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A NON-STANDARD EMPIRICAL LIKELIHOOD FOR TIME SERIES

By Daniel J. Nordman*, Helle Bunzel and Soumendra N. Lahiri†

Iowa State University and North Carolina State University

Standard blockwise empirical likelihood (BEL) for stationary, weakly dependent time series requires specifying a fixed block length as a tuning parameter for setting confidence regions. This aspect can be difficult and impacts coverage accuracy. As an alternative, this paper proposes a new version of BEL based on a simple, though non-standard, data-blocking rule which uses a data block of every possible length. Consequently, the method does not involve the usual block selection issues and is also anticipated to exhibit better coverage performance. Its non-standard blocking scheme, however, induces non-standard asymptotics and requires a significantly different development compared to standard BEL. We establish the large-sample distribution of log-ratio statistics from the new BEL method for calibrating confidence regions for mean or smooth function parameters of time series. This limit law is not the usual chi-square one, but is distribution-free and can be reproduced through straightforward simulations. Numerical studies indicate that the proposed method generally exhibits better coverage accuracy than standard BEL.

1. Introduction. For independent, identically distributed data (iid), Owen [24, 25] introduced empirical likelihood (EL) as a general methodology for re-creating likelihood-type inference without a joint distribution for the data, as typically specified in parametric likelihood. However, the iid formulation of EL fails for dependent data by ignoring the underlying dependence structure. As a remedy, Kitamura [15] proposed so-called blockwise empirical likelihood (BEL) methodology for stationary, weakly dependent processes, which has been shown to provide valid inference in various scenarios with time series (cf. [3, 4, 7, 18, 22, 34]). Similarly to the iid EL version, BEL creates an EL log-ratio statistic having a chi-square limit for inference, but the BEL construction crucially involves blocks of consecutive observations in time, rather than individual observations. This data-blocking

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serves to capture the underlying time dependence and related concepts have also proven important in defining resampling methodologies for dependent data, such as block bootstrap [9, 16, 19] and time subsampling methods [6, 27, 28]. However, the coverage accuracy of BEL can depend crucially on the block length selection, which is a fixed value $1 \leq b \leq n$ for a given sample size n, and appropriate choices can vary with the underlying process (a point briefly illustrated at the end of this section).

To advance the BEL methodology in a direction away from block selection with a goal of improved coverage accuracy, we propose an alternative version of BEL for stationary, weakly dependent time series, called an expansive block empirical likelihood (EBEL). The EBEL method involves a non-standard, but simple, data-blocking rule where a data block of every possible length is used. Consequently, the method does not involve a block length choice in the standard sense. We investigate EBEL in the prototypical problem of inference about the process mean or a smooth function of means. For setting confidence regions for such parameters, we establish the limiting distribution of log-likelihood ratio statistics from the EBEL method. Because of the non-standard blocking scheme, the justification of this limit distribution requires a new and substantially different treatment compared to that of standard BEL (which closely resembles that of EL for iid data in its large-sample development, cf. [25, 30]). In fact, unlike with standard BEL or EL for iid data, the limiting distribution involved is non-standard and *not* chi-square. However, the EBEL limit law is distribution-free, corresponding to a special integral of standard Brownian motion on $[0, 1]$, and so can be easily approximated through simulation to obtain appropriate quantiles for calibrating confidence regions. In addition, we anticipate that the EBEL method may have generally better coverage accuracy than standard BEL methods, though formally establishing and comparing convergence rates is beyond the scope of this manuscript (and, in fact, optimal rates and block sizes for even standard BEL remain to be determined). Simulation studies, though, suggest that interval estimates from the EBEL method can perform much better than the standard BEL approach, especially when the later employs a poor block choice, and be less sensitive to the dependence strength of the underlying process.

The rest of manuscript is organized as follows. We end this section by briefly recalling the standard BEL construction with overlapping blocks and its distributional features. In Section 2, we separately describe the EBEL method for inference on process means and smooth function model parameters, and establish the main distributional results in both cases. These results require introducing a new type of limit law based on Brownian motion, which is also given in Section 2. Additionally, Section 2.1 describes how the usual EL theory developed by Owen [24, 25], and often underlying many EL arguments including the time series extensions of BEL [15], fails here and requires new technical developments; consequently, the theory provided may be useful for future developments of EL (with an example given in Section 2.4). Section 3 provides a numerical study of the coverage accuracy of the EBEL method and comparisons to standard BEL. Section 4 offers some concluding remarks and heuristic arguments on the expected performance of EBEL. Proofs of the main results appear in Section 5 and in supplementary materials [23], where the latter also presents some additional simulation summaries.

To motivate what follows, we briefly recall the BEL construction, considering, for concreteness, inference about the mean $\mathrm{E}X_t = \mu \in \mathbb{R}^d$ of a vector-valued stationary stretch $X_1, \ldots, X_n$. Upon choosing an integer block length $1 \leq b \leq n$, a collection of maximally overlapping (OL) blocks of length b is given by $\{(X_i, \ldots, X_{i+b-1}) : i = 1, \ldots, N_b \equiv n - b + 1\}$. For a given $\mu \in \mathbb{R}^d$ value, each block in the collection provides a centered block sum $B_{i,\mu} \equiv \sum_{j=i}^{i+b-1}(X_j - \mu)$ for defining a BEL function

$$L_{\mathrm{BEL},n}(\mu) = \sup\left\{\prod_{i=1}^{N_b} p_i : p_i \geq 0, \sum_{i=1}^{N_b} p_i = 1, \sum_{i=1}^{N_b} p_i B_{i,\mu} = 0_d\right\} \tag{1}$$

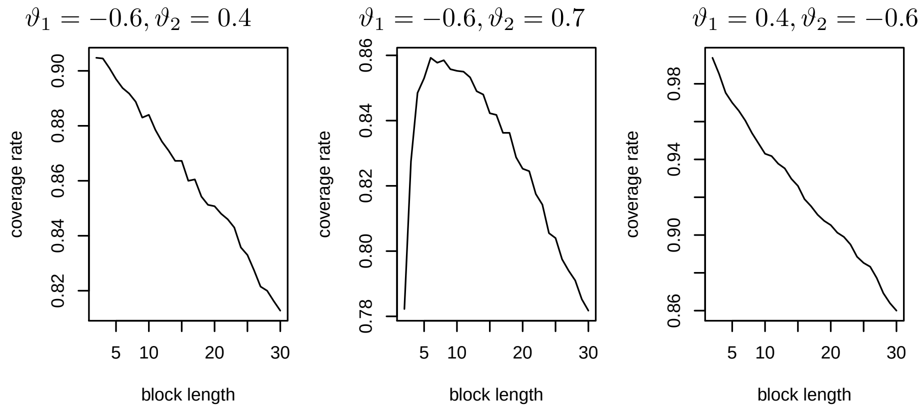
and corresponding BEL ratio $R_{\mathrm{BEL},n}(\mu) = L_n(\mu)/N_b^{-N_b}$, where above $0_d = (0, \ldots, 0)' \in \mathbb{R}^d$. The function $L_{\mathrm{BEL},n}(\mu)$ assesses the plausibility of a value μ by maximizing a multinomial likelihood from probabilities $\{p_i\}_{i=1}^{N_b}$ assigned to the centered block sums $B_{i,\mu}$ under a zero-expectation constraint. Without the linear mean constraint in (1), the multinomial product is maximized when each $p_i = 1/N_b$ (i.e., the empirical distribution on blocks), defining the ratio $R_{\mathrm{BEL},n}(\mu)$. Under certain mixing and moment conditions entailing weak dependence, and if the block b grows with the sample size n but at a smaller rate (i.e., $b^{-1} + b^2/n \to 0$ as $n \to \infty$), the log-EL ratio of the standard BEL has chi-square limit

$$-\frac{2}{b} \log R_{\mathrm{BEL},n}(\mu_0) \xrightarrow{d} \chi^2_d, \tag{2}$$

at the true mean parameter $\mu_0 \in \mathbb{R}^d$ (cf. Kitamura [15]). Here b^{-1} represents an adjustment in (2) to account for OL blocks and, for iid data, a block length $b = 1$ above produces the EL distributional result of Owen [24, 25]. To illustrate the connection between block selection and performance, Figure 1 shows the coverage rate of nominal 90% BEL confidence intervals $\{\mu \in$

$\mathbb{R} : -2/b \log R_{\mathrm{BEL},n}(\mu_0) \leq \chi^2_{1,0.9}\}$, as a function of the block size b, for estimating the mean of three different MA(2) processes based on samples of size $n = 100$. One observes that the coverage accuracy of BEL varies with the block length and and that the best block size can depend on the underlying process. The EBEL method described next is a type of modification of the OL BEL version, without a particular fixed block length selection b.

FIG 1. *Plot of coverage rates for 90% BEL intervals for the process mean* $\mathrm{E}X_t = \mu$ *over various blocks* $b = 2, \ldots, 30$*, based on samples of size* $n = 100$ *from three MA*(2) *processes* $X_t = Z_t + \vartheta_1 Z_{t-2} + \vartheta_2$ *with iid standard normal innovations* $\{Z_t\}$ *(from* $4,000$ *simulations).*



2. Expansive block empirical likelihood.

2.1. *Mean inference.* Suppose $X_1, \ldots, X_n$ represents a sample from a strictly stationary process $\{X_t : t \in \mathbb{Z}\}$ taking values in $\mathbb{R}^d$ and consider problem about inference on the process mean $\mathrm{E}X_t = \mu \in \mathbb{R}^d$. While the BEL uses data blocks of a fixed length b for a given sample size n, the EBEL uses overlapping data blocks $\{(X_1), (X_1, X_2), \ldots, (X_1, \ldots, X_n)\}$ that vary in length up to the longest block consisting of the entire time series. Hence, this block collection, which constitutes a type of forward "scan" in the block subsampling language of McElroy and Politis [21], contains a data block of every possible length b for a given sample size n. This block sequence also appears in fixed-b asymptotic schemes [13] and related self-normalization approaches (cf. Shao [31], sec. 2); see Section 4 here. In this sense, these blocks are interesting and novel to consider in a BEL framework. Other block schemes may be possible and used for practical gain (e.g., improved power), where the theoretical results of this paper could also directly apply. We leave this largely for future research but we shall give one example of a modified, though related, blocking scheme in Section 2.4 while focusing the exposition on the block collection above (i.e., the alternative blocking incorporates a backward scan $\{(X_n), (X_n, X_{n-1}), \ldots, (X_n, \ldots, X_1)\}$ with similar theoretical development).

Let $w : [0,1] \to [0,\infty)$ denote a nonnegative weighting function. To assess the likelihood of a given value of μ, we create centered block sums $T_{i,\mu} = w(i/n)\sum_{j=1}^{i}(X_j - \mu)$, $i = 1, \ldots, n$, and define a EBEL function

$$L_n(\mu) = \sup\left\{\prod_{i=1}^{n} p_i : p_i \geq 0, \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i T_{i,\mu} = 0_d\right\} \tag{3}$$

and ratio $R_n(\mu) = n^{-n}L_n(\mu)$. After defining the block sums, the computation of $L_n(\mu)$ is analogous to the BEL version and essentially the same as that described by Owen [24, 25] for iid data. Namely, when the zero 0_d vector lies in the interior convex hull of $\{T_{i,\mu} : i = 1, \ldots, n\}$, then $L_n(\mu)$ is the uniquely achieved maximum at probabilities $p_i = 1/[n(1 + \lambda_{n,\mu}' T_{i,\mu})] > 0$, $i = 1, \ldots, n$, with a Lagrange multiplier $\lambda_{n,\mu} \in \mathbb{R}^d$ satisfying

$$\sum_{i=1}^{n} \frac{T_{i,\mu}}{n(1 + \lambda_{n,\mu}' T_{i,\mu})} = 0_d; \tag{4}$$

see [25] for these and other computational details. Regarding the weight function above in the EBEL formulation, more details are provided below and in Section 2.2.

The next section establishes the limiting distribution of the log-EL ratio from the EBEL method for setting confidence regions for the process mean μ parameter. However, it is helpful to initially describe how the subsequent developments of EL differ from previous ones with iid or weakly dependent data (cf. [15] for BEL). The standard arguments for developing EL results, due to Owen [25] (p. 101), typically begin from algebraically re-writing (4) to expand the Lagrange multiplier. If we consider the real-valued case $d = 1$ for simplicity, this becomes

$$\lambda_{n,\mu} = \frac{\sum_{i=1}^{n} T_{i,\mu}}{\sum_{i=1}^{n} T_{i,\mu}^2} + \frac{\lambda_{n,\mu}^2}{\sum_{i=1}^{n} T_{i,\mu}^2} \sum_{i=1}^{n} \frac{T_{i,\mu}^3}{1 + \lambda_{n,\mu}' T_{i,\mu}}.$$

In the usual independence or weak dependence cases of EL (e.g., where $B_{i,\mu}$ from (1) replaces $T_{i,\mu}$ in the Lagrange multiplier above), the first right-side term dominates the second, which gives a substantive form for $\lambda_{n,\mu}$ as a ratio of sample means and consequently drives the large sample results (i.e., producing chi-square limits). However, in the EBEL case here, both terms on the right side above have the *same* order, implying that the standard approach to developing EL results breaks down under the EBEL blocking scheme. The proofs here use a different EL argument than the standard one mentioned above [15, 25], involving no asymptotic expansions of the Lagrange multiplier or Taylor expansions of the EL ratio based on these.

The large sample results for the EBEL method require two mild assumptions stated below. Let $\mathcal{C}_d[0,1]$ denote the metric space of all $\mathbb{R}^d$-valued continuous functions on $[0,1]$ with the supremum metric $\rho(g_1, g_2) \equiv \sup_{0\le t\le 1}\|g_1(t) - g_2(t)\|$, and let $B(t) = (B_1(t), \ldots, B_d(t))'$, $0 \le t \le 1$, denote a $\mathcal{C}_d[0,1]$-valued random variable where $B_1(t), \ldots, B_d(t)$ are iid copies of standard Brownian motion on $[0,1]$.

Assumptions

(A.1) The weight function $w : [0,1] \to [0,\infty)$ is continuous on $[0,1]$ and is strictly positive on an interval $(0,c)$ for some $c \in (0,1]$.

(A.2) Let $\mathrm{E}X_t = \mu_0 \in \mathbb{R}^d$ denote the true mean of the stationary process $\{X_t\}$ and suppose $d \times d$ matrix $\Sigma = \sum_{j=-\infty}^{\infty} \mathrm{Cov}(X_0, X_j)$ is positive definite. For the empirical process $S_n(t)$ on $t \in [0,1]$ defined by linear interpolation of $\{S_n(i/n) = \sum_{j=1}^{i}(X_j - \mu_0) : i = 0, \ldots, n\}$ with $S_n(0) = 0$, it holds that $S_n(\cdot)/n^{1/2} \xrightarrow{d} \Sigma^{1/2}B(\cdot)$ in $\mathcal{C}_d[0,1]$.

Assumption (A.1) is used to guarantee that, in probability, the EBEL ratio $R_n(\mu_0)$ positively exists at the true mean, which holds for uniformly weighted blocks $w(t) = 1$, $t \in [0,1]$, for example. Assumption (A.2) is a functional central limit theorem for weakly dependent data, which holds under appropriate mixing and moment conditions on $\{X_t\}$ [12].

2.2. *Main distributional results.* To state the limit law for the log-EBEL ratio (3), we require a result regarding a vector $B(t) = (B_1(t), \ldots, B_d(t))'$, $0 \le t \le 1$, of iid copies $B_1(t), \ldots, B_d(t)$ of standard Brownian motion on $[0,1]$. Indeed, the limit distribution of $-2\log R_n(\mu_0)$ is a non-standard functional of the vector of Brownian motion $B(\cdot)$. Theorem 1 identifies key elements of the limit law and describes some of its basic structural properties.

Theorem 1 *Suppose that $B(t) = (B_1(t), \ldots, B_d(t))'$, $0 \le t \le 1$, is defined on a probability space and let $f(t) = w(t)B(t)$, $0 \le t \le 1$, where $w(\cdot)$ satisfies Assumption (A.1). Then, with probability 1 (w.p.1), there exists an $\mathbb{R}^d$-valued random vector Y_d satisfying the following:*

(i) Y_d is the unique minimizer of

$$g_d(a) \equiv -\int_0^1 \log(1 + a'f(t))dt \quad \text{for } a \in \overline{K}_d,$$

where $\overline{K}_d \equiv \{y \in \mathbb{R}^d : \min_{0\le t\le 1}(1+y'f(t)) \ge 0\}$ *is the closure of* $K_d \equiv \{y \in \mathbb{R}^d : \min_{0\le t\le 1}(1+y'f(t)) > 0\}$*; the latter set is open, bounded and convex in* $\mathbb{R}^d$ *(w.p.1). On* K_d*,* g_d *is also real-valued, strictly convex, and infinitely differentiable (w.p.1).*

(ii) $-\infty < g_d(Y_d) < 0, \qquad Y_d' \int_0^1 f(t)dt > 0, \qquad 0 \le \int_0^1 \frac{Y_d' f(t)}{1+Y_d' f(t)} dt < \infty.$

(iii) If $Y_d \in K_d$*, then* Y_d *is the unique solution to* $\int_0^1 \frac{f(t)}{1+a'f(t)}dt = 0_d$ *for* $a \in K_d$*; and if* $\int_0^1 \frac{f(t)}{1+a'f(t)}dt = 0_d$ *has a solution* $a \in K_d$*, then this solution is uniquely* Y_d*.*

We use the subscript d in Theorem 1 to denote the dimension of either the random vector Y_d, the space K_d or the arguments of g_d. The function g_d is well-defined and convex on $\overline{K}_d$, though possibly $g_d(a) = +\infty$ for some $a \in \partial K_d = \{y \in \mathbb{R}^d : \min_{0\le t\le 1}(1+y'f(t)) = 0\}$ on the boundary of K_d; a minimizer of $g_d(\cdot)$ may also occur on $\partial K_d \cap \{y \in \mathbb{R}^d : g_d(y) \le 0\}$. Importantly, the probability law of $g_d(Y_1)$ is distribution-free and, because standard Brownian motion is fast and straightforward to simulate, the distribution of $g_d(Y_d)$ can be approximately numerically. Parts (ii) and (iii) provide properties for characterizing and identifying the minimizer Y_d. For example, considering the real-valued case $d = 1$, it holds that $K_1 = (m, M)$ where $m = -[\max_{0\le t\le 1} f(t)]^{-1} < 0 < M = -[\min_{0\le t\le 1} f(t)]^{-1}$ and the derivative $dg_1(a)/da$ is strictly increasing on K_1 by convexity. Because the derivative of g_1 at 0 is $-\int_0^1 f(x)dx$, parts (ii)-(iii) imply that if $-\int_0^1 f(x)dx < 0$ then either $Y_1 = m$ or Y_1 solves $dg_1(a)/da = 0$ on $m < a \le 0$; alternatively, if $-\int_0^1 f(x)dx > 0$, then $Y_1 = M$ or Y_1 solves $dg_1(a)/da = 0$ on $0 \le a < M$. Additionally, while the weight function $w(\cdot)$ influences the distribution of $g_d(Y_d)$, the scale of $w(\cdot)$ does not; defining f with w or cw, for a non-zero $c \in \mathbb{R}$, produces the same minimized value $g_d(Y_d)$.

We may now state the main result on the large-sample behavior of the EBEL log-ratio evaluated at the true process mean $\mathrm{E}X_t = \mu_0 \in \mathbb{R}^d$. Recall that, when $L_n(\mu_0) > 0$ in (3), the EBEL log-ratio admits a representation (4) at μ_0 in terms of the Lagrange multiplier $\lambda_{n,\mu_0} \in \mathbb{R}^d$.

Theorem 2 *Under Assumptions A.1-A.2, as* $n \to \infty$*,*

(i) $n^{1/2}\Sigma^{1/2}\lambda_{n,\mu_0} \xrightarrow{d} Y_d$*,*

(ii) $-\frac{1}{n}\log R_n(\mu_0) \xrightarrow{d} -g_d(Y_d)$*,*

for Y_d *and* $g_d(Y_d)$ *defined as in Theorem 1, and* $\Sigma = \sum_{j=-\infty}^{\infty} \mathrm{Cov}(X_0, X_j)$*.*

From Theorem 2(i), the Lagrange multiplier in the EBEL method has a limiting distribution which is not the typical normal one, as in the standard BEL case. This has a direct impact on the limit law of the EBEL ratio statistic. As Theorem 2(ii) shows, the negative logarithm of the EBEL ratio statistic, scaled by the inverse of the sample size, has a non-standard limit, given by the functional $-g_d(Y_d)$ of the vector of Brownian motion $B(\cdot)$ (cf. Theorem 1), that critically depends on the limit Y_d of the scaled Lagrange multiplier. The distribution of $-g_d(Y_d)$ is free of any population parameters so that quantiles of $-g_d(Y_d)$, which are easy to compute numerically, can be used to calibrate the EBEL confidence regions. As $-g_d(Y_d)$ is a strictly positive random variable, an approximate $100(1-\alpha)\%$ confidence region for μ_0 can be computed as

$$\{\mu \in \mathbb{R}^d : -n^{-1} \log R_n(\mu_0) \leq a_{d,1-\alpha}\},$$

where $a_{d,1-\alpha}$ is the lower $(1-\alpha)$ percentile of $-g_d(Y_d)$. When $d = 1$, the confidence region is an interval; for $d > 2$, the region is guaranteed to be connected without voids in $\mathbb{R}^d$. In contrast to the standard BEL (2), EBEL confidence regions do not require a similar fixed choice of block size.

We next provide additional results that give the limit distribution of the log-EBEL ratio statistic under a sequence of local alternatives and that also show the size of a EBEL confidence region will be no larger than $O_p(n^{-1/2})$ in diameter around the true mean $\mathrm{E}X_t = \mu_0$. Let

$$G_n \equiv \{\mu \in \mathbb{R}^d : R_n(\mu) \geq R_n(\mu_0) > 0\} \tag{5}$$

be the collection of mean parameter values which are at least as likely as μ_0, and therefore elements of a EBEL confidence region whenever the true mean is.

Corollary 1 *Suppose the assumptions of Theorem 2 hold. For $c \in \mathbb{R}^d$, define $f_c(t) = w(t)[B(t) + t\Sigma^{-1/2}c]$, $t \in [0,1]$, in terms of the vector of Brownian motion $B(t)$.*

(i) Then, as $n \to \infty$, $-n^{-1} \log R_n(\mu_0 \pm n^{-1/2}c) \xrightarrow{d}$

$$-\min\left\{-\int_0^1 \log(1 + a' f_c(t))dt : a \in \mathbb{R}^d, \min_{0\leq t\leq 1}(1 + a' f_c(t)) \geq 0\right\};$$

(ii) $\sup\{\|\mu - \mu_0\| : \mu \in G_n\} = O_p(n^{-1/2})$, for G_n in (5).

Hence, along a sequence of local alternatives ($n^{-1/2}$ away from the true mean), the log-EBEL ratio converges to a random variable, defined as the optimizer of an integral involving Brownian motion; this resembles Theorem 2 (involving $f(t) = w(t)B(t)$ there), but the integrated function $f_c(\cdot)$ has an addition term $w(t)t\Sigma^{-1/2}c$ under the alternative. With respect to Corollary 1(i), the involved limit distribution can be described with similar properties as in Theorem 1 upon replacing $f(t)$ with $f_c(t)$ there. In particular, the limiting distribution under the scaled alternatives depends on $\Sigma^{-1/2}c$, similarly to the normal theory case (e.g., with standard BEL) where $\Sigma^{-1/2}c$ determines the non-centrality parameter of a non-central chi-square distribution.

We note that Theorem 2 remains valid for potentially negative-valued weight functions $w(\cdot)$ as well. Simulations have shown that, with weight functions oscillating between positive and negative values on $[0, 1]$ (e.g., $w(t) = \sin(2\pi t)$), EBEL intervals for the process mean perform consistently well in terms of coverage accuracy. However, with weight functions $w(\cdot)$ that vary in sign, a result as in Corollary 1(ii) fails to hold. Hence, the weight functions $w(\cdot)$ considered are non-negative as stated in Assumption A.1.

Remark 1: The EBEL results in Theorem 2 also extend to certain parameters described by general estimating functions; for examples and similar EL results in the iid and time series cases, respectively, see [30] and [15]. Suppose $\theta \in \mathbb{R}^p$ represents a parameter of interest and $G(\cdot;\cdot) \in \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^p$, is a vector of p estimating functions such that $\mathrm{E}G(X_t;\theta_0) = 0_p$ holds at the true parameter value θ_0. The previous process mean case corresponds to $G(X_t;\mu) = X_t - \mu$ with $X_t, \mu \in \mathbb{R}^d$, $d = p$. A EBEL ratio statistic $R_n(\theta)$ for θ results by replacing $T_{u,i} = w(i/n)\sum_{j=1}^{i}(X_j - \mu)$ and 0_d with $T_{\theta,i} = w(i/n)\sum_{j=1}^{i} G(X_j;\theta)$ and 0_p in (3). Under the Theorem 2 conditions (substituting $G(X_j;\theta_0)$ for $X_j - \mu_0$ in Assumption A.2),

$$-\frac{1}{n}\log R_n(\theta_0) \xrightarrow{d} -g_p(Y_p)$$

holds as $n \to \infty$ with Y_p and $g_p(Y_p)$ as defined in Theorem 1, generalizing Theorem 2 and following by the same proof. The next section considers extensions of the EBEL approach to a different class of time series parameters.

2.3. *Smooth function model parameters.* We next consider extending the EBEL method for inference on a broad class of parameters under the so-called "smooth function model" (cf. [2, 10]). For independent and time series data, respectively, Hall and La Scala [11] and Kitamura [15] have considered EL inference for similar parameters; see also [25] (sec. 4).

If $\mathrm{E}X_t = \mu_0 \in \mathbb{R}^d$ again denotes the true mean of the process, the target parameter of interest is given by

$$\theta_0 = H(\mu_0) \in \mathbb{R}^p, \tag{6}$$

based on a smooth function $H(\mu) = (H_1(\mu), \ldots, H_p(\mu))'$ of the mean parameter μ, where $H_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, p$ and $p \leq d$. This framework allows a large variety of parameters to be considered such as sums, differences, products and ratios of means, which can be used, for example, to formulate parameters such as covariances and autocorrelations as functions of the m-dimensional moment structure (for a fixed m) of a time series. For a univariate stationary series $U_1, \ldots, U_n$, for instance, one can define a multivariate series X_t based on transformations of $(U_t, \ldots, U_{t+m-1})$ and estimate parameters for the process $\{U_t\}$ based on appropriate functions H of the mean of X_t. The correlations $\theta_0 = H(\mu_0)$ of $\{U_t\}$ at lags m and $m_1 < m$, for example, can be formulated in (6) by $H(x_1, x_2, x_3, x_4) = (x_3 - x_1^2, x_4 - x_1^2)'/[x_2 - x_1^2]$ and $\mathrm{E}X_t = \mu_0$ for $X_t = (U_t, U_t^2, U_t U_{t+m_1}, U_t U_{t+m})' \in \mathbb{R}^4$. [16] and [17] (Ch. 4) provide further examples of smooth function parameters.

For inference on the parameter $\theta = H(\mu)$, the EBEL ratio is defined as

$$R_n(\theta) = \sup \left\{ \prod_{i=1}^n p_i : p_i \geq 0, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i T_{i,\mu} = 0_d, \mu \in \mathbb{R}^d, H(\mu) = \theta \right\},$$

and its limit distribution is provided next.

Theorem 3 *In addition to the assumptions of Theorem 2, suppose H from (6) is continuously differentiable in a neighborhood of μ_0 and that ∇_{μ_0} has rank $p \leq d$, where $\nabla_\mu \equiv [\partial H_i(\mu)/\partial \mu_j]_{i=1,\ldots,p; j=1,\ldots,d}$ denotes the $p \times d$ matrix of first-order partial derivatives of H. Then, at the true parameter $\theta_0 = H(\mu_0)$, as $n \to \infty$*

$$-\frac{1}{n} \log R_n(\theta_0) \xrightarrow{d} -g_p(Y_p)$$

with Y_p and $g_p(Y_p)$ as defined in Theorem 1.

Theorem 3 shows that the log-EBEL ratio statistic for the parameter $\theta_0 = H(\mu_0) \in \mathbb{R}^p$ under the smooth function model continues to have a limit of the same form as that in the case of the EBEL for the mean parameter $\mu_0 \in \mathbb{R}^d$ itself. The main difference is that the functional $g_p(Y_p)$ is now defined in terms of a p-dimensional Brownian motion as in Theorem 1, but with $p \leq d$, where p denotes the dimension of the parameter θ_0 (see also Remark 1). It is interesting to note that, similarly to the traditional profile likelihood theory in a parametric set-up with iid observations, the limit law here does not depend on the function H as long as the matrix ∇_{μ_0} of the first order partial derivatives of H at $\mu = \mu_0$ has full rank p. Due to the non-standard blocking, the proof of this EBEL result again requires a different development compared to the one for standard BEL (cf. [15]) that mimics the iid EL case (cf. [11, 25]) involving expansion of Lagrange multipliers.

2.4. *Extensions to other data blocking.* As mentioned in Section 2.1, other versions of EBEL may be possible with other data blocking schemes, which likewise involve no fixed block selection in the usual BEL sense and have a related theoretical development. We give one example here. Recall the EBEL function (3) for the mean $L_n(\mu)$, $\mu \in \mathbb{R}^d$, involves centered block sums $T_{i,\mu} = w(i/n)\sum_{j=1}^{i}(X_j - \mu)$, $i = 1, \ldots, n$, based on blocks $\{(X_1), (X_1, X_2), \ldots, (X_1, \ldots, X_n)\}$. Reversed blocks for example, given by $\{(X_n), (X_n, X_{n-1}), \ldots, (X_n, \ldots, X_1)\}$, can also be additionally incorporated by defining further block sums $T_{n+i,\mu} = w(i/n)\sum_{j=1}^{i}(X_{n-j+1} - \mu)$, $i = 1, \ldots, n$ and a corresponding EBEL function

$$\tilde{L}_n(\mu) = \sup\left\{\prod_{i=1}^{2n} p_i : p_i \geq 0, \sum_{i=1}^{2n} p_i = 1, \sum_{i=1}^{2n} p_i T_{i,\mu} = 0_d\right\}$$

and ratio $\tilde{R}_n(\mu) = (2n)^{-2n}\tilde{L}_n(\mu)$. At the true mean $\mu_0 \in \mathbb{R}^d$, the log-ratio $-\log \tilde{R}_n(\mu_0) = \sum_{i=1}^{2n}\log[1+\tilde{\lambda}'_{n,\mu_0}T_{i,\mu_0}]$ can similarly be re-written in terms of a Lagrange multiplier $\tilde{\lambda}_{n,\mu_0} \in \mathbb{R}^d$ satisfying $0_d = \sum_{i=1}^{2n} T_{i,\mu_0}/[1 + \tilde{\lambda}'_{n,\mu_0}T_{i,\mu_0}]$. The EL distributional results of the previous subsections then extend in a natural manner, as described below for the mean inference case (cf. Theorem 2). For $0 \leq t \leq 1$, recall $f(t) = w(t)B(t)$ (cf. Theorem 1), for $B(t) = (B_1(t), \ldots, B_d(t))'$ denoting a vector of iid copies $B_1(t), \ldots, B_d(t)$ of standard Brownian motion on $[0, 1]$, and define additionally $\tilde{f}(t) = w(t)[B(1) - B(1 - t)]$.

Theorem 4 *Under Assumptions A.1-A.2, as $n \to \infty$,*

$$n^{1/2}\Sigma^{1/2}\tilde{\lambda}_{n,\mu_0} \xrightarrow{d} \tilde{Y}_d, \qquad -\frac{1}{n}\log \tilde{R}_n(\mu_0) \xrightarrow{d} -\tilde{g}_d(\tilde{Y}_d) \in (0, \infty),$$

for a $\mathbb{R}^d$-valued random vector $\tilde{Y}_d$ defined as the unique minimizer of

$$\tilde{g}_d(a) \equiv -\int_0^1 \log(1 + a'f(t))dt - \int_0^1 \log(1 + a'\tilde{f}(t))dt$$

for $a \in \overline{K}_d \equiv \{y \in \mathbb{R}^d : \min_{0\leq t\leq 1}(1 + y'f(t)) \geq 0, \min_{0\leq t\leq 1}(1 + y'\tilde{f}(t)) \geq 0\}$.

As in Theorem 2, the limit law of the log EBEL ratio above is similarly distribution-free and easily simulated from Brownian motion. The main difference between Theorems 2 and 4 is that the reversed data blocks in the

EL construction contribute a further integral component based on (reversed) Brownian motion in the limit. Straightforward analog versions of Theorem 1 (regarding $\tilde{Y}_d$ and $\tilde{g}_d(\cdot)$) as well as Corollary 1 and Theorem 3 (with respect to $\tilde{R}_n(\cdot)$) also hold; we state these in the supplementary materials for completeness.

3. Numerical studies. Here we summarize the results of a simulation study to investigate the performance of the EBEL method, considering the coverage accuracy of confidence intervals (CIs) for the process mean. We considered several real-valued ARMA processes, allowing a variety of dependence structures with ranges of weak and strong dependence, defined with respect to an underlying iid centered χ^2_1-distributed innovation series; these processes appear in Table 2 in the following. Other iid innovation types (e.g., normal, Bernoulli, Pareto) produced qualitatively similar results.

For each process, we generated 2000 samples of size $n = 250, 500, 1000$ for comparing the coverage accuracy of 90% CIs from various EL procedures. We applied the EBEL method with forward expansive data blocks, as in Section 2.1, as well as forward/backward data blocks, as in Section 2.4; we denote these methods as EBEL1/EBEL2, respectively, in summarizing results. In addition to a constant weight $w(t) = 1$, we implemented these methods with several other choices of weight functions $w(t)$ on $[0, 1]$, each down-weighting the initial (smaller) data blocks in the EBEL construction and differing in their shapes. The resulting coverages were very similar across non-constant weight functions and we provide results for two weight choices: linear $w(t) = t$ and cosine-bell $w(t) = [1-\cos(2\pi t)]/2$. Additionally, for each weight function $w(t)$, the limiting distribution of the EBEL ratio was approximated by 50000 simulations to determine its 90th percentile for calibrating intervals, as listed in Table 1 with Monte Carlo error bounds.

TABLE 1

Approximated 90th percentiles of the limit law of the log-EBEL ratio ($-g_1(Y_1)$ under Theorem 2 for EBEL1 and $-\tilde{g}_1(\tilde{Y}_1)$ under Theorem 4 for EBEL2) for weight functions $w(t)$. Approximation $\pm$ parenthetical quantity gives a 95% CI for true percentile.

$w(t)$, $t \in [0,1]$	$-g_1(Y_1)$		$-\tilde{g}_1(\tilde{Y}_1)$	
$w(t) = 1$	2.51	(0.03)	2.50	(0.03)
$w(t) = t$	5.64	(0.09)	4.37	(0.06)
$w(t) = (1-\cos(2\pi t))/2$	7.00	(0.15)	3.42	(0.09)

For comparison, we also include coverage results for the standard BEL method with OL blocks (denoted as BEL). Kitamura [15] (p. 2093) considered a block order $n^{1/3}$ for BEL as the method involves a block-based variance estimator in its asymptotic studentization mechanics (see Section 4), which is asymptotically equivalent to the Bartlett kernel spectral density estimator at zero having $n^{1/3}$ at its optimal block/lag order (cf. [26]). Based this correspondence, we considered two data-driven block selection rules from the spectral kernel literature, which estimate the coefficient $\hat{C}$ in the theoretical optimal block length expression $Cn^{1/3}$ known from spectral estimation. One block estimation approach (denoted FTK) is based on flat-top kernels and results in block estimates for BEL due to a procedure in Politis and White [29] (p. 60); we used a flat-top kernel bandwidth $n^{1/5}$ for generally consistent estimation as described by [29]. The second block estimation approach (denoted AAR) is due to Andrews [1] (p. 834-835), producing block estimates for BEL based on bandwidth estimates for the Bartlett spectral kernel assuming an approximating AR(1) process.

Table 2 lists the realized coverage accuracy of 90% EL CIs for the mean. From the table, the linear weight function $w(t) = t$ generally produced slightly more accurate coverages for both EBEL1/EBEL2 methods than the constant weight $w(t) = 1$; additionally and interestingly, despite their shape differences, the coverage rates for both the linear and cosine-bell weight functions closely matched (to the extent that we defer the cosine-bell results to the supplementary materials [23]). For all sample sizes and processes in Table 2, the EBEL2 method with linear weight typically and consistently emerged as having the most accurate coverage properties, often exhibiting less sensitivity to the underlying dependence while most closely achieving the nominal coverage level. Additionally, linear weight-based EBEL1 generally performed similarly to, or somewhat better than, the *best* BEL method based on a data-driven block selection from among the FTK/AAR block rules and, at times, much better than the worst performer among the BEL methods with estimated blocks. Note as well that, while that the two block selection rules for BEL can produce similar coverages, their relative effectiveness often depends crucially on the underlying process, with no resulting clear best block selection for BEL. In the case of the strong positive AR(1) dependence model in Table 2, the AAR block selection for BEL performed well (i.e., better than EBEL1 or BEL/FTK approaches), but similar advantages in coverage accuracy did not necessarily carry over to other processes. In particular, for a process not approximated well by an AR(1) model, the BEL coverage rates from AAR block estimates may exhibit extreme over- or under-coverage under negative or positive dependence, respectively, and FTK block selections for BEL may prove better.

Because of the blocking scheme in EBEL method and some of the method's other connections to fixed-b asymptotics (see Section 4), one might anticipate that there exist trade-offs in coverage accuracy (i.e., good size control

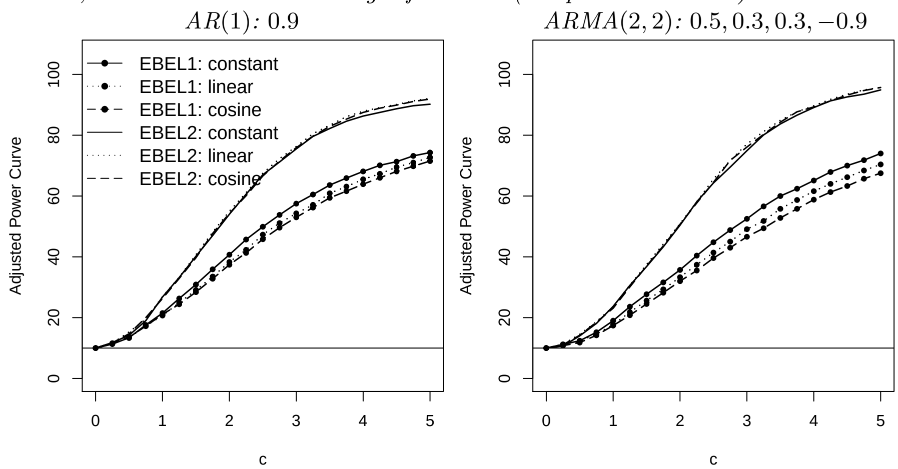
TABLE 2

Coverage percentages of 90% intervals for the process mean over several ARMA processes (with listed AR/MA components) and sample sizes n. EBEL1/EBEL2 use constant $w(t) = 1$ or linear $w(t) = t$ weights; BEL uses FTK or AAR data-based block selections. (MA$(1)^*$ has a discrete component $X_t = \varepsilon_t + 0.5\mathbb{I}(\varepsilon_{t-1} < \chi^2_{1,0.8}) - 1.4$, iid $\varepsilon_t \sim \chi^2_1$.)

		EBEL1, $w(t)$		EBEL2, $w(t)$		BEL	
Process	n	1	t	1	t	FTK	AAR
MA(2)	250	90.6	91.1	91.4	91.4	93.7	98.3
$0.4, -0.6$	500	91.0	91.2	91.7	91.5	93.4	98.0
	1000	90.0	90.0	90.4	90.0	90.6	96.6
MA$(1)^*$	250	87.4	89.4	90.2	90.5	91.3	94.2
	500	89.4	90.8	90.4	90.2	90.9	92.7
	1000	89.6	89.8	90.8	90.2	91.3	92.9
MA(3)	250	87.4	88.5	90.4	90.8	93.6	92.7
$-1, -1, -1$	500	87.8	88.6	90.0	90.2	93.4	92.0
	1000	89.7	89.2	89.2	89.8	92.2	91.9
ARMA(1,2)	250	84.4	86.0	89.1	89.8	93.8	94.6
$0.9, -0.6, -0.3$	500	87.2	88.7	90.4	90.3	95.5	95.2
	1000	89.4	89.9	91.6	91.6	95.6	96.2
AR(1)	250	89.2	90.0	92.0	91.4	95.8	91.8
-0.7	500	89.4	90.6	90.9	90.8	95.2	91.0
	1000	90.4	90.2	90.4	90.8	92.4	92.0
AR(1)	250	67.0	70.5	79.0	80.0	61.1	76.4
0.9	500	73.4	77.0	82.4	83.4	66.0	81.4
	1000	77.4	80.1	86.2	87.2	74.6	85.6
ARMA(1,1)	250	79.5	81.8	86.3	86.2	81.0	80.2
$0.7, -0.5$	500	82.0	84.6	86.3	86.9	82.2	82.0
	1000	85.0	87.0	87.9	89.0	85.4	84.0
ARMA(2,2)	250	78.3	81.0	84.0	84.6	77.2	73.0
$0.3, 0.3, -0.3, -0.1$	500	81.5	83.6	86.2	87.2	81.0	74.4
	1000	84.4	85.4	88.4	88.7	84.7	75.3
ARMA(2,2)	250	81.2	83.9	85.5	86.2	79.4	81.8
$0.5, 0.3, 0.3, -0.9$	500	84.2	86.0	87.4	88.0	82.8	84.6
	1000	85.4	86.2	88.0	88.2	84.0	85.5
MA(2)	250	83.2	85.0	87.4	87.6	86.0	79.2
$0.1, 2$	500	84.6	86.0	87.5	88.6	86.8	81.2
	1000	86.2	87.2	89.2	90.2	87.5	80.4

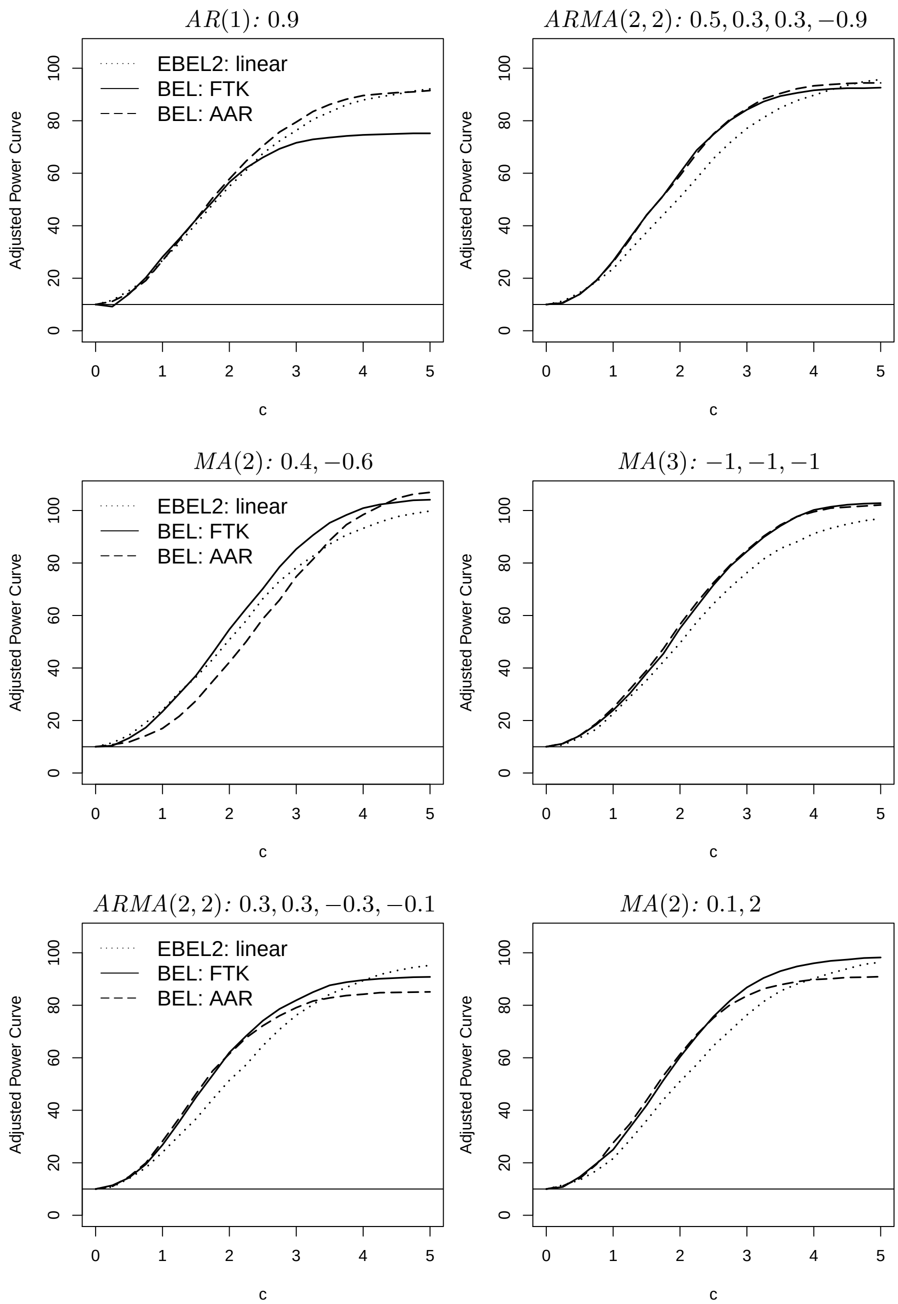
properties) at the expense of power in testing, a phenomenon also associated with fixed-b asymptotics (cf. [5, 32]). This does seem to be the case. To illustrate, for various sample sizes n and processes, we approximated power curves for EBEL/BEL tests at the 10% level (based on the 90th percentile of the associated null limit law) along a sequence of local alternatives $c_n = \mu_0 + n^{-1/2}\Sigma^{1/2}c$, $c = 0, 0.25, \ldots, 5$ where $\Sigma = \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_0, X_k)$;

FIG 2. *Adjusted power curves for tests at 10% level using EBEL1/EBEL2 methods with constant, linear and cosine-bell weight functions (sample size $n = 500$).*



for example, with EBEL1, the power curves correspond to the rejection probabilities $P(-n^{-1} \log R_n(c_n) > q_{0.90})$ where $q_{0.90}$ is a percentile from Table 1. The alternative sequence c_n was formulated to make power curves roughly comparable across processes with varying sample sizes, and so that the power curves can be plotted as a function of $c = 0, 0.25, \ldots, 5$; for instance, by Corollary 1(i), the asymptotic power curve of EBEL1 will be a function of c, as will the curves for BEL/EBEL2. Figures 2 and 3 display *size adjusted* power curves (APCs) for samples of size $n = 500$ based on 2000 simulations (curves are similar for $n = 250, 1000$ with additional results given in [23]). If a percentage $\hat{\alpha}_n$ denotes the actual size of the test for a given method and process (i.e., $\hat{\alpha}_n = 100\%-$ coverage percentage in Table 2), the APC is calibrated to have size 10% by vertically shifting the true power curve by $10\% - \hat{\alpha}_n$; this allows the shapes of power curves to be more easily compared across methods. Figure 2 shows APCs for EBEL1/EBEL2 methods, where EBEL2 curves exhibit more power apparently as a result of combining two data block sets (i.e., forward/backward) in the EBEL construction rather than one; additionally, while EBEL2 power curves are quite similar across different weights, EBEL1 curves exhibit slightly more power for the constant weight function. Figure 3 shows APCs in comparing the linear weight-based EBEL2 method with BEL methods based on FTK/AAR block estimates. The APCs for EBEL2 generally tend to be smaller than those of BEL, though the APC of a block estimate-based BEL may not always dominate the associated curve of EBEL2.

Fig 3. *Adjusted power curves for tests at 10% level using BEL with FTK/AAR block selections and linear weight-based EBEL2 (sample size $n = 500$).*



4. Conclusions. The proposed expansive block empirical likelihood (EBEL) is a type of variation on standard blockwise empirical likelihood (BEL) for time series which, instead of using a fixed block length b for a given sample size n, involves a non-standard blocking scheme to capture the dependence structure. While the coverage accuracy of standard BEL methods can depend intricately on the block choice b (where the best b can vary with the underlying process), the EBEL method does not involve this type of block selection. As mentioned in the Introduction, we also anticipate that the EBEL method will generally have better rates of coverage accuracy compared to BEL. The simulations of Section 3 lend support to this notion, along with suggesting that the EBEL can be less sensitive to the strength of the underlying time dependence. While asymptotic coverage rates for BEL methods remain to be determined, we may offer the following heuristic based on analogs drawn to so-called "fixed-b asymptotics" (cf. Keifer, Vogelsang and Bunzel [14]; Bunzel et al [5]; Kiefer and Vogelsang [13]), or related "self-normalization" (cf. Lobato [20]; Shao [31]) schemes.

In asymptotic expansions of log-likelihood statistics from standard BEL formulations, the data blocks serve to provide a type of block-based variance estimator (cf. [6, 27]) for purposes of normalizing scale and obtaining chi-square limits for log-BEL ratio statistics. Such variance estimators are consistent, requiring block sizes b which grow at a smaller rate than the sample size n (i.e., $b^{-1} + b/n \to \infty$ as $n \to \infty$), and are known to have equivalences to variance estimators formulated as lag window estimates involving kernel functions and bandwidths b with similar behavior to block lengths $b^{-1} + b/n \to \infty$ (cf. [16, 26]). That is, standard BEL intervals have parallels with normal theory intervals based on normalization with consistent lag window estimates. However, considering hypothesis testing with sample means for example, there is some numerical and theoretical evidence (cf. [5, 32]) that normalizing scale with inconsistent lag window estimates having fixed bandwidth ratios (e.g., $b/n = C$ for some $C \in (0, 1]$) results in better size and lower power compared to normalization with consistent ones, though the former case requires calibrating intervals with non-normal limit laws. Shao [31] (sec 2.1) provides a nice summary of these points as well as the form of some of these distribution-free limit laws, which typically involve ratios of random variables defined by Brownian motion (cf. [13]). While the EBEL method is not immediately analogous to normalizing with inconsistent variance estimators (as mentioned in Section 2.1, the usual EL expansions do not hold for EBEL), there are parallels in that the EBEL method does not use block lengths satisfying standard bandwidth conditions (cf. Sec. 2.1), its blocking scheme itself appears in self-normalization literature (cf. Shao [31], sec. 2), and confidence region calibration involves non-normal limits based on Brownian motion. This heuristic in the mean case suggests that better coverage rates (and lower power) associated with fixed-b asymptotics over standard normal theory asymptotics may be anticipated to carry over to comparisons of EBEL to standard BEL formulations.

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SUPPLEMENTARY MATERIAL

Additional Proofs and Results for a Non-standard Empirical Likelihood for Time Series
(). A supplement [23] provides proofs of the remaining main results omitted here, namely Corollary 1 (properties of confidence regions), Theorem 3 (smooth function model results) and Theorem 4 (forward/backward block EL version); additional numerical summaries are included as well.

5. Proofs of main results. To establish Theorem 1, we first require a lemma regarding a standard Brownian motion. For concreteness, suppose $B(t) \equiv B(\omega, t) = (B_1(\omega, t), \ldots, B_d(\omega, t))'$, $\omega \in \Omega$, $t \in [0, 1]$ is a random $\mathcal{C}_d[0, 1]$-valued element defined on some probability space $(\Omega, \mathcal{F}, P)$, where $B_1, \ldots, B_d$ are again distributed as iid copies of standard Brownian motion on $[0, 1]$. In the following, we use the basic fact that each $B_i(\cdot)$ is continuous on $[0, 1]$ with probability 1 (w.p.1) along with the fact that increments of standard Brownian motion are independent (cf. [8]).

Lemma 1 *With probability 1, it holds that*

- *(i)* $\min_{0 \le t < \epsilon} a' B(t) < 0 < \max_{0 \le t < \epsilon} a' B(t)$ *for all* $\epsilon > 0$ *and* $a \in \mathbb{R}^d$, $\|a\| = 1$.
- *(ii)* 0_d *is in the interior of the convex hull of* $B(t)$, $0 \le t \le 1$.
- *(iii) There exists a positive random variable* M *such that, for all* $a \in \mathbb{R}^d$, *it holds that* $\min_{0 \le t \le 1} a' B(t) \le -M\|a\|$ *and* $M\|a\| \le \max_{0 \le t \le 1} a' B(t)$.
- *(iv) If Assumption A.1 holds in addition, (i), (ii), (iii) above hold upon replacing* $B(t)$ *with* $f(t) = w(t) B(t)$, $t \in [0, 1]$.

Proof of Lemma 1. For real-valued Brownian motion, it is known that $\min_{0 \le t < \epsilon} B_i(t) < 0 < \max_{0 \le t < \epsilon} B_i(t)$ holds for all $\epsilon > 0$ w.p.1. (cf. [8], Lemma 55); we modify the proof of this. Let $\{t_n\} \subset (0, 1)$ be a decreasing sequence where $t_n \downarrow 0$ as $n \to \infty$. Pick and fix $c_1, \ldots, c_d \in \{-1, 1\}$ and define the event $A_n \equiv A_{n, c_1, \ldots, c_d} = \{\omega \in \Omega : c_i B_i(\omega, t_n) > 0, i = 1, \ldots, d\}$. Then,

$P(A_n) = 2^{-d}$ for all $n \geq 1$ by normality and independence. As the events $B_n = \bigcup_{k=n}^{\infty} A_k$, $n \geq 1$, are decreasing, it holds that

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) \geq \lim_{n\to\infty} P(A_n) = 2^{-d}.$$

Since $\bigcap_{n=1}^{\infty} B_n$ is a tail event generated by the independent random variables $B_i(t_1) - B_i(t_2), B_i(t_2) - B_i(t_3), \ldots$ for $i = 1, \ldots, d$, (i.e., increments of Brownian motion are independent and $B_i(0) = 0$), it follows from Kolmogorov's 0-1 law that $1 = P\left(\bigcap_{n=1}^{\infty} B_n\right) = P\left(A_n \text{ infinitely often (i.o.)}\right)$. Hence, $P\left(A_{n,c_1,\ldots,c_d} \text{ i.o. for any } c_i \in \{1,-1\}, i = 1, \ldots, d\right) = 1$ must hold, which implies part (i).

For part (ii), if 0_d is not in the interior convex hull of $B(t)$, $t \in [0,1]$, then the supporting/separating hyperplane theorem would imply that, for some $a \in \mathbb{R}^d$, $\|a\| = 1$, it holds that $a'B(t) \geq 0$ for all $t \in [0,1]$, which contradicts part (i).

To show part (iii), we use the events developed in part (i) and define $n_{c_1,\ldots,c_d} = \min\{n : A_{n,c_1,\ldots,c_d} \text{ holds}\}$. Define $M = \min\{|B_i(t_{n_{c_1,\ldots,c_d}})| : c_1, \ldots, c_d \in \{-1,1\}, i = 1, \ldots, d\} > 0$. For $a = (a_1, \ldots, a_d)' \in \mathbb{R}^d$, let $c_i^a = \max\{-\text{sign}(a_i), 1\}$, $i = 1, \ldots, d$. Then, $a'B(t_{n_{c_1^a,\ldots,c_d^a}}) = -\sum_{i=1}^d |a_i B_i(t_{n_{c_1^a,\ldots,c_d^a}})| \leq -M\|a\|$, and likewise $a'B(t_{n_{-c_1^a,\ldots,-c_d^a}}) = \sum_{i=1}^d |a_i B_i(t_{n_{-c_1^a,\ldots,-c_d^a}})| \geq M\|a\|$. This establishes (iii).

Part (iv) follows from the fact that $w(t) > 0$ for $t \in (0,c)$ and we may make take the positive sequence $\{t_n\} \subset (0,c)$ in the proof of part (i). Then, the results for $B(t)$ imply the same hold upon substituting $f(t) = w(t)B(t)$, $t \in [0,1]$.

Proof of Theorem 1. The set $K_d = \{a \in \mathbb{R}^d : \min_{0\leq t\leq 1}(1 + a'f(t)) > 0\}$ is open, bounded and convex (w.p.1), where boundedness follows from Lemma 1(iii,iv). Likewise, the closure $\overline{K}_d = \{a \in \mathbb{R}^d : \min_{0\leq t\leq 1}(1+a'f(t)) \geq 0\}$ is convex and bounded. Since $\min_{0\leq t\leq 1}(1+a'f(t))$ is a continuous function in $a \in \mathbb{R}^d$, one may apply the Dominated Convergence Theorem (DCT) (with the fact that $\min_{0\leq t\leq 1}(1 + a'f(t))$ is bounded away from 0 on closed balls inside K_d around a) to show that partial derivatives of $g_d(\cdot)$ at $a \in K_d$ (of all orders) exist, with first and second partial derivatives given by

$$\frac{\partial g_d(a)}{\partial a} = -\int_0^1 \frac{f(t)}{1 + a'f(t)} dt, \qquad \frac{\partial^2 g_d(a)}{\partial a \partial a'} = \int_0^1 \frac{f(t)f(t)'}{[1 + a'f(t)]^2} dt.$$

Because $\int_0^1 f(t)f(t)'dt$ is positive definite by Lemma 1(i,iv) and the continuity of f, the matrix $\partial^2 g_d(a)/\partial a\partial a'$ is also positive definitive for all $a \in K_d$, implying g_d is strictly convex on K_d. By Jensen's inequality, it also holds that g_d is convex on $\overline{K}_d$.

Note for $a \in \overline{K}_d$, $g_d(a) \geq -\int_0^1 \log(1 + \sup_{a\in\overline{K}_d} \|a\| \cdot \sup_{0\leq t\leq 1} \|f(t)\|) > -\infty$ holds, so that $I \equiv \inf_{a\in\overline{K}_d} g_d(a)$ exists. Additionally, $0_d \in K_d$ with $g_d(0_d) = 0$ and $\partial g_d(0_d)/\partial a = -\int_0^1 f(t)dt$, where the components of $\int_0^1 f(t)dt$ are all non-zero (w.p.1) by normality and independence; by the continuity of partial derivatives on the open set K_d, there then exists $\bar{a} \in K_d$ such that $\bar{a}' \int_0^1 f(t)dt > 0$ holds with the components of $-\int_0^1 f(t)$ and $\partial g_d(\bar{a})/\partial a$ having the same sign. By strict convexity, $g_d(0_d) - g_d(\bar{a}) > [\partial g_d(\bar{a})/\partial a]'(0_d - \bar{a}) > 0$ follows, implying $I < 0$ and $I = \inf_{a\in\tilde{K}_d} g_d(a)$ for the level set $\tilde{K}_d \equiv \{a \in \overline{K}_d : g_d(a) \leq 0\}$.

Then, there exists a sequence $a_n \in \tilde{K}_d$ such that $g_d(a_n) < I + n^{-1}$ for $n \geq 1$. Since $\{a_n\}$ is bounded, we may extract a subsequence such that $a_{n_k} \to Y_d \in \tilde{K}_d$, for some $Y_d \in \tilde{K}_d$. Pick $\delta \in (0,1)$. Then, by the DCT,

$$\begin{aligned}
\underline{\lim} g_d(a_{n_k}) &\geq \underline{\lim} \int_{\{t:a'_{n_k}f(t)>-1+\delta\}} -\log(1 + a'_{n_k}f(t))dt \\
&= \int_{\{t:Y'_d f(t)>-1+\delta\}} -\log(1 + Y'_d f(t))dt \\
&= g_d(Y_d) + \int_{\{t:Y'_d f(t)\leq -1+\delta\}} \log(1 + Y'_d f(t))dt.
\end{aligned}$$

Note that because $g_d(Y_d) \in (-\infty, 0]$, it follows that $-\int_{\{t:Y'_d f(t)<0\}} \log(1 + Y'_d f(t))dt < \infty$ and $\{t \in [0,1] : Y'_d f(t) = -1\}$ has Lebesgue measure zero. Hence, the DCT yields

$$\lim_{\delta\to 0} -\int_{\{t:Y'_d f(t)\leq -1+\delta\}} \log(1 + Y'_d f(t))dt = 0.$$

Consequently,

$$I \geq \overline{\lim}\, g_d(a_{n_k}) \geq \underline{\lim} g_d(a_{n_k}) \geq g_d(Y_d) \geq I,$$

establishing the existence of a minimizer Y_d of g_d on $\overline{K}_d$ such that $-\infty < I = g_d(Y_d) < 0$.

For part (ii) of Theorem 1, note $y_n = (1 - n^{-1})Y_d + n^{-1}0_d \in K_d$, $n \geq 1$, by convex geometry, as K_d is the convex interior of $\overline{K}_d$. Then, $g_d(y_n) \leq (1 - n^{-1})g_d(Y_d)$ holds by convexity of g_d and $g_d(0_d) = 0$, implying $0 \leq n[g_d(y_n) - g_d(Y_d)] \leq -g_d(Y_d) < \infty$, from which it follows that $g_d(y_n) \to g_d(Y_d)$ and, by the mean value theorem,

$$0 \leq n[g_d(y_n) - g_d(Y_d)] = \int_0^1 \frac{Y'_d f(t)}{1 + c_n Y'_d f(t)} dt \leq -g_d(Y_d)$$

holds for some $(1-n^{-1}) < c_n < 1$ (note $c_nY_d \in K_d$ so $\min_{0\le t\le 1}(1+c_nY_d'f(t)) > 0$ for all n); the latter implies $0 \le \int_{\{t:Y_d'f(t)<0\}} -Y_d'f(t)/[1+c_nY_d'f(t)]dt \le \int_{\{t:Y_d'f(t)>0\}} Y_d'f(t) < \infty$ so that Fatou's lemma yields

$$0 \le \int_{\{t:Y_d'f(t)<0\}} -\frac{Y_d'f(t)}{1+Y_d'f(t)}dt < \infty$$

as $n\to\infty$, and consequently $\int_0^1 1/[1+Y_d'f(t)]dt < \infty$. We may then apply the DCT to find

$$\lim_{n\to\infty}\int_0^1 \frac{Y_d'f(t)}{1+c_nY_d'f(t)}dt = \int_0^1 \frac{Y_d'f(t)}{1+Y_d'f(t)}dt \in [0,\infty).$$

Also by convexity and $0_d \in K_d$, $0 > g_d(Y_d) - g_d(0_d) > [\partial g_d(0_d)/\partial a]'(Y_d - 0_d)$ holds (w.p.1), implying $Y_d'\int_0^1 f(t)dt > 0$ from $\partial g_d(0_d)/\partial a = -\int_0^1 f(t)dt$. This establishes part (ii) of Theorem 1.

To show uniqueness of the minimizer, we shall construct sequences with the same properties in the proof of part (ii) above. Suppose $x \in \tilde{K}_d$ such that $g_d(x) = I = g_d(Y_d)$. Defining $x_n = (1-n^{-1})x + n^{-1}0_d \in K_d$ and $y_n = (1-n^{-1})Y_d + n^{-1}0_d \in K_d$ for $n \ge 1$, by convexity we have $0 \ge g_d(x) - g_d(y_n) > [\partial g_d(y_n)/\partial a]'(x-y_n)$, so that taking limits yields $0 \ge -\int_0^1 (x-Y_d)'f(t)/[1+Y_d'f(t)]dt$, and, by symmetry, $0 \ge -\int_0^1 (Y_d-x)'f(t)/[1+x'f(t)]dt$ as well. Adding these terms gives

$$0 \ge \int_0^1 \frac{[(x-Y_d)'f(t)]^2}{(1+x'f(t))(1+Y_d'f(t))}dt,$$

implying that $x = Y_d$ by Lemma 1(iv) and the continuity of f.

Finally, to establish part (iii), if $Y_d \in K_d$, then $0_d = \partial g_d(Y_d)/\partial a = -\int_0^1 f(t)/[1+Y_d'f(t)]dt$ must hold. If there exists another $b \in \overline{K}_d$ satisfying $\int_0^1 f(t)/[1+b'f(t)]dt = 0_d$, then adding $\partial g_d(Y_d)/\partial a$ to this integral and multiplying by $(Y_d - b)'$ yields $0 = \int_0^1 [(b-Y_d)'f(t)]^2/[(1+b'f(t))(1+Y_d'f(t))]dt$, implying that $b = Y_d$. Also, if $0_d = \int_0^1 f(t)/[1+b'f(t)]dt = -\partial g_d(b)/\partial a$ holds for some $b \in K_d$, then strict convexity implies $g_d(a) - g_d(b) > [\partial g_d(b)/\partial a]'(a-b) = 0$ for all $a \in \overline{K}_d$, implying $b = Y_d$ is the unique minimizer of g_d.

Proof of Theorem 2. Under Assumption A.2, we use Skorohod's embedding theorem (cf. [33], Theorem 1.1.04) to embed $\{S_n(\cdot)\}$ and $\{B(\cdot)\}$ in a larger probability space $(\Omega, \mathcal{F}, P)$ such that $\sup_{0\le t\le 1}\|\Sigma^{-1/2}S_n(t)/n^{1/2} - B(t)\| \to 0$ w.p.1(P). Defining $T_n(t) = w(t)S_n(t)$ and $f(t) = w(t)B(t)$, $t\in[0,1]$, the continuity of w under Assumption A.1 then implies

$$\sup_{0\le t\le 1}\left\|\frac{\Sigma^{-1/2}T_n(t)}{n^{1/2}} - f(t)\right\| \to 0 \quad \text{w.p.1.} \tag{7}$$

Note that $T_{i,\mu_0} = w(i/n)\sum_{j=1}^{i}(X_j - \mu_0) = T_n(i/n)$, $i = 1,\ldots,n$. By (7) and Lemma 1, 0_d is in the interior convex hull of $\{T_{i,\mu_0} : i = 1,\ldots,n\}$ eventually (w.p.1) so that $L_n(\mu_0) > 0$ eventually (w.p.1.). (That is, by Lemma 1(iv), there exists $A \in \mathcal{F}$ with $P(A) = 1$ and, for $\omega \in A$, $\min_{0\le t\le} a'f(\omega,t) \le -M(\omega)$ and $\max_{0\le t\le} a'f(\omega,t) \ge M(\omega)$ hold for some $M(\omega) > 0$ and all $a \in \mathbb{R}^d$, $\|a\| = 1$. Then, (7) implies $\min_{1\le i\le n} a'\Sigma^{-1/2}T_n(\omega, i/n) < 0 < a'\max_{1\le i\le n}\Sigma^{-1/2}T_n(\omega, i/n)$ holds for all $a \in \mathbb{R}^d$, $\|a\| = 1$ eventually, implying 0_d is in the interior convex hull of $\{\Sigma^{-1/2}T_n(i/n) : i = 1,\ldots,n\}$.) Hence, eventually (w.p.1), as in (4), we can write

$$\frac{1}{n}R_n(\mu_0) = -\frac{1}{n}\sum_{i=1}^{n}\log(1 + \lambda'_{n,\mu_0}T_{i,\mu_0}) = \frac{1}{n}\sum_{i=1}^{n}\log(1 + \ell'_n T_{i,n})$$

where $T_{i,n} \equiv \Sigma^{-1/2}T_n(i/n)/n^{1/2}$, $i = 1,\ldots,n$ and $\ell_n = n^{1/2}\Sigma^{1/2}\lambda_{n,\mu_0}$ and (8)

$$\min_{i=1,\ldots,n}(1+\ell'_n T_{i,n}) > 0, \qquad \sum_{i=1}^{n}\frac{1}{n(1+\ell'_n T_{i,n})} = 1, \qquad \sum_{i=1}^{n}\frac{T_{i,n}}{n(1+\ell'_n T_{i,n})} = 0_d.$$

From here, all considered convergence will be pointwise along some fixed $\omega \in A$ where $P(A) = 1$, and we suppress the dependence of terms f, T_n, etc. on ω. Then, (8) [i.e., $\min_{i=0,\ldots,n}\ell'_n(\Sigma^{-1/2}T_n(i/n)/n^{1/2}) > -1$] with (7) and Lemma 1(iv) implies that $\|\ell_n\|$ is bounded eventually. For any subsequence $\{n_j\}$ of $\{n\}$, we may extract a further subsequence $\{n_k\} \subset \{n_j\}$ such that $\ell_{n_k} \to b$ for some $b \in \overline{K}_d$. For simplicity, write $n_k \equiv k$ in the following. We will show below that $k^{-1}\log R_k(\mu_0) \to g_d(Y_d)$ and that $\ell_k \to Y_d$, where $Y_d \in \overline{K}_d$ denotes the minimizer of $g_d(a) = -\int_0^1 \log(1 + a'f(t))dt$, $a \in \overline{K}_d$ under Theorem 1. Since the subsequence $\{n_j\}$ is arbitrary, we then have $n^{-1}\log R_n(\mu_0) \to g_d(Y_d)$ and $\ell_n \to Y_d$ w.p.1, implying the distributional convergence in Theorem 2.

Define $Y_\epsilon = (1-\epsilon)Y_d + \epsilon 0_d \in K_d$ (since $0_d \in K_d$, the interior of $\overline{K}_d$) for $\epsilon \in (0,1)$. From $Y_\epsilon \in K_d$, $\min_{0\le t\le 1}(1 + Y'_\epsilon f(t)) > \delta$ holds for some $\delta > 0$ (dependent on ϵ) so that $\min_{1\le i\le k}(1 + Y'_\epsilon T_{i,k}) > \delta$ holds eventually by (7). Then, because

$$g_{d,k}(a) \equiv -\frac{1}{k}\sum_{i=1}^{k}\log(1 + a'T_{i,k})$$

is strictly convex on $a \in \{y \in \mathbb{R}^d : \min_{1\le i\le k}(1 + y'T_{i,n}) > 0\}$ with a unique minimizer at ℓ_k by (8) (i.e., $\partial g_{d,k}(\ell_k)/\partial a = 0_d$ holds and strict convexity follows when $k^{-1}\sum_{i=1}^{k}T_{i,k}T'_{i,k}$ is positive definite, which holds eventually from $k^{-1}\sum_{i=1}^{k}T_{i,k}T'_{i,k} \to \int_0^1 f(t)f(t)'dt$ by (7) and the DCT, with the latter matrix being positive definite w.p.1 by Lemma 1(iv) and continuity of f), we have that

$$g_{d,k}(Y_\epsilon) \geq g_{d,k}(\ell_k) = \frac{1}{k}\log R_k(\mu_0).$$

Define $\bar{g}_{d,k}(a) \equiv -k^{-1}\sum_{i=1}^k \log(1+a'f(i/k))$, $a \in K_d$. Then, by Taylor expansion (recalling $\min_{0\leq t\leq 1}(1+Y_\epsilon' f(t)) > \delta$, $\min_{1\leq i\leq k}(1+Y_\epsilon' T_{i,k}) > \delta$),

$$\begin{aligned}
& |g_{d,k}(Y_\epsilon) - \bar{g}_{d,k}(Y_\epsilon)| \\
\leq\ & \frac{1}{k}\sum_{i=1}^k |Y_\epsilon'(T_{i,k} - f(i/k))| \left(\frac{1}{1+Y_\epsilon' T_{i,k}} + \frac{1}{1+Y_\epsilon' f(i/k)}\right) \\
\leq\ & \|Y_d\| 2\delta^{-1} \max_{1\leq i\leq k} \|T_{i,k} - f(i/k)\| \to 0
\end{aligned}$$

from (7) and Theorem 1. Also, by the DCT, $\bar{g}_{d,k}(Y_\epsilon) \to g_d(Y_\epsilon)$ as $k\to\infty$. Hence, $g_d(Y_\epsilon) \geq \overline{\lim}\, g_{d,k}(\ell_k)$ holds and, since $g_d(Y_\epsilon) \leq (1-\epsilon)g_d(Y_d)$ by convexity and $g_d(0_d) = 0$, we have, letting $\epsilon \to 0$, that

$$g_d(Y_d) \geq \overline{\lim}\, g_{d,k}(\ell_k). \tag{9}$$

Recalling $\ell_k \to b \in \overline{K}_d$, define $b_\epsilon = (1-\epsilon)b + \epsilon 0_d \in K_d$, so that $\min_{0\leq t\leq 1}(1+b_\epsilon' f(t)) > 0$. Then, $\bar{g}_{d,k}(b_\epsilon) \to g_d(b_\epsilon)$ by (7) and the DCT. And, by Taylor expansion and using (8),

$$\begin{aligned}
& \overline{\lim}\, |g_{d,k}(\ell_k) - \bar{g}_{d,k}(b_\epsilon)| \\
\leq\ & \overline{\lim} \max_{1\leq i\leq k} |\ell_k' T_{i,k} - b_\epsilon' f(i/k)| \left(1 + \frac{1}{k}\sum_{i=1}^k \frac{1}{1+b_\epsilon' f(i/k)}\right) \\
\leq\ & \epsilon \sup_{0\leq t\leq 1} |b' f(t)| \left(1 + \int_0^1 \frac{1}{1+b_\epsilon' f(t)} dt\right) \equiv C(\epsilon),
\end{aligned}$$

following from (7) and the DCT. Hence, we have

$$\underline{\lim}\, g_{d,k}(\ell_k) \geq g_d(b_\epsilon) - C(\epsilon). \tag{10}$$

We will show below that

$$\int_0^1 \frac{1}{1+b' f(t)} dt < \infty \tag{11}$$

holds. In which case, $\lim_{\epsilon\to 0}\int_0^1 [1+b_\epsilon' f(t)]^{-1} dt = \int_0^1 [1+b' f(t)]^{-1} dt < \infty$ by the DCT and so that $C(\epsilon) \to 0$ as $\epsilon \to 0$ (noting $\sup_{0\leq t\leq 1}|b' f(t)| < \infty$ since f is continuous and $\overline{K}_d$ is bounded by Theorem 1). By Fatou's lemma and the DCT, $\underline{\lim}_{\epsilon\to 0}\, g_d(b_\epsilon) \geq g_d(b)$ holds also. Hence, by (9)-(10), we then have

$$g_d(Y_d) \geq \overline{\lim}\, g_{d,k}(\ell_k) \geq \underline{\lim}\, g_{d,k}(\ell_k) \geq g_d(b) \geq g_d(Y_d),$$

implying $b = Y_d$ by the uniqueness of the minimizer and $\lim_{k\to\infty} k^{-1} \log R_k(\mu_0) = g_d(Y_d)$.

To finally show (11), let $A = \{t \in [0,1] : 1 + b'f(t) \le d\}$ for some $0 < d \le 1/2$ chosen so that $\{t \in [0,1] : 1 + b'f(t) = d\}$ has Lebesgue measure zero (since f is continuous). Let $A^c = [0,1] \setminus A$. Using the indicator function $\mathbb{I}(\cdot)$, define a simple function

$$h_k(t) \equiv \sum_{i=1}^{k} \frac{\ell_k' T_{i,k}}{1 + \ell_k' T_{i,k}} \mathbb{I}\left(t \in \left(\frac{i-1}{k}, \frac{i}{k}\right]\right), \quad t \in [0,1].$$

From (8), note that

$$\int_A h_k(t)dt + \int_{A^c} h_k(t)dt = \frac{1}{k}\sum_{i=1}^{k} \frac{\ell_k' T_{i,k}}{1 + \ell_k' T_{i,k}} = 0_d.$$

From (7), $\mathbb{I}(t \in A^c)h_k(t) \to \mathbb{I}(t \in A^c)b'f(t)/(1 + b'f(t))$ (almost everywhere (a.e.) Lebesgue measure) and for large k, $\mathbb{I}(t \in A^c)|h_k(t)| \le 2C/d$ holds for $t \in [0,1]$, since eventually $\max_{1\le i\le k} |\ell_k' T_{i,k}|$ is bounded by a constant $C > 0$ and also $1 + b'f(t) + (\ell_k' T_{i,k} - b'f(t)) > d/2$ for $t \in A^c$, $(i-1)/k < t \le i/k$. Then, by the DCT, $\int_{A^c} h_k(t)dt \to \int_{A^c} b'f(t)/(1+b'f(t))dt$. And for $\delta \in (0,1)$, note $-\mathbb{I}(t \in A)h_k(t) \ge h_{1,k}(t)$ for

$$h_{1,k}(t) \equiv \sum_{i=1}^{k} \frac{-\ell_k' T_{i,k}}{1 + \ell_k' T_{i,k} + \delta\mathbb{I}(\mathrm{sign}(\ell_k' T_{i,k}) < 0)} \mathbb{I}\left(t \in \left(\frac{i-1}{k}, \frac{i}{k}\right] \cap A\right).$$

Since $|h_{1,k}(t)| \le C/\delta$ and $h_{1,k}(t) \to -\mathbb{I}(t \in A)b'f(t)/(1+b'f(t)+\delta)$ (a.e. Lebesgue measure), by the DCT

$$0 \le \int_A \frac{-b'f(t)}{1 + b'f(t) + \delta}dt = \lim_{k\to\infty}\int_A h_{1,k}(t)dt \le \lim_{k\to\infty}\int_A -h_k(t)dt = \int_{A^c} \frac{b'f(t)}{1 + b'f(t)}dt$$

using $\int_A -h_k(t)dt = \int_{A^c} h_k(t)dt$. Letting $\delta \to 0$, Fatou's lemma gives

$$0 \le \int_A \frac{-b'f(t)}{1 + b'f(t)}dt \le \int_{A^c} \frac{b'f(t)}{1 + b'f(t)}dt < \infty.$$

Because $-b'f(t) \ge 1/2$ on A, $\int_A [1 + b'f(t)]^{-1}dt < \infty$ holds, implying (11).

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Department of Statistics
Iowa State University
Ames, IA USA 50011
E-mail: dnordman@iastate.edu

Department of Economics
Iowa State University
Ames, IA USA 50011
CREATES, Aarhus University
Aarhus, DK-8000
E-mail: hbunzel@iastate.edu

Department of Statistics
North Carolina State University
Raleigh, NC USA 27695-8203
E-mail: snlahiri@ncsu.edu