^{(n)} \leq \max\left\{\delta\hat{X}_0^{(n)} + \max_{0\leq l\leq N}\left\{\sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + \delta\theta_l^{(n)}\right\} + v_N^{(n)}(t), \delta + \delta\theta_N^{(n)}\right\}.$$

Thus, for any $M_0 > 0$,

$$\begin{aligned}
&P_x(\delta \hat{X}_t^{(n)} \geq M_0) \\
&\leq \sum_{l=0}^{N-1} P_x\left(v_N^{(n)}(t) + \sum_{j=l+1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) + \delta\theta_l^{(n)} + \delta + \delta\hat{X}_0^{(n)} \geq M_0\right) + P_x(\delta\theta_N^{(n)} + \delta \geq M_0) \\
&\leq e^{\delta(1+x)-M_0}\Bigg(\sum_{l=0}^{N-1}\left[E_x\Big(\exp\Big[\delta\theta_l^{(n)} + \sum_{j=l+1}^{N-1} v_j^{(n)} + v_N^{(n)}(t)\Big]\Big)e^{-\delta\kappa\rho(N-l-1)}\right] + E_x(e^{\delta\theta_N^{(n)}})\Bigg).
\end{aligned}$$

Recalling $U^{(n)}$ from (5.32) and using its martingale property, we get

$$\begin{aligned}
P_x(\delta\hat{X}_t^{(n)} \geq M_0) &\leq e^{-M_0}e^{\delta(1+x)}\left(E_x\left(e^{\delta\theta_N^{(n)}}\right) + \sum_{l=0}^{N-1} e^{-\delta\kappa\rho(N-l-1)}E_x\left(e^{\delta\theta_l^{(n)}}\right)\right) \\
&\leq e^{-M_0}e^{(1+x)\delta}d(\delta,\rho,1)\left(1 + \frac{1}{1-e^{-\delta\kappa\rho}}\right), \qquad (5.36)
\end{aligned}$$

where the last inequality follows from Lemma 5.3 and the observation that

$$\sup_{n\in\mathbb{N}} E_x(e^{\delta\theta_l^{(n)}}) \leq \sup_{n\in\mathbb{N}} E_1\left(\sup_{0\leq t\leq\rho} e^{\delta\hat{X}_t^{(n)}}\right) \leq d(\delta,\rho,1) < \infty, \qquad (5.37)$$

where $\theta_l^{(n)}$ is as in (5.34). Finally, from (5.36), we get that for all $t \geq 0$ and $n \in \mathbb{N}$

$$\begin{aligned}
E_x\left(e^{\frac{\delta}{2}\hat{X}_t^{(n)}}\right) &= \int_0^\infty P_x\left(\delta\hat{X}_t^{(n)} > 2\ln(y)\right)dy \\
&\leq 1 + e^{(1+x)\delta}d(\delta,\rho,1)\left(1 + \frac{1}{1-e^{-\delta\kappa\rho}}\right)\int_1^\infty e^{-2\ln(y)}dy \leq \tilde{d}e^{\delta x}, \qquad (5.38)
\end{aligned}$$

where $\tilde{d} = 1 + e^{\delta}d(\delta,\rho,1)\left(1 + \frac{1}{1-e^{-\delta\kappa\rho}}\right)$. The result follows. □

Proof of Proposition 5.1. We will first consider the second part of the proposition. We begin by showing that $\{\hat{N}_{s+\cdot}^{(n)} - \hat{N}_s^{(n)}\}_{s,n}$ is tight. For that, in view of (5.13), it suffices to show that the following condition (Aldous-Kurtz criterion) holds: For each $M > 0, \epsilon > 0$, and $\gamma > 0$ there are $\delta_0 > 0$ and n_0 such that for all stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \leq M$, we have

$$\sup_{s\in\mathbb{R}_+, n\geq n_0}\sup_{\theta\leq\delta_0} P(|\hat{N}_{s+\tau_n+\theta}^{(n)} - \hat{N}_{s+\tau_n}^{(n)}| \geq \gamma) \leq \epsilon. \qquad (5.39)$$

Let $M, \epsilon, \gamma \in (0,\infty)$ be given. Note that

$$\begin{aligned}
&P(|\hat{N}_{s+\tau_n+\theta}^{(n)} - \hat{N}_{s+\tau_n}^{(n)}| \geq \gamma) \\
&\qquad \leq P\big(|c_1^{(n)}\lambda_1^{(n)}\int_{s+\tau_n}^{s+\tau_n+\theta}\hat{X}_u^{(n)}du| \geq \frac{\gamma}{2}\big) + P\big(|M_{s+\tau_n+\theta}^{(n)}(\phi_1) - M_{s+\tau_n}^{(n)}(\phi_1)| \geq \frac{\gamma}{2}\big).
\end{aligned}$$

By (5.12) we have, for δ_0 sufficiently small,

$$\sup_{s\in\mathbb{R}_+,n\in\mathbb{N}}\sup_{\theta\le\delta_0} P\left(|c_1^{(n)}\lambda_1^{(n)}\int_{s+\tau_n}^{s+\tau_n+\theta}\hat{X}_u^{(n)}du|\ge\frac{\gamma}{2}\right)<\frac{\epsilon}{2}.$$

It remains to prove that, for some $\delta_0>0$,

$$\sup_{s\in\mathbb{R}_+,n\in\mathbb{N}}\sup_{\theta\le\delta_0} P\left(|M^{(n)}_{s+\tau_n+\theta}(\phi_1)-M^{(n)}_{s+\tau_n}(\phi_1)|\ge\frac{\gamma}{2}\right)<\frac{\epsilon}{2}. \tag{5.40}$$

Using the martingale property of $M^{(n)}(\phi_1)$,

$$\begin{aligned}P\Big(|M^{(n)}_{s+\tau_n+\theta}(\phi_1)-M^{(n)}_{s+\tau_n}(\phi_1)|\ge\frac{\gamma}{2}\Big)&\le\frac{E(|M^{(n)}_{s+\tau_n+\theta}(\phi_1)-M^{(n)}_{s+\tau_n}(\phi_1)|^2)}{(\gamma/2)^2}\\ =\frac{E((M^{(n)}_{s+\tau_n+\theta}(\phi_1))^2)-E((M^{(n)}_{s+\tau_n}(\phi_1))^2)}{(\gamma/2)^2}&=\frac{E\langle M^{(n)}(\phi_1)\rangle_{s+\tau_n+\theta}-E\langle M^{(n)}(\phi_1)\rangle_{s+\tau_n}}{(\gamma/2)^2},\end{aligned}$$

and, using (5.5),

$$E\langle M^{(n)}(\phi_1)\rangle_{s+\tau_n+\theta}-E\langle M^{(n)}(\phi_1)\rangle_{s+\tau_n}\le E\left(\lambda_1^{(n)}\alpha_1^{(n)}\int_{s+\tau_n}^{s+\tau_n+\theta}\hat{X}_u^{(n)}du\right).$$

Now, using (5.12) once more, we can choose $\delta_0>0$ such that (5.40) holds. This proves tightness of $\{\hat{N}^{(n)}_{s+\cdot}-\hat{N}^{(n)}_s\}_{s,n}$ and, using the continuity property of the Skorohod map (from $D_1(\mathbb{R}_+:\mathbb{R})$ to $D(\mathbb{R}_+:[1,\infty))$), that of $\{\hat{X}^{(n)}_{s+\cdot}-\hat{X}^{(n)}_s\}_{s,n}$ and $\{\hat{\eta}^{(n)}_{s+\cdot}-\hat{\eta}^{(n)}_s\}_{s,n}$. Tightness of $\{\hat{X}^{(n)}_{s+\cdot}\}_{s,n}$ now follows by using the uniform estimate in Lemma 5.4.

Now we consider the first part of the proposition. Tightness of $(\hat{X}^{(n)},\hat{\eta}^{(n)})$ follows as before. We now consider $\hat{Y}^{(n)}$. Fix $\epsilon>0$. Using (5.9), we get, for $\tilde{K}\in(0,\infty)$,

$$\begin{aligned}P\Big(\sup_{t\le T}(\hat{Y}^{(n)}_t)>\tilde{K}\Big)&\le P\Big(\sup_{t\le T}(\hat{Y}^{(n)}_{\sigma_k^{(n)}\wedge t})>\tilde{K}\text{ and }\sigma_k^{(n)}>T\Big)+P\left(\sigma_k^{(n)}\le T\right)\\ \le\frac{E\big(\sup_{t\le T}(\hat{Y}^{(n)}_{\sigma_k^{(n)}\wedge t})^2\big)}{\tilde{K}^2}&+P\Big(\sup_{t\le T}(\hat{X}^{(n)}_t)\ge k\Big)\le\frac{\exp(KT^2k^2)(y_0^{(n)})^2}{\tilde{K}^2}+\frac{E\big(\sup_{t\le T}(\hat{X}^{(n)}_t)^2\big)}{k^2}.\end{aligned}$$

Using (5.8), we can choose k such that

$$\sup_{n\in\mathbb{N}}\frac{E\left(\sup_{t\le T}(\hat{X}^{(n)}_t)^2\right)}{k^2}<\frac{\epsilon}{2}. \tag{5.41}$$

Now choose $\tilde{K}$ such that

$$\sup_{n\in\mathbb{N}}\frac{\exp(KT^2k^2)(y_0^{(n)})^2}{\tilde{K}^2}<\frac{\epsilon}{2}.$$

The last two displays imply $\sup_{n\in\mathbb{N}}P(\sup_{t\le T}(\hat{Y}^{(n)}_t)>\tilde{K})<\epsilon$, and since $\epsilon>0$ is arbitrary, the tightness of the random variables $\{\hat{Y}^{(n)}_t\}_{n\in\mathbb{N}}$, for each $t\ge0$, follows. To establish the tightness

of the processes $\{\hat{Y}^{n)}\}_{n\in\mathbb{N}}$, it now suffices to show that for each $M>0, \epsilon>0$, and $\gamma>0$ there are $\delta_0>0$ and n_0 such that for all stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n \le M$, we have

$$\sup_{n\ge n_0}\sup_{\theta\le\delta_0} P\left(|\hat{Y}^{(n)}_{\tau_n+\theta}-\hat{Y}^{(n)}_{\tau_n}|\ge\gamma\right)\le\epsilon. \tag{5.42}$$

Fix $M,\epsilon,\gamma\in(0,\infty)$. Then, for any $\theta\in(0,1)$,

$$P\left(|\hat{Y}^{(n)}_{\tau_n+\theta}-\hat{Y}^{(n)}_{\tau_n}|\ge\gamma\right)\le P\Big(|\hat{Y}^{(n)}_{(\tau_n+\theta)\wedge\sigma_k^{(n)}}-\hat{Y}^{(n)}_{\tau_n\wedge\sigma_k^{(n)}}|\ge\gamma\Big)+P\left(\sigma_k^{(n)}\le M+1\right).$$

Taking $T=M+1$ and k as in (5.41), we have $P(\sigma_k^{(n)}<M+1)<\epsilon/2$ for all $n\in\mathbb{N}$. For the first term on the right hand side of the last display, we get, using (5.4) and that $\sup_{t\le T\wedge\sigma_k^{(n)}}\hat{X}_t^{(n)}\le k$,

$$\begin{aligned}&P\left(|\hat{Y}^{(n)}_{(\tau_n+\theta)\wedge\sigma_k^{(n)}}-\hat{Y}^{(n)}_{\tau_n\wedge\sigma_k^{(n)}}|\ge\gamma\right) \qquad (5.43)\\ &\le P\Big(|c_2^{(n)}\lambda_2^{(n)}\int_{\tau_n\wedge\sigma_k^{(n)}}^{(\tau_n+\theta)\wedge\sigma_k^{(n)}}\hat{Y}_s^{(n)}ds|\ge\frac{\gamma}{2k}\Big)+P\Big(|M^{(n)}_{(\tau_n+\theta)\wedge\sigma_k^{(n)}}(\phi_2)-M^{(n)}_{\tau_n\wedge\sigma_k^{(n)}}(\phi_2)|\ge\frac{\gamma}{2}\Big).\end{aligned}$$

The first term on the right hand side can be bounded as follows:

$$\begin{aligned}P\Big(\Big|c_2^{(n)}\lambda_2^{(n)}\int_{\tau_n\wedge\sigma_k^{(n)}}^{(\tau_n+\theta)\wedge\sigma_k^{(n)}}\hat{Y}_s^{(n)}ds\Big|\ge\frac{\gamma}{2k}\Big)&\le\left(\frac{2kc_2^{(n)}\lambda_2^{(n)}}{\gamma}\right)^2E\Big(\Big(\int_{\tau_n\wedge\sigma_k^{(n)}}^{(\tau_n+\theta)\wedge\sigma_k^{(n)}}\hat{Y}_s^{(n)}ds\Big)^2\Big)\\ &\le\theta\left(\frac{2kc_2^{(n)}\lambda_2^{(n)}}{\gamma}\right)^2\exp\left(K(M+1)^2k^2\right)(y_0^{(n)})^2,\end{aligned}$$

where K is the constant from (5.9). Thus, for δ_0 sufficiently small, we get

$$\sup_{n\in\mathbb{N}}\sup_{\theta\le\delta_0}P\left(\left|c_2^{(n)}\lambda_2^{(n)}\int_{\tau_n\wedge\sigma_k^{(n)}}^{(\tau_n+\theta)\wedge\sigma_k^{(n)}}\hat{Y}_s^{(n)}ds\right|\ge\frac{\gamma}{2k}\right)<\epsilon/4.$$

The second term on the right hand side of (5.43) can be bounded as follows:

$$\begin{aligned}P\big(|M^{(n)}_{(\tau_n+\theta)\wedge\sigma_k^{(n)}}(\phi_2)-M^{(n)}_{\tau_n\wedge\sigma_k^{(n)}}(\phi_2)|\ge\frac{\gamma}{2}\big)&\le\frac{E\left(\langle M^{(n)}(\phi_2)\rangle_{(\tau_n+\theta)\wedge\sigma_k^{(n)}}-\langle M^{(n)}(\phi_2)\rangle_{\tau_n\wedge\sigma_k^{(n)}}\right)}{(\gamma/2)^2}\\ &\le\frac{4}{\gamma^2}\lambda_2^{(n)}\alpha_2^{(n)}E\left(\int_{\tau_n\wedge\sigma_k^{(n)}}^{(\tau_n+\theta)\wedge\sigma_k^{(n)}}\hat{X}_s^{(n)}\hat{Y}_s^{(n)}ds\right)\le\frac{4}{\gamma^2}\lambda_2^{(n)}\alpha_2^{(n)}k\theta E\left(\sup_{0\le s\le M+1}\hat{Y}^{(n)}_{s\wedge\sigma_k^{(n)}}\right).\end{aligned}$$

Using (5.9) once more, we have that, for δ_0 sufficiently small, the second term in (5.43) is bounded by $\frac{\epsilon}{4}$. Combining the above estimates, we now see that (5.42) holds, and thus tightness of $\{\hat{Y}^{(n)}\}_{n\in\mathbb{N}}$ follows. □

Proof of Lemma 5.6. Consider (5.15). Let $N^{(n)}_{s,T}$ be the number of deaths of particles of the (unscaled) process $X^{(n)}$ in the time interval $[s, s+T]$. Fix $\epsilon, \delta > 0$. Then

$$P\big(\sup_{0\le t\le T} |\Delta \hat{X}^{(n)}_{s+t}| \ge \epsilon\big) \le P\big(\sup_{0\le t\le T} |\Delta \hat{X}^{(n)}_{s+t}| \ge \epsilon; \sup_{0\le t\le T} X^{(n)}_{s+t} \le nL\big) + P\big(\sup_{0\le t\le T} X^{(n)}_{s+t} > nL\big).$$

By Corollary 5.1 we can choose $L \in (0,\infty)$ such that $P\big(\sup_{0\le t\le T} X^{(n)}_{s+t} > nL\big) < \frac{\delta}{3}$, for $s \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Next, consider

$$\begin{aligned} &P\big(\sup_{0\le t\le T} |\Delta \hat{X}^{(n)}_{s+t}| \ge \epsilon; \sup_{0\le t\le T} X^{(n)}_{s+t} \le nL\big) \\ &\le P\big(\sup_{0\le t\le T} |\Delta \hat{X}^{(n)}_{s+t}| \ge \epsilon; N^{(n)}_{s,T} < nCL\big) + P\big(\sup_{0\le t\le T} X^{(n)}_{s+t} \le nL; N^{(n)}_{s,T} \ge nCL\big). \end{aligned}$$

Note that on the set $\{\sup_{0\le t\le T} X^{(n)}_{s+t} \le nL\}$ the branching rates of $X^{(n)}$ are bounded during the time interval $[s, s+T]$, uniformly in s and n, and thus we can choose a $C \in (0,\infty)$ such that for $s \in \mathbb{R}_+$ and $n \in \mathbb{N}$

$$P\big(\sup_{0\le t\le T} X^{(n)}_{s+t} \le nL; N^{(n)}_{s,T} \ge nCL\big) < \frac{\delta}{3}.$$

Finally, let, for $n \in \mathbb{N}$, $\{\xi^{(n)}_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables distributed as $\mu^{(n)}_1$. Then, since the variance of the offspring distribution converges, we have for n_0 sufficiently large and all $n \ge n_0$,

$$\begin{aligned} P\left(\sup_{0\le t\le T} |\Delta \hat{X}^{(n)}_{s+t}| \ge \epsilon; N^{(n)}_{s,T} < nCL\right) &\le P\left(\max_{1\le i<nCL} \frac{|\xi^{(n)}_i - 1|}{n} \ge \epsilon\right) \\ &\le \sum_{i=1}^{nCL-1} P\left(|\xi^{(n)}_i - 1| \ge n\epsilon\right) \le \sum_{i=1}^{nCL-1} \frac{E\left(|\xi^{(n)}_i - 1|^2\right)}{(n\epsilon)^2} < \frac{\delta}{3}. \end{aligned}$$

Combining the above estimates, (5.15) follows. The limit in (5.14) can be established similarly, using Lemma 5.2 instead of Corollary 5.1; the proof is therefore omitted. □

6. Proof of Theorem 2.1

The following martingale characterization result will be useful in the proof of Theorem 2.1. The proof is standard and is omitted ([19, 13, 14], see also Theorem 5.3 of [4]).

For $\phi \in C^\infty_c([1,\infty)\times\mathbb{R}_+)$, let

$$\mathcal{L}\phi(x,y) := c_1\lambda_1 x\frac{\partial}{\partial x}\phi(x,y) + \frac{1}{2}\alpha_1\lambda_1 x\frac{\partial^2}{\partial x^2}\phi(x,y) + c_2\lambda_2 xy\frac{\partial}{\partial y}\phi(x,y) + \frac{1}{2}\alpha_2\lambda_2 xy\frac{\partial^2}{\partial y^2}\phi(x,y).$$

Let $\tilde{\Omega} := D(\mathbb{R}_+ : [1,\infty)\times\mathbb{R}^2_+)$ and $\tilde{\mathcal{F}}$ be the corresponding Borel σ-field (with respect to the Skorohod topology). Denote by $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ the canonical filtration on $(\tilde{\Omega}, \tilde{\mathcal{F}})$, i.e. $\mathcal{F}_t = \sigma(\pi_s | s \le t)$, where $\pi_s(\tilde{\omega}) = \tilde{\omega}_s = \tilde{\omega}(s)$ for $\tilde{\omega} \in \tilde{\Omega}$. Finally, let $\pi^{(i)}$, $i = 1,2,3$, be the coordinate processes, i.e. $(\pi^{(1)}(\tilde{\omega}), \pi^{(2)}(\tilde{\omega}), \pi^{(3)}(\tilde{\omega})) = \pi(\tilde{\omega})$.

Theorem 6.1. *Let $\tilde{P}$ be a probability measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ under which the following hold a.s.:*

(i) $\pi^{(3)}$ *is a non-decreasing, continuous process, and* $\pi_0^{(3)} = 0$,
(ii) $(\pi^{(1)}, \pi^{(2)})$ *is an* $([1,\infty) \times \mathbb{R}_+)$ *valued continuous process,*
(iii) $\int_0^\infty 1_{(1,\infty)}(\pi_s^{(1)}) d\pi_s^{(3)} = 0$,
(iv) for all $\phi \in C_c^\infty([1,\infty) \times \mathbb{R}_+)$

$$\phi(\pi_t^{(1)}, \pi_t^{(2)}) - \int_0^t \mathcal{L}\phi(\pi_s^{(1)}, \pi_s^{(2)}) ds - \int_0^t \frac{\partial \phi}{\partial x}(1, \pi_s^{(2)}) d\pi_s^{(3)}$$

is an $\{\mathcal{F}_t\}$ *martingale, and*
(v) $\tilde{P} \circ (\pi_0^{(1)}, \pi_0^{(2)})^{-1} = \bar{P} \circ (X_0, Y_0)^{-1}$*,where* X, Y*, and* $\bar{P}$ *are as in Proposition 2.1.*
Then $\tilde{P} \circ (\pi^{(1)}, \pi^{(2)})^{-1} = \bar{P} \circ (X, Y)^{-1}$.

Proof of Theorem 2.1. Recall that for $\phi \in C_c^\infty([1,\infty) \times \mathbb{R}_+)$, we have

$$\phi(\hat{X}_t^{(n)}, \hat{Y}_t^{(n)}) = \phi(\hat{X}_0^{(n)}, \hat{Y}_0^{(n)}) + \int_0^t \hat{\mathcal{A}}^{(n)} \phi(\hat{X}_s^{(n)}, \hat{Y}_s^{(n)}) ds + M_t^{(n)}(\phi), \tag{6.1}$$

where $M_t^{(n)}(\phi)$ is a martingale, and $\hat{\mathcal{A}}^{(n)}$ is as defined in (2.2). Also note that $\hat{\mathcal{A}}^{(n)}$ can be rewritten as

$$\hat{\mathcal{A}}^{(n)} \phi(x,y) = \mathcal{L}^{(n)} \phi(x,y) + \mathcal{D}^{(n)} \phi(y) n \lambda_1^{(n)} \mu_1^{(n)}(0) 1_{\{x=1\}},$$

where

$$\begin{aligned} \mathcal{L}^{(n)} \phi(x,y) := \lambda_1^{(n)} n^2 x \sum_{k=0}^\infty \left[\phi\left(x + \frac{k-1}{n}, y\right) - \phi(x,y) \right] \mu_1^{(n)}(k) \\ + \lambda_2^{(n)} n^2 xy \sum_{k=0}^\infty \left[\phi\left(x, y + \frac{k-1}{n}\right) - \phi(x,y) \right] \mu_2^{(n)}(k) \end{aligned}$$

and

$$\mathcal{D}^{(n)} \phi(y) := n \left(\phi(1,y) - \phi\left(1 - \frac{1}{n}, y\right) \right).$$

Thus, using (5.2), (6.1) can be rewritten as

$$\begin{aligned} \phi(\hat{X}_t^{(n)}, \hat{Y}_t^{(n)}) = \phi(\hat{X}_0^{(n)}, \hat{Y}_0^{(n)}) + \int_0^t \mathcal{L}^{(n)} \phi(\hat{X}_s^{(n)}, \hat{Y}_s^{(n)}) ds \\ + \int_0^t \mathcal{D}^{(n)} \phi(\hat{Y}_s^{(n)}) d\hat{\eta}_s^{(n)} + M_t^{(n)}(\phi). \end{aligned}$$

Recall the path space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ introduced above Theorem 6.1. Denote by $\tilde{P}^{(n)}$ the measure induced by $(\hat{X}^{(n)}, \hat{Y}^{(n)}, \hat{\eta}^{(n)})$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ and by $\tilde{E}^{(n)}$ the corresponding expectation.

From Proposition 5.1, $\tilde{P}^{(n)}$ is tight. Let $\tilde{P}$ be a limit point of $\{\tilde{P}^{(n)}\}$ along some subsequence $\{n_k\}$. In order to complete the proof, it suffices to show that under $\tilde{P}$ properties (i) – (v) in

Theorem 6.1 hold almost surely. Property (i) is immediate from the fact that $\hat{\eta}^{(n)}$ is non-decreasing and continuous with initial value 0 for each n. Also, property (v) is immediate from the fact that $(\hat{X}_0^{(n)}, \hat{Y}_0^{(n)}) = (1,1)$, a.s., for each n. Next, consider property (ii). The continuity of $\pi^{(1)}$ and $\pi^{(2)}$ follows by (5.14) (see [9], Proposition VI.3.26, p. 315).

To see (iii), consider, for $\delta > 0$, continuous bounded test functions $f_\delta : [1,\infty) \longrightarrow \mathbb{R}_+$ such that

$$f_\delta(x) = \begin{cases} 1, & \text{if } x \geq 1+2\delta \\ 0, & \text{if } x \leq 1+\delta. \end{cases} \tag{6.2}$$

Note that, for each $n \in \mathbb{N}$, $\int_0^\infty f_\delta(\hat{X}_s^{(n)}) d\hat{\eta}_s^{(n)} = 0$ and thus, for each $\delta > 0$,

$$0 = \lim_{k\to\infty} \tilde{E}^{(n_k)} \left(\int_0^\infty f_\delta(\pi_s^{(1)}) d\pi_s^{(3)} \wedge 1 \right) = \tilde{E} \left(\int_0^\infty f_\delta(\pi_s^{(1)}) d\pi_s^{(3)} \wedge 1 \right).$$

Consequently, for each $\delta > 0$, $\int_0^\infty 1_{[1+2\delta,\infty)}(\pi_s^{(1)}) d\pi_s^{(3)} = 0$, almost surely w.r.t. $\tilde{P}$. The property in (iii) now follows on sending $\delta \to 0$.

Finally, we consider part (iv). It suffices to show that for every $0 \leq s \leq t < \infty$

$$\tilde{E}\Big(\psi(\cdot) \Big(\phi(\pi_t^{(1)}, \pi_t^{(2)}) - \phi(\pi_s^{(1)}, \pi_s^{(2)}) - \int_s^t \mathcal{L}\phi(\pi_u^{(1)}, \pi_u^{(2)}) du - \int_s^t \frac{\partial \phi}{\partial x}(1, \pi_u^{(2)}) d\pi_u^{(3)} \Big) \Big) = 0,$$

where $\psi : \tilde{\Omega} \to \mathbb{R}$ is an arbitrary bounded, continuous, $\mathcal{F}_s$ measurable map. Now fix such s, t, and ψ. Then by weak convergence of $\tilde{P}^{(n_k)}$ to $\tilde{P}$ and using the moment bound in Lemma 5.2,

$$\begin{aligned} &\lim_{k\to\infty} \tilde{E}^{(n_k)} \big(\psi(\cdot) \big(\phi(\pi_t^{(1)}, \pi_t^{(2)}) - \phi(\pi_s^{(1)}, \pi_s^{(2)}) - \int_s^t \mathcal{L}\phi(\pi_u^{(1)}, \pi_u^{(2)}) du - \int_s^t \frac{\partial \phi}{\partial x}(1, \pi_u^{(2)}) d\pi_u^{(3)} \big) \big) \\ &\quad = \tilde{E} \big(\psi(\cdot) \big(\phi(\pi_t^{(1)}, \pi_t^{(2)}) - \phi(\pi_s^{(1)}, \pi_s^{(2)}) - \int_s^t \mathcal{L}\phi(\pi_u^{(1)}, \pi_u^{(2)}) du - \int_s^t \frac{\partial \phi}{\partial x}(1, \pi_u^{(2)}) d\pi_u^{(3)} \big) \big). \end{aligned}$$

To complete the proof, it suffices to show that the limit on the left side above is 0. In view of the martingale property in (6.1), to show this, it suffices to prove that for $\phi \in C_c^\infty([1,\infty) \times \mathbb{R}_+)$

$$\lim_{n\to\infty} E \left| \int_0^t \left(\mathcal{L}^{(n)} \phi(\hat{X}_s^{(n)}, \hat{Y}_s^{(n)}) - \mathcal{L}\phi(\hat{X}_s^{(n)}, \hat{Y}_s^{(n)}) \right) ds \right| = 0 \tag{6.3}$$

and

$$\lim_{n\to\infty} E \left| \int_0^t \left(D^{(n)} \phi(\hat{Y}_s^{(n)}) - \frac{\partial \phi}{\partial x}(1, \hat{Y}_s^{(n)}) \right) d\hat{\eta}_s^{(n)} \right| = 0. \tag{6.4}$$

The latter is immediate upon using the smoothness of ϕ and the moment estimate for $\hat{\eta}^{(n)}$ in

(5.8). For (6.3), we rewrite $\mathcal{L}^{(n)}\phi$ using a Taylor expansion as follows:

$$\begin{aligned}\mathcal{L}^{(n)}\phi(x,y) &= \lambda_1^{(n)} n^2 x \sum_{k=0}^{\infty}\left[\frac{k-1}{n}\frac{\partial}{\partial x}\phi(x,y) + \frac{1}{2}\left(\frac{k-1}{n}\right)^2 \frac{\partial^2}{\partial x^2}\phi(x,y)\right]\mu_1^{(n)}(k)\\ &\quad + \lambda_2^{(n)} n^2 xy \sum_{k=0}^{\infty}\left[\frac{k-1}{n}\frac{\partial}{\partial y}\phi(x,y) + \frac{1}{2}\left(\frac{k-1}{n}\right)^2 \frac{\partial^2}{\partial y^2}\phi(x,y)\right]\mu_2^{(n)}(k) + R^{(n)}(x,y)\\ &= c_1^{(n)}\lambda_1^{(n)} x\frac{\partial}{\partial x}\phi(x,y) + \frac{1}{2}\alpha_1^{(n)}\lambda_1^{(n)} x\frac{\partial^2}{\partial x^2}\phi(x,y)\\ &\quad + c_2^{(n)}\lambda_2^{(n)} xy\frac{\partial}{\partial y}\phi(x,y) + \frac{1}{2}\alpha_2^{(n)}\lambda_2^{(n)} xy\frac{\partial^2}{\partial y^2}\phi(x,y) + R^{(n)}(x,y),\end{aligned}$$

where the term $R^{(n)}(x,y)$ is a remainder term, which, using part (iii) of Condition 2.1, is seen to satisfy $\sup_{|x|,|y|\le L}|R^{(n)}(x,y)| \to 0$ as $n\to\infty$, for any $L\in(0,\infty)$. Furthermore, using the compact support property of ϕ, it follows that $\lim_{n\to\infty} E\int_0^t |R^{(n)}(\hat{X}_s^{(n)},\hat{Y}_s^{(n)})|ds = 0$. Next note that

$$\begin{aligned}&\mathcal{L}^{(n)}\phi(x,y) - R^{(n)}(x,y) - \mathcal{L}\phi(x,y)\\ &= (\lambda_1^{(n)}c_1^{(n)} - \lambda_1 c_1)x\frac{\partial}{\partial x}\phi(x,y) + \frac{1}{2}(\lambda_1^{(n)}\alpha_1^{(n)} - \lambda_1\alpha_1)x\frac{\partial^2}{\partial x^2}\phi(x,y)\\ &\quad + (\lambda_2^{(n)}c_2^{(n)} - \lambda_2 c_2)xy\frac{\partial}{\partial y}\phi(x,y) + \frac{1}{2}(\lambda_2^{(n)}\alpha_2^{(n)} - \lambda_2\alpha_2)xy\frac{\partial^2}{\partial y^2}\phi(x,y),\end{aligned}$$

and therefore, in view of Condition 2.1,

$$\sup_{|x|,|y|\le L}\left|\mathcal{L}^{(n)}\phi(x,y) - R^{(n)}(x,y) - \mathcal{L}\phi(x,y)\right| \to 0 \quad \text{as } n\to\infty.$$

Once more using the compact support property of ϕ, it follows that

$$\lim_{n\to\infty} E\int_0^t \left|\mathcal{L}^{(n)}\phi(\hat{X}_s^{(n)},\hat{Y}_s^{(n)}) - R^{(n)}(\hat{X}_s^{(n)},\hat{Y}_s^{(n)}) - \mathcal{L}\phi(\hat{X}_s^{(n)},\hat{Y}_s^{(n)})\right| ds = 0.$$

Combining the above estimates, we have (6.3), and the result follows. □

7. Proofs of Results from Section 3

7.1. Proof of Proposition 3.1

Uniqueness of the invariant measure of X is an immediate consequence of the non-degeneracy of the diffusion coefficient (note that $\alpha_2\lambda_2 x \ge \alpha_2\lambda_2 > 0$). For existence, we will apply an extension of the well known Echeverria criterion for invariant measures of Markov processes ([14], see also Theorem 5.7 of [4]). This criterion, in the current context, says that in order to

establish that a probability measure $\bar{\nu}_1$ is an invariant measure for X, it suffices to verify that for some $C \geq 0$ and all $\phi \in C_c^\infty([1,\infty))$

$$\int_{[1,\infty)} \mathcal{L}_1\phi(x)\bar{\nu}_1(dx) + C\alpha_1\lambda_1\phi'(1) = 0, \tag{7.1}$$

where

$$\mathcal{L}_1\phi(x) = c_1\lambda_1 x\phi'(x) + \frac{1}{2}\alpha_1\lambda_1 x\phi''(x). \tag{7.2}$$

We now show that (7.1) holds with $\bar{\nu}_1 = \nu_1$ and $C = \frac{p(1)}{2}$. For $\phi \in C_c^\infty([1,\infty))$ and p as in (3.1),

$$\begin{aligned}
&\int_1^\infty \left(c_1\lambda_1 x\phi'(x) + \frac{1}{2}\alpha_1\lambda_1 x\phi''(x)\right)p(x)dx \\
&\quad = c_1\lambda_1\theta e^{2\frac{c_1}{\alpha_1}x}\phi(x)\Big|_1^\infty - \int_1^\infty 2c_1\lambda_1\theta\frac{c_1}{\alpha_1}e^{2\frac{c_1}{\alpha_1}x}\phi(x)dx \\
&\qquad + \frac{1}{2}\alpha_1\lambda_1\theta e^{2\frac{c_1}{\alpha_1}x}\phi'(x)\Big|_1^\infty - \int_1^\infty \alpha_1\lambda_1\theta\frac{c_1}{\alpha_1}e^{2\frac{c_1}{\alpha_1}x}\phi'(x)dx \\
&\quad = -\frac{1}{2}\alpha_1\lambda_1\theta e^{2\frac{c_1}{\alpha_1}}\phi'(1) = -\frac{p(1)}{2}\alpha_1\lambda_1\phi'(1).
\end{aligned}$$

Thus (7.1) follows. □

7.2. Proof of Theorem 3.1

Throughout this section we assume that Conditions 2.1, 3.1, and 3.2 hold. This will not be explicitly noted in the statements of the results.

Existence of a stationary distribution $\nu_1^{(n)}$ of the $\mathbb{S}_X^{(n)} = \{\frac{l}{n} | l \in \{n, n+1, \ldots\}\}$ valued Markov process $\hat{X}^{(n)}$ follows from the tightness of $\{\hat{X}_t^{(n)}\}_{t\geq 0}$, which is a consequence of Lemma 5.4. The uniqueness of the stationary distribution follows from the irreducibility of $\hat{X}^{(n)}$.

In order to establish the tightness of the sequence $\{\nu_1^{(n)}\}_{n\in\mathbb{N}}$, we will use the following uniform in n moment stability estimate for $\hat{X}^{(n)}$.

Theorem 7.1. *There is a $t_0 \in \mathbb{R}_+$ such that for all $t \geq t_0$ and $p > 0$*

$$\lim_{x\to\infty}\sup_{n\in\mathbb{N}} \frac{1}{x^p}E_x\left(\left(\hat{X}_{tx}^{(n)}\right)^p\right) = 0. \tag{7.3}$$

Proof. Fix an $L > 1$, and let $\tau^{(n)} := \inf\{t : \hat{X}_t^{(n)} \leq L\}$. Observe that if $t \in [(N-1)\rho, N\rho)$ for

some $N \in \mathbb{N}$, then, following arguments as in the proof of Lemma 5.4, for $x > L$,

$$P_x(\tau^{(n)} > t) \leq P_x\left(\sum_{j=1}^{N-1}(v_j^{(n)} - \delta\kappa\rho) > \delta(L-x)\right) \leq e^{\delta(x-L-\delta\kappa\rho(N-1))}.$$

Thus we have that

$$\sup_{n\in\mathbb{N}} P_x(\tau^{(n)} > t) \leq \gamma_1 e^{\delta x} e^{-\gamma_2 t},$$

where $\gamma_i \in (0,\infty)$, $i = 1,2$. The above estimate along with Lemma 5.4 implies, for $n \in \mathbb{N}$,

$$\begin{aligned} E_x e^{\frac{\delta}{2}\hat{X}_t^{(n)}} &= E_x\left(1_{\{\tau^{(n)}\leq t\}} e^{\frac{\delta}{2}\hat{X}_t^{(n)}}\right) + E_x\left(1_{\{\tau^{(n)}>t\}} e^{\frac{\delta}{2}\hat{X}_t^{(n)}}\right) \\ &\leq \tilde{d}e^{\delta L} + \left(\gamma_1 e^{\delta x} e^{-\gamma_2 t}\right)^{\frac{1}{2}} \left(E_x\left(e^{\delta \hat{X}_t^{(n)}}\right)\right)^{\frac{1}{2}} \leq \tilde{d}e^{\delta L} + \left(\gamma_1 e^{\delta x} e^{-\gamma_2 t}\right)^{\frac{1}{2}} (\tilde{d}e^{\delta x})^{\frac{1}{2}} \\ &\leq d_1(1 + e^{\delta x} e^{-\frac{\gamma_2}{2}t}), \end{aligned}$$

where $\tilde{d}$ is as in Lemma 5.4 and $d_1 \in (0,\infty)$ is some constant, independent of n. Fix $p > 0$. Then, for some $d_2 \in (0,\infty)$, we have

$$\sup_{n\in\mathbb{N}} \frac{E_x(\hat{X}_{tx}^{(n)})^p}{x^p} \leq \sup_{n\in\mathbb{N}} \frac{d_2 E_x e^{\frac{\delta}{2}\hat{X}_{tx}^{(n)}}}{x^p} \leq \sup_{n\in\mathbb{N}} \frac{d_1 d_2(1 + e^{\delta x} e^{-\frac{\gamma_2}{2}tx})}{x^p}.$$

Choose t_0 large enough such that $\frac{\gamma_2}{2}t_0 > \delta$. Then for $t \geq t_0$

$$\lim_{x\to\infty} \sup_{n\in\mathbb{N}} \frac{E_x\left((\hat{X}_{tx}^{(n)})^p\right)}{x^p} = 0.$$

The result follows. □

As a consequence of Theorem 7.1, we have the following result. For $\delta \in (0,\infty)$, define the return time to a compact set $C \subset [1,\infty)$ by $\tau_C^{(n)}(\delta) := \inf\{t \geq \delta | \hat{X}_t^{(n)} \in C\}$.

Theorem 7.2. *There are $\tilde{c}, \hat{\delta} \in (0,\infty)$ and a compact set $C \subset [1,\infty)$ such that*

$$\sup_n E_x\left(\int_0^{\tau_C^{(n)}(\hat{\delta})} \ln(\hat{X}_t^{(n)})dt\right) \leq \tilde{c}x^3, \quad x \geq 1.$$

Proof. Note that $\ln(\hat{X}_t^{(n)}) \geq 0$ since $\hat{X}_t^{(n)} \geq 1$. Applying Theorem 7.1 with $p = 3$, we have that there is an $L \in (1,\infty)$ such that with $C := \{x \in \mathbb{R}_+ | x \leq L\}$, for all $x \in C^c$,

$$\sup_n E_x\left(\left(\hat{X}_{t_0 x}^{(n)}\right)^3\right) \leq \frac{1}{2}x^3, \tag{7.4}$$

where t_0 is as in Theorem 7.1. Let $\hat{\delta} := t_0 L$ and $\tau^{(n)} := \tau_C^{(n)}(\hat{\delta}) = \inf\{t \geq \hat{\delta} | \hat{X}_t^{(n)} \leq L\}$. Consider a sequence of stopping times defined as follows:

$$\sigma_0^{(n)} := 0, \quad \sigma_m^{(n)} := \sigma_{m-1}^{(n)} + t_0 \left(\hat{X}^{(n)}_{\sigma_{m-1}^{(n)}} \vee L \right), \quad m \in \mathbb{N}.$$

Let $m_0^{(n)} = \min\{m \geq 1 | \hat{X}^{(n)}_{\sigma_m^{(n)}} \leq L\}$, and

$$\hat{V}^{(n)}(x) := E_x \left(\int_0^{\tau^{(n)}} \ln(\hat{X}_t^{(n)}) dt \right). \tag{7.5}$$

Then

$$\hat{V}^{(n)}(x) \leq E_x \left(\int_0^{\sigma^{(n)}_{m_0^{(n)}}} \ln(\hat{X}_t^{(n)}) dt \right) = \sum_{k=0}^{\infty} E_x \left(\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}} \ln(\hat{X}_t^{(n)}) dt 1_{\{k < m_0^{(n)}\}} \right).$$

Let $\mathcal{F}_t^{(n)} := \sigma\{\hat{X}_s^{(n)} | 0 \leq s \leq t\}$. We claim that there is a $c_0 \in (0, \infty)$ such that for all $n, k \in \mathbb{N}$, $x \geq 1$

$$E_x \left(\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}} \ln(\hat{X}_t^{(n)}) dt | \mathcal{F}^{(n)}_{\sigma_k^{(n)}} \right) 1_{\{k < m_0^{(n)}\}} \leq c_0 \left(\hat{X}^{(n)}_{\sigma_k^{(n)}} \right)^3 1_{\{k < m_0^{(n)}\}}. \tag{7.6}$$

Due to the strong Markov property, to prove the claim it suffices to show that for some $c_0 \in (0, \infty)$ and for all $n \in \mathbb{N}, x \geq 1$

$$E_x \left(\int_0^{\sigma_1^{(n)}} \ln(\hat{X}_t^{(n)}) dt \right) \leq c_0 x^3.$$

Note that for $x \geq 1$, $\sigma_1^{(n)} = t_0(x \vee L) \leq \tilde{c}_0 x$, where $\tilde{c}_0 = t_0 L$. Using this bound along with Lemma 5.2, we get, for some $\hat{c}_0 \in (0, \infty)$,

$$E_x \left(\sup_{t \leq \sigma_1^{(n)}} \ln(\hat{X}_t^{(n)}) \right) \leq \ln \left(E_x \left(\sup_{t \leq \sigma_1^{(n)}} \hat{X}_t^{(n)} \right) \right) \leq \ln \left(x^2 e^{K(\tilde{c}_0 x)^2} \right) \leq \hat{c}_0 x^2.$$

The claim follows.

From the estimate (7.6), we now have

$$\sup_n \hat{V}^{(n)}(x) \leq c_0 \sup_n E_x \left(\sum_{k=0}^{m_0^{(n)}-1} \left(\hat{X}^{(n)}_{\sigma_k^{(n)}} \right)^3 \right). \tag{7.7}$$

Note that $\left\{ \hat{X}^{(n)}_{\sigma_k^{(n)}} \right\}_{k \in \mathbb{N}_0}$ is a Markov chain with transition probability kernel

$$\check{P}^{(n)}(x, A) := P^{(n)}_{t_0(x \vee L)}(x, A), \quad x \in [1, \infty), \quad A \in \mathcal{B}([1, \infty)),$$

where $P_t^{(n)}$ is the transition probability kernel for $\hat{X}^{(n)}$. Using (7.4) and Lemma 5.4, we get that for any $L \in (1,\infty)$ there exists a $\tilde{b} \in (0,\infty)$ such that for all $x \in [1,\infty)$

$$\sup_n \int_1^\infty y^3 \check{P}^{(n)}(x,dy) = \sup_n \int_1^\infty y^3 P^{(n)}_{t_0(x\vee L)}(x,dy) \tag{7.8}$$
$$= \sup_n \int_1^\infty y^3 P^{(n)}_{t_0 x}(x,dy) 1_{\{x>L\}} + \sup_n \int_1^\infty y^3 P^{(n)}_{t_0 L}(x,dy) 1_{\{x\le L\}} \le x^3 - \frac{1}{2}x^3 + \tilde{b} 1_{[0,L]}(x).$$

The above inequality along with Theorem 14.2.2 of [16] yields

$$\sup_n E_x\Big(\sum_{k=0}^{m_0^{(n)}-1} \Big(\hat{X}^{(n)}_{\sigma_k^{(n)}}\Big)^3\Big) \le 2\Big(x^3 + \sup_n E_x\Big(\sum_{k=0}^{m_0^{(n)}-1} \tilde{b} 1_{[0,L]}\Big(\hat{X}^{(n)}_{\sigma_k^{(n)}}\Big)\Big)\Big)$$
$$= 2\left(x^3 + \tilde{b} 1_{[0,L]}(x)\right) \le \tilde{c} x^3,$$

where the equality in the last display follows from the fact that $\hat{X}^{(n)}_{\sigma_k^{(n)}} > L$ for $1 \le k < m_0^{(n)}$. The result follows on combining the last estimate with (7.7). □

The following theorem is proved exactly as Proposition 5.4 of [5]. The proof is omitted.

Theorem 7.3. *Let $f : [1,\infty) \to \mathbb{R}_+$ be a measurable function. Define for $\hat{\delta} \in (0,\infty)$ and a compact set $C \subset [1,\infty)$*

$$V^{(n)}(x) := E_x\left(\int_0^{\tau_C^{(n)}(\hat{\delta})} f(\hat{X}_t^{(n)})dt\right), \quad x \in [1,\infty).$$

If $\sup_{n\in\mathbb{N}} V^{(n)}$ is everywhere finite and uniformly bounded on C, then there exists a $\hat{\kappa} \in (0,\infty)$ such that for all $n \in \mathbb{N}, t > 0, x \in [1,\infty)$

$$\frac{1}{t}E_x\left(V^{(n)}(\hat{X}_t^{(n)})\right) + \frac{1}{t}\int_0^t E_x\left(f(\hat{X}_s^{(n)})\right) ds \le \frac{1}{t}V^{(n)}(x) + \hat{\kappa}.$$

We now return to the proof of Theorem 3.1 and establish the tightness of $\{\nu_1^{(n)}\}_{n\in\mathbb{N}}$. We will apply Theorem 7.3 with $f(x) := \ln(x)$, and $\hat{\delta}, C$ as in Theorem 7.2. Since $\nu_1^{(n)}$ is an invariant measure for $\hat{X}^{(n)}$, we have for non-negative, real valued, measurable functions Φ on $[1,\infty)$

$$\int_1^\infty E_x\left(\Phi(\hat{X}_t^{(n)})\right)\nu_1^{(n)}(dx) = \int_1^\infty \Phi(x)\nu_1^{(n)}(dx). \tag{7.9}$$

Fix $k \in \mathbb{N}$ and let $V_k^{(n)}(x) := V^{(n)}(x) \wedge k$. Let

$$\Psi_k^{(n)}(x) := \frac{1}{t}V_k^{(n)}(x) - \frac{1}{t}E_x\left(V_k^{(n)}(\hat{X}_t^{(n)})\right).$$

By (7.9), we have that $\int_1^\infty \Psi_k^{(n)}(x)\nu_1^{(n)}(dx) = 0$. Let

$$\Psi^{(n)}(x) := \frac{1}{t}V^{(n)}(x) - \frac{1}{t}E_x\left(V^{(n)}(\hat{X}_t^{(n)})\right).$$

By the monotone convergence theorem $\Psi_k^{(n)}(x) \to \Psi^{(n)}(x)$ as $k \to \infty$. We next show that $\Psi_k^{(n)}(x)$ is bounded from below for all $x \in [1, \infty)$: If $V^{(n)}(x) \leq k$

$$\Psi_k^{(n)}(x) = \frac{1}{t} V_k^{(n)}(x) - \frac{1}{t} E_x \left(V_k^{(n)}(\hat{X}_t^{(n)}) \right) \geq \frac{1}{t} V^{(n)}(x) - \frac{1}{t} E_x \left(V^{(n)}(\hat{X}_t^{(n)}) \right) \geq -\hat{\kappa},$$

where the last inequality follows from Theorem 7.3. If $V^{(n)}(x) \geq k$

$$\Psi_k^{(n)}(x) = \frac{1}{t} k - \frac{1}{t} E_x \left(V_k^{(n)}(\hat{X}_t^{(n)}) \right) \geq 0.$$

Thus $\Psi_k^{(n)}(x) \geq -\hat{\kappa}$ for all $x \geq 1$. By Fatou's lemma, we have

$$\int_1^\infty \Psi^{(n)}(x) \nu_1^{(n)}(dx) \leq \liminf_{k \to \infty} \int_1^\infty \Psi_k^{(n)}(x) \nu_1^{(n)}(dx) = 0.$$

By Theorem 7.3 we have $\Psi^{(n)}(x) \geq \frac{1}{t} \int_0^t E_x \left(f(\hat{X}_s^{(n)}) \right) ds - \hat{\kappa}$. Combining this with the last display, we have

$$0 \geq \int_1^\infty \Psi^{(n)}(x) \nu_1^{(n)}(dx) \geq \frac{1}{t} \int_0^t \int_1^\infty E_x \left(f(\hat{X}_s^{(n)}) \right) \nu_1^{(n)}(dx) ds - \hat{\kappa}.$$

Using the invariance property of $\nu_1^{(n)}$ once more, we see that the first term on the right side above equals $\int f(x) \nu_1^{(n)}(dx)$, and therefore $\int f(x) \nu_1^{(n)}(dx) \leq \hat{\kappa}$. This completes the proof of tightness.

The tightness of $\{\nu_1^{(n)}\}_{n \in \mathbb{N}}$ implies that every subsequence of $\{\nu_1^{(n)}\}$ has a convergent subsequence. Call such a limit ν_1^*. Theorem 2.1 and the stationarity of $\nu_1^{(n)}$ imply that ν_1^* is a stationary distribution of X. Since the stationary distribution of X is unique, we have $\nu_1^* = \nu_1$, which completes the proof. □

8. Proofs of Theorems 4.1 and 4.2

8.1. Proof of Theorem 4.1

In order to prove the result, we will verify that the assumptions of Theorem II.1 (more precisely, those in the remark following Theorem II.1) in [18], pp. 78-79, hold. For this, it suffices to show that for all $k \in \mathbb{N}$, $\Phi \in \mathrm{BM}(\mathbb{R}_+^k)$, $\phi \in C_c^\infty(\mathbb{R}_+)$, and $0 \leq t_1 < t_2 < \cdots < t_{k+1} < T < \infty$, there exists a sequence h_n with $\lim_{n \to \infty} h_n = 0$ and

$$\sup_{t \in [t_{k+1}, T]} \left| E \left[\Phi \left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)} \right) \left(\phi \left(\check{Y}_{t+h_n}^{(n)} \right) - \phi \left(\check{Y}_t^{(n)} \right) - h_n \check{L} \phi \left(\check{Y}_t^{(n)} \right) \right) \right] \right| = o(h_n), \quad (8.1)$$

where $\check{\mathcal{L}}$ is given as

$$\check{\mathcal{L}}\phi(y) := c_2\lambda_2 m_X y\phi'(y) + \frac{1}{2}\alpha_2\lambda_2 m_X y\phi''(y), \quad \phi \in C_c^\infty(\mathbb{R}_+). \tag{8.2}$$

Letting $X_t^\circ := \check{X}^{(n)}_{t/a_n}$, $t \geq 0$, we see, using scaling properties of the Skorohod map and straight forward martingale characterization results, that X° has the same probability law as the process X that was introduced in Proposition 2.1 with initial value $X_0 = x_0$. The following uniform moment bound will be used in the proof of Theorem 4.1.

Lemma 8.1. *There exists a $\delta_0 \in (0,\infty)$, such that whenever X is as in Proposition 2.1 with initial value $X_0 = x$, for some $x \in [1,\infty)$, we have*

$$\sup_{0\leq t<\infty} E_x\left(e^{\delta_0 X_t}\right) =: d(\delta_0, x) < \infty.$$

Proof of Lemma 8.1. We begin by establishing exponential moment estimates for the increase of X over time intervals of length $l\rho$ when the process is away from the boundary 1, where $\rho > 0$ and $l \geq 1$. Fix $\rho \in (0,\infty)$. Let $a := c_1\lambda_1$, $b := \alpha_1\lambda_1$, and $\delta \in (0, -\frac{a}{b} \wedge 1)$. Note that in view of Condition 3.1, $a < 0$. Define $\sigma_r := \inf\{t \in [0,\infty) | \int_0^t X_s ds > r\}$ and $\rho_{l,r} := l\rho \wedge \sigma_r$. Then

$$\begin{aligned}
&E_x\left(\exp\left(\delta a\int_0^{\rho_{l,r}} X_s ds + \delta\sqrt{b}\int_0^{\rho_{l,r}} \sqrt{X_s}dB_s^X\right)\right)\\
&= E_x\left(\exp\left((\delta a + \delta^2 b)\int_0^{\rho_{l,r}} X_s ds + \delta\sqrt{b}\int_0^{\rho_{l,r}} \sqrt{X_s}dB_s^X - \delta^2 b\int_0^{\rho_{l,r}} X_s ds\right)\right)\\
&\leq \left(E_x e^{2\rho_{l,r}(\delta a+\delta^2 b)}\right)^{\frac{1}{2}}\left(E_x\exp\left(2\delta\sqrt{b}\int_0^{\rho_{l,r}} \sqrt{X_s}dB_s^X - \frac{(2\delta\sqrt{b})^2}{2}\int_0^{\rho_{l,r}} X_s ds\right)\right)^{\frac{1}{2}},
\end{aligned}$$

where the inequality follows on noting that $\rho_{l,r} \leq \int_0^{\rho_{l,r}} X_s ds$ and $\delta \in (0, -\frac{a}{b})$. Using the super martingale property of the stochastic exponential, we have that the second term on the right hand side of the last display is bounded by 1. Thus, sending $r \to \infty$, we have, with $-\theta := \delta a + \delta^2 b < 0$,

$$E_x\left(\exp\left(\delta a\int_0^{l\rho} X_s ds + \delta\sqrt{b}\int_0^{l\rho} \sqrt{X_s}dB_s^X\right)\right) \leq e^{l\rho(\delta a+\delta^2 b)} = e^{-\theta l\rho}. \tag{8.3}$$

Next, for $x \in [1,\infty)$, we have by application of Itô's formula that for $t \leq \rho$ and $\tilde{\delta} \leq \delta$

$$E_x\left(e^{\tilde{\delta}X_t}\right) \leq e^{\tilde{\delta}x} + \tilde{\delta}e^{\tilde{\delta}}E_x\eta_\rho \leq e^{\tilde{\delta}x} + xC_1(\rho,\delta), \tag{8.4}$$

where $C_1(\rho,\delta) \in (0,\infty)$ and the last inequality follows by an application of Gronwall's lemma and the Lipschitz property of the Skorohod map (see (2.4)).

Using the above estimates, we will now establish certain uniform estimates on the tail probabilities of $X_{k\rho}$, which will lead to exponential moment estimates at these time points. Fix $L > 1$, and let

$$\tau_j := \inf\{t \geq (j-1)\rho | X_t \leq L\} \wedge j\rho \quad \text{and} \quad e_j := X_{j\rho} - X_{\tau_j}, \quad j \geq 1,$$

$e_0 = 0$. Fix $k \in \mathbb{N}$, and let

$$M := \max\Big\{ j = 1, \ldots, k \Big| \inf_{(j-1)\rho \leq s \leq j\rho} X_s \leq L \Big\}$$

if there is an $s \in [0, k\rho]$ such that $X_s \leq L$ and set M equal to 0 otherwise. Let

$$v_j := \int_{(j-1)\rho}^{j\rho} aX_s ds + \int_{(j-1)\rho}^{j\rho} \sqrt{bX_s} dB_s^X, \quad j \geq 1.$$

Then $X_{k\rho} = X_{M\rho} + \sum_{j=M+1}^k v_j$. Letting $\zeta_i := e_i + \sum_{j=i+1}^k v_j$, we have, using (8.3),

$$\begin{aligned} P_x(X_{k\rho} > K) &\leq P_x\Big(X_{M\rho} + \sum_{j=M+1}^k v_j > K\Big) \leq P_x\left(\max_{0\leq i\leq k} \zeta_i > K - L\right) \leq \sum_{i=0}^k P_x(\zeta_i > K - L) \\ &\leq \sum_{i=0}^k E_x\left(e^{\frac{\delta}{2}\zeta_i}\right) e^{-\frac{\delta}{2}(K-L)} \leq \sum_{i=0}^k \left(E_x e^{\delta e_i}\right)^{\frac{1}{2}} \left(E_x e^{\delta \sum_{j=i+1}^k v_j}\right)^{\frac{1}{2}} e^{-\frac{\delta}{2}(K-L)} \\ &\leq \sum_{i=0}^k \left(E_x e^{\delta e_i}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(k-i)\theta\rho} e^{-\frac{\delta}{2}(K-L)}. \end{aligned}$$

Next note that, from (8.4),

$$\begin{aligned} E_x e^{\delta e_i} &\leq E_x\left(e^{\delta[X_{i\rho} - X_{\tau_i}]} 1_{\tau_i < i\rho}\right) + 1 = E_x\left(e^{-\delta X_{\tau_i}} 1_{\tau_i < i\rho} E_{X_{\tau_i}}\left(e^{\delta X_{i\rho}}\right)\right) + 1 \\ &\leq E_x\left[e^{-\delta X_{\tau_i}} 1_{\tau_i < i\rho}\left(e^{\delta X_{\tau_i}} + X_{\tau_i} C_1(\rho, \delta)\right)\right] + 1 \equiv C_2(\rho, \delta) + 1. \end{aligned}$$

Hence,

$$P_x(X_{k\rho} > K) \leq (C_2(\rho, \delta) + 1)^{\frac{1}{2}} e^{-\frac{\delta}{2}(K-L)} \sum_{l=0}^k e^{-\frac{1}{2}l\rho\theta} \leq (C_2(\rho, \delta) + 1)^{\frac{1}{2}} \frac{e^{-\frac{\delta}{2}(K-L)}}{1 - e^{-\frac{1}{2}\rho\theta}}.$$

The last estimate yields, analogously to (5.38),

$$\sup_{k \in \mathbb{N}_0} E_x(e^{\frac{\delta}{4}X_{k\rho}}) \leq Ce^{\frac{\delta}{2}x}, \quad \text{for all } x \in [1, \infty), \tag{8.5}$$

for some $C \in (0, \infty)$. Finally, letting $\delta_0 := \frac{\delta}{4}$, we have from (8.4) for $t \in ((k-1)\rho, k\rho]$, $k \geq 1$,

$$E_x(e^{\delta_0 X_t}) = E_x\big(E_{X_{(k-1)\rho}}(e^{\delta_0 X_t})\big) \leq E_x\big(e^{\delta_0 X_{(k-1)\rho}} + X_{(k-1)\rho} C_1(\rho, \delta)\big) \leq Ce^{\frac{\delta}{2}x}\big(1 + \frac{1}{\delta_0} C_1(\rho, \delta)\big).$$

The result follows. □

Remark 8.1. *Note that Lemma 8.1 and the scaling property noted above that lemma say that for all $x \in [1,\infty)$*

$$\sup_{n\in\mathbb{N}}\sup_{0\le t<\infty} E_x\left(e^{\delta_0 \check{X}_t^{(n)}}\right) < \infty.$$

We now prove Theorem 4.1 by showing (8.1). Let, for $\phi \in C_c^\infty(\mathbb{R}_+)$,

$$\mathcal{L}_x\phi(y) := c_2\lambda_2 xy\phi'(y) + \frac{1}{2}\alpha_2\lambda_2 xy\phi''(y), \quad (x,y)\in[1,\infty)\times\mathbb{R}_+. \tag{8.6}$$

Then

$$\begin{aligned}
&E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\left(\phi(\check{Y}_{t+h_n}^{(n)})-\phi(\check{Y}_t^{(n)})\right)\right]\\
&= E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\int_t^{t+h_n}\mathcal{L}_{\check{X}_s^{(n)}}\phi(\check{Y}_t^{(n)})ds\right]\\
&\quad + E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\int_t^{t+h_n}\left(\mathcal{L}_{\check{X}_s^{(n)}}\phi(\check{Y}_s^{(n)})-\mathcal{L}_{\check{X}_s^{(n)}}\phi(\check{Y}_t^{(n)})\right)ds\right].
\end{aligned} \tag{8.7}$$

For the second term, we have, using Remark 8.1 and the fact that the function Φ is bounded and ϕ as well as its derivatives are continuous with bounded support, that

$$\begin{aligned}
&\sup_{t\in[t_{k+1},T]} E\left|\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\int_t^{t+h_n}\left(\mathcal{L}_{\check{X}_s^{(n)}}\phi(\check{Y}_s^{(n)})-\mathcal{L}_{\check{X}_s^{(n)}}\phi(\check{Y}_t^{(n)})\right)ds\right|\\
&= \sup_{t\in[t_{k+1},T]} E\left|\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\int_t^{t+h_n}\left(c_2\lambda_2\check{X}_s^{(n)}\left(\check{Y}_s^{(n)}\phi'(\check{Y}_s^{(n)})-\check{Y}_t^{(n)}\phi'(\check{Y}_t^{(n)})\right)\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\alpha_2\lambda_2\check{X}_s^{(n)}\left(\check{Y}_s^{(n)}\phi''(\check{Y}_s^{(n)})-\check{Y}_t^{(n)}\phi''(\check{Y}_t^{(n)})\right)\right)ds\right| = o(h_n).
\end{aligned} \tag{8.8}$$

Recalling the definition of X° above Lemma 8.1, the first expected value on the right hand side in (8.7) equals

$$h_n E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\frac{1}{h_n a_n}\int_{ta_n}^{ta_n+h_na_n}\mathcal{L}_{X_s^\circ}\phi(\check{Y}_t^{(n)})ds\right].$$

Thus

$$\begin{aligned}
&E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)\left(\phi(\check{Y}_{t+h_n}^{(n)})-\phi(\check{Y}_t^{(n)})-h_n\check{\mathcal{L}}\phi(\check{Y}_t^{(n)})\right)\right]\\
&= E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)h_n\left(\frac{1}{h_na_n}\int_{ta_n}^{ta_n+h_na_n}\mathcal{L}_{X_s^\circ}\phi(\check{Y}_t^{(n)})ds-\check{\mathcal{L}}\phi(\check{Y}_t^{(n)})\right)\right]+o(h_n)\\
&= E\left[\Phi\left(\check{Y}_{t_1}^{(n)},\ldots,\check{Y}_{t_k}^{(n)}\right)h_n\left(c_2\lambda_2\check{Y}_t^{(n)}\phi'(\check{Y}_t^{(n)})+\frac{1}{2}\alpha_2\lambda_2\check{Y}_t^{(n)}\phi''(\check{Y}_t^{(n)})\right)\right.\\
&\qquad\qquad\left.\left(\frac{1}{h_na_n}\int_{ta_n}^{ta_n+h_na_n}X_s^\circ ds-m_X\right)\right]+o(h_n).
\end{aligned}$$

To complete the proof, it thus remains to show that for some sequence $\{h_n\}$ with $\lim_{n\to\infty} h_n = 0$

$$E\left|\frac{1}{h_n a_n}\int_{ta_n}^{ta_n+h_n a_n} X_s^\circ ds - m_X\right| = E\left|\frac{1}{h_n a_n}\int_{ta_n}^{ta_n+h_n a_n} X_s ds - m_X\right| \tag{8.9}$$

converges to 0 uniformly in $t \in [t_{k+1}, T]$. From the ergodicity of X and the moment estimate in Lemma 8.1, it follows that

$$E\left|\frac{1}{ta_n}\int_0^{ta_n} X_s ds - m_X\right| \to 0, \quad \text{as } n \to \infty. \tag{8.10}$$

The above result, along with Lemma 8.2, below, implies that there is a sequence $\{h_n\}$ such that $\lim_{n\to\infty} h_n = 0$ and the expression in (8.9) converges to 0 uniformly in $t \in [t_{k+1}, T]$. This completes the proof.

The proof of the following lemma is adapted from Lemma II.9, p. 137, in [18].

Lemma 8.2. *Let $0 \le t_{k+1} < T < \infty$ and $a_n \to \infty$ monotonically as $n \to \infty$. If for all $t \in [t_{k+1}, T]$*

$$E\left|\frac{1}{ta_n}\int_0^{ta_n} X_s ds - m_X\right| \to 0, \quad \textit{as } n \to \infty,$$

then there is a sequence $\{h_n\}$ such that $h_n \to 0$ as $n \to \infty$, and

$$\sup_{t\in[t_{k+1},T]} E\left|\frac{1}{h_n a_n}\int_{ta_n}^{ta_n+h_n a_n} X_s ds - m_X\right| \to 0, \quad \textit{as } n \to \infty.$$

Proof. Let $\alpha(\tau) := \sup_{u>\tau} E\left|\frac{1}{u}\int_0^u X_s ds - m_X\right|$. Note that $\alpha(\tau)$ converges monotonically to 0 as $\tau \to \infty$. For $t \in [t_{k+1}, T]$ we have

$$\begin{aligned}
&E\left|\frac{1}{h_n a_n}\int_{ta_n}^{ta_n+h_n a_n} X_s ds - m_X\right| \\
&= E\left|\frac{ta_n + h_n a_n}{h_n a_n}\frac{1}{ta_n + h_n a_n}\int_0^{ta_n+h_n a_n} X_s ds - \frac{ta_n}{h_n a_n}\frac{1}{ta_n}\int_0^{ta_n} X_s ds - m_X\right| \\
&\le \frac{ta_n + h_n a_n}{h_n a_n}\alpha(ta_n + h_n a_n) + \frac{ta_n}{h_n a_n}\alpha(ta_n) \le \frac{3T}{h_n}\alpha(t_{k+1}a_n)
\end{aligned}$$

for all n such that $h_n \le T$. Note that the right hand side of the last display is independent of $t \in [t_{k+1}, T]$. Choosing $h_n = \sqrt{\alpha(t_{k+1}a_n)}$, the lemma follows. □

8.2. Proof of Theorem 4.2

As in the proof of Theorem 4.1, it suffices to show that for all $k \in \mathbb{N}$, $\Phi \in \mathrm{BM}(\mathbb{R}_+^k)$, $\phi \in C_c^\infty(\mathbb{R}_+)$, and $0 \le t_1 < t_2 < \cdots < t_{k+1} < T < \infty$, there exists a sequence h_n with $\lim_{n\to\infty} h_n = 0$ and

$$\sup_{t\in[t_{k+1},T]}\left|E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)\left(\phi\left(\check{Y}_{t+h_n}^{(n)}\right) - \phi\left(\check{Y}_t^{(n)}\right) - h_n\check{L}\phi\left(\check{Y}_t^{(n)}\right)\right)\right]\right| = o(h_n),$$

where $\check{\mathcal{L}}$ is given as in (8.2).

Let for $\phi \in C_c^\infty(\mathbb{R}_+)$ and $(x,y) \in [1,\infty) \times \mathbb{R}_+$

$$\mathcal{L}_x^{(n)}\phi(y) := \lambda_2^{(n)} n^2 xy \sum_{k=0}^{\infty} \left[\phi\left(y + \frac{k-1}{n}\right) - \phi(y)\right] \mu_2^{(n)}(k),$$

and recall $\mathcal{L}_x$ from (8.6). Then

$$\begin{aligned} &E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)\left(\phi(\check{Y}_{t+h_n}^{(n)}) - \phi(\check{Y}_t^{(n)})\right)\right] \\ &= E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)\int_t^{t+h_n} \mathcal{L}_{\check{X}_s^{(n)}}^{(n)}\phi(\check{Y}_t^{(n)})ds\right] \\ &\quad + E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)\int_t^{t+h_n}\left[\mathcal{L}_{\check{X}_s^{(n)}}^{(n)}\phi\left(\check{Y}_s^{(n)}\right) - \mathcal{L}_{\check{X}_s^{(n)}}^{(n)}\phi\left(\check{Y}_t^{(n)}\right)\right]ds\right]. \end{aligned} \tag{8.11}$$

Using Lemma 5.4, we get, as in (8.8), that the second term in the last display is $o(h_n)$ uniformly in $t \in [t_{k+1}, T]$. Thus

$$\begin{aligned} &E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)\left(\phi(\check{Y}_{t+h_n}^{(n)}) - \phi(\check{Y}_t^{(n)}) - h_n\check{\mathcal{L}}\phi(\check{Y}_t^{(n)})\right)\right] \\ &= E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)\left(\int_t^{t+h_n}\left(\mathcal{L}_{\check{X}_s^{(n)}}^{(n)}\phi\left(\check{Y}_t^{(n)}\right) - \mathcal{L}_{\check{X}_s^{(n)}}\phi\left(\check{Y}_t^{(n)}\right)\right)ds\right)\right] \\ &\quad + E\left[\Phi\left(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\right)h_n\left(\frac{1}{h_n}\int_t^{t+h_n}\mathcal{L}_{\check{X}_s^{(n)}}\phi(\check{Y}_t^{(n)})ds - \check{\mathcal{L}}\phi(\check{Y}_t^{(n)})\right) + o(h_n). \end{aligned} \tag{8.12}$$

Calculations similar to those in the proof of Theorem 2.1 show that the first term on the right hand side in the last display is $o(h_n)$ uniformly in $t \in [t_{k+1}, T]$ (see proof of (6.3)), while the second term can be written as

$$E\Big[\Phi\Big(\check{Y}_{t_1}^{(n)}, \ldots, \check{Y}_{t_k}^{(n)}\Big)h_n\Big(c_2\lambda_2\check{Y}_t^{(n)}\phi'(\check{Y}_t^{(n)}) + \frac{1}{2}\alpha_2\lambda_2\check{Y}_t^{(n)}\phi''(\check{Y}_t^{(n)})\Big)\Big(\frac{1}{h_n}\int_t^{t+h_n}\check{X}_s^{(n)}ds - m_X\Big)\Big].$$

To show that the latter term is $o(h_n)$ uniformly in $t \in [t_{k+1}, T]$, it suffices to show the following result.

Theorem 8.1. *As $n \to \infty$*

$$\sup_{t\in[t_{k+1},T]} E\left|E_{\check{\mathcal{F}}_t^{(n)}}\left(\frac{1}{h_n}\int_t^{t+h_n}\check{X}_s^{(n)}ds - m_X\right)\right| \to 0,$$

where $\check{\mathcal{F}}_t^{(n)} := \sigma\{(\check{X}_s^{(n)}, \check{Y}_s^{(n)}) : s \le t\}$, and $E_{\check{\mathcal{F}}_t^{(n)}}(\cdot) = E(\cdot \mid \check{\mathcal{F}}_t^{(n)})$.

In order to prove this theorem, we need the following three results. Let $S := D(\mathbb{R}_+ : [1,\infty) \times \mathbb{R}_+)$, $\mathcal{P}(S)$ be the space of probability measures on S, and, given a sequence $\{t_n\} \subset [t_{k+1}, T]$, μ_n be a sequence of $\mathcal{P}(S)$ valued random variables defined as follows. For $A \in \mathcal{B}(S)$,

$$\mu_n(A) = \frac{1}{a_n h_n}\int_{a_n t_n}^{a_n t_n + a_n h_n} P[(\hat{X}_{s+\cdot}^{(n)}, \hat{\eta}_{s+\cdot}^{(n)} - \hat{\eta}_s^{(n)}) \in A | \check{\mathcal{F}}_{t_n}^{(n)}]ds. \tag{8.13}$$

Let $S_0 := C(\mathbb{R}_+ : [1,\infty) \times \mathbb{R}_+)$.

Lemma 8.3. *The family of $\mathcal{P}(S)$ valued random variables $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, and any weak limit point is a $\mathcal{P}(S_0)$ valued random variable.*

Let $\pi = (\pi^{(1)}, \pi^{(2)})$ with $\pi^{(1)}$ and $\pi^{(2)}$ being the canonical coordinate processes on S_0.

Lemma 8.4. *Let μ be a weak limit point of $\{\mu_n\}$ given on some probability space $(\Omega_0, \mathcal{F}_0, P_0)$. Then for P_0 a.e. $\omega \in \Omega_0$, $\mu(\omega)$ satisfies the following:*

(a) $\mu(w)(\pi^{(1)}(t+\cdot) \in F) = \mu(\omega)(\pi^{(1)} \in F)$, for all $t \geq 0$, $F \in \mathcal{B}(C(\mathbb{R}_+ : [1,\infty)))$,
(b) $\pi^{(2)}$ is non-decreasing and $\pi_0^{(2)} = 0$ a.s. $\mu(\omega)$,
(c) $\int_0^\infty 1_{(1,\infty)}(\pi_u^{(1)})d\pi_u^{(2)} = 0$ a.s. $\mu(\omega)$, and
(d) under $\mu(\omega)$, for all $\phi \in C_c^\infty([1,\infty))$

$$\phi(\pi_t^{(1)}) - \phi(\pi_0^{(1)}) - \int_0^t \mathcal{L}_1\phi(\pi_s^{(1)})ds - \phi'(1)\pi_t^{(2)}$$

is a $\{\mathcal{G}_t\}$-martingale, where $\mathcal{L}_1$ is as in (7.2) and $\mathcal{G}_t := \sigma\{(\pi_s^{(1)}, \pi_s^{(2)}) : s \leq t\}$.

We postpone the proofs of Lemmas 8.3 and 8.4 until after the proof of Theorem 8.1. The following is immediate from the above two lemmas, Proposition 3.1 and the martingale characterization of the probability law of the process in (2.5) (see Theorem 6.1).

Corollary 8.1. *Let (X, η) be as in Proposition 2.1 with $X_0 \sim \nu_1$ and ν_1 given as in Proposition 3.1. Let μ_0 be the probability measure on S_0 induced by (X, η). Then μ_n converges weakly to μ_0.*

Proof of Theorem 8.1. It suffices to show that for an arbitrary sequence $\{t_n\} \subset [t_{k+1}, T]$ we have, as $n \to \infty$,

$$E\left|E_{\check{\mathcal{F}}_{t_n}^{(n)}}\left(\frac{1}{h_n}\int_{t_n}^{t_n+h_n} \check{X}_s^{(n)}ds - m_X\right)\right| = E\left|E_{\check{\mathcal{F}}_{t_n}^{(n)}}\left(\frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n} \hat{X}_s^{(n)}ds - m_X\right)\right| \to 0.$$

Since

$$E_{\check{\mathcal{F}}_{t_n}^{(n)}}\left(\frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n} \hat{X}_s^{(n)}ds\right) = \int \pi_0^{(1)}d\mu_n,$$

it suffices to show that

$$E\left|\int \pi_0^{(1)}d\mu_n - \int \pi_0^{(1)}d\mu_0\right| \to 0 \quad \text{as } n \to \infty. \tag{8.14}$$

For any $c > 0$, let ψ_c be the following continuous function:

$$\psi_c(x) = \begin{cases} 1, & \text{if } x \leq \frac{c}{2} \\ 0, & \text{if } x \geq c \end{cases}$$

and ψ_c is linearly interpolated on $[\frac{c}{2}, c]$. By Corollary 8.1 μ_n converges weakly to μ_0, and therefore, for every $c > 0$,

$$E\left|\int \pi_0^{(1)}\psi_c(\pi_0^{(1)})d\mu_n - \int \pi_0^{(1)}\psi_c(\pi_0^{(1)})d\mu_0\right| \to 0 \quad \text{as } n \to \infty.$$

Moreover, using the estimate in Lemma 5.4,

$$\begin{aligned}&\sup_{n\in\mathbb{N}}\left(E\left|\int \pi_0^{(1)}\left(1-\psi_c(\pi_0^{(1)})\right)d\mu_n\right|\right)\\ &\quad\le \sup_{n\in\mathbb{N}}\left(\frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n} E\left(\hat{X}_s^{(n)}1_{|\hat{X}_s^{(n)}|\ge\frac{c}{2}}\right)ds\right) \to 0 \quad \text{as } c\to\infty,\end{aligned}$$

and

$$E\left|\int \pi_0^{(1)}(1-\psi_c(\pi_0^{(1)}))d\mu_0\right| \le E\left(X_0 1_{|X_0|\ge\frac{c}{2}}\right) \to 0 \quad \text{as } c\to\infty.$$

The last three displays imply the convergence in (8.14), and thus the result follows. □

Proof of Lemma 8.3. To show the tightness of $\{\mu_n\}$, it suffices to show that $\{\nu_n\}$ is tight, where

$$\nu_n(A) := E\mu_n(A) = \frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n} P[(\hat{X}_{s+\cdot}^{(n)}, \hat{\eta}_{s+\cdot}^{(n)} - \hat{\eta}_s^{(n)}) \in A]ds, \quad A \in \mathcal{B}(S).$$

However, the tightness of ν_n is immediate in view of the tightness of $\{(\hat{X}_{s+\cdot}^{(n)}, \hat{\eta}_{s+\cdot}^{(n)} - \hat{\eta}_s^{(n)})\}_{n\in\mathbb{N}, s\in\mathbb{R}_+}$, which was proved in Proposition 5.1.

Let μ be a weak limit point of μ_n and $J : S \to \mathbb{R}_+$ be defined by

$$J(\pi) := \int_0^\infty e^{-u}[J(\pi, u) \wedge 1]du,$$

where

$$J(\pi, u) := \sup_{0\le t\le u}\left(|\Delta(\pi_t^{(1)})| + |\Delta\pi_t^{(2)}|\right).$$

Then J is continuous and bounded on S, and in order to show that μ is supported on S_0, it suffices to show that $\mu(J(\pi) = 0) = 1$ (see [7], p. 147). In turn, for the latter equality to hold, it suffices to show that for all $\epsilon > 0$, $E\mu_n(J(\pi) > \epsilon) \to 0$, as $n \to \infty$. Now

$$E\mu_n(J(\pi) > \epsilon) = \frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n} P\Big(\int_0^\infty e^{-u}\big(\sup_{0\le t\le u}[|\Delta\hat{X}_{s+t}^{(n)}| + |\Delta(\hat{\eta}_{s+t}^{(n)} - \hat{\eta}_s^{(n)})|] \wedge 1\big)du > \epsilon\Big)ds.$$

Finally, noting that $\hat{\eta}_{s+\cdot}^{(n)} - \hat{\eta}_s^{(n)}$ is continuous and using Lemma 5.6, we now have that the right hand side of the latter equation converges to 0 as $n \to \infty$. The result follows. □

Proof of Lemma 8.4. For a measure $\nu \in \mathcal{P}(S)$, let $\tilde{E}^{\nu}$ denote the expectation operator. For (a), we show that

$$\tilde{E}^{\mu(\omega)}\left(f(\pi^{(1)}_{t+\cdot})\right) - \tilde{E}^{\mu(\omega)}\left(f(\pi^{(1)})\right) = 0 \quad a.s. \text{ for all bounded continuous } f \text{ on } S. \tag{8.15}$$

Note that

$$\begin{aligned}\left|\tilde{E}^{\mu_n}(f(\pi^{(1)}_{t+\cdot})) - \tilde{E}^{\mu_n}(f(\pi^{(1)}))\right| &= \left|\frac{1}{a_n h_n}\int_{a_n t_n}^{a_n t_n + a_n h_n}\left(E_{\check{\mathcal{F}}^{(n)}_{t_n}} f(\hat{X}^{(n)}_{s+t+\cdot}) - E_{\check{\mathcal{F}}^{(n)}_{t_n}} f(\hat{X}^{(n)}_{s+\cdot})\right) ds\right| \\ &\le \frac{2t}{a_n h_n}||f||_{\sup} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This proves (8.15) since we can choose h_n such that $a_n h_n \to \infty$, and thus (a) follows.

Property (b) is immediate from the fact that $\hat{\eta}^{(n)}_{s+\cdot} - \hat{\eta}^{(n)}_s$ is non-decreasing and continuous with initial value 0 for each n.

To prove (c), it suffices to show that for a.e. ω and for every $T, \delta > 0$

$$\tilde{E}^{\mu(\omega)}\left(\int_0^T f_\delta(\pi^{(1)}_s) d\pi^{(2)}_s \wedge 1\right) = 0,$$

where f_δ is defined in (6.2). In turn, for the above equality to hold, it suffices to show that

$$E\left(\tilde{E}^{\mu}\left(\int_0^T f_\delta(\pi^{(1)}_s) d\pi^{(2)}_s \wedge 1\right)\right) = 0.$$

The latter equality is immediate on noting that for every $T, \delta > 0$

$$\begin{aligned}&E\left(\tilde{E}^{\mu_n}\left(\int_0^T f_\delta(\pi^{(1)}_u) d\pi^{(2)}_u \wedge 1\right)\right) \\ &= \frac{1}{a_n h_n}\int_{a_n t_n}^{a_n t_n + a_n h_n} E\left(\int_0^T f_\delta(\hat{X}^{(n)}_{s+u}) d(\hat{\eta}^{(n)}_{s+\cdot} - \hat{\eta}^{(n)}_s)(u) \wedge 1\right) ds = 0\end{aligned}$$

and thus

$$E\left[\tilde{E}^{\mu}\left(\int_0^T f_\delta(\pi^{(1)}_s) d\pi^{(2)}_s \wedge 1\right)\right] = \lim_{n\to\infty} E\left[\tilde{E}^{\mu_n}\left(\int_0^T f_\delta(\pi^{(1)}_s) d\pi^{(2)}_s \wedge 1\right)\right] = 0.$$

Finally, consider (d). It suffices to show that for every $0 \le r \le t < \infty$

$$E\left|\tilde{E}^{\mu}\big(\psi(\pi^{(1)}, \pi^{(2)})\big(\phi(\pi^{(1)}_t) - \phi(\pi^{(1)}_r) - \int_r^t \mathcal{L}_1\phi(\pi^{(1)}_u) du - \phi'(1)[\pi^{(2)}_t - \pi^{(2)}_r]\big)\big)\right| = 0,$$

where $\psi : S \to \mathbb{R}$ is an arbitrary bounded, continuous, $\mathcal{G}_s$ measurable map.

Now fix such r, t, and ψ. Assume without loss of generality that μ_n converges to μ. Combining this weak convergence with Lemma 5.5, we see that the left hand side of the last display is the

limit of

$$E\left|\tilde{E}^{\mu_n}\big(\psi(\pi^{(1)},\pi^{(2)})\big(\phi(\pi_t^{(1)})-\phi(\pi_r^{(1)})-\int_r^t \mathcal{L}_1\phi(\pi_u^{(1)})du-\phi'(1)[\pi_t^{(2)}-\pi_r^{(2)}]\big)\right|$$
$$=\frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n} E\Big|E_{\check{\mathcal{F}}_{t_n}}\Big(\psi(\hat{X}^{(n)}_{s+\cdot},\hat{\eta}^{(n)}_{s+\cdot}-\hat{\eta}^{(n)}_s)$$
$$\Big[\phi(\hat{X}^{(n)}_{s+t})-\phi(\hat{X}^{(n)}_{s+r})-\int_r^t\mathcal{L}_1\phi(\hat{X}^{(n)}_{s+u})du-\phi'(1)[\hat{\eta}^{(n)}_{s+t}-\hat{\eta}^{(n)}_{s+r}]\Big]\Big)\Big|ds.$$

To complete the proof, it suffices to show that the limit of the expression in the last display is 0. Note that for $\phi\in C_c^\infty([1,\infty))$

$$\phi(\hat{X}^{(n)}_t)-\phi(\hat{X}^{(n)}_0)-\int_0^t\mathcal{L}_1^{(n)}\phi(\hat{X}^{(n)}_s)ds-\mathcal{D}_1^{(n)}\phi(1)\hat{\eta}^{(n)}_t$$

is a martingale, where $\mathcal{D}_1^{(n)}\phi(1):=n[\phi(1)-\phi(1-\frac{1}{n})]$ and

$$\mathcal{L}_1^{(n)}\phi(x):=\lambda_1^{(n)}n^2x\sum_{k=0}^\infty\left[\phi\left(x+\frac{k-1}{n}\right)-\phi(x)\right]\mu_1^{(n)}(k).$$

Thus, it suffices to prove that

$$\lim_{n\to\infty}\frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n}E\left|\int_r^t\left(\mathcal{L}_1^{(n)}\phi(\hat{X}^{(n)}_{s+u})-\mathcal{L}_1\phi(\hat{X}^{(n)}_{s+u})\right)du\right|ds=0$$

and

$$\lim_{n\to\infty}\frac{1}{a_nh_n}\int_{a_nt_n}^{a_nt_n+a_nh_n}\left|\mathcal{D}_1^{(n)}\phi(1)-\phi'(1)\right|E\left(\left|\hat{\eta}^{(n)}_{s+t}-\hat{\eta}^{(n)}_{s+r}\right|\right)ds=0.$$

The proofs for the last two equalities are completed as those for (6.3) and (6.4) upon using the uniform estimates in Corollary 5.1 and Lemma 5.5. □

9. Appendix

Proof of Proposition 2.1. We will only consider the case where $(X_0,Y_0)\equiv(x,y)$ for some $(x,y)\in[1,\infty)\times[0,\infty)$. The general case can be treated similarly. The unique solvability of (2.5) is an immediate consequence of the Lipschitz property of the Skorohod map, Lipschitz coefficients (note that $f(x)=\sqrt{x}$ is a Lipschitz function on $[1,\infty)$), and a standard Picard iteration scheme (see e.g. Proposition 1 in [1]).

We next argue the unique solvability of (2.6). For $n\in\mathbb{N}$, let $\sigma^{(n)}:=\inf\{t>0|X_t\geq n\}$, $\bar{X}^{(n)}_t:=X_{t\wedge\sigma^{(n)}}$, and $f^{(n)}(y):=y\vee\frac{1}{n}$. Consider the equation

$$\bar{Y}^{(n)}_t=Y_0+c_2\lambda_2\int_0^t\bar{X}^{(n)}_sf^{(n)}(\bar{Y}^{(n)}_s)ds+\sqrt{\alpha_2\lambda_2}\int_0^t\sqrt{\bar{X}^{(n)}_sf^{(n)}(\bar{Y}^{(n)}_s)}dB^Y_s. \tag{9.1}$$

From the Lipschitz property of $f^{(n)}$ and $\sqrt{f^{(n)}}$ it follows that, for each n, the above equation has a unique pathwise solution. Let $\tau^{(n)} := \inf\{t > 0 | \bar{Y}_t^{(n)} = \frac{1}{n}\}$ and $\theta^{(n)} := \tau^{(n)} \wedge \sigma^{(n)}$. Note that $\bar{Y}^{(n)}$ solves (2.6) on $[0, \theta^{(n)}]$. Also, by unique solvability of (9.1), we have for all $n \in \mathbb{N}$, $\bar{Y}^{(n+1)}(\cdot \wedge \theta^{(n)}) = \bar{Y}^{(n)}(\cdot \wedge \theta^{(n)})$. Finally, letting $\theta^{(\infty)} := \lim_{n\to\infty} \theta^{(n)}$, the unique solution of (2.6) is given by the following:

$$Y_t(\omega) = \begin{cases} \bar{Y}_t^{(n)}(\omega), & \text{if } 0 \le t \le \theta^{(n)}(\omega) \text{ for some } n \in \mathbb{N} \\ 0, & \text{if } t \ge \theta^{\infty}(\omega). \end{cases}$$

□

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