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# Statistics, Causality and Bell's Theorem

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*Abstract.* Bell's (1964) theorem is popularly supposed to establish the non-locality of quantum physics. Violation of Bell's inequality in experiments such as that of Aspect et al. (1982) provides empirical proof of non-locality in the real world. This paper reviews recent work on Bell's theorem, linking it to issues in causality as understood by statisticians. The paper starts with a proof of a strong, finite sample, version of Bell's inequality and thereby also of Bell's theorem, which states that quantum theory is incompatible with the conjunction of three formerly uncontroversial physical principles, here referred to as *locality*, *realism*, and *freedom*.

Locality is the principle that the direction of causality matches the direction of time, and that causal influences need time to propagate spatially. Realism and freedom are directly connected to statistical thinking on causality: they relate to counterfactual reasoning, and to randomisation, respectively. Experimental loopholes in state-of-the-art Bell type experiments are related to statistical issues of post-selection in observational studies, and the missing at random assumption. They can be avoided by properly matching the statistical analysis to the actual experimental design, instead of by making untestable assumptions of independence between observed and unobserved variables. Methodological and statistical issues in the design of quantum Randi challenges (QRC) are discussed.

The paper argues that Bell's theorem (and its experimental confirmation) should lead us to relinquish not locality, but realism.

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## 1. INTRODUCTION

Bell's (1964) theorem states that certain predictions of quantum mechanics are incompatible with the conjunction of three fundamental principles of classical physics which are sometimes given the short names "realism", "locality" and "freedom". Corresponding real world experiments, Bell experiments, are supposed to demonstrate that this incompatibility is a property not just of the theory of quantum mechanics, but also of Nature itself. The consequence is that we are forced to reject at least one of these three principles.

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Both theorem and experiment hinge around an inequality constraining probability distributions of outcomes of measurements on spatially separated physical systems; an inequality which must hold if all three fundamental principles are true. In a nutshell, the inequality is an empirically verifiable consequence of the idea that the outcome of one measurement on one system cannot depend on which measurement is performed on the other. This idea, called *locality* or, more precisely, *relativistic local causality*, is just one of the three principles. Its formulation refers to outcomes of measurements which are not actually performed, so we have to assume their existence, alongside of the outcomes of those actually performed: the principle of *realism*, or more precisely, *counterfactual definiteness*. Finally we need to assume that we have complete *freedom* to choose *which* of several measurements to perform – this is the third principle, also called the *no-conspiracy* principle or *no super-determinism*. (As we shall see, super-determinism is a conspiratorial form of determinism!)

We shall implement the freedom assumption as the assumption of statistical independence between the randomisation in a randomised experimental design, and the set of outcomes of each experimental unit under all possible treatments. This set consists of the "counterfactual" outcomes of those treatments which were not actually applied, as well as the "factual" outcome belonging to the treatment chosen by the randomisation.

By existence of the outcomes of not actually performed experiments, we mean their mathematical existence within some mathematical-physical theory of the phenomenon in question. So "realism" actually refers to models of reality, not to reality itself. Moreover, it could be thought of as a somewhat idealistic position. If we already have an adequate mathematical physical model of reality, there would not seem to be a pressing need to add into this theory some mathematical description of outcomes of experiments which are not performed; and even if we do that, why should we demand that these counterfactual objects satisfy the same kind of physical constraints as the factual objects? However, it is a fact that prior to quantum physics, realism was a completely natural property of all physical theories.

The concepts of realism and locality together are often considered as one principle called *local realism*. Local realism is implied by the existence of *local hidden variables*, whether deterministic or stochastic. In a precise mathematical sense, the reverse implication is also true: local realism implies that we can construct a local hidden variable (LHV) model for the phenomenon under study. However one likes to think of this assumption (or pair of assumptions), the important thing to realize is that it is a completely unproblematic feature of all classical physical theories; freedom (no conspiracy) even more so.

The connection between Bell's theorem and statistical notions of causality has been noted many times in the past. For instance, in a short note, Robins, VanderWeele, and Gill (2014) derive Bell's inequality using the statistical language of *causal interactions*. The *causal graph* (DAG) of observed and unobserved variables corresponding to a classical physical description of one run of a standard Bell experiment is given in Figure 1. Alice and Bob's settings are binary: they independently use randomisation (a coin toss) to choose between one of two settings on a measurement device. The outcome of each measurement is also binary. Observed variables are represented by grey rectangles; unobserved (there is only

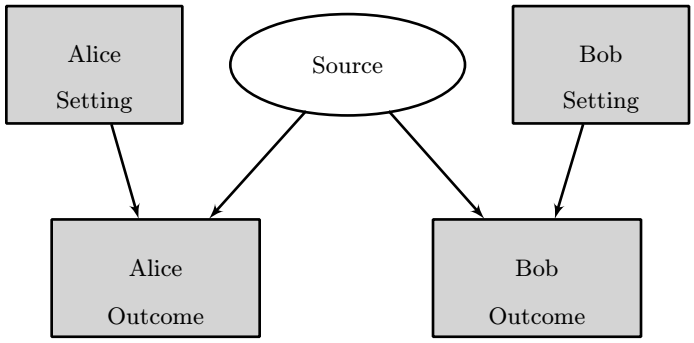


FIG 1. *A classical description of a Bell-CHSH type experiment entails the validity of the graphical model described by this simple causal graph. Rectangles: observed variables; ellipse: unobserved. Settings and outcomes are both binary. Experimental results arguably (via Bell's theorem) show that the classical description has to be abandoned. The probability distribution of experimental data is far outside the class of probability distributions allowed by the model.*

one, but of course it might be of arbitrarily complex nature) by a white oval. The validity of this causal model places restrictions on the joint distribution of the observed variables, see for instance Ver Steeg and Galstyan (2011).

In view of the experimental support for violation of Bell's inequality, the present writer prefers to imagine a world in which "realism" is not a fundamental principle of physics but only an emergent property in the familiar realm of daily life. In this way we can keep quantum mechanics, locality and freedom. This position does entail taking quantum randomness very seriously: it becomes an irreducible feature of the physical world, a "primitive notion"; it is not "merely" an emergent feature. He believes that within this position, the measurement problem (Schrödinger cat problem) has a decent mathematical solution, in which causality is the guiding principle (Slava Belavkin's "eventum mechanics").

Many practical minded physicists claim to be adherents of the so-called Many Worlds interpretation (MWI) of quantum mechanics. In the writer's opinion (but also of many writers on quantum foundations), this interpretation also entails a rejection of "realism", but now in a very strong sense: the reality of an actual random path taken by Nature through space-time is denied. The only reality is the ensemble of all possible paths. Devilish experiments lead to dead cats turning up on some paths, and alive cats on others. According to MWI the only reality is the quantum wave-function. The reality of the death (or not) of the cat is an illusion.

## 2. BELL'S INEQUALITY

To begin with I will establish a new version of the famous Bell inequality (more precisely: Bell-CHSH inequality). My version is not an inequality about theoretical expectation values, but is a probabilistic inequality about experimentally observed averages. Probability derives purely from randomisation in the experimental design.

Consider a spreadsheet containing an $N \times 4$ table of numbers $\pm 1$. The rows will be labelled by an index $j = 1, \ldots, N$. The columns are labelled with names $A$, $A'$, $B$ and $B'$. I will denote the four numbers in the $j$th row of the table by $A_j$, $A'_j$, $B_j$ and $B'_j$. Denote by $\langle AB \rangle = (1/N) \sum_{j=1}^{N} A_j B_j$, the average over the

$N$ rows of the product of the elements in the $A$ and $B$ columns. Define $\langle AB' \rangle$, $\langle A'B \rangle$, $\langle A'B' \rangle$ similarly.

Suppose that for each row of the spreadsheet, two fair coins are tossed independently of one another, independently over all the rows. Suppose that depending on the outcomes of the two coins, we either get to see the value of $A$ or $A'$, and either the value of $B$ or $B'$. We can therefore determine the value of just one of the four products $AB$, $AB'$, $A'B$, and $A'B'$, each with equal probability $\frac{1}{4}$, for each row of the table. Denote by $\langle AB \rangle_{\text{obs}}$ the average of the observed products of $A$ and $B$ ("undefined" if the sample size is zero). Define $\langle AB' \rangle_{\text{obs}}$, $\langle A'B \rangle_{\text{obs}}$ and $\langle A'B' \rangle_{\text{obs}}$ similarly.

**Fact 1** *For any four numbers $A$, $A'$, $B$, $B'$ each equal to $\pm 1$,*

$$AB + AB' + A'B - A'B' \;=\; \pm 2. \tag{1}$$

**Proof.** Notice that

$$AB + AB' + A'B - A'B' \;=\; A(B + B') + A'(B - B').$$

$B$ and $B'$ are either equal to one another or unequal. In the former case, $B - B' = 0$ and $B + B' = \pm 2$; in the latter case $B - B' = \pm 2$ and $B + B' = 0$. Thus $AB + AB' + A'B - A'B'$ equals either $A$ or $A'$, both of which equal $\pm 1$, times $\pm 2$. All possibilities lead to $AB + AB' + A'B - A'B' = \pm 2$. $\square$

**Fact 2**

$$\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \;\leq 2. \tag{2}$$

**Proof.** By (1),

$$\begin{aligned} &\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \\ &= \langle AB + AB' + A'B - A'B' \rangle \;\in\; [-2, 2]. \qquad \square \end{aligned}$$

Formula (2) is known as the CHSH inequality (Clauser, Horne, Shimony and Holt, 1969). It is a generalisation of the original Bell (1964) inequality.

When $N$ is large one would expect $\langle AB \rangle_{\text{obs}}$ to be close to $\langle AB \rangle$, and the same for the other three averages of observed products. Hence, equation (2) should remain approximately true when we replace the averages of the four products over all $N$ rows with the averages of the four products in each of four disjoint subsamples of expected size $N/4$ each. The following theorem expresses this intuition in a precise and useful way. Its straightforward proof, given in the appendix, uses two Hoeffding (1963) inequalities (exponential bounds on the tail of binomial and hypergeometric distributions) to probabilistically bound the difference between $\langle AB \rangle_{\text{obs}}$ and $\langle AB \rangle$, etc.

**Theorem 1** *Given an $N \times 4$ spreadsheet of numbers $\pm 1$ with columns $A$, $A'$, $B$ and $B'$, suppose that, completely at random, just one of $A$ and $A'$ is observed and just one of $B$ and $B'$ are observed in every row. Then, for any $\eta \geq 0$,*

$$\Pr\big( \langle AB \rangle_{\text{obs}} + \langle AB' \rangle_{\text{obs}} + \langle A'B \rangle_{\text{obs}} - \langle A'B' \rangle_{\text{obs}} \;\leq\; 2 + \eta \big) \;\geq\; 1 - 8e^{-N(\frac{\eta}{16})^2}. \tag{3}$$

Traditional presentations of Bell's theorem derive the large $N$ limit of this result. If for $N \to \infty$, experimental averages converge to theoretical mean values, then by (3) these must satisfy

$$\langle AB \rangle_{\text{lim}} + \langle AB' \rangle_{\text{lim}} + \langle A'B \rangle_{\text{lim}} - \langle A'B' \rangle_{\text{lim}} \ \leq \ 2. \tag{4}$$

Like (2), this inequality is also called the CHSH inequality.

I conclude this section with an open problem. An analysis by Vongehr (2013) of the original Bell inequality, which is "just" the CHSH inequality in the situation that one of the four correlations is identically equal to $\pm 1$, suggests that the following conjecture might be true. I come back to this in the last section of the paper.

**Conjecture 1** *Under the assumptions of Theorem 1,*

$$\Pr\big( \langle AB \rangle_{\text{obs}} + \langle AB' \rangle_{\text{obs}} + \langle A'B \rangle_{\text{obs}} - \langle A'B' \rangle_{\text{obs}} \ > 2 \big) \ \leq \ \frac{1}{2}. \tag{5}$$

## 3. BELL'S THEOREM

Both the original Bell inequality, and Bell-CHSH inequality (4), can be used to prove *Bell's theorem*: quantum mechanics is incompatible with the principles of realism, locality and freedom. If we want to hold on to all three principles, quantum mechanics must be rejected. Alternatively, if we want to hold on to quantum theory, we have to relinquish at least one of those three principles.

An *executive summary* of the proof of Bell's theorem consists purely of the following one-liner: certain models in quantum physics, referring to an experiment with the layout of Figure 1, predict

$$\langle AB \rangle_{\text{lim}} + \langle AB' \rangle_{\text{lim}} + \langle A'B \rangle_{\text{lim}} - \langle A'B' \rangle_{\text{lim}} \ = \ 2\sqrt{2}. \tag{6}$$

More details will be given in a moment.

If we accept quantum mechanics, should we reject locality, realism, or freedom? Almost no-one is prepared to abandon freedom. It seems to be a matter of changing fashion whether one blames locality or realism. I will argue that we must place the blame on realism, and not in the weak sense of the Copenhagen interpretation which is a kind of dogmatic assertion that it doesn't make any sense to ask "what is actually going on behind the scenes", but in a more positive sense: the positive assertion that quantum randomness is both real and fundamental. In classical physics, randomness is *merely* the result of dependence on uncontrollable initial conditions. Variation in those conditions, or uncertainty about them, leads to variation, or uncertainty, in the final result. However, there is no such explanation for quantum randomness. Quantum randomness is intrinsic, non-classical, irreducible. It is not an emergent phenomenon. It is the bottom line. It is a fundamental feature of the fabric of reality,

For present purposes we do not need to understand any of the quantum mechanics behind (6): we just need to know the specific statistical predictions which follow from a particular model in quantum physics called the EPR-B model. The initials refer here to the celebrated paradox of Einstein, Podolsky and Rosen (1935) in a version introduced by David Bohm (1951). The EPR-B model predicts the statistics of measurements of spin on each of an entangled pair of spin-half

quantum systems in the singlet state. Fortunately, we do not need to understand any of these words in order to understand what an EPR-B experiment looks like (see Figure 1, again).

In one run of this stylised experiment, two particles are generated together at a source, and then travel to two distant locations. Here, they are measured by two experimenters Alice and Bob. Alice and Bob are each in possession of a measurement apparatus which can "measure the spin of a particle in any chosen direction". Alice (and similarly, Bob) can freely choose (and set) a *setting* on her measurement apparatus. Alice's setting is an arbitrary direction in real three-dimensional space represented by a unit vector $\mathbf{a}$. Her apparatus will then register an observed outcome $\pm 1$ which is called the observed spin of Alice's particle in direction $\mathbf{a}$. At the same time, far away, Bob chooses a direction $\mathbf{b}$ and also gets to observe an outcome $\pm 1$. This is repeated many times – the complete experiment will consist of a total of $N$ runs. We'll imagine Alice and Bob repeatedly choosing new settings for each new run, in the same fashion as in Section 2: each tossing a fair coin to make a binary choice between just two possible settings, $\mathbf{a}$ and $\mathbf{a}'$ for Alice, $\mathbf{b}$ and $\mathbf{b}'$ for Bob.

First we will complete our description of the quantum mechanical predictions for each run separately. For pairs of particles in the singlet state, the prediction of quantum mechanics is that in whatever directions Alice and Bob perform their measurements, their outcome $\pm 1$ is completely random; i.e., both marginal distributions are uniform. The outcomes are not, however, independent. They are correlated, with correlation depending on the two settings. To be precise, the expected value of the product of the outcomes is equal to $-\mathbf{a}\cdot\mathbf{b} = -\cos(\theta)$ where $\theta$ is the angle between the two directions.

With this information we can write down the $2\times 2$ table for the joint probability distribution of the outcomes at the two locations, given two settings differing in direction by the angle $\theta$:

| | $+1$ | $-1$ |
|---|---|---|
| $+1$ | $\frac{1}{4}(1-\cos(\theta))$ | $\frac{1}{4}(1+\cos(\theta))$ |
| $-1$ | $\frac{1}{4}(1+\cos(\theta))$ | $\frac{1}{4}(1-\cos(\theta))$ |

Both marginals of the table are uniform. The expectation of the product of the outcomes equals the probability that they are equal minus the probability they are different $\frac{2}{4}(1-\cos(\theta)) - \frac{2}{4}(1+\cos(\theta)) = -\cos(\theta)$. Physicists use the word "correlation" to refer to the raw (uncentered, unnormalised) product moment but in this case the physicist's and the statistician's correlation coincide.

As mentioned before, Alice and Bob now perform $N$ runs of the experiment according to the following randomised experimental design. Alice has fixed in advance two particular directions $\mathbf{a}$ and $\mathbf{a}'$; Bob has fixed in advance two particular directions $\mathbf{b}$ and $\mathbf{b}'$. In each run, Alice and Bob are each sent one of a new pair of particles in the singlet state. While their particles are *en route* to them, they each toss a fair coin in order to choose one of their two measurement directions. In total $N$ times, Alice observes either $A = \pm 1$ or $A' = \pm 1$ say, and Bob observes either $B = \pm 1$ or $B' = \pm 1$. At the end of the experiment, four "correlations" are calculated: the four sample averages of the products $AB$, $AB'$, $A'B$ and $A'B'$.

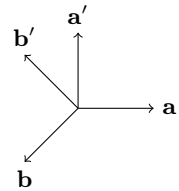


FIG 2. *Four measurement directions, all in the same plane. Alice's settings are the two orthogonal directions* $\mathbf{a}$, $\mathbf{a}'$, *and Bob's settings are the orthogonal* $\mathbf{b}$, $\mathbf{b}'$. *Relative to one another, the pairs are arranged so that* $\mathbf{a}'$ *and* $\mathbf{b}'$ *are close to pointing in the same direction, while at the same time the other three pairs of one of Alice's and one of Bob's settings (*$\mathbf{a}$ *and* $\mathbf{b}$, $\mathbf{a}$ *and* $\mathbf{b}'$, $\mathbf{a}'$ *and* $\mathbf{b}$*) are all equally close to pointing in opposite directions.*

Each correlation is based on a different subset of runs, of expected size $N/4$, determined by the $2N$ fair coin tosses.

Under *realism* we can imagine, for each run, alongside of the outcomes of the actually measured pair of variables, also the outcomes of the not measured pair. Under *locality*, the outcomes in Alice's wing cannot depend on the choice of which variable is measured in Bob's wing. Thus, for each run there is a suite of potential outcomes $A$, $A'$, $B$ and $B'$, but only one of $A$ and $A'$, and only one of $B$ and $B'$ actually gets to be observed. By *freedom* the choices are statistically independent of the actual values of the four.

I'll assume furthermore that the suite of counterfactual outcomes in the $j$th run does not actually depend on which particular variables were observed in previous runs. This *memoryless* assumption can be completely avoided by using the martingale version of Hoeffding's inequality, Gill (2003). But the present analysis is already applicable if we imagine $N$ copies of the experiment each with only a single run, all being done simultaneously in different laboratories.

The assumptions of realism, locality and freedom have put us firmly in the situation of the previous section. Therefore by Theorem 1, the four sample correlations (empirical raw product moments) satisfy (3).

Let's contrast this prediction with the quantum mechanical predictions obtained with a certain clever selection of directions. We'll take the four vectors $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$ to lie in the same plane. It's then enough to specify the angles $\alpha$, $\alpha'$, $\beta$, $\beta' \in [0, 2\pi]$ which they make with respect to some fixed vector in this plane. Consider the choice $\alpha = 0$, $\alpha' = \pi/2$, $\beta = 5\pi/4$, $\beta' = 3\pi/4$, see Figure 2. The differences $|\alpha-\beta|$, $|\alpha-\beta'|$, $|\alpha'-\beta|$ are all equal to $\pi\pm\pi/4$: these pairs of angles are only 45 degrees away from being opposite to one another; the corresponding measurements are quite strongly positively correlated. On the other hand, $|\alpha'-\beta'| = \pi/4$: these two angles are 45 degrees away from being equal and the corresponding measurements are as strongly anti-correlated, as the other pairs were strongly correlated. Three of the correlations are equal to $-\cos(3\pi/4) = -(-1/\sqrt{2}) = 1/\sqrt{2}$ and the fourth is equal to $-\cos(\pi/4) = -1/\sqrt{2}$. Thus we would expect to see, up to statistical variation,

$$\langle AB\rangle_{\text{obs}} + \langle AB'\rangle_{\text{obs}} + \langle A'B\rangle_{\text{obs}} - \langle A'B'\rangle_{\text{obs}} \approx 4/\sqrt{2} \;=\; 2\sqrt{2} \;\approx\; 2.828,$$

cf. (6). By Tsirelson's inequality (Cirel'son, 1980), this is actually the largest absolute deviation from the CHSH inequality which is allowed by quantum mechanics.

Many experiments have been performed confirming these predictions. Two particularly notable ones are those of Aspect et al. (1982) in Orsay, Paris, and of Weihs et al. (1998) in Innsbruck (later I will discuss two of the most recent).

In these experiments, the choices of which direction to measure were not literally made with coin tosses, but by rather more practical physical systems. In Alain Aspect's Orsay and Gregor Weihs' Innsbruck experiments the separation between the locations of Alice and Bob was large; the time it took from initiating the choice of random direction to measure to completion of the measurement was small: so small, that Alice's measurement is complete before a signal traveling at the speed of light could possibly transmit Bob's choice to Alice's location. However, note that this depends on what one considers to be the time each randomisation starts happening. Weihs' experiment improves on Aspect's in this respect.

The data gathered from the Innsbruck experiment is available online. It had $N \approx 15\,000$; and found $\langle AB\rangle_{\text{obs}} + \langle AB'\rangle_{\text{obs}} + \langle A'B\rangle_{\text{obs}} - \langle A'B'\rangle_{\text{obs}} = 2.73 \pm 0.022$, the statistical accuracy (standard deviation) following from a standard delta-method calculation assuming i.i.d. observations per setting pair. The reader can check that this corresponds to accuracy obtained by a standard computation using binomial variances of the counts for each of the four roughly equal subsamples. By (3), under realism, locality and freedom, the chance that $\langle AB\rangle_{\text{obs}} + \langle AB'\rangle_{\text{obs}} + \langle A'B\rangle_{\text{obs}} - \langle A'B'\rangle_{\text{obs}}$ would exceed 2.73 is less than $10^{-12}$.

The experiment deviates in several ways from what has been described so far, and I will summarise them here.

An unimportant difference is the physical system used: polarisation of entangled photons rather than spin of entangled spin-half particles (e.g., electrons).

An important difference between the idealisation and the truth concerns the picture of Alice and Bob repeating some actions $N$ times with $N$ fixed in advance. The experimenters do not control when a pair of photons will leave the source nor how many times this happens. Even talking about "pairs of photons" is using classical physical language which can be acutely misleading. In actual fact, all we observe are individual detection events (time, current setting, outcome) at each of the two detectors, i.e., at each measurement apparatus.

Complicating this still further is the fact that many particles fail to be detected at all. One could say that the outcome of measuring one particle is not binary but ternary: $+$, $-$, or *no detection*. If neither particle of a pair is detected, then we do not even know there is a pair at all. $N$ was not only not fixed in advance: it is not even known. The data cannot be summarised in a list of pairs of settings and pairs of outcomes (whether binary of ternary), but consists of two lists of the *random times* of definite measurement outcomes in each wing of the experiment together with the settings in force at the time of the measurements. The settings are being extremely rapidly, randomly switched, between the two alternative values. When detection events occur close together in time they are treated as belonging to a pair of photons.

In Weihs' experiment, only 1 in 20 of the events in each wing of the experiment seemed to be paired with an event in the other. If all detections correspond to emissions of pairs from the source, then for every 400 pairs of photons, just one pair leads to a paired event, $2 \times 19$ lead to unpaired events, and the remaining 361 to no observed event at all.

We will return to the issue of whether the idealised picture of $N$ pairs of particles, each separately being measured, each particle in just one of two ways, is really appropriate, in a later section; we will also take a look then at two more, very recent, experiments. However, the point is that quantum mechanics does seem to promise that experiments of this nature could in principle be done, and if so, there seems no reason to doubt they could violate the CHSH inequality. Three correlations more or less equal to $1/\sqrt{2}$ and one equal to $-1/\sqrt{2}$ have been measured in the lab. Not to mention that the whole curve $-\cos(\theta)$ has been experimentally recovered.

Right now the situation is that at least four major experimental groups (Singapore, Brisbane, Vienna, Illinois) seem to be vying to be the first to perform a successful and completely "loophole-free" experiment, predictions being that this is no more than five years away (cf. Marek Żukowski, quoted in Merali, 2010). It will be a major achievement, the crown of more than fifty years' labour.

## 4. REALISM, LOCALITY, FREEDOM

This section and the next are about metaphysics and can safely be skipped by the reader impatient to learn more about statistical aspects of Bell experiments.

The EPR-B correlations have a second message beyond the fact that they violate the CHSH inequality. They also exhibit perfect anti-correlation in the case that the two directions of measurement are exactly equal – and perfect correlation in the case that they are exactly opposite. This brings us straight to the EPR argument not for the non-locality of quantum mechanics, but for the incompleteness of quantum mechanics.

Einstein, Podolsky and Rosen (1935) were revulsed by the idea that the "last word" in physics would be a "merely" statistical theory. Physics should explain why, in each individual instance, what actually happens does happen. The belief that every "effect" *must* have a "cause" has driven Western science since Aristotle. Now according to the singlet correlations, if Alice were to measure the spin of her particle in direction $\mathbf{a}$, it's certain that if Bob were to do the same, he would find exactly the opposite outcome. Since it is inconceivable that Alice's choice has any immediate influence on the particle over at Bob's place, it must be that the outcome of measuring Bob's particle in the direction $\mathbf{a}$ is predetermined "in the particle" as it were. The measurement outcomes from measuring spin in all conceivable directions on both particles must be predetermined properties of those particles. The observed correlation is merely caused by their origin at a common source.

Thus Einstein used locality, together with the predictions of quantum mechanics itself, to infer realism or counterfactual definiteness in the strong sense that the outcomes of measurements on physical systems are predefined properties of those systems, carried in them, and merely revealed by the act of measurement. From this he argued the *incompleteness* of quantum mechanics – it describes some aggregate properties of collectives of physical systems, but does not even deign to talk about physically definitely existing properties of individual systems.

Whether it needed external support or not, the notion of counterfactual definiteness is nothing strange in all of physics (prior to the invention of quantum mechanics). It comes for free with a deterministic view of the world as a collection of objects blindly obeying definite rules.

Instead of assuming quantum mechanics and deriving counterfactual definiteness, Bell turned the EPR argument on its head. He assumes three principles which Einstein would have endorsed anyway, and uses them to get a contradiction with quantum mechanics; and the first is counterfactual definiteness. We must first agree that though, say, only $A$ and $B$ are actually measured in one particular run, still, in a mathematical sense, $A'$ and $B'$ also exist (or at least may be constructed) alongside of the other two; and moreover they may be thought to be located in space and time just where one would imagine. Only after that does it make sense to discuss *locality*: the assumption that *which* variable is being observed at Alice's location does not influence the values taken by the other two at Bob's location.

Having assumed realism and locality we can bring the freedom assumption into play. As we have seen, it allowed us to analyse our experiment with classical probability tools based on a classically randomised design. Some writers like to associate the freedom assumption with the free will of the experimenter, others with the existence of "true" randomness in other physical processes: either way, one metaphysical assumption is justified by another. I would rather see it in a practical way: we understand pseudo-randomness very well, and its principles underly coin tossing just as much as pseudo random generators. We use randomisation effectively in all kinds of contexts (randomised algorithms, randomised clinical trials, randomised designs). Do we really want to believe that the observed correlations $\pm 0.7071$ (three positive, one negative) come about through a physical mechanism by which the outcomes of two coin tosses and two polarisation measurements are all exquisitely dependent on one another through *all four* being jointly *pre-determined* by events in the deep past? When such a hypothesis is otherwise completely unnecessary? (Otherwise, we never see spatial-temporal correlations following this sign pattern, larger in absolute value than 0.5). A mechanism which is completely unknown? A mechanism which ensures in effect that Alice's photon knows how Bob's photon is being measured? Yet, a mechanism which cannot make any of those correlations larger in absolute value than 0.7071, though if it really were the case that Alice's photon knows Bob's setting, three positive and one negative correlation $\pm 1.0$ could have been achieved.

I think that Occam's razor tells us to discard this flavour of super-determinism, also known as conspiracy. In fact, to abandon freedom means to abandon Science: we may discard all empirical (observational) data. Everything is explained but nothing can be predicted.

Keeping freedom, we have to make a choice between two other inconceivable possibilities: do we reject locality, or do we reject realism?

Here I would like to call on Occam's principle again. Suppose realism is true. Instead of invoking the fact that a collection of four coin toss outcomes and photo-detector clicks were jointly predetermined in the deep past, we now have to invoke instantaneous communication across large distances of the outcomes of these processes, by as yet unknown processes, and again with only the extremely subtle and special effects which quantum mechanics seems to predict. Alice cannot communicate with Bob through this phenomenon. There is no observable action-at-a-distance. The surface predictions of quantum mechanics are perfectly compatible with relativistic causality. It is only when we hypothesise a hidden layer that we run into difficulties.

It seems to me that we are pretty much forced into rejecting *realism*, which, please remember, is actually an idealistic concept: outcomes "exist" of measurements which were *not* performed. However, I admit it goes against all instinct. In the case of equal settings, how can it be that the outcomes are equal and opposite, if they were not predetermined at the source?

Though it is perhaps only a comfort blanket, I would like here to appeal to the limitations of our own brains, the limitations we experience in our "understanding" of physics due to our own rather special position in the universe. In philosophy this notion is called *embodied cognition*. There is also hard empirical evidence for this idea.

According to cognitive scientists (see for instance Spelke and Kinzler, 2007) our brains are at birth hardwired with various basic conceptions about the world. These "modules" are called *systems of core knowledge*. The idea is that we cannot acquire new knowledge from our sensory experiences (including learning from experiments: we cry, and food and/or comfort is provided!) without having a prior framework in which to interpret the data of experience and experiment. It seems that we have modules for algebra and modules for geometry: basic notions of number and of space. Most interestingly in the present context, we also have modules for causality. We distinguish between *objects* and *agents* (we learn that we ourselves are agents). Objects are acted on by agents. Objects have continuous existence in space time, they are local. Agents can act on objects, also at a distance. Together this seems to me to be a built-in assumption of determinism: we have been created (by evolution) to operate in an Aristotelian world, a world in which every effect has a cause.

The argument (from physics, and by Occam's razor, not from neuroscience!) for abandoning realism is made eloquently by Boris Tsirelson in an internet encyclopaedia article on entanglement (Citizendium: entanglement). It was Tsirelson from whom I borrowed the terms counterfactual definiteness, relativistic local causality, and no-conspiracy. He points out that it is a mathematical fact that quantum physics is consistent with relativistic local causality and with no-conspiracy. In all of physics, there is no evidence *against* either of these two principles.

I would like to close this section by mentioning a beautiful paper by Masanes, Acin and Gisin (2006) who argue in a very general setting (i.e., not assuming quantum theory, or local realism, or anything) that *quantum non-locality*, by which they mean the violation of Bell inequalities, together with *non-signalling*, which is the property that the marginal probability distribution seen by Alice of $A$ does not depend on whether Bob measures $B$ of $B'$, together imply indeterminism: that is to say: that the world is stochastic, not deterministic.

## 5. RESOLUTION OF THE MEASUREMENT PROBLEM

The measurement problem, also known as Schrödinger's cat problem) is the problem of how to reconcile two apparently mutually contradictory parts of quantum mechanics. When a quantum system is isolated from the rest of the world, its quantum state (a vector, normalised to have unit length, in Hilbert space) evolves unitarily, deterministically. When we look at a quantum system from outside, by making a measurement on it in a laboratory, the state collapses to one of the eigenvectors of an operator corresponding to the particular measurement, and it does so with probabilities equal to the squared lengths of the projections of the

original state vector into the eigenspaces. Yet the system being measured together with the measurement apparatus used to probe it form together a much larger quantum system, supposedly evolving unitarily and deterministically in time.

Accepting that quantum theory is intrinsically stochastic, and accepting the reality of measurement outcomes, led Slava Belavkin (2000) to a mathematical framework which he called eventum mechanics which (in my opinion) indeed reconciles the two faces of quantum physics (Schrödinger evolution, von Neumann collapse) by a most simple device. Moreover, it is based on ideas of causality with respect to time. I have attempted to explain this model in as simple terms as possible in Gill (2009). The following words will only make sense to those with some familiarity with quantum mechanics.

The idea is to model the world in the conventional way with a Hilbert space, a quantum state on that space, and a unitary evolution. Inside this framework we look for a collection of bounded operators on the Hilbert space which all commute with one another, and which are *causally* compatible with the unitary evolution of the space, in the sense that they all commute with past copies of themselves (in the Heisenberg picture, one thinks of the quantum observables as changing, the state as fixed; each observable corresponds to a time indexed family of bounded operators). We call this special family of operators the *beables*: they correspond to physical properties in a classical-like world which can coexist, all having definite values at the same time, and definite values in the past too. The state and the unitary evolution together determine a joint probability distribution of these time-indexed variables, i.e., a stochastic process. At any fixed time we can condition the state of the system on the past trajectories of the beables. This leads to a quantum state over all bounded operators which commute with all the beables.

The result is a theory in which the deterministic and stochastic parts of traditional quantum theory are combined into one harmonious whole. In fact, the notion of restricting attention to a subclass of all observables goes back a long way in quantum theory under the name *superselection rule*; and abstract quantum theory (and quantum field theory) has long worked with arbitrary algebras of observables, not necessarily the full algebra of a specific Hilbert space. With respect to those traditional approaches the only novelty is to suppose that the unitary evolution when restricted to the sub-algebra is not invertible. It is an endomorphism, not an isomorphism. There is an arrow of time.

It turns out that the theory is mathematically equivalent to important versions of the continuous spontaneous localisation (CSL) model, a way to solve the measurement problem by adding an explicit stochastic collapse term to the Schrödinger equation. (Initially the two theories seem quite different in nature). The problem of crafting a relativistically invariant version of CSL remained open for many years (and was a major obstruction to its acceptance) yet just recently this problem has been solved by Daniel Beddingham (2011). See Pearle (1997, 2012) for further details.

CSL has been eloquently championed over the years by Philip Pearle and I refer the reader to his many works, in particular the two just cited, both explaining CSL and explaining why it does solve the measurement problem, while MWI does not.

## 6. LOOPHOLES

In real world experiments, the ideal experimental protocol of particles leaving a source at definite times, and being measured at distant locations according to locally randomly chosen settings cannot be implemented.

Experiments have been done with pairs of entangled ions, separated only by a short distance. The measurement of each ion takes a relatively long time, but at least it is almost always successful. Such experiments are obviously blemished by the so-called *communication* or *locality* loophole. Each particle can know very well how the other one is being measured.

Many very impressive experiments have been performed with pairs of entangled photons. Here, the measurement of each photon can be performed very rapidly and at huge distance from one another. However, many photons fail to be detected at all. For many events in one wing of the experiment, there is often no event at all in the other wing, even though the physicists are pretty sure that almost all detection events do correspond to (members of) entangled pairs of photons. This is called the *detection* loophole. Popularly it is thought to be merely connected to the efficiency of photo-detectors and that it will be easily overcome by the development of better and better photodetectors. Certainly that is necessary, but not sufficient, as I'll explain.

In Weihs' experiment mentioned earlier, only 1 in 20 of the events in each wing of the experiment is paired with an event in the other wing. Thus of every 400 pairs of photons – if we assume that detection and non-detection occur independently of one another in the two wings of the experiment – only 1 pair results in a successful measurement of both the photons; there are 19 further unpaired events in each wing of the experiment; and there were 361 pairs of photons not observed at all.

Imagine (anthropocentrically) classical particles about to leave the source and aiming to fake the singlet correlations. If they are allowed to go undetected often enough, they can engineer any correlations they like, as follows. Consider two new photons about to leave the source. They agree between one another with what pair of settings they would like to be measured. Having decided on the desired setting pair, they next generate outcomes $\pm 1$ by drawing them from the joint probability distribution of outcomes given settings, which they want the experimenter to see. Only then do they each travel to their corresponding detector. There, each particle compares the setting it had chosen in advance with the setting chosen by Alice or Bob. If they are not the same, it decides to go undetected. With probability $1/4$ we will have successful detections in both wings of the experiment. For those detections, the pair of settings according to which the particles are being measured is identical to the pair of settings they had aimed at in advance.

This example illustrates that if one wants to experimentally *prove* a violation of local realism without making the untestable statistical assumption of "missing at random", known as the *fair-sampling assumption* in this context, one has to put limits on the amount of "non-detection". There is a long history and big literature on this topic. I will just mention one of such results.

Jan-Åke Larsson (1998, 1999) has proved variants of the CHSH inequality which take account of the possibility of non-detections. The idea is that under local realism, as the proportion of "missing" measurements increases from zero,

the upper bound "2" in the CHSH inequality (4) increases too. We introduce a quantity $\gamma$ called the *efficiency of the experiment*: this is the minimum over all setting pairs of the probability that Alice sees an outcome given Bob sees an outcome (and vice versa). It is not to be confused with "detector efficiency". It turns out that the (sharp) bound on $\langle AB\rangle_{\lim} + \langle AB'\rangle_{\lim} + \langle A'B\rangle_{\lim} - \langle A'B'\rangle_{\lim}$ set by local realism is no longer 2 as in (4), but $2+\delta$, where $\delta = \delta(\gamma) = 4(\gamma^{-1}-1)$.

As long as $\gamma \geq 1/\sqrt{2} \approx 0.7071$, the bound $2+\delta$ is smaller than $2\sqrt{2}$. Weihs' experiment has an efficiency of 5%. If only we could increase it to above 71% *and* simultaneously keep the state and measurements close to perfection, we could have definitive experimental proof of Bell's theorem.

This would be correct for a "clocked" experiment. Suppose now particles determine themselves the times that they are measured. Thus a local realist pair of particles trying to fake the singlet correlations could arrange between themselves that their measurement times are delayed by smaller or greater amounts depending on whether the setting they see at the detector is the setting they want to see, or not. It turns out that this gives our devious particles even more scope for faking correlations. Larsson and Gill (2004) called this the *coincidence loophole*, and derived the sharp bound on $\langle AB\rangle_{\lim} + \langle AB'\rangle_{\lim} + \langle A'B\rangle_{\lim} - \langle A'B'\rangle_{\lim}$ set by local realism is $2+\delta$, where now $\delta = \delta(\gamma) = 6(\gamma^{-1}-1)$. As long as $\gamma \geq 3(1-1/\sqrt{2}) \approx 0.8787$, the bound $2+\delta$ is smaller than $2\sqrt{2}$. We need to get experimental efficiency above 88%, and keep everything else close to perfect at the very limits allowed by quantum physics.

How far is there still to go? In 2013 the Vienna group published a paper in the journal *Nature* entitled "Bell violation using entangled photons without the fair-sampling assumption" (Giustina et al., 2013). The authors write "this is the very first time that an experiment has been done using photons which does not suffer from the detection loophole", and moreover the experiment "makes the photon the first physical system for which each of the main loopholes has been closed, albeit in different experiments."

It was however rapidly pointed out that the experiment was actually vulnerable to the coincidence loophole, not "just" to the detection loophole. Now, it actually should be possible to simply reanalyse the data from that experiment, defining coincidences with respect to an externally defined lattice of time intervals instead of relative to observed detection times only. Ideally, this will only slightly increase the "singles rate" and slightly decrease the number of coincidences, thereby slightly decreasing both size and statistical significance of the Bell violation, but hopefully without altering the substantive conclusion. A more stringent re-analysis of the data (Larsson et al., 2013) has confirmed that the initial claims were justified.

In the meantime, exploiting this gap between results and claims, a consortium led by researchers from Illinois have published their own experimental results, also reporting that theirs is "the first experiment that fully closes the detection loophole with photons, which are then the only system in which both loopholes have been closed, albeit not simultaneously" (Christensen et al., 2013). They used the Larsson and Gill (2004) inequality.

Is this just a question of prestige? No: various new quantum technologies depend on quantum entanglement, and in particular, various cryptographic communication protocols are not secure as long as it is possible to "fake" violation

of Bell inequalities with classical systems.

It is *logically possible* that quantum mechanics itself could prevent one ever from performing a both successful and loophole-free experiment. Quantum uncertainty relations could in principle prevent the creation of a multipartite quantum system, whose components can be measured in well-separated space-time regions, while simultaneously those components are in the required joint entangled state. I christened this possibility "Bell's fifth position" in Gill (2003). Here I just mention that the possibility had already been championed for many years by Emilios Santos, whose paper Santos (2005) is well worth reading.

On the other hand, continuous improvement of experimental techniques over more than fifty years has seen continuous pushing of detection efficiency towards the critical boundaries, without any attenuation of the quantum correlations of the singlet state. Now that we are getting very close indeed to the boundary, it would seem very unlikely that we won't be able to go past it.

I conclude this section with mention of some recent work on the conspiracy loophole. Recently Gallicchio, Friedman, and Kaiser (2014) have made the novel suggestion to rule out conspiracy by the experimental device of choosing settings with the help of detection times of photons arriving from widely separated, and very distant galaxies from the dawn of time. This idea will probably be implemented soon in an experiment by Zeilinger. I am not however convinced by this idea: though Alice's setting choice is triggered by a photon which cannot yet have interacted with Bob's measurement device or the source, still the setting itself is also partially determined by Alice's detection apparatus, and it certainly has. And if there is dependence between subsequent settings on Alice's side, then on Bob's side it soon becomes possible to predict future settings (this is known as the memory loophole).

In my opinion, we have to rule out conspiracy (ensure freedom) by choosing settings by a cascade of classical randomness (coins, pseudo RNGs, etc.). We can never logically rule out conspiratorial super determinism, but we can make appeal to this escape clause ludicrous.

## 7. BELL'S THEOREM WITHOUT INEQUALITIES

In recent years new proofs of Bell's theorem have been invented which appear to avoid probability or statistics altogether, such as the famous GHZ (Greenberger, Horne, Zeilinger) proof. Experiments have already been done implementing the set-up of these proofs, and physicists have claimed that these experiments prove quantum-nonlocality by the outcomes of a finite number of runs: no statistics, no inequalities (yet their papers do exhibit error bars!).

Such a proof runs along the following lines. Suppose local realism is true. Suppose also that some event $\mathcal{A}$ is certain. Suppose that it then follows from local realism that another event $\mathcal{B}$ has probability zero, while under quantum mechanics it can be arranged that the same event $\mathcal{B}$ has probability one. Paradoxical, but not a contradiction in terms: the catch is that events $\mathcal{A}$ and $\mathcal{B}$ are events under different experimental conditions: it is only under local realism and freedom that the events $\mathcal{A}$ and $\mathcal{B}$ can be situated in the same sample space. Freedom is needed here to equate the probabilities of observable events with those of unobservable events, just as in our own proof of Bell's theorem. We need to be able to assume that the subset of runs of the experiment in which the events were observable are

a random sample of the set of all repetitions.

When we use randomisation in experimental design, we are assuming that randomisation is independent of pre-existing unobserved characteristics of the experimental units. Is it plausible that the outcomes of coin tosses used to create a randomised experimental design were predetermined together with properties of the experimental units under study, so that our random subsamples of units being given a particular treatment, are actually heavily biased with regard to the properties we measure on them? Of course, under a purely deterministic world view (super-determinism) everything that was ever going to happen was determined in advance, at the dawn of creation. But even in a deterministic world, pseudo-randomness exists and is well understood.

As an example, consider the following scenario, generalizing the Bell-CHSH scenario to the situation where the outcome of the measurements on the two particles is not binary, but an arbitrary real number. This situation has been studied by Zohren and Gill (2008), Zohren, Reska, Gill and Westra (2010).

Just as before, settings are chosen at random in the two wings of the experiment. Under local realism we can introduce variables $A$, $A'$, $B$ and $B'$ representing the outcomes (real numbers) in one run of the experiment, both of the actually observed variables, and of those not observed.

It turns out that it is possible under quantum mechanics to arrange that $\Pr\{B' \leq A\} = \Pr\{A \leq B\} = \Pr\{B \leq A'\} = 1$ while $\Pr\{B' \leq A'\} = 0$. On the other hand, under local realism, $\Pr\{B' \leq A\} = \Pr\{A \leq B\} = \Pr\{B \leq A'\} = 1$ implies $\Pr\{B' \leq A'\} = 1$.

Note that the four probability measures under which, under quantum mechanics, $\Pr\{A \leq B\}$, $\Pr\{A \geq B'\}$, $\Pr\{A' \geq B\}$, $\Pr\{A' \geq B'\}$ are defined, refer to four different experimental set-ups, according to which of the four pairs $(A, B)$ etc. we are measuring.

The experiment to verify these quantum mechanical predictions has not yet been performed though some colleagues are interested. Interestingly, though it requires a quantum entangled state, that state should *not* be the maximally entangled state (the amount of entanglement of a state can be quantified in many ways, for instance through entropy notions, but it would take us too far into the quantum formalism to explain that here). Maximal "quantum non-locality" is quite different from maximal entanglement. And this is not an isolated example of the phenomenon.

Note that even if the experiment is repeated a large number of times, it can never prove that probabilities like $\Pr\{A \leq B\}$ are exactly equal to 1. It can only give strong statistical evidence, at best, that the probability in question is very close to 1 indeed. But actually experiments are never perfect and more likely is that after a number of repetitions, one discovers that $\{A > B\}$ actually has positive probability – that event will happen a few times. The experimenter cannot create the required quantum state exactly, measurements are not perfect. Thus the logical conclusion from the experiment is that nothing has been proved.

To be sure, one can give a proof of Bell's theorem that the theory of quantum mechanics is in conflict with local realism, which relies only on logic, not on probability. But if we want to use the set-up of the proof as a set-up for an experiment, we move to a different ball-park. We want to perform an experiment which gives strong evidence that Nature is incompatible with local realism. It

turns out that whatever experimental set-up we take, we will necessarily find ourselves explicitly or implicitly in the business of statistically proving violation of inequalities, as the next section will make clear.

## 8. BETTER BELL INEQUALITIES

Why all the attention to the CHSH inequality? There are others around, aren't there? And are there alternatives to "inequalities" altogether? I will argue here that the whole story is "just" a collection of inequalities, and the reason behind this can be expressed in a simple geometric picture.

In a precise sense the CHSH inequality is the only Bell inequality worth mentioning in the scenario of two parties, two measurements per party, two outcomes per measurement. Let's generalise this scenario and consider $p$ parties, each choosing between one of $q$ measurements, where each measurement has $r$ possible outcomes (further generalisations are possible to unbalanced experiments, multistage experiments, and so on). I want to explain why CHSH plays a very central role in the $2 \times 2 \times 2$ case, and why in general, *generalised Bell inequalities* are all there is when studying the $p \times q \times r$ case. The short answer is: these inequalities are the bounding hyperplanes of a convex polytope of "everything allowed by local realism". The vertices of the polytope are deterministic local realistic models. An arbitrary local realist model is a mixture of the models corresponding to the vertices. Such a mixture is a *hidden variables model*, the hidden variable being the particular random vertex chosen by the mixing distribution in a specific instance.

From quantum mechanics, after we have fixed a joint $p$-partite quantum state, and sets of $q$ $r$-valued measurements per party, we will be able to write down probability tables $p(a, b, ...|x, y, ...)$ where the variables $x$, $y$, etc. take values in $1, \ldots, q$, and label the measurement used by the first, second, ... party. The variables $a$, $b$, etc., take values in $1, \ldots, r$ and label the possible outcomes of the measurements. Altogether, there are $q^p r^p$ "elementary probabilities" in this list of tables. More generally, any specific instance of a theory, whether local-realist, quantum mechanical, or beyond, generates such a list of probability tables, and defines thereby a point in $q^p r^p$-dimensional Euclidean space.

We can therefore envisage the sets of all local-realist models, all quantum models, and so on, as subsets of $q^p r^p$-dimensional Euclidean space. Now, whatever the theory, for any values of $x$, $y$, etc., the sum of the probabilities $p(a, b, \ldots |x, y, \ldots)$ must equal 1. These are called *normalisation constraints*. Moreover, whatever the theory, all probabilities must be nonnegative: *positivity constraints*. Quantum mechanics is certainly local in the sense that the marginal distribution of the outcome of any one of the measurements of any one of the parties does not depend on which measurements are performed by the other parties. Since marginalization corresponds again to summation of probabilities, these so-called *no-signalling constraints* are expressed by linear equalities in the elements in the probability tables corresponding to a specific model. Not surprisingly, local-realist models also satisfy the no-signalling constraints.

We will call a list of probability tables restricted only by positivity, normalisation and no-signalling, but otherwise completely arbitrary, a *no-signalling* model. The positivity constraints are linear inequalities which place us in the positive orthant of Euclidean space. Normalisation and no-signalling are linear equalities which place us in a certain affine subspace of Euclidean space. Intersection of

orthant and affine subspace creates a convex polytope: the set of all no-signalling models. We want to study the sets of local-realist models, of quantum models, and of no-signalling models. We already know that local-realist and quantum are contained in no-signalling. It turns out that these sets are successively larger, and strictly so: *quantum* includes all *local-realist* and more (that's Bell's theorem); *no-signalling* includes all *quantum* and more (that's Tsirelson's inequality combined with an example of a no-signalling model which violates Tsirelson's inequality).

Let's investigate the local-realist models in more detail. A special class of local-realist models are the *local-deterministic* models. A local-deterministic model is a model in which all of the probabilities $p(a, b, \dots | x, y, \dots)$ equal 0 or 1 *and* the no-signalling constraints are all satisfied. This implies that for each possible measurement by each party, the outcome is prescribed, independently of what measurements are made by the other parties. Now, it is easy to see that any local-realist model corresponds to a probability mixture of local-deterministic models. After all, it "is" a joint probability distribution of simultaneous outcomes of each possible measurement on each system, and thus it "is" a probability mixture of degenerate distributions: fix the random element $\omega$, and each outcome of each possible measurement of each party is fixed; we recover their joint distribution by picking $\omega$ at random.

This makes the set of local-realist models a convex polytope: all mixtures of a finite set of extreme points. Therefore it can also be described as the intersection of a finite collection of half-spaces, each half-space corresponding to a boundary hyperplane.

It can also be shown that the set of quantum models is closed and convex, but its boundary is very difficult to describe.

Let's think of these three models from "within" the affine subspace of no-signalling and normalisation. Relative to this subspace, the no-signalling models form a full (non-empty interior) closed convex polytope. The quantum models form a strictly smaller closed, convex, full set. The local-realist models form a strictly smaller still, closed, convex, full polytope.

Slowly we have arrived at a rather simple picture, Figure 3. Imagine a square, with a circle inscribed in it, and with another smaller square inscribed within the circle. The outer square represents the boundary of the set of all no-signalling models. The circle is the boundary of the convex set of all quantum models. The square inscribed within the circle is the boundary of the set of all local-realist models. The picture is oversimplified. For instance, the vertices of the local-realist polytope are also extreme points of the quantum body and vertices of the no-signalling polytope.

A generalised Bell inequality is simply a boundary hyperplane, or face, of the local-realist polytope, relative to the normalisation and no-signalling affine subspace, and excluding boundaries corresponding to the positivity constraints. I will call these interesting boundary hyperplanes "non-trivial". In the $2 \times 2 \times 2$ case, for which the affine subspace where all the action lies is 8 dimensional, the local-realist polytope has exactly 8 non-trivial boundary hyperplanes. They correspond exactly to all possible CHSH inequalities (obtained by permuting outcomes, measurements and parties). Thus in the $2 \times 2 \times 2$ case, the Bell-CHSH inequality is indeed "all there is".

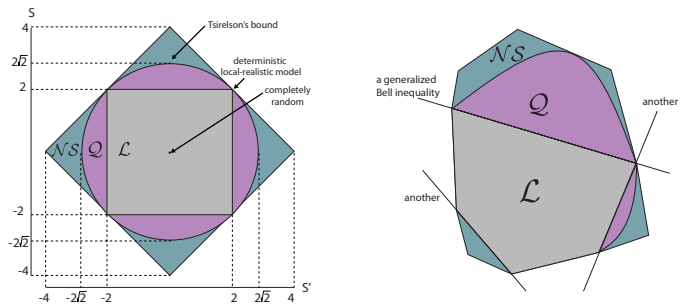


FIG 3. *Left: a caricature of the* $2 \times 2 \times 2$ *case. It actually lives in 8, not 2 dimensions. Right: caricature of the general case in which (bottom left) a further possibility is allowed: no purple between the green and grey. Artwork: Daniel Cavalcanti*

When we increase $p$, $q$ or $r$, new Bell inequalities turn up, and moreover keep turning up ("new" means not obtainable from "old" by omitting parties or measurements or grouping outcomes). It seems a hopeless (and probably pointless) exercise to try to classify them.

A natural question is whether every non-trivial generalised Bell inequality can actually be violated by quantum mechanics. I posed this as an open question a long time ago, and for a long time the answer seemed probably be "yes". However a nice counter-example has recently been discovered, see Almeida et al. (2010).

Quite a few generalised Bell inequalities have turned out to be of particular interest, for instance, the work of Zohren and Gill concerned the $2 \times 2 \times r$ case and discussed a class of inequalities, one for each $r$, whose asymptotic properties could be studied as $r$ increased to infinity. Further statistical connections to missing data problems and optimal experimental design, have been exploited by van Dam, Gill and Grunwald (2005) and Gill (2007).

Much of the material of this section is covered in an excellent survey paper by Brunner et al. (2013), from which I took, with the permission both of the authors and of the artist Daniel Cavalcanti, the two illustrations in Figure 3: the first is a cartoon of the $2 \times 2 \times 2$ case, the second of the general case.

## 9. QUANTUM RANDI CHALLENGES

A second reason for the specific form of the proof of Bell's theorem which started this paper is that it lends itself well to design of computer challenges. Every year, new researchers publish, or try to publish, papers in which they claim that Bell made some fundamental errors, and in which they put forward a specific local realist model which allegedly reproduces the quantum correlations. The papers are long and complicated; the author finds it hard to get the work published, and suspects a conspiracy by "The Establishment". The claims regularly succeed in attracting media attention, occasionally becoming head-line news in serious science journalism; some papers are published, too, and not only in obscure journals.

Extraordinary claims require extraordinary evidence. I used to find it useful in debates with "Bell-deniers" to challenge them to implement their local realist model as computer programs for a network of classical computers, connected so as to mimic the time and space separations of the Bell-CHSH experiments.

The protocol of the challenge I issued in the past is the following. Bell-denier is to write computer programs for three personal computers, which are to play the roles of source $\mathcal{S}$, measurement station $\mathcal{A}$, and measurement station $\mathcal{B}$. The following is to be repeated say 15 000 times. First, $\mathcal{S}$ sends messages to $\mathcal{A}$ and $\mathcal{B}$. Next, connections between $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ are severed. Next, from the outside world so to speak, I deliver the results of two coin tosses (performed by myself), separately of course, as input setting to $\mathcal{A}$ and to $\mathcal{B}$. Heads or tails correspond to a request for $A$ or $A'$ at $\mathcal{A}$, and for $B$ or $B'$ at $\mathcal{B}$. The two measurement stations $\mathcal{A}$ and $\mathcal{B}$ now each output an outcome $\pm 1$. Settings and outcomes are collected for later data analysis, Bell-denier's computers are re-connected; next run.

Bell-denier's computers can contain huge tables of random numbers, shared between the three, and of course they can use pseudo-random number number generators of any kind. By sharing the pseudo-random keys in advance, they have resources to any amount of shared randomness they like.

In Gill (2003) I showed how a martingale Hoeffding inequality gives an exponential bound like (3) in the situation just described. This enabled me to choose $N$, and a criterion for win/lose (say, halfway between 2 and $2\sqrt{2}$), and a guarantee to Bell-denier (at least so many runs with each combination of settings), such that I would happily bet 3000 Euros any day that Bell-denier's computers will fail the challenge.

The point (for me) was not to win money for myself, but to enable the Bell-denier who considers accepting the challenge (a personal challenge between the two of us, with adjudicators to enforce the protocol) to discover for him or herself that "it cannot be done". It's important that the adjudicators do not need to look inside the programs written by the Bell-denier, and preferably don't even need to look inside his computers. They are black boxes. The only thing that has to be enforced are the communication rules. However, there are difficulties here. What if Bell-denier's computers are using a wireless network which the adjudicators can't detect?

A new kind of computer challenge, called the "quantum Randi challenge", was proposed in 2011 by Sascha Vongehr (`Science2.0: QRC`). It is inspired by the well known challenge to "paranormal phenomena" by James Randi (scientific sceptic and fighter against pseudo-science, see `Wikipedia: James Randi`). Vongehr's challenge (see Vongehr 2012, 2013) differs in a number of fundamental respects from mine, which indeed was not a quantum Randi challenge in Vongehr's sense.

Sascha Vongehr's QRC completely cuts out any necessity for communication, protocol verification, adjudication. In fact, the Bell-denier no longer has to cooperate with myself or with any other member of the establishment. They simply have to write a program which should perform a certain task. They post their program on internet. If others find that it does indeed perform that task, the news will spread like wildfire.

Vongehr prefers Bell's original inequality, and I prefer CHSH, so I will here present an (unauthorised) "CHSH style" modification of his QRC.

Suppose someone has invented a local hidden variables theory. He can use it to simulate $N = 800$ runs of a CHSH experiment. Typically he will simulate the source, the photons, the detectors, all in one program. Let us suppose that his computer code produces reproducible results, which means that the code or the application is reasonably portable, and will give *identical* output when run on

another computer with the same inputs. In particular, if it makes use of a pseudo random number generator (RNG), it must have the usual "save" and "restore" facilities for the seed of the RNG. Let's suppose that the program calls the RNG the same number of times for each run, and that the program does not make use in any way of memory of past measurement settings. The program must accept any legal stream of pairs of binary measurement settings of any length $N$.

In particular then, the program can be run with $N = 1$ and all four possible pairs of measurement settings, and the same initial random seed, and it will thereby generate successively four pairs $(A, B)$, $(A', B)$, $(A, B')$, $(A', B')$. If the programmer neither cheated nor made any errors, in other words, if the program is a correct implementation of a genuine LHV model, then both values of $A$ are the same, and so are both values of $A'$, both values of $B$, and both values of $B'$. We now have the first row of the $N \times 4$ spreadsheet of Section 2 of this paper.

The random seed at the *end* of the previous phase is now used as the initial seed for another phase, the second run, generating a second row of the spreadsheet. This is where the prohibition of exploiting memory comes into force. The second row of counterfactual outcomes has to be completed without knowing which particular setting pair Alice and Bob will actually pick for the first row.

Notice that the LHV model is allowed to use *time*, since the saved random seeds could also include the current run number and the initial random seed value, too: in other words, when doing the calculations for the $n$th run, the LHV model has access to everything it did in the previous $n - 1$ runs.

My claim is that a correct implementation of a bona-fide LHV model which does not exploit the memory loophole can be used to fill in the $N \times 4$ spreadsheet of Section 2. When we now generate random settings and calculate the correlations, we get the same results as if they had been submitted in a single stream to the same program, run once with the same initial seed.

My new CHSH-style QRC to any local realist out there who is interested, is that they program their LHV model, modified so that it simply accepts a random seed and value of $N$, and outputs an $N \times 4$ spreadsheet. They should post it on internet and draw attention to it on any of the many internet fora devoted to discussions of quantum foundations. Anyone interested runs the program, generates $N \times 2$ settings, and calculates CHSH. If the program reproducibly, repeatedly (significantly more than half the time, cf. Conjecture 1 of Section 2), violates CHSH, then the creator has created a classical physical system which systematically violates the CHSH inequalities, thereby disproving Bell's theorem. No establishment conspiracy can stop this news from spreading round the world, everyone can replicate the experiment. The creator will get the Nobel prize and there will be incredible repercussions throughout physics.

Some local realists will however insist on using memory. They cannot rewrite their programs to create one $N \times 4$ spreadsheet. Instead, $N$ rounds of communication are needed between themselves and some trusted neutral vetting agency. To borrow an idea I learnt from Han Geurdes, we should think of some kind of rating agency such as those for banks, an independent agency which carries out "stress tests", on demand, but at a reasonable price, to anyone who is interested and will pay. The procedure is almost as before: it ensures yet again that the LHV model is legitimate, or more precisely, is legitimate in its implemented form. The agency generates a first run of settings (i.e., one setting pair), but keeps it secret for the

moment. The LHV theorist supplies a first run-set of values of $(A, A', B, B')$. The agency reveals the first setting pair, the LHV theorist generates a second run set $(A, A', B, B')$. This is repeated $N = 800$ times. The whole procedure can be repeated any number of times, the results are published on internet, everyone can judge for themselves.

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## APPENDIX: PROOF OF THEOREM 1

The proof of (3) will use the following two Hoeffding inequalities:

**Fact 3 (Binomial)** *Suppose $X \sim Bin(n, p)$ and $t > 0$. Then*

$$\Pr(X/n \;\geq\; p + t) \;\leq\; \exp(-2nt^2).$$

**Fact 4 (Hypergeometric)** *Suppose $X$ is the number of red balls found in a sample without replacement of size $n$ from a vase containing $pM$ red balls and $(1-p)M$ blue balls and $t > 0$. Then*

$$\Pr(X/n \;\geq\; p + t) \;\leq\; \exp(-2nt^2).$$

**Proof of Theorem 1** In each row of our $N \times 4$ table of numbers $\pm 1$, the product $AB$ equals $\pm 1$. For each row, with probability $1/4$, the product is either observed or not observed. Let $N_{AB}^{\text{obs}}$ denote the number of rows in which both $A$ and $B$ are observed. Then $N_{AB}^{\text{obs}} \sim \text{Bin}(N, 1/4)$, and hence by Fact 3, for any $\delta > 0$,

$$\Pr\Big(\frac{N_{AB}^{\text{obs}}}{N} \;\leq\; \tfrac{1}{4} - \delta\Big) \;\leq\; \exp(-2N\delta^2).$$

Let $N_{AB}^{+}$ denote the total number of rows (i.e., out of $N$) for which $AB = +1$, define $N_{AB}^{-}$ similarly. Let $N_{AB}^{\text{obs},+}$ denote the number of rows such that $AB = +1$ among those selected for observation of $A$ and $B$. Conditional on $N_{AB}^{\text{obs}} = n$, $N_{AB}^{\text{obs},+}$ is distributed as the number of red balls in a sample without replacement of size $n$ from a vase containing $N$ balls of which $N_{AB}^{+}$ are red and $N_{AB}^{-}$ are blue. Therefore by Fact 4, conditional on $N_{AB}^{\text{obs}} = n$, for any $\epsilon > 0$,

$$\Pr\Big(\frac{N_{AB}^{\text{obs},+}}{N_{AB}^{\text{obs}}} \geq \frac{N_{AB}^{+}}{N} + \epsilon\Big) \leq \exp(-2n\epsilon^2).$$

Recall that $\langle AB \rangle$ stands for the average of the product $AB$ over the whole table; this can be rewritten as

$$\langle AB \rangle \;=\; \frac{N_{AB}^{+} - N_{AB}^{-}}{N} \;=\; 2\frac{N_{AB}^{+}}{N} - 1.$$

Similarly, $\langle AB\rangle_{\rm obs}$ denotes the average of the product $AB$ just over the rows of the table for which both $A$ and $B$ are observed; this can be rewritten as

$$\langle AB\rangle_{\rm obs} \;=\; \frac{N_{AB}^{\rm obs,+} - N_{AB}^{\rm obs,-}}{N_{AB}^{\rm obs}} \;=\; 2\frac{N_{AB}^{\rm obs,+}}{N_{AB}^{\rm obs}} - 1.$$

For given $\delta > 0$ and $\epsilon > 0$, all of $N_{AB}^{\rm obs}$, $N_{AB'}^{\rm obs}$, $N_{A'B}^{\rm obs}$ and $N_{A'B'}^{\rm obs}$ are at least $(\frac{1}{4}-\delta)N$ with probability at least $1-4\exp(-2N\delta^2)$. On the event where this happens, the conditional probability that $\langle AB\rangle_{\rm obs}$ exceeds $\langle AB\rangle + 2\epsilon$ is bounded by

$$\exp(-2N_{AB}^{\rm obs}\epsilon^2) \le \exp(-2(\tfrac{1}{4}-\delta)N\epsilon^2).$$

The same is true for the other three averages (for the last one we first exchange the roles of $+$ and $-$ to get a bound on $\langle -A'B'\rangle_{\rm obs}$). Combining everything, we get that

$$\langle AB\rangle_{\rm obs} + \langle AB'\rangle_{\rm obs} + \langle A'B\rangle_{\rm obs} - \langle A'B'\rangle_{\rm obs} \;\le\; 2 + 8\epsilon,$$

except possibly on an event of probability at most

$$p \;=\; 4\exp(-2N\delta^2) + 4\exp(-2(\tfrac{1}{4}-\delta)N\epsilon^2).$$

We want to bound $p$ by $8\exp(-N(\eta/16)^2)$ where $\eta = 8\epsilon$, making $(\eta/16)^2 = (\epsilon/2)^2$. Choosing $8\delta^2 = \epsilon^2$, we find $2\delta^2 = (\epsilon/2)^2 = (\eta/16)^2$. If $8(\frac{1}{4}-\delta) \ge 1$, then $p \le 8\exp(-2N\delta^2)$ and we are home. The restriction on $\delta$ translates to $\delta \le \frac{1}{8}$ and thence to $\eta \le 2\sqrt{2}$. But for $\eta > 2$, (3) is trivially true anyway, so the restriction on $\eta$ can be forgotten. □

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