Diffusion-Limited Aggregation on the Hyperbolic Plane

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Abstract

We consider an analogous version of the Diffusion-Limited Aggregation model defined on the Hyperbolic plane. We prove that almost surely, the aggregate viewed at time infinity will have a positive density.

1 Introduction

The celebrated *Diffusion-limited aggregation* (in short, **DLA**) model is a probabilistic model where particles undergoing a certain diffusion stick together and form up into clusters. Most commonly, the aggregate begins with a single particle at a fixed point, and in every iteration a new particle arrives via a Brownian motion (or some random walk) starting from infinity and stops at the moment it hits the existing cluster, thus expanding it. This model was first introduced by Witten and Sander ([WS]) in 1981, as a model which could be used to represent several physical phenomena related to systems where the principle mean of transport of particles is by diffusion. Some examples of systems which appear to have DLA-like behaviour are electro-deposition, mineral deposits, and dielectric breakdown systems.

The most interesting settings for the DLA model are naturally the two and three dimensional Euclidean spaces (or the grids $\mathbb{Z}^2$ and $\mathbb{Z}^3$). In these spaces, determining some of the most basic properties of this model seem to be notoriously hard problems. For example, it is not known whether the rate of growth of the diameter of the aggregate is not $O(n^{1/d})$ where n is the number of particles and d is the dimension, or whether or not the density of the cluster at time infinity is zero. It is conjectured by physicists that the answers to both these questions are positive. One of the only known facts about DLA in Euclidean space is the result of Kesten, [K], who obtained the upper bound $O(n^{2/\max(d,3)})$ for the speed of growth of the diameter of the DLA in $\mathbb{Z}^d$. We would also like to mention a paper of Barlow, Pemantle and Perkins ([BPP]) in which the DLA model on a tree is studied.

Roughly speaking, an analogous version of this model can be defined in any space where the notion of diffusion exists. If the Poisson boundary consists of one point (or, in other words, the definition of "a particle released at infinity" makes sense) and the diffusion is recurrent, the growth process can be defined so that law of the location of a new particle is the harmonic measure of the existing aggregate with pole at infinity. If the diffusion is transient (such as in the case of $\mathbb{Z}^3$) one can consider the harmonic measure with a pole far away from the aggregate, let the pole go to infinity and take limits (i.e., conditioning on a random walk coming from infinity to hit the cluster).

Another way to define the law of growth in settings where the diffusion is transient is to use the *time-reversibility* property of the random walk. According to this property, the harmonic measure of a set, with pole at infinity, is proportional to the so-called *equilibrium measure* associated to the set. For sets with sufficient smoothness properties, this measure is absolutely continuous with respect to the Hausdorff measure on the boundary of the set and its density is proportional to the gradient, in the normal direction to the boundary, of the solution of the Dirichlet problem with boundary conditions 1 on the set and 0 at infinity. From a probabilistic point of view, this density is roughly proportional to the probability that a particle released close to the boundary of the set reaches infinity before hitting the aggregate. Fortunately, this definition also makes sense in settings where the Poisson boundary consists of more than one point. A more detailed description of this will be given in the next section.

Our aim in this paper is to study a DLA model defined on the *hyperbolic plane*, showing that in this case, the cluster at time infinity almost surely admits a positive upper density. Our results suggest that in the Hyperbolic setting the behavior of the aggregate is simpler to analyze than the Euclidean one. However, simulations point that its geometry is still fairly complicated: it seems that the so-called "rich-get-richer" behavior takes place also in this setting and the aggregates look far from having a certain limit shape. Our results may therefore be viewed as a modest attempt to rigorously study certain properties of a model whose complexity is somewhat similar to that of the Euclidean DLA. Diffusion-limited growth on general Riemannian manifolds and specifically on the hyperbolic plane was already considered in the physics literature, see [CCB]; The physical motivation for this study is that natural phenomena of DLA-like behavior such as mineral dendrites, cell colonies and cancerous tumors usually grow on curved surfaces.

In our construction, the particles will be metric balls of radius 1. We define A_0 to be a fixed point p_0 and recursively $A_{i+1} = A_i \cup \{x\}$ where the point x (thought of as the center of a disc-shaped particle) will be picked from the set of points whose distance from A_i is exactly 2 (which means exactly that the corresponding discs will be tangent to each other) and will be distributed in this set proportionally to the probability of escape to infinity, described in the previous paragraph. We will also write $A_\infty = \cup_{i=1}^{\infty} A_i$. The precise construction appears in the next section.

In a metric measure space X whose diameter is infinite, we say that a locally-finite set $A \subset X$ has an *upper density* greater or equal to c if there exists a point

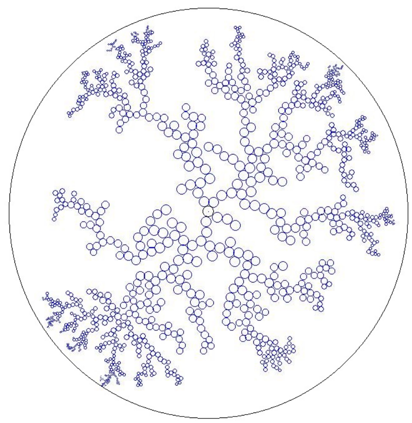

Figure 1: *A simulation of the DLA model with 1000 particles, viewed on the Poincaré disc model.*

$p \in X$ and a sequence $R_1 < R_2 < ...$ such that $R_i \to \infty$ as $i \to \infty$, such that

$$\#\left(A \cap B(p, R_i)\right) \geq c\mu(B(p, R_i)), \quad \forall i \in \mathbb{N}.$$

where $B(p,r)$ is a metric ball centered at p with radius r and μ is the measure defined on X. We can use this definition in the hyperbolic plane, using the standard hyperbolic distance as a metric and the standard Riemannian volume of a set as a measure.

Our main theorem reads,

Theorem 1.1 *The set $A_\infty = \bigcup_{i=1}^{\infty} A_i$ almost surely has an upper density greater than c, where $c > 0$ is a universal constant.*

Remark 1.2 *The reader may suspect that the above theorem follows from a general geometric fact about the hyperbolic plane and does not use any of the randomness in the model. Alas, there is an example of a connected set which is a union of balls of radius 1, whose convex hull is the entire plane, but whose upper density is zero. Indeed, consider the following "spiral" set: take a point $p \in \mathbb{H}^2$ and $\theta_0 \in T_p$ (where T_p is the tangent space at p) and consider the exponential map $e : T_p \to \mathbb{H}^2$. Define*

$$A = \bigcup_{\theta \in [0,\infty)} B_H(\exp_p(X_\theta R(\theta)), 1)$$

where X_θ is a unit vector in T_p whose angle with θ_0 is θ, $B_H(p,r)$ is a geodesic ball of radius r centered at p and $R(\theta)$ is an increasing function. It is not hard to verify that if the function $R(\theta)$ goes to infinity fast enough, the set A will have the properties described above.

In the vaguest sense, the intuition behind the fact that the behavior of the DLA model in the hyperbolic plane is different from the conjectured behavior in Euclidean space is related to the rate of decay of the harmonic potential. Consider two particles located at distance L apart. The probability for two Brownian paths released from the two particles to intersect at some point is exponentially decreasing with L which, in turn, roughly means that when growing an aggregate from those two points simultaneously, these two aggregates will hardly interact. In particular, the new particles added to any two given "arms" of our aggregate will grow farther away from each other at linear speed. This means that the growth law of the aggregate is almost "local" in the sense that the subtree related to each new particle added to the aggregate will only ever be affected by its immediate neighborhood and, moreover, their interaction will decrease exponentially with time. The absence of long-range interactions will prevent the multi-scale phenomena, expected in the Euclidean case, from occurring in our case.

Specifically, the geometry of the hyperbolic plane makes it much harder to isolate certain parts of the DLA and disallowing them to grow further by creating *fjords* which are too narrow for particles to come through, which in turn means that the DLA will locally keep growing at most of its parts and will eventually fill the whole space.

Let us now review the general plan of our proof, while trying to explain how the aforementioned properties of hyperbolic geometry come into play.

The main step of the proof will be to show that there exists a universal constant $R_0 > 0$ such that for any metric ball B of radius R_0, there is a probability of at least 0.99 that the aggregate will intersect this ball, no matter how far the ball is from the starting point of the aggregate.

The proof of this step relies heavily on the fact that the upper half-plane, $\mathbb{R} \times (0, \infty)$, is isometric to $\mathbb{H}^2$ via a conformal mapping (using the so-called Poincaré metric). Regarding our aggregate on the upper half plane and choosing the correct embedding, this is easily reduced to showing that an aggregate which begins at the point $(0, \varepsilon)$ reaches, with a non-negligible probability, any rectangle of the form $\Psi = [-C, C] \times [1, 2]$ where $C > 0$ is a universal constant and ε is an arbitrarily small positive number.

At this point, let us now try to further illustrate the difference between Euclidean and hyperbolic geometry which we are going to exploit: in order for the aggregate to never reach the rectangle Ψ, it has to encompass Ψ, at least in the sense that Ψ will be contained in the convex hull of the aggregate before any point of the Ψ has a chance to be reached by it. In particular, the aggregate has to reach one of the lines $\{x = \pm C\}$. Now, note that any geodesic line connecting the starting points with these two lines actually passes through Ψ.

In other words, the rectangle Ψ acts as bottleneck which prevents the aggregate from encompassing it. It is easy to see that no analogous phenomenon takes place in the Euclidean space.

Remark 1.3 *As mentioned above, in the paper of Barlow, Pemantle and Perkins ([BPP]), a diffusion-limited aggregation on an infinite regular tree is studied. The fact that the hyperbolic space has a tree-like structure may mislead the reader to think that the model studied in their paper is closely related to our model, and that the two are therefore expected to behave in the same way. While these two models are superficially similar and both called DLA, their behavior is nevertheless quite different. Remark that on the discrete tree, each connected component of the complement of a given subtree looks exactly the same. Thus, the tree counterpart of our process would be defined such that the rate of growth of the aggregate is constant on all points of its boundary, regardless of its geometry. By definition, this aggregate will eventually fill the entire tree and it is not hard to see that it would do it in a rather uniform way.*

Let us try to explain our strategy to formally establish the fact that Ψ is likely to be reached by the aggregate before one of the lines $\{x = \pm C\}$ is reached.

The idea will be to establish bounds on the rate of growth of the minimum encompassing rectangle of the aggregate, hence the maximal x-coordinate of the aggregate at time t, denoted by $X(t)$, and the maximal y-coordinate, denoted by $Y(t)$ (see figure 2 below). In order to prove that the aggregate reaches the rectangle Ψ, it will be enough to show that $X(t)$ does not grow much faster than $Y(t)$. We will work with a continuous time $t \in \mathbb{R}^+$, so that growth of the cluster is according to an exponential clock whose rate is proportional to the capacity, which ensures us that in small time intervals the expected rate of growth in different parts of the cluster is roughly independent (this is defined in section 2).

Two key geometric lemmas proven in section 3 will provide an upper bound for the rate of growth of $X(t)$ and a lower bound for the rate of growth of $Y(t)$. The former bound, whose proof uses the easy fact that in the half-plane model the y coordinate of the center of metric circle of radius 1 is proportional to its Euclidean radius, roughly says that $\frac{d}{dt}\mathbb{E}[X(t)] < CY(t)$. According to the latter bound, which makes use of the conformal invariance, the probability of $Y(t)$ to multiply itself by a constant during a unit time interval is at least of the order $cY(t)/(X(t)+Y(t))$, or in other words, roughly $dY(t) > cY^2(t)/X(t)$. Here c, C are universal constants.

Next, we note that (very informally) these bounds combined give

$$d\frac{X(t)}{Y(t)} = \frac{dX(t)}{Y(t)} - \frac{X(t)dY(t)}{Y(t)^2} \leq C - c.$$

One would expect that by integrating those two bounds it should be possible to attain an estimate of the form $Y(t) > X(t)^\alpha$ where α is a positive constant which depends on the ratio C/c, at least in expectation. However, it seems like the above bounds cannot be pushed to give constants which would yield $\alpha \geq 1$.

Because of this, we have to do something a little more complicated. We define $\tilde{Y}(t)$ as the height of the cluster close to the edge where x attains its maximum (as in figure 2 below), and consider two difference cases: if $\tilde{Y}(t)$ is much smaller than $Y(t)$, we get that $dX(t)$ is small enough so that the two bounds above can be integrated to attain that $d\frac{X(t)}{Y(t)}$ is negative. On the other hand, if $\tilde{Y}(t)$ and $Y(t)$ are comparable, it turns out that we expect $X(t)/Y(t)$ to decrease due to a completely different reason (provided that it is not too small). We know that there is a non-negligible probability that the height of the cluster will grow rather rapidly close to its edge (hence close to the place where $X(t)$ is attained), and therefore $Y(t)$ can multiply itself by a constant within a constant amount of time. All of this is carried out in section 4.

Once we have those two bounds, which can be combined into a unified bound on the (expected) rate of growth of $R(t) = X(t)/Y(t)$ the proof of the main step is just a matter of defining the correct martingale and using the optional stopping theorem. Note, however, that the process $X(t)/Y(t)$ cannot actually be a super-martingale as we know that it is always positive, and it clearly does not converge. Ideologically, this process should be regarded as a super-martingale reflecting at zero, and for such processes, the optional stopping theorem cannot help (it is not hard to see that Brownian motion with a strong drift towards zero and reflection at zero can be almost surely stopped at arbitrarily large values with a stopping time of finite expectation). With a little extra work, we show that the process $x \to R(\min\{t; X(t) > x\})$ is also a super-martingale with reflection at zero and a strong enough drift, which turns out to be enough. In section 5, we tie up the loose ends, showing how the main step can be used complete the proof.

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2 Preliminaries

2.1 The Poincaré half-plane model

We denote the hyperbolic plane by $\mathbb{H}^2$. For two points $p_1, p_2 \in \mathbb{H}^2$, we define the hyperbolic distance between them by $d_H(p_1, p_2)$. In many cases, we will view the hyperbolic plane using the Poincaré half-plane model, which is the usual open half plane $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ (sometimes called the Poincaré half-plane) equipped with an embedding $H : \mathbb{R}^2_+ \to \mathbb{H}^2$ and a distance function defined by

$$d_H((x_1, y_1), (x_2, y_2)) = d_H(H(x_1, y_1), H(x_2, y_2)) = \tag{1}$$

$$\textbf{Arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2}\right).$$

By slight abuse of notation, throughout this note we will sometimes allow ourselves to interchange freely between the roles of p and $H(p)$, whenever the intention is clear from the context.

For a point $p \in \mathbb{R}^2_+$ let $B_H(p,r) \subset \mathbb{R}^2_+$ be the closed d_H-ball centred at p with radius r and let $B_E(p,r) \subset \mathbb{R}^2_+$ be the closed Euclidean-ball centred at p with radius r. We will often use the following elementary estimate, which follows immediately from formula (1).

Lemma 2.1 *For any* $(x,y) \in \mathbb{R}^2_+$*, one has*

$$B_E((x,y), 0.5y) \subseteq B_H((x,y),1) \subseteq B_H((x,y),2) \subseteq B_E((x,y),7y)$$

Another basic fact of which we will make use quite often is the invariance of the model to Möbius transformations leaving $\mathbb{R}^2_+$ intact:

Fact 2.2 *For any constants* $\alpha \in \mathbb{R}$ *and* $\beta > 0$ *consider the transformation*

$$T : (x,y) \to (\beta x + \alpha, \beta y)$$

Then d_H *is invariant under* T*, namely,*

$$d_H((x_0,y_0),(x_1,y_1)) = d_H(T(x_0,y_0), T(x_1,y_1))$$

for all $(x_0,y_0),(x_1,y_1) \in \mathbb{R}^2_+$.

We denote by $\mathbb{H}^2(\infty)$ the set of ideal points (or omega points) of the hyperbolic plane. We also define

$$\mathbb{R}^2_+(\infty) = \mathbb{R} \times \{0\} \cup \{\infty\}.$$

By continuity, we can extend an embedding $H : \mathbb{R}^2_+ \to \mathbb{H}^2$ to the set $\mathbb{R}^2_+(\infty)$.

One last property of the Poincaré model which we will exploit is its *conformality*, namely, the fact that the map $H : \mathbb{H}^2 \to \mathbb{R}^2_+$ is a conformal map. Thanks to this fact and since, according to a theorem of P. Lévy, the path of a Brownian motion is invariant under conformal maps, we have the following.

Fact 2.3 (conformal invariance) *Let* $A \subset \mathbb{H}^2$ *be a measurable set and let* $x \in \mathbb{H}^2$ *be any point. The path of a hyperbolic Brownian motion starting at* x *and stopped when it reaches* $A \cup \mathbb{H}^2(\infty)$ *has the same distribution as the image under the map* H *of the path of the usual Euclidean Brownian motion defined on* $\mathbb{R}^2_+$ *started at* $H^{-1}(x)$ *and stopped at* $H^{-1}(A) \cup \mathbb{R}^2_+(\infty)$.

2.2 The harmonic measure

As explained above, in Euclidean space, the DLA is usually defined via particles arriving from infinity, or equivalently, the place of the particle added to the aggregate is has a distribution whose law is the harmonic measure on the boundary of the existing aggregate, with a pole at infinity. Unfortunately, in the hyperbolic space there is no natural analogous definition, as the harmonic measure actually depends on the point in $H(\infty)$ from which the particle is released (or, in other words, the Poisson boundary contains more than one point). In order to find a definition of a DLA growth model on the hyperbolic plane that makes sense, we use the following fact which is a consequence of the time reversibility of the Brownian motion (for a proof see [IM], p.252 and [MP, Theorem 8.33])

Fact 2.4 *For any smooth set $A \subset \mathbb{R}^n$, $n \geq 3$, there exists a constant C_A such that for any $x \in \partial A$, one has*

$$C_A m_{A,\infty}(x) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P} \begin{pmatrix} \textit{A brownian motion released from } x + \vec{n}\epsilon \\ \textit{reaches } \infty \textit{ before hitting } A \end{pmatrix}$$

where $\vec{n}$ is the normal direction to ∂A at x, pointing outwards and $m_{A,\infty}(x)$ is the density of the harmonic measure of the domain A with pole at ∞ evaluated at the point x.

Fortunately, the right hand side of the above formula can be defined just the same in the hyperbolic plane. Fix two measurable subsets $A, B \subset \mathbb{H}^2 \cup \mathbb{H}^2(\infty)$ such that $\mathbb{H}^2(\infty) \subset A \cup B$ and fix a point $x \in \partial A \setminus H(\infty)$ such that ∂A is smooth at x. Denote by T_x be the tangent space of $\mathbb{H}^2$ at x and let $v \in T_x$ be the outward normal to ∂A at x. Consider the exponential map $\exp_x : T_x \to \mathbb{H}^2$. We define,

$$m_{A,B}(x) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P} \begin{pmatrix} \text{A brownian motion released from } \exp_x(\epsilon v) \\ \text{reaches } B \text{ before hitting } A \end{pmatrix}.$$

For all measurable $D \subset \partial A \setminus \mathbb{H}^2(\infty)$, we define

$$\mathcal{M}_{A,B}(D) = \int_D m_{A,B}(x) d\ell(x)$$

where $\ell(\cdot)$ is the standard length measure in the hyperbolic plane. We claim that the above integral is well-defined and finite whenever A is a finite union of metric balls. Indeed, the boundary of such a set is smooth up to a finite set of points, which means that the above integral is well defined. Moreover, it is evident from the above definition that $m_{A,B}$ admits the following monotonicity property: for two sets $A' \subset A$ such that $x \in \partial A' \cap \partial A$, one has $m_{A',B}(x) \geq m_{A,B}(x)$. Consequently, the function $m_{A,B}(x)$ is bounded on ∂A and the integral is finite.

Remark 2.5 *In fact, this definition is valid for any set whose boundary is a rectifiable curve (see [P, Example 1.2]).*

Finally, when $A \cap \mathbb{H}^2(\infty) = \emptyset$, we also abbreviate

$$\mathcal{M}_A(D) = \mathcal{M}_{A,\mathbb{H}^2(\infty)}(D). \tag{2}$$

In view of fact 2.4, it seems natural to construct our DLA cluster using this measure.

2.3 Construction of the DLA

The evolution of our aggregate will be represented via a sequence of random finite sets $A_1 \subset A_2 \subset ...$, each element of which is a point in $\mathbb{H}^2$ represents a single particle. The particles are assumed to be metric balls of radius 1, and the elements of the above sets are the centers of those metric balls, hence the actual aggregate takes the form

$$\bigcup_{p \in A_i} B_H(p, 1).$$

We fix a point $p_0 \in \mathbb{H}^2$ which we regard as the origin of the aggregate. We begin with the set $A_0 = \{p_0\}$. The set A_{i+1} will be the existing aggregate A_i with the addition of one point representing the center of the new particle. In order to define the law according to which this new point is distributed, will need some more definitions.
For a finite set $A \subset \mathbb{H}^2$, we define

$$\mathcal{B}(A) = \bigcup_{x \in A} B_H(x, 2).$$

The point of taking balls of radius 2 is that any ball centered at a point in $\partial\mathcal{B}(A)$ whose radius is 1 will be tangent to the aggregate (which is assumed to be a union of balls of radius 1). Define

$$\mu_A(\cdot) = Cap(A)^{-1}\mathcal{M}_{\mathcal{B}(A)}(\cdot)$$

where

$$Cap(A) := \mathcal{M}_{\mathcal{B}(A)}(\partial\mathcal{B}(A)).$$

is a normalizing constant to which we will refer to as the *capacity* of A and where the measure $\mathcal{M}_{\mathcal{B}(A)}$ is defined in equation (2). Note that by definition, the measure μ_A is a probability measure.

Remark 2.6 *The quantity $Cap(A)$ is sometimes referred to as the inverse* Riemann modulus *of A. It is a well known fact, which is a consequence of Schottky's theorem, that it is invariant under conformal maps of the hyperbolic plane.*

We can finally define by recursion,

$$A_{i+1} = A_i \cup \{X_i\}$$

where X_i is a random point in $\partial\mathcal{B}(A_i)$ distributed according to the law μ_{A_i}.

Throughout this note, we will usually allow ourselves to interchange freely between A_i and $H^{-1}(A_i)$ (when this does not cause any confusion), thus sometimes considering A_i as a subset of $\mathbb{R}^2_+$.

2.4 Continuous time

In our proofs, it will be more convenient to regard our process in continuous time. We define a sequence of times $t_0, t_1, t_2, ...$ by the following inductive law: Define $t_0 = 0$, and for all $i \geq 0$, let $t_{i+1} - t_i$ be an exponentially-distributed variable whose expectation is $Cap(A_i)^{-1}$, independent from all the rest. Finally, we define

$$A(t) = A_{i(t)}$$

where

$$i(t) = \max\{i;\ t_i \leq t\}.$$

We denote by $\mathcal{F}_t$ the filtration corresponding to the process. The next fact will be useful to us

Fact 2.7 *The process $A(t)$ is a Markov process, hence for every random variable X measurable with respect to $\mathcal{F}_\infty$ and every $t \geq 0$,*

$$\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X|A(t)].$$

Moreover, for any t and for any measurable $B \subset \partial\mathcal{B}(A(t))$, one has

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}\Big(B \cap A(t+\epsilon) \neq \emptyset \Big| A(t)\Big) = \mathcal{M}_{\mathcal{B}(A(t))}(B) \tag{3}$$

and for all B such that $B \cap \partial\mathcal{B}(A(t)) = \emptyset$,

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}\Big(B \cap \big(A(t+\epsilon) \setminus A(t)\big) \neq \emptyset \Big| A(t)\Big) = 0. \tag{4}$$

Proof: The Markov property follows immediately from the definition of the process. In order to prove formula (3), we make note that for all $i \in \mathbb{N}$,

$$Cap(A_i) \leq \sum_{p\in A_i} \mathcal{M}_{\mathcal{B}(A_i)}(\partial B_H(p,2)) \leq$$

$$\sum_{p\in A_i} \mathcal{M}_{B_H(p,2)}(\partial B_H(p,2)) = P_0 i$$

for some constant $P_0 > 0$. Therefore, we can estimate

$$\mathbb{P}(i(t+\epsilon) \geq i(t) + 2\ |A(t)) \leq$$

$$\mathbb{P}(t_{i(t)+1} \leq t + \epsilon\ |A(t))\mathbb{P}(t_{i(t)+2} < t_{i(t)+1} + \epsilon\ |A(t)) \leq$$

$$\mathbb{P}(E(1/(P_0 i(t))) < \epsilon)\mathbb{P}(E(1/(P_0(i(t)+1))) < \epsilon) = O(\epsilon^2)$$

where $E(v)$ denotes an exponential variable with expectation v. We deduce that the probability that more than one particle is added to the cluster in an interval of the form $[t, t+\epsilon]$ is of the order ϵ^2. Since by definition, the next particle added must be at $\partial\mathcal{B}(A(t))$, equation (4) follows. Next, we have

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}\left(A(t+\epsilon)\cap B \neq \emptyset \mid A(t)\right) =$$

$$\mathbb{P}(A_{i(t)+1}\cap B \neq \emptyset) \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}(t_{i(t)+1} \leq t+\epsilon | A(t)) =$$

$$\mu_{A(t)}(B) \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}(1-\exp(-\epsilon Cap(A(t)))) = \mu_{A(t)}(B) Cap(A(t))$$

which proves (3). The proof is complete. □

3 Geometric lemmas

The goal of this section is to prove two geometric lemmas which will serve as central ingredients in the proof. Throughout this section we assume that the embedding of $\mathbb{H}^2$ in $\mathbb{R}^2_+$ has been fixed, and consider the aggregate $A(t)$ as a subset of $\mathbb{R}^2_+$. We begin with some definitions which will be frequently used later on.

For every time $t \geq 0$, we define

$$X(t) = \sup\{|x|;\ \exists y \text{ such that } (x,y)\in A(t)\}$$

and

$$Y(t) = \sup\{y; \exists x \text{ such that } (x,y)\in A(t)\}.$$

We define also,

$$Y_L^+(t) = \sup\{y;\ \exists x \geq L \text{ such that } (x,y)\in A(t)\}$$

and

$$Y_L^-(t) = \sup\{y;\ \exists x \leq L \text{ such that } (x,y)\in A(t)\}.$$

For a particle $b\in A(t)$, we say that b is in the *front* of $A(t)$ and denote $b \in \mathbf{Fr}(A(t))$ if there exists a point $p=(x,y)\in\mathbb{R}^2_+$ having $d_H(b,p)\leq 1$ and $|x| \geq X(t)$. Lastly, we define

$$\tilde{Y}(t) = \sup\{y;\ (x,y)\in \mathbf{Fr}(A(t))\}.$$

These definitions are illustrated in figure 2.

We begin with the following upper bound for the rate of growth of $X(t)$, which turns out to be controlled by $\tilde{Y}(t)$ in expectation.

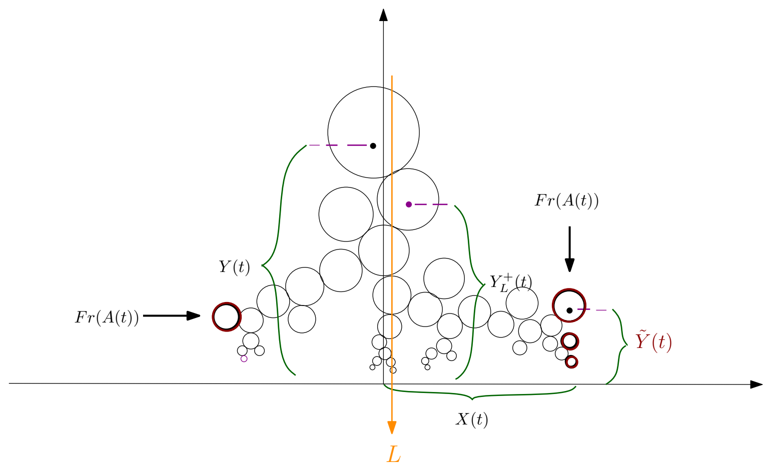

Figure 2: *The definitions* $X(t), Y(t), Y_L^+(t), \mathbf{Fr}(A(t))$ *and* $\tilde{Y}(t)$ *illustrated.*

Lemma 3.1 *There exists a universal constant* $C > 0$ *such that for all* $t \geq 0$*, one has almost surely*

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\big(\mathbb{E}[X(t+\epsilon)|\mathcal{F}_t] - X(t)\big) \leq C\tilde{Y}(t) \tag{5}$$

The geometric intuition behind this lemma is the following: first of all, by the nature of the harmonic measure, if each particle of the aggregate would be allowed to duplicate itself with a constant rate, regardless of the other existing particles, this would result in a faster expected growth of $X(t)$. Consequently, it is enough to prove this lemma for the simpler model in which the harmonic measure is replaced with the usual length measure on the boundary of the aggregate. By definition of the *front* of the aggregate, we may only consider particles in $\mathbf{Fr}(A(t))$ since only these can cause $X(t)$ to increase by duplicating. Lemma 2.1 shows us that a particle whose height is y is expected to duplicate to a particle at horizontal distance Cy for some fixed $C > 0$, which implies that the total expected horizontal growth of the aggregate at unit time is bounded by the sum $\sum_{p \in \mathbf{Fr}(A(t))} Cy(p)$. The geometry of the front of the aggregate only allows a constant number of particles at a given height, which will allow us to bound this sum by that of a geometric sequence, which only depends on the largest summand. In other words the expected growth will be bounded by the height of $\mathbf{Fr}(A(t))$.

We will first need the following intermediate, technical result, whose proof is postponed to the end of the section.

Lemma 3.2 *For all* $t \geq 0$ *and given any aggregate* $A(t)$*, there exist constants*

$C, \varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$

$$\mathbb{P}\big(X(t+\varepsilon) - X(t) > \alpha \mid \mathcal{F}_t\big) \leq C\varepsilon \min\left(\alpha^{-2}, 1\right), \quad \forall \alpha > 0.$$

Proof of lemma 3.1:
Fix a time $t > 0$ and an aggregate $A(t)$. For all $s > 0$, define the set

$$B_s = \big\{(x,y) \in \partial\mathcal{B}(A(t));\ |x| - X(t) \geq s\big\}.$$

According to formulas (3) and (4), one has

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\mathbb{P}\big(X(t+\epsilon) - X(t) \geq s \mid \mathcal{F}_t\big) = \mathcal{M}_{\mathcal{B}(A(t))}(B_s).$$

Using lemma 3.2, we know that there exist constants $\varepsilon_0, C > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\int_{s=0}^{\infty} \frac{1}{\epsilon}\mathbb{P}\big(X(t+\epsilon) - X(t) \geq s \mid \mathcal{F}_t\big) ds < C.$$

Consequently, we may use the dominated convergence theorem to get

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\mathbb{E}\big(X(t+\epsilon) - X(t) \mid \mathcal{F}_t\big) = \tag{6}$$

$$\lim_{\epsilon \to 0^+} \int_{s=0}^{\infty} \frac{1}{\epsilon}\mathbb{P}\big(X(t+\epsilon) - X(t) \geq s \mid \mathcal{F}_t\big) ds =$$

$$\int_{s=0}^{\infty} \mathcal{M}_{\mathcal{B}(A(t))}(B_s) ds.$$

Next, using lemma 2.1, we learn that for two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+^2$ one has,

$$d_H((x_1, y_1), (x_2, y_2)) \leq 2 \Rightarrow |x_1 - x_2| \leq C_1 y_1. \tag{7}$$

It follows that, using the definition of $\mathbf{Fr}(A(t))$,

$$B_s \subset \bigcup_{\substack{(x,y) \in \mathbf{Fr}(A(t)) \\ |x| + C_1 y \geq X(t) + s}} \partial\mathcal{B}\left(\{(x,y)\}\right)$$

for all $s > 0$. Next, observe that for all $(x,y) \in \mathbb{R}_+^2$, one has by definition

$$\mathcal{M}_{\mathcal{B}(A(t))}(\partial\mathcal{B}(\{x,y\})) \leq \mathcal{M}_{\mathcal{B}(\{x,y\})}(\partial\mathcal{B}(\{x,y\})) =: P_0. \tag{8}$$

where $P_0 > 0$ is a universal constant (in particular, it does not depend on (x,y)). A combination of the two above equations teaches us that

$$\mathcal{M}_{\mathcal{B}(A(t))}(B_s) \leq \sum_{\substack{(x,y) \in \mathbf{Fr}(A(t)) \\ |x| + C_1 y \geq X(t) + s}} \mathcal{M}_{\mathcal{B}(\{x,y\})}(\partial\mathcal{B}(\{x,y\})) \leq$$

$$\#\big\{(x,y) \in \mathbf{Fr}(A(t));\ \ |x| + C_1 y \geq X(t) + s\big\} P_0 \leq$$

$$\#\big\{(x,y)\in \mathbf{Fr}(A(t));\quad C_1 y \geq s\big\}P_0.$$

A combination of the above inequality with (6) yields

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\mathbb{E}\big(X(t+\epsilon)-X(t)\mid \mathcal{F}_t\big)\leq \tag{9}$$

$$P_0\int_{s=0}^{\infty}\#\big\{(x,y)\in \mathbf{Fr}(A(t));\quad C_1 y\geq s\big\}ds =$$

$$P_0 C_1 \sum_{(x,y)\in \mathbf{Fr}(A(t))} y.$$

We turn to estimate the above sum. Recall the definition of $\mathbf{Fr}(A(t))$ and observe that lemma 2.1 also implies

$$(x,y)\in \mathbf{Fr}(A(t)) \Rightarrow X(t)-C_1 y \leq |x| \leq X(t). \tag{10}$$

Now, for any number $K>0$, define

$$F(K)=\Big\{(x,y)\in \mathbb{R}^2_+;\quad X(t)-C_1 y\leq |x|\leq X(t) \text{ and } K/2\leq y\leq K\Big\}.$$

Fact 2.2 teaches us that the hyperbolic volume of $F(K)$ does not depend on K, as a dilation of the number K corresponds to rescaling of each connected component of $F(K)$ about a point on the x-axis. Since these sets are compact and separated from the X axis, they have a finite volume. It is thus clear that the cardinality of any set of disjoint d_H-balls of radius 1 whose centers are in $F(K)$ is bounded by some universal constant C_2 (which does not depend of K). Consequently,

$$\sum_{(x,y)\in \mathbf{Fr}(A(t))\cap F(K)} y \leq C_2 K. \tag{11}$$

Note that by equation (10), we have

$$\mathbf{Fr}(A(t))\subset \bigcup_{j=0}^{\infty} F(\tilde{Y}(t)2^{-j}).$$

Using this fact with (9) and (11) finally gives

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\mathbb{E}\big(X(t+\epsilon)-X(t)\mid \mathcal{F}_t\big)\leq P_0C_1\sum_{(x,y)\in \mathbf{Fr}(A(t))} y =$$

$$P_0C_1\sum_{j=0}^{\infty}\sum_{\substack{(x,y)\in\\ \mathbf{Fr}(A(t))\cap F(\tilde{Y}(t)2^{-j})}} y \leq 2P_0C_1C_2\tilde{Y}(t).$$

and the lemma is complete. □

The next bound can be regarded as a lower bound for the rate of growth of $Y(t)$, whose proof relies heavily on the conformity of the map H. This bound is a consequence of a rather straightforward geometric fact about the harmonic measure: given a rectangle of the form $K = [-M, M] \times [0, 1]$, consider the harmonic measure $\mathcal{M}_{K,\mathbb{R}\times\{0\}}$ evaluated on different points of its upper edge $[-M, M] \times \{1\}$. The density of this measure at a point $(x, 1) \in \partial K$ is bounded from below by $c(M - |x| + 1)^{-1}$. Recall that, by definition, the aggregate $A(t)$ is contained in the rectangle $[-X(t), X(t)] \times [0, Y(t)]$. This means that the probability of the aggregate's top-most particle (the one attaining $Y(t)$) to duplicate itself upwards and thus increase $Y(t)$ by a constant multiplicative factor is bounded from below by $cY(t)/(X(t) + Y(t))$.

We will need a bound that deals with a slightly more general scenario, in which one has the additional information that a constant fraction of the aggregate's height is attained at a point close to the front of the aggregate, say located at $X(t) - L$. In this case, the above estimate on the harmonic measure gives a rate of growth of cL^{-1}. However, since we do not assume here that the aggregate is entirely contained in the corresponding rectangle the argument will have to be slightly more delicate.

Lemma 3.3 *There exists a constant $c > 0$ such that for all $t \geq 0$ one has*

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}[Y(t+\epsilon) > (1+c)Y(t)|\ A(t)] > c\frac{Y(t)}{Y(t)+X(t)}. \tag{12}$$

Furthermore, for any constant $\Delta \geq 1$, there exists a constant $c(\Delta)$ (which depends only on Δ) such that the following holds: Let $L \in \mathbb{R}$ and suppose that $Y(t) \leq \Delta Y_L^+(t)$. Then,

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}[Y^+_{L-10Y(t)}(t+\epsilon) \geq (1+c)Y_L^+(t)|\ A(t)] > c(\Delta)\frac{Y(t)}{Y(t)+X(t)-L}. \tag{13}$$

Likewise, if $Y(t) \leq \Delta Y_L^-(t)$ then

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\mathbb{P}[Y^-_{L+10Y(t)}(t+\epsilon) \geq (1+c)Y_L^-(t)|\ A(t)] > c(\Delta)\frac{Y(t)}{Y(t)+X(t)+L}. \tag{14}$$

Proof: We will prove formula (13). The proof of (14) is completely analogous, and the fact that (12) is true will follow immediately from (13) by taking $L = -X(t)$ and $\Delta = 1$.

Let (x_0, y_0) be the point attaining the maximum $y_0 = Y_L^+(t)$. Denote $B = B_H((x_0, y_0), 2)$ and $y_1 = \max\{y; \exists x \text{ s.t. } (x, y) \in B\}$.

Fix a constant $c > 0$, which will be the universal constant in (13), whose value will be chosen later. If there exists a point $(x, y) \in A(t)$ such that $x \geq L - 10Y(t)$ and $y \geq (1+c)y_0$ then the event in (13) holds almost surely, and we're done. Therefore, we may assume from this point on that this is not the case, hence, we can assume from now on that

$$A(t) \cap [L - 10Y(t), \infty) \times [(1+c)y_0, \infty) = \emptyset. \tag{15}$$

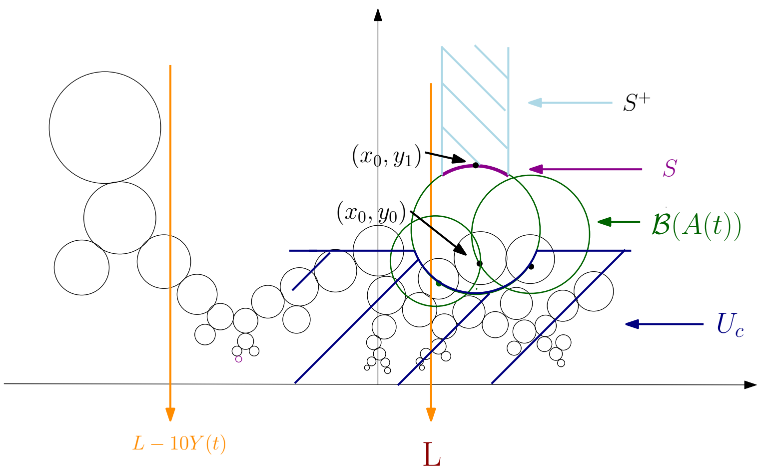

Figure 3: *Some of the definitions in the proof of lemma 3.3 .*

Define the set

$$U_c = \Big([L - 10Y(t), \infty) \times [0, (1+c)y_0]\Big) \setminus B$$

It is easy to verify that $d_H((x_0, y_1), U_0) \geq 2 + c_1$ for a universal constant $c_1 \geq 0$ (recall that $d_H(x_0, y_0), (x_0, y_1)) = 2$ and see figure 3). Therefore, by continuity by the invariance of the metric to rescaling around the point $(x_0, 0)$ (which follows from fact 2.2), we can choose the constant $c > 0$ to be a small enough universal constant so that

$$d_H((x_0, y_1), U_c) \geq 2 + c_2 \tag{16}$$

for some universal constant $c_2 > 0$. Define,

$$S = \{(x, y) \in \partial B;\ d_H((x, y), (x_0, y_1)) \leq c_2/2\}$$

(also see figure 3). Equations (15) and (16) imply that $S \subset \partial\mathcal{B}(A(t))$. Note that this fact does cease to be true if we make the constants c, c_2 smaller. Therefore, by decreasing the value of these constants if necessary, we can also assert that

$$S \subset [L - 10y_0, \infty) \times [(1+c)y_0, \infty) \tag{17}$$

(here we also used the fact that $x_0 \geq L$). Thanks to the last equation and in view of equations (3) and (4), we have

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}[Y^+_{L-10y_0}(t+\epsilon) \geq (1+c)y_0 |\ A(t)] =$$

$$\mathcal{M}_{\mathcal{B}(A(t))}\Big(\big\{(x, y) \in \partial\mathcal{B}(A(t));\ \ y \geq (1+c)y_0 \text{ and } x \geq L - 10y_0\big\}\Big) \geq$$

$$\mathcal{M}_{\mathcal{B}(A(t))}(S).$$

It is therefore enough to prove that,

$$\mathcal{M}_{\mathcal{B}(A(t))}(S) \geq c(\Delta)\frac{Y(t)}{Y(t)+X(t)-L}. \tag{18}$$

Our next goal thus to give a lower bound for $\mathcal{M}_{\mathcal{B}(A(t))}(S)$. We do this in three steps.

Step 1: Define the set

$$F = [x_0, \infty) \times (2\Delta y_1, \infty).$$

in this step we aim at showing that,

$$\mathcal{M}_{\mathcal{B}(A(t)),F}(S) \geq c(\Delta) \tag{19}$$

for some $c(\Delta) > 0$ which is a constant only depending on Δ. Define,

$$S^+ := \bigcup_{(x,y)\in S} \{x\} \times (y, \infty)$$

and

$$E = \mathbb{R}_+^2 \setminus S^+,$$

The assumptions (15) and (16) along with lemma 2.1 ensure that

$$S^+ \cap \mathcal{B}(A(t)) = \emptyset,$$

(see figure 3) which implies that

$$\mathcal{M}_{\mathcal{B}(A(t)),F}(S) \geq \mathcal{M}_{E,F}(S). \tag{20}$$

In order to give a bound for the right hand side, we consider the transformation

$$T : (x, y) \to ((x - x_0)/y_0, y/y_0).$$

By fact 2.2, we know that T is an isometry. Now, it is not hard to verify that the sets $T(E)$ and $T(F)$ do not actually depend on the aggregate $A(t)$, they only depend on the constant Δ. It follows that there exists some constant $c(\Delta)$ such that,

$$\mathcal{M}_{E,F}(S) = \mathcal{M}_{T(E),T(F)}(S) = c(\Delta) > 0.$$

It is also easy to verify (by drawing a picture) that $c(\Delta) > 0$ for all $\Delta \geq 1$. By combining this with (20), equation (19) is proven.

Step 2: Define,

$$G = (L, \infty) \times (2(\Delta y_1 + X(t) - L), \infty),$$

and

$$f(x,y) = \mathbb{P}\begin{pmatrix} \text{Brownian motion started at } (x,y) \\ \text{reaches } G \text{ before reaching } \mathcal{B}(A(t)) \end{pmatrix}.$$

The aim of this step is to estimate $\inf_{(x,y)\in F} f(x,y)$. Along with the previous step this will give us a bound for $\mathcal{M}_{\mathcal{B}(A(t)),G}(S)$.

In order to do this, we use the fact that y coordinate of the Brownian motion is a martingale whose starting value is at least $2\Delta y_1$, together with the optional stopping theorem, to deduce that the y coordinate of the Brownian motion hits the set $2(\Delta y_1 + X(t) - L)$ before hitting the set $[0, \Delta y_1]$ with probability at least $p' := \frac{\Delta y_1}{2(\Delta y_1 + X(t) - L)}$. Now since, by definition, $x_0 \geq L$, it follows from the symmetry of the x coordinate of the Brownian motion and from the independence between the two coordinates, that

$$\inf_{(x,y)\in F} f(x,y) \geq \frac{\Delta y_1}{4(\Delta y_1 + X(t) - L)} \geq c' \frac{Y(t)}{Y(t) + X(t) - L}. \tag{21}$$

where $c' > 0$ is a universal constant.

Step 3: In view of that last step, it is enough to estimate the probability that a Brownian motion starting from any point in G will hit the set $\mathbb{H}^2(\infty)$ before hitting $\mathcal{B}(A(t))$. To show that, we define

$$H = (-\infty, \Delta y_1 + X(t)) \times [0, \Delta y_1 + X(t) - L],$$

Note that $A(t) \subset H$, so it is enough to estimate the probability of reaching $\mathbb{R}^2_+(\infty)$ before hitting H. The key in this step is to define,

$$T : (x,y) \to ((x-L)/(\Delta y_1 + X(t) - L), y/(\Delta y_1 + X(t) - L)).$$

Again, by fact 2.2, we know that T is an isometry. Moreover,

$$T(G) = [0,\infty) \times [2,\infty), \quad T(H) = (-\infty, 1] \times [0,1].$$

Viewed this way, it is clear that thanks to the conformal invariance there exists a universal constant $c_3 > 0$ such that the probability of a Brownian motion starting from any point in G to hit to x axis before hitting H is greater than c_3. Plugging this fact together with (19) and (21) finally gives,

$$\mathcal{M}_{\mathcal{B}(A(t))}(S) \geq c_3 c(\Delta) c' \frac{Y(t)}{Y(t) + X(t) - L} \tag{22}$$

which is exactly (18), and the proof is complete. □

Remark 3.4 *It is not hard to verify that the above proof gives us a rather poor dependence of the constant $c(\Delta)$ on Δ, namely, $c(\Delta) \sim \exp(-\Delta^2)$. However, it is possible to prove that, in fact, one can have the dependence $c(\Delta) \sim \Delta^{-1}$. Since this difference will only affect the magnitude of the universal constant we get in our main theorem, we choose to only present the above proof, which is simpler.*

Finally, we will need the following lemma which will allow us to use the optional stopping theorem.

Lemma 3.5 *Fix an aggregate $A(t)$ at time t, and fix a number $x_0 > 0$. Define the stopping time,*

$$T = \min\{s \geq t; X(s) > x_0 \ |A(t)\}$$

Then,

$$\mathbb{E}[T] \leq \infty.$$

The proof is not hard but rather technical, and we only provide a sketch. One way to explain the reason behind this fact is that the equilibrium measure on a geodesic line in the hyperbolic plane exists, and is a constant multiple of the length measure. As a result it follows that the convex hull of the aggregate encapsulates any ball within a time whose expectation is finite.
Proof: (sketch)
Consider the domain

$$L = \{(x,y) \in \mathbb{R}^2_+; \ x^2 + y^2 > 1\}$$

It is well known that for any two geodesic curves, there exists an isometry of the hyperbolic plane sending the first to the second. Consequently there is a bijective isometry T such that

$$T\big(\{(x,y) \in \mathbb{R}^2_+; \ x \geq x_0\}\big) = L.$$

Therefore, by considering the initial aggregate $T(A(t))$, without loss of generality we may assume that

$$T' = \min\{t \geq 0; A(t) \cap L \neq \emptyset \ |A(0)\}$$

and prove that $\mathbb{E}[T'] \leq \infty$ for an arbitrary initial aggregate $A(0)$. Defining,

$$T_1 = \min\{t; \ X(t) \geq 1 \text{ or } Y(t) \geq 1\}$$

It is clear that $T' \leq T_1$, therefore it is enough to show that $\mathbb{E}[T_1] < \infty$. Lemma 3.3 teaches us that for any $t \leq T_1$ one has,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{P}(Y(t+\epsilon) \geq (1+c)Y(t)|\mathcal{F}_t) \geq cY(0)$$

for a universal constant $c > 0$. It is not hard to check that the last equation implies that there exists a constant c_1 which only depends on $Y(0)$ such that,

$$\mathbb{P}(Y(t+1) \geq 1 \text{ or } X(t+1) \geq 1|\mathcal{F}_t) \geq c_1$$

for all $t > 0$. In other words,

$$\mathbb{P}(T_1 < t+1|\mathcal{F}_t) \geq c_1, \quad \forall t \geq 0.$$

The above equation implies that T_1 has a sub-exponential tail, and therefore has a finite expectation. $\square$

Proof of lemma 3.2:
Fix $t \geq 0$ and fix an aggregate $A(t)$. Define $n_0 = i(t) = \#A(t) + 1$. We begin with noting that lemma 2.1 teaches us that

$$A_{n_0+n} \subset \mathbb{R} \times [0, Y(t)10^n], \quad \forall n \geq 1$$

and therefore

$$X(t_{n_0+n}) \leq X(t) + 7Y(t)10^n, \quad \forall n \geq 1 \tag{23}$$

almost surely. We claim that, in order to conclude the lemma, it will be enough to show that there exist constants $C', \varepsilon_0 > 0$ (which may depend on $A(t)$) such that

$$\mathbb{P}(t_{n_0+n} - t < \varepsilon | A(t)) < C' \varepsilon 10^{-2n}, \quad \forall \varepsilon < \varepsilon_0. \tag{24}$$

Indeed, for all $\alpha > 0$, write

$$n = \max \left(\left\lfloor \frac{\log(\alpha/7Y(t))}{\log 10} \right\rfloor, 1 \right).$$

Then, thanks to (23),

$$\mathbb{P}\big(X(t+\varepsilon) - X(t) > \alpha \mid A(t)\big) \leq \mathbb{P}(t_{n_0+n} \leq t + \epsilon | A(t))$$

and plugging (24) to this would prove the lemma.

We therefore move on to the proof of (24). Recall that for all j, the difference $t_j - t_{j-1}$ is an exponentially-distributed random variable whose expectation is $Cap(A_j)^{-1}$. Moreover, we clearly have by the definition of the harmonic measure

$$Cap(A_j) = \mathcal{M}_{\mathcal{B}(A_j)}(\partial \mathcal{B}(A_j)) = \sum_{p \in A_j} \mathcal{M}_{\mathcal{B}(A_j)}(\partial \mathcal{B}(\{p\})) \leq$$

$$\sum_{p \in A_j} \mathcal{M}_{\mathcal{B}(\{p\})}(\partial \mathcal{B}(\{p\})) = (j+1)C_0, \quad \forall j \geq 0.$$

where $C_0 > 0$ is some universal constant. It follows that for all $j < n$, the expectation of $t_{j+1} - t_j$ is at least $\frac{1}{nC_0}$. An elementary fact about exponentially-distributed variables is that

$$0 < a < b \Rightarrow \mathbb{P}(E[b] < t) < \mathbb{P}(U([0,a]) < t), \quad \forall t > 0.$$

where $U([0,a])$ represents a uniformly-distributed point in the interval $[0,a]$. It follows that

$$\mathbb{P}(t_{n_0+n} - t_{n_0+1} < \varepsilon) \leq \mathbb{P}\left(\sum_{i=1}^{n-1} X_i < \varepsilon \right), \quad \forall \varepsilon > 0, \quad \forall n \geq 1$$

where X_i are independent variables whose distribution is uniform over the interval $\left[0, \frac{1}{C_0(n_0+n)}\right]$. An application of a standard large-deviation principle teaches us that there exists some $\varepsilon_0 > 0$ (which may depend on n_0) such that

$$\mathbb{P}(t_{n_0+n} - t_{n_0+1} < \varepsilon \ |A(t_{n_0+1})) \leq 10^{-2n}$$

for all $\varepsilon < \varepsilon_0$ and for all $n > 1$. Moreover, since the density of the exponential distribution is bounded, we have

$$\mathbb{P}(t_{n_0+1} - t < \varepsilon | A(t)) \leq C_2 \varepsilon, \quad \forall \varepsilon > 0$$

for some constant C_2. Plugging the two above estimates finally establishes equation (24) and the lemma is complete. □

4 The process of ratios

For all $t \geq 0$, define $R(t) = X(t)/Y(t)$. The goal of this section is to prove the following theorem:

Theorem 4.1 *There exists a universal constant $C > 0$ such that the following holds:*
Let $t \geq 0$ be a time and fix any initial configuration $A(t)$. In addition, fix a number X_0 such that $X_0 \geq X(t)$. Define, for every non-negative integer i,

$$\tau_i = \min\{s; \ X(s) \geq 2^i X_0\}.$$

Then one has for all i,

$$\mathbb{E}[R(\tau_{i+1})|A(t)] \leq C + 0.9 \mathbb{E}[R(\tau_i)|A(t)].$$

The next lemma, which is one of the two main ingredients in the proof of the theorem, gives upper bounds on the expected growth of $R(t)$. Its proof relies on a combination of lemmas 3.1 and 3.3.

Lemma 4.2 *There exist universal constants $\delta, c_1, c_2 > 0$ such that one has for all $t \geq 0$,*

$$\limsup_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{E}[R(t+\epsilon) - R(t)|A(t)] < +c_1. \tag{25}$$

Moreover, defining the following event,

$$E(t) := \{\tilde{Y}(t) < \delta Y(t)\}, \tag{26}$$

whenever the event $E(t)$ holds one has

$$\limsup_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{E}[R(t+\epsilon) - R(t)|A(t)] < -c_2. \tag{27}$$

Proof: Denote,

$$F(\epsilon) = \{Y(t+\epsilon) \geq (1+c)Y(t)\}$$

where c is the constant from equation (12). According to lemma 3.3 we have

$$\mathbb{P}(F(\epsilon)|A(t)) \geq c\epsilon \frac{Y(t)}{X(t)+Y(t)} + o(\epsilon). \tag{28}$$

Next, we use lemma 3.1 to deduce that

$$\mathbb{E}[X(t+\epsilon) - X(t)|\ A(t)] \leq C\tilde{Y}(t)\epsilon + o(\epsilon) \tag{29}$$

We write,

$$\mathbb{E}[X(t+\epsilon)/Y(t+\epsilon)|A(t)] =$$
$$\mathbb{E}\left[\frac{X(t+\epsilon)}{Y(t+\epsilon)}\mathbf{1}_{F(\epsilon)^C}\middle|\mathcal{F}_t\right] + \mathbb{E}\left[\frac{X(t+\epsilon)}{Y(t+\epsilon)}\mathbf{1}_{F(\epsilon)}\middle|\mathcal{F}_t\right] \leq$$
$$\frac{1}{Y(t)}\mathbb{E}[X(t+\epsilon)\mathbf{1}_{F(\epsilon)^C}|\mathcal{F}_t] + \frac{1}{(1+c)Y(t)}\mathbb{E}[X(t+\epsilon)\mathbf{1}_{F(\epsilon)}|\mathcal{F}_t] =$$
$$\frac{1}{Y(t)}\mathbb{E}[X(t+\epsilon)|\mathcal{F}_t] - \left(1 - \frac{1}{1+c}\right)\frac{1}{Y(t)}\mathbb{E}[X(t+\epsilon)\mathbf{1}_{F(\epsilon)}|\mathcal{F}_t] =$$
$$\frac{X(t)}{Y(t)} + \frac{1}{Y(t)}\mathbb{E}[X(t+\epsilon) - X(t)|\mathcal{F}_t] - \frac{c}{1+c}\frac{1}{Y(t)}\mathbb{E}[X(t+\epsilon)\mathbf{1}_{F(\epsilon)}|\mathcal{F}_t] \leq$$
$$\frac{X(t)}{Y(t)} + \frac{1}{Y(t)}\mathbb{E}[X(t+\epsilon) - X(t)|\mathcal{F}_t] - \frac{c}{1+c}\frac{X(t)}{Y(t)}\mathbb{P}(F(\epsilon)|\mathcal{F}_t).$$

Plugging equations (28) and (29) into this formula gives

$$\mathbb{E}[X(t+\epsilon)/Y(t+\epsilon)|\mathcal{F}_t] \leq \frac{X(t)}{Y(t)} + C\frac{\tilde{Y}(t)}{Y(t)}\epsilon - c_3\epsilon\frac{X(t)}{X(t)+Y(t)} + o(\epsilon)$$

for a universal constant $c_3 > 0$. Since $\tilde{Y}(t) \leq Y(t)$ by definition, equation (25) follows. To prove the second part of the lemma, the reader may easily verify that by the definition of the event $E(t)$, whenever $E(t)$ holds with $\delta < 1$, one has

$$X(t) > c_4 Y(t) \tag{30}$$

for a universal constant $c_4 > 0$. Moreover, by definition of the event $E(t)$ one has

$$\frac{\tilde{Y}(t)}{Y(t)} \leq \delta.$$

Plugging these two facts gives,

$$\mathbb{E}[X(t+\epsilon)/Y(t+\epsilon)|\mathcal{F}_t] \leq \frac{X(t)}{Y(t)} + \epsilon\left(C\delta - c_3\frac{1}{1+c_4^{-1}}\right) + o(\epsilon).$$

Thus, by choosing δ to be a small enough universal constant, the second part of the lemma is also established. $\square$

As a corollary, we get

Corollary 4.3 *There is a universal $\delta > 0$ such that if we define the event $E(t)$ as in (26), then the following holds; suppose $A(t)$ is such that $E(t)$ holds. Define,*

$$T = \min\{s > t;\ E(s) \text{ does not hold or } X(s) > 1.1X(t) \text{ or } Y(s) > 100Y(t)\}$$

Then one has,

$$\mathbb{P}(X(T) \geq 1.1X(t)|\ \mathcal{F}_t) \leq 0.01.$$

Proof: Using the optional stopping theorem (which is justified thanks to lemma 3.5) with the result of the previous lemma, we have for a small enough choice of δ,

$$\mathbb{E}[X(T)/Y(T)|\ \mathcal{F}_t] \leq X(t)/Y(t) - c_2\mathbb{E}[T-t].$$

Since the left hand side cannot be negative,

$$\mathbb{E}[T-t] \leq \frac{1}{c_2}\frac{X(t)}{Y(t)}.$$

According to lemma 3.1, the following process is a super-martingale:

$$s \to X(t+s) - X(t) - \int_t^{t+s} C\tilde{Y}(r)dr. \tag{31}$$

Therefore, by the optional stopping theorem, and since for every $t \leq s < T$ we have by definition $\tilde{Y}(s) \leq \delta Y(s) \leq 100\delta Y(t)$,

$$\mathbb{E}[X(T) - X(t)|\ \mathcal{F}_t] \leq C\mathbb{E}\left[\int_t^T \tilde{Y}(s)ds\right] \leq 100C\delta Y(t)\mathbb{E}[T-t] \leq \tag{32}$$

$$C'\delta Y(t)\frac{X(t)}{Y(t)} \leq C'\delta X(t).$$

Again, by choosing δ small enough (note that it can always be made smaller without affecting the result of the previous lemma), we can make sure that,

$$\mathbb{E}[X(T) - X(t)|\ \mathcal{F}_t] \leq 0.001X(t),$$

and since $X(t)$ is increasing it follows by Markov's inequality that

$$\mathbb{P}(X(T) \geq 1.1X(t)|\ \mathcal{F}_t) \leq 0.01$$

which is the promised result. □

From this point on, we assume that the event $E(t)$ is defined as in equation (26), and the constant δ is a fixed positive universal constant taken to be small enough such that the above corollary holds true.

In view of the above corollary, the only times we have to worry about are whenever $E(t)$ does not hold. The next lemma in some sense complements the previous one, ensuring us that also if $E(t)$ does not hold, we should expect $X(t)/Y(t)$ to decrease after a while (due to completely different reasons), providing that it is not too small.

Lemma 4.4 *There exists a universal constant* $\Gamma > 0$ *such that the following holds: Assume that for some* $t_0 \geq 0$*,* $E(t_0)$ *doesn't hold and* $X(t_0)/Y(t_0) > \Gamma$*, then*

$$\mathbb{E}[X(t_1)/Y(t_1)|A(t_0)] < \frac{1}{4}X(t)/Y(t)$$

where $t_1 = \min\{s;\ X(s) \geq 1.1X(t_0)\}$*.*

Before we move on to the proof, let us try to explain why this bound should be correct. Whenever the event $E(t)$ does not hold, we know that there is a particle p located close to the front of the aggregate which, up to a constant, attains the vertical height of the entire aggregate, $Y(t)$. In this case, we can effectively "restart" the growth process by only considering the part close to the front of the aggregate, while ignoring the rest of it: as a consequence of lemma 3.3, we know that parts of the aggregate located close to the front have a vertical growth rate which is proportional only to the distance from the front. This means that when considering only the latter part of the aggregate, the growth rate will no longer be a function of $X(t)$. Now, as a result of Lemma 2.1, the vertical growth of the particles is multiplicative in the sense that in order for $Y(t)$ to multiply itself by a constant, it is enough for the particle p to duplicate itself upwards a constant number of times. From this point on the proof relies on a compactness-type argument: we know that the top particle has to duplicate a constant number of times, while the rate of duplication is independent of $X(t)$. Therefore, it is enough to establish that the universal rate of growth is such that any number of duplications will occur eventually, with high probability. The time that it takes, which affects the increment of $X(t)$, can then be absorbed into the constant Γ; When this constant is big enough, a prescribed additive growth of $X(t)$ results in a small multiplicative growth which does not significantly affect $R(t)$.

The proof will be divided into a few steps. In the first step we demonstrate that it suffices to show that there exists a constant $C > 0$ such that $Y(t)$ multiplies itself by some constant, say 5, before $X(t)$ grows (additively) by C. The second and third steps deal with the rate of duplications of the particle p mentioned above. It is shown that within any time interval in which $X(t) - X(t_0)$ multiplies itself by two, there is at least a constant probability for the particle p to duplicate itself once. This is the "compactness" to which we were referring above, as this rate does not depend on $X(t_0)$. In the fourth and last step, we iteratively use this fact to conclude that there is a probability bounded from below for any constant number of multiplications when the time interval is large enough.

Proof of lemma 4.4: Since the claim is invariant to rescaling around the origin, we may assume that $Y(t_0) = 1$. Define

$$T = \min\{s; Y(s) > 5\}.$$

Step 1: We claim that it is enough to show that there exists a universal constant $C > 0$ such that

$$\mathbb{P}(X(T) < X(t_0) + C) > 0.99. \tag{33}$$

Let us explain why this fact suffices in order to complete the proof. Since almost surely only one particle can be added at a time and assuming that Γ is a large enough constant, an application of lemma 2.1 gives

$$X(t_1) \leq 1.1X(t_0) + 10Y(t_0) \leq 1.11X(t_0).$$

Also, if Γ is large enough then we can assume that $X(t_0)+C < 1.1X(t_0)$ which implies that

$$\mathbb{P}(T < t_1) > 0.99.$$

Using these two facts, we can thus estimate,

$$\mathbb{E}[X(t_1)/Y(t_1)|\mathcal{F}_{t_0}] =$$

$$\mathbb{E}[X(t_1)/Y(t_1)\mathbf{1}_{\{T<t_1\}}|\mathcal{F}_{t_0}] + \mathbb{E}[X(t_1)/Y(t_1)\mathbf{1}_{\{t_1 \leq T\}}|\mathcal{F}_{t_0}] \leq$$

$$\mathbb{E}[1.11X(t_0)/(5Y(t_0))\mathbf{1}_{\{T<t_1\}}|\mathcal{F}_{t_0}] + \mathbb{E}[1.11X(t_0)/Y(t_0)\mathbf{1}_{\{t_1 \leq T\}}|\mathcal{F}_{t_0}] \leq$$

$$\left(\frac{1.11}{5} + 1.11 \cdot 0.01\right) X(t_0)/Y(t_0) < \frac{1}{4}X(t_0)/Y(t_0).$$

which is the result.

Step 2: Define $Z(s) = X(s) - X(t_0) + 5$. According to the assumption that $E(t_0)$ does not hold and by definition of $\mathbf{Fr}(A(t))$, we know that either $Y^+_{X(t_0)-5}(t_0)$ or $Y^-_{-X(t_0)+5}(t_0)$ are greater than the universal constant $\delta > 0$. Assume without loss of generality that

$$Y^+_{X(s)-5}(t_0) \geq \delta. \tag{34}$$

(the assumption is legitimate since the model is invariant under reflection around the y axis). Define $\Delta = 100\delta^{-1}$. The assumption (34), together with the definitions of $Y^+_L(s)$ and T, implies that for any $L < X(t_0) - 5$ and for any $t_0 \leq s \leq T$ one has $Y(s) \leq \Delta Y^+_L(s)$. Therefore, we can use the second part of lemma 3.3 to deduce that there exists a universal constant $c_1 > 0$ such that for all $t_0 \leq s < T$ and for all $L < X(t_0) - 5$ one has

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\mathbb{P}[Y_{L-50}(s+\epsilon) > (1+c_1)Y_L(s)|\ A(s)] > c_1/(5 + X(s) - L). \tag{35}$$

Here, we used the assumption that for $s < T$, one has $Y(s) \leq 5$.

Define,

$$L_0 = 5 + 50\log_{(1+c_1)} \Delta.$$

At this point the reader may regard L_0 as some large universal constant, its significance will become clear later on. Let L be a number satisfying

$$X(t_0) - L_0 \leq L \leq X(t_0) - 5. \tag{36}$$

Also, fix a time $t_0 \leq t < T$ and define

$$T_1 = \min\{s|\ Z(s) > 2Z(t)\}.$$

Let $N(s)$ be a random variable counting the number of "jumps" up to time s hence,

$$N(s) = \#\left\{ r \in [t, s]; Y_{L-50}^{+}(r) \geq (1 + c_1) \lim_{\epsilon \to 0^+} Y_L^{+}(r - \epsilon) \right\}.$$

Our next goal will be to show that there exists a universal constant $c > 0$ such that

$$\mathbb{P}(T < T_1 \text{ or } N(T_1) \geq 1 \; | \mathcal{F}_t) \geq c. \tag{37}$$

which will be done in the next step.

Step 3: To prove the last formula, we begin by defining

$$M(s) = N(s) - c_1(s - t)/(2Z(t) + L_0).$$

By equation (35) and by the fact that $L \geq X(t_0) - L_0$, we learn that $M(s)$ is a sub-martingale in the interval $[t, T_1 \wedge T]$. Thus, by the optional stopping theorem (which we can use thanks to lemma 3.5), one has

$$\mathbb{E}[N(T_1 \wedge \tau) | \mathcal{F}_t] \geq \mathbb{E}[(T_1 \wedge \tau - t) | \mathcal{F}_t] c_1 / (2Z(t) + L_0)$$

where $\tau = \min\{t | N(t) \geq 1\} \wedge T$. Consequently, for all $\alpha > 0$, we may calculate

$$\mathbb{P}(\tau < T_1 | \mathcal{F}_t) \geq \mathbb{P}(N(T_1) \geq 1 | \mathcal{F}_t) \geq \mathbb{E}[N(T_1 \wedge \tau) | \mathcal{F}_t] \geq$$

$$\mathbb{E}[(T_1 \wedge \tau - t) | \mathcal{F}_t] c_1 / (2Z(t) + L_0) \geq$$

$$\mathbb{E}[(T_1 - t) \mathbf{1}_{\{\tau > T_1\}} | \mathcal{F}_t] c_1 / (2Z(t) + L_0) \geq$$

$$(\mathbb{P}(T_1 - t > 2\alpha Z(t) | \mathcal{F}_t) - \mathbb{P}(T_1 > \tau | \mathcal{F}_t)) 2\alpha Z(t) c_1 / (2Z(t) + L_0) \geq$$

(using the assumption $Z(t) \geq 5$)

$$(\mathbb{P}(T_1 - t > \alpha 2 Z(t) | \mathcal{F}_t) - \mathbb{P}(T_1 > \tau | \mathcal{F}_t)) \alpha c_2,$$

for some universal constant $c_2 > 0$. Thus,

$$\mathbb{P}(\tau < T_1 | \mathcal{F}_t) \geq \alpha c_2 \mathbb{P}(T_1 - t > 2\alpha Z(t) | \mathcal{F}_t) / (1 + c_2 \alpha). \tag{38}$$

We now use lemma 3.1, combined with the fact that $Y(s) < 5$ for all $t \leq s \leq T$, according to which

$$\mathbb{E}(Z(s \wedge T) - Z(t) | \mathcal{F}_t) < C_1(s - t)$$

for a universal constant $C_1 > 0$. Taking $s = t + 2\alpha Z(t)$ and using Markov's inequality, we get

$$\mathbb{P}\Big(Z\big((t + 2\alpha Z(t)) \wedge T\big) > 2Z(t) \;\; | \; \mathcal{F}_t \Big) < 2C_1 \alpha.$$

Now, by the definition of T_1,

$$\Big\{Z\big((t+2\alpha Z(t))\wedge T\big) < 2Z(t)\Big\} \subseteq \Big\{Z(t+2\alpha Z(t)) < 2Z(t)\Big\} \cup \Big\{T < T_1\Big\}$$

so a union bound gives

$$\mathbb{P}(Z(t+2\alpha Z(t)) < 2Z(t)|\mathcal{F}_t) > 1 - 2C_1\alpha - \mathbb{P}(T < T_1|\mathcal{F}_t).$$

But, using the definition of T_1 once more, we know that

$$Z(t+2\alpha Z(t)) < 2Z(t) \Rightarrow T_1 \geq t + 2\alpha Z(t)o$$

and the last equation becomes

$$\mathbb{P}(T_1 - t > 2\alpha Z(t)|\mathcal{F}_t) \geq 1 - 2C_1\alpha - \mathbb{P}(T < T_1|\mathcal{F}_t).$$

Choosing α to be a small enough universal constant and plugging the above into (38) gives,

$$\mathbb{P}(\tau < T_1|\mathcal{F}_t) \geq \alpha c_3(1 - 2C_1\alpha - \mathbb{P}(T < T_1|\mathcal{F}_t)) \geq c_4(1 - \mathbb{P}(\tau < T_1|\mathcal{F}_t)),$$

where c_3, c_4 are universal constants. In other words, we have that

$$\mathbb{P}(\tau < T_1|\mathcal{F}_t) \geq c. \tag{39}$$

and equation (37) is proven.

Step 4: At this point, the strategy we will use in order to prove (33) is to repeat this argument again and again, for a sequence of times Q_i, until we accumulate enough "jumps" so that T is surely reached. Define,

$$Q_1 = \min\{s \geq t_0; Z(s) > 2Z(0)\}$$

and inductively,

$$Q_{i+1} = \min\{s \geq t_0; Z(s) > 2Z(Q_i)\}.$$

Also define I to be the largest integer i such that $Q_i < T$. By the definition of T and by lemma 2.1, we know that the (Euclidean) radius of any added ball is smaller than a constant, so we can easily deduce the "continuity" in the following sense,

$$Z(Q_{i+1}) < RZ(Q_i), \forall 1 \leq i < I,$$

where R is a universal constant. It follows that,

$$Z(Q_i) < 5R^i, \quad \forall 1 \leq i < I. \tag{40}$$

For all $i \in \mathbb{N}$ define N_i to be the number of "jumps" so far. In other words, define $N_0 = 0$ and (recursively)

$$N_i = \#\left\{j < i; \exists r \in (Q_j, Q_{j+1}] \text{ such that } Y^+_{L_j - 50}(r) \geq (1 + c_1) \lim_{\epsilon \to 0^+} Y^+_{L_j}(r - \epsilon)\right\}$$

where

$$L_j := X(t_0) - 5 - 50N_j.$$

Define also $Y_i = Y^+_{L_i}(Q_i)$, $Z_i = Z(Q_i)$ and $\mathcal{F}_i$ to be the σ-algebra generated by $A(Q_i)$. Observe that, by (34), the number of jumps needed in order to reach T is smaller than $\log_{(1+c_1)} \Delta$. So, by definition,

$$N_I \leq \log_{(1+c_1)} \Delta$$

which implies, by the definition of L_0 that

$$X(t_0) - L_0 \leq L_i \leq X(t_0) - 5, \quad \forall i \leq I.$$

The above equation asserts that (36) is fulfilled, so we may use equation (37) which translates to

$$\mathbb{P}(Y_{i+1} > (1 + c_1)Y_i | \mathcal{F}_i) > c, \ \forall 1 \leq i < I. \tag{41}$$

An application of, say, Hoeffding's inequality gives

$$\mathbb{P}\left(Y_k < (1 + c_1)^{\frac{kc}{2}} Y_0 \text{ and } I > k \middle| \mathcal{F}_{t_0}\right) < C_3 \exp(-c_3 k) \tag{42}$$

where $C_3, c_3 > 0$ are universal constants. Define $\alpha = \frac{2}{c} \log_{(1+c_1)} \Delta$. By (34), we know that

$$(1 + c_1)^{\frac{kc}{2}} Y_0 > 5, \quad \forall k > \alpha.$$

which by definition means that

$$I > k \Rightarrow Y_k \leq (1 + c_1)^{\frac{kc}{2}} Y_0, \quad \forall k > \alpha.$$

Equation (42) becomes,

$$\mathbb{P}(I > k | \mathcal{F}_{t_0}) < C_3 \exp(-c_3 k), \quad \forall k > \alpha.$$

Now choose k large enough universal constant such that $k > \alpha$ and also the right hand side of the above equation is smaller than 0.01 (this is possible since Δ and c_1 have been fixed as universal constants, so α is a universal constant). We get

$$\mathbb{P}(I > k | \mathcal{F}_{t_0}) < 0.01$$

and along with (40) this yields

$$\mathbb{P}\left(Z(T) > 5R^k | \mathcal{F}_{t_0}\right) < 0.01.$$

equation (33) follows and the proof is complete. □

The next proposition combines the results of the previous two lemmas together into a unified bound on the behaviour of the process $R(t) = X(t)/Y(t)$.

Proposition 4.5 *There exists a universal constant $C > 0$ such that the following holds: Assume that for some $t_0 \geq 0$, $X(t_0)/Y(t_0) > C$. Then*

$$\mathbb{E}[X(T)/Y(T)|A(t_0)] < \frac{1}{2} X(t_0)/Y(t_0)$$

where $T = \min\{s;\ X(s) \geq 1.3X(t_0))$.

Once we have established the above lemmas, the idea of the proof is very simple: just split into two cases, determined by whether or not there is a point in time at which $X(t)$ has not yet reached the value $1.1X(t_0)$ and the event $E(t)$ does not hold. If such a point exists, we use lemma 4.4, otherwise, we use corollary 4.3.

Proof: If the event $E(t_0)$ does not hold, just use lemma 4.4 with the legitimate assumption that $C > \Gamma$ and we're done. Otherwise, denote

$$T_1 = \min\{s > t_0;\ E(s) \text{ does not hold or } X(s) > 1.1X(t_0) \text{ or } Y(s) > 100Y(t_0)\}$$

and,

$$T_2 = \min\{s > T_1;\ X(s) > 1.1X(T_1) \text{ or } Y(s) > 100Y(t_0)\}$$

Using lemma 2.1 and since we're stopping before $Y(s)$ has reached the height $100Y(t_0) \leq 100C^{-1}X(t_0)$, we see that by taking the constant C to be large enough, we can make sure that any particle added to the aggregate before time T_2 can increase $X(t)$ by no more than $0.01X(t_0)$. Since almost surely only one particle can be added at a time, and assuming that C is a large enough constant, we get

$$X(T_2) \leq 1.3X(T_0). \tag{43}$$

Denote by F the event that $E(T_1)$ holds. By corollary 4.3, we know that

$$\mathbb{P}(X(T_1) \geq 1.1X(t)|\ \mathcal{F}_t) \leq 0.01.$$

We can estimate,

$$\mathbb{E}[X(T_2)/Y(T_2)\mathbf{1}_F|\mathcal{F}_{t_0}] \leq \tag{44}$$
$$0.01 \cdot 1.3X(t_0)/Y(t_0) + \frac{1}{100} 1.3X(t_0)/Y(t_0)$$

where we have used that fact that by definition of T_1 whenever $X(t_0) < 1.1$ and F holds, then necessarily $Y(T_1) > 100Y(t_0)$.

Next, we handle the case that F doesn't hold. By assuming that C is large enough, we can assume that $X(T_1)/Y(T_1) > \Gamma$ (the universal constant in the formulation of lemma 4.4). An application of lemma 4.4 gives,

$$\mathbb{E}[X(T_2)/Y(T_2)\mathbf{1}_{F^C}|\mathcal{F}_{t_0}] \leq \frac{1}{4}\mathbb{E}[X(T_1)/Y(T_1)\mathbf{1}_{F^C}|\mathcal{F}_{t_0}] \leq \frac{1.3}{4} X(t_0)/Y(t_0).$$

Combining this bound with (44) gives us,

$$\mathbb{E}[X(T_2)/Y(T_2)|\mathcal{F}_{t_0}] \leq$$

$$\left(0.013 + \frac{1.3}{100} + \frac{1.3}{4}\right) X(t_0)/Y(t_0) \leq 0.36 X(t_0)/Y(t_0),$$

by the same argument as the one preceding (43), one has $X(T) < 1.2X(T_2)$, which gives us the desired result. □

We are finally in a position to prove the main theorem of this section.
Proof of theorem 4.1:
Define

$$E_i = \{X(\tau_i)/Y(\tau_i) > C\}$$

and $E_0 = \{X(t)/Y(t) > C\}$, where C is a universal constant whose value will be determined later on. Observe that if for some i, we have $X(\tau_i) > 1.01 \times 2^i X_0$, it means that the last jump in $X(t)$ must have been rather big, namely that for the smallest integer j such that $X(\tau_j) = X(\tau_i)$, one has

$$X(\tau_j) > 1.01 \lim_{\epsilon \to 0^+} X(\tau_j - \epsilon)$$

(here we used the assumption that $X_0 \geq X(t)$). This, in turn, means that the radius of the last ball added was proportional to $X(\tau_i)$. By lemma 2.1 we learn that in that case, $X(\tau_i)/Y(\tau_i)$ cannot be larger than some universal constant, say C_1. In other words, by picking the constant C to be large enough, we can ensure that

$$E_i \text{ holds } \Rightarrow X(\tau_i) < 1.01 \times 2^i X_0 \tag{45}$$

Otherwise, if $X(\tau_i) \leq 1.01 \times 2^i X_0$ then we necessarily have $X(\tau_i) \leq 2.02 X(\tau_{i-1})$. It follows that for all $i \geq 1$, either E_i does not hold or $R(\tau_i) \leq 2.02 R(\tau_{i-1})$, and consequently

$$\mathbb{E}[R(\tau_i) \mathbf{1}_{E_{i-1}^C} | A(t)] \leq C + 2.02C \leq 4C. \tag{46}$$

Next we deal with the case that E_{i-1} holds. By choosing the constant C to be large enough, we can use proposition 4.5 to get

$$\mathbb{E}[R(T) \mathbf{1}_{E_{i-1}} | A(t)] \leq \frac{1}{2} \mathbb{E}[R(\tau_{i-1}) | A(t)].$$

where

$$T = \min\{s \geq \tau_{i-1};\ X(s) \geq 1.3 X(\tau_{i-1})\}.$$

Now, equation (45) teaches us that

$$\mathbb{E}[R(\tau_i) \mathbf{1}_{E_i} \mathbf{1}_{E_{i-1}} | A(t)] \leq$$

$$\frac{1.01 \times 2}{1.3} \mathbb{E}[R(T) | A(t)] \leq 0.9 \mathbb{E}[R(\tau_{i-1}) | A(t)],$$

and therefore

$$\mathbb{E}[R(\tau_i) \mathbf{1}_{E_{i-1}} | A(t)] = \mathbb{E}[R(\tau_i) \mathbf{1}_{E_i^C} \mathbf{1}_{E_{i-1}} | A(t)] + \mathbb{E}[R(\tau_i) \mathbf{1}_{E_i} \mathbf{1}_{E_{i-1}} | A(t)] \leq$$

$$C + 0.9 \mathbb{E}[R(\tau_{i-1}) | A(t)].$$

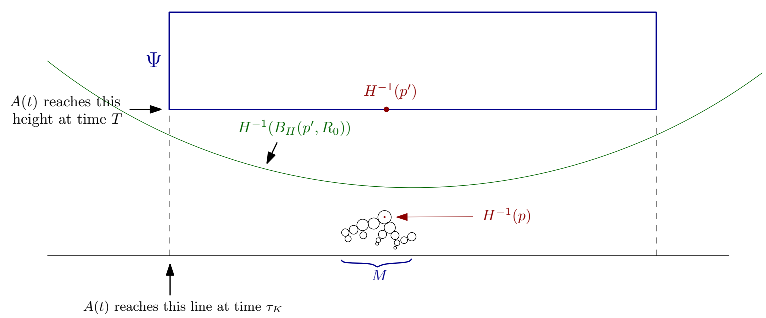

Figure 4: *The geometric definitions of Lemma 5.1.*

Together with (46), we get

$$\mathbb{E}[R(\tau_i)|A(t)] \leq 5C + 0.9\mathbb{E}[R(\tau_{i-1})|A(t)].$$

This completes the proof of the theorem. □

5 Proof of the main theorem

In this section we finally prove theorem 1.1. We begin with a lemma which roughly claims that the probability of the aggregate to intersect a any metric ball whose radius is large enough, is close to 1, no matter how far the ball is from the origin of the aggregate. The proof is a consequence of the tools developed in the previous section; we show that by choosing a suitable embedding of the aggregate into the Poincaré half-plane, the question of intersecting a specific metric ball boils down to the fact that $Y(t)$ grows rapidly enough compared to $X(t)$.

Lemma 5.1 *There exists a universal constant $R_0 > 0$ such that the following holds: Given any time $t \geq 0$ and any finite starting aggregate, $A(t)$, which started from a point $p \in \mathbb{H}^2$, there exists a number $L > 0$ such that for any point p' with $d(p, p') \geq L$ one has,*

$$\mathbb{P}(A(\infty) \cap B_H(p', R_0) \neq \emptyset \; |A(t)) \geq 0.99.$$

Proof: Denote by D the d_H-diameter of $A(t)$, and define

$$M = \max\{x; \; \exists y > 0 \text{ such that } d_H((x,y),(0,1)) < D\}.$$

For any two points $p_1, p_2 \in \mathbb{H}^2$, there is a (unique up to orientation) isometric embedding $\phi : \mathbb{H}^2 \to \mathbb{R}^2_+$ such that $\phi(p_1) = (0,1)$ and $\phi(p_2) = (0,S)$ for some $S \geq 1$. So given the starting point of the aggregate, p, and an arbitrary point p' satisfying $d(p,p') \geq L$ (where L is a constant whose value will be determined later on), we may therefore assume without loss of generality that $H^{-1}(p) = (0,1)$ and that $H^{-1}(p') = (0,S)$. Consider the metric ball,

$$B = B_H((0,S), R_0).$$

Clearly, if R_0 is a large enough universal constant, this ball will contain a rectangle of the form

$$\Psi = [-3R_1 S, 3R_1 S] \times [S, 2S] \subset \mathbb{R}^2_+.$$

where R_1 is a universal constant whose value will be chosen later on (see figure 4 for an illustration). Also consider the stopping times,

$$\tau_i = \min\{s;\ X(s) \geq 2^i M\}, \quad \forall i \in \mathbb{N}.$$

By the definition of M we have $X(t) \leq M$, and thus by theorem 4.1 we know that for all $i \geq 1$,

$$\mathbb{E}[R(\tau_{i+1})|A(t)] < C + 0.9\mathbb{E}[R(\tau_i)|A(t)]. \tag{47}$$

for a universal constant $C > 0$, which implies that

$$\mathbb{E}[R(\tau_i)|A(t)] \geq C_1 \Rightarrow \mathbb{E}[R(\tau_{i+1})|A(t)] \leq 0.95\mathbb{E}[R(\tau_i)|A(t)] \tag{48}$$

for $C_1 = 20C$. Next, if $R(\tau_0) > 2M$, then necessarily by lemma 2.1 it means that $Y(\tau_0) > cX(\tau_0)$ for a universal constant $c > 0$ (since almost surely only one particle is added at a time, and the increment in $X(s)$ is not larger than a constant times $Y(s)$). We deduce that

$$\mathbb{E}[R(\tau_0)|A(t)] = \mathbb{E}[R(\tau_0)\mathbf{1}_{R(\tau_0)<2M}|A(t)] + \mathbb{E}[R(\tau_0)\mathbf{1}_{R(\tau_0)\geq 2M}|A(t)] \leq$$

$$2M/Y(t) + c^{-1} \leq 2M + C_2.$$

for a universal constant $C_2 > 0$. Together with (47) and (48), it gives

$$\mathbb{E}[R(\tau_i)|A(t)] \leq C_1, \quad \forall i \geq \Theta$$

with $\Theta = \max\left(\log_{0.95} \frac{C_1}{2M+C_2}, 1\right)$. Denote $K = \lceil \log_2(R_1 S/M) \rceil$. Recall that we are free to take the constant L as large as we want, which ensures us that the number S can be as large as we like thanks to the assumption $d_H(p,p') \geq L$. Now, since the number M does not depend on the point p' (but only on the aggregate $A(t)$), by taking L to be large enough, it is legitimate to assume that

$$K \geq \Theta.$$

With this assumption, we get

$$\mathbb{E}[R(\tau_K)|A(t)] \leq C_1,$$

and also by the definition τ_i,

$$X(\tau_K) \geq M2^K \geq R_1 S.$$

These two equations combined yield

$$\mathbb{P}[Y(\tau_K) < S | A(t)] < C_1/R_1 < 0.01$$

where the last inequality can be attained by making sure that R_1 is a large enough universal constant (note that the value of C_1 has already been fixed and thus does not depend on R_1). Defining

$$T = \min\{s;\ Y(s) \geq S\},$$

the previous equation becomes

$$\mathbb{P}(T > \tau_K | A(t)) < 0.01. \tag{49}$$

On the other hand, another application of lemma 2.1 with the fact that only one particle is added at a time almost surely, teaches us that

$$X(T \wedge \tau_K) \leq 2R_1 S + C_3 S$$

for a universal constant $C_3 > 0$, and by choosing that R_1 to be large enough we can assert that

$$X(T \wedge \tau_K) \leq 3R_1 S$$

almost surely, without affecting the correctness of the above. Using the last equation and the definition of Ψ, it is easy to check that we have the implication,

$$T < \tau_K \Rightarrow A(T) \cap \Psi \neq \emptyset \Rightarrow A(T) \cap B_H(p', R_0) \neq \emptyset.$$

In light of equation (49), this finishes the proof. □

We are finally ready to prove the main theorem.

Proof of theorem 1.1

The main idea of the proof is to use the previous lemma iteratively, in order to prove that there exists a random sequence of radii $L_1 \leq L_2 \leq ...$ such that $L_i \to \infty$ almost surely and a random sequence of stopping times $T_1 \leq T_2 \leq ...$ such that for all $i \geq 1$, almost surely

$$\mathbb{P}\Big(\#\big(A(\infty) \cap B_H(p_0, L_{i+1})\big) \geq cVol_H(B_H(p_0, L_{i+1}))\ \ |\ A(T_i), L_i\Big) \geq c \tag{50}$$

where $c > 0$ is a universal constant and p_0 is the starting point of the aggregate. This will clearly finish the proof, since it implies that with probability one there exists a sub-sequence of radii $\{L_{i_k}\}_{k=1}^{\infty}$ such that

$$\#\big(A(\infty) \cap B_H(p_0, L_{i_k})\big) \geq cVol_H(B_H(p_0, L_{i_k})),\ \forall k \in \mathbb{N}.$$

for a universal constant $c > 0$.
We build these sequences inductively. We begin with $L_1 = 1$ and $T_1 = 0$. Suppose L_i, T_i and $A(T_i)$ are known. We use the previous lemma with $A(t) = A(T_i)$ as a starting aggregate. The result of the lemma ensures the existence of a number L such that

$$\mathbb{P}(A(\infty) \cap B_H(p, R_0) \neq \emptyset \; |A(T_i)) \geq 0.99. \tag{51}$$

for all p such that $d_H(p_0, p) \geq L$. Take $L_{i+1} = \max\{2L_i, 2L\}$. Now consider a maximal set of disjoint metric balls of radius R_0 whose centers lie within the annulus $B_H(p_0, L_{i+1}) \setminus B_H(p_0, L_{i+1}/2)$. Denote the centres of these balls by $p_1, ..., p_N$ so that N is the number of balls in this packing. By the maximality of this set, it is obvious that we have

$$B_H(p_0, L_{i+1}) \setminus B_H(p_0, L_{i+1}/2) \subset \bigcup_{i=1}^{N} B_H(p_i, 2R_0).$$

Consequently,

$$N \geq \frac{Vol_H\big(B_H(p_0, L_{i+1}) \setminus B_H(p_0, L_{i+1}/2)\big)}{Vol_H(p_0, 2R_0)} \geq c_1 Vol_H(B_H(p_0, L_{i+1})) \tag{52}$$

for a universal constant $c_1 > 0$. Define,

$$M(t) = \#\big\{j \in \{1, .., N\};\; B_H(p_j, R_0) \cap A(t) \neq \emptyset\big\}$$

and note that, since the balls $B_H(p_j, R_0)$ are disjoint, we have that

$$\#\big(A(t) \cap B_H(p_0, L_{i+1})\big) \geq M(t), \quad \forall t \geq T_i. \tag{53}$$

Equation (51) ensures that $\mathbb{E}[M(\infty)|A(T_i)] \geq 0.99N$. It then follows from Markov's inequality that

$$\mathbb{P}(M(\infty) > N/2|A(T_i)) > \frac{1}{2}.$$

By σ-additivity, there exists a number $T > 0$ such that

$$\mathbb{P}(M(T) > N/2|A(T_i)) > \frac{1}{2}.$$

Set $T_{i+1} = T$. Together with equations (52) and (53), this establishes (50). Note that L_{i+1} and T_{i+1} only depended on L_i, T_i and $A(T_i)$, and therefore the conditioning on $A(T_i)$ and L_i in formula (50) is legitimate.
The proof is complete. □

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