

# Nonlinear noise excitation of intermittent stochastic PDEs and the topology of LCA groups\*

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## Abstract

Consider the stochastic heat equation  $\partial_t u = \mathcal{L}u + \lambda\sigma(u)\xi$ , where  $\mathcal{L}$  denotes the generator of a Lévy process on a locally compact Hausdorff abelian group  $G$ ,  $\sigma : \mathbf{R} \rightarrow \mathbf{R}$  is Lipschitz continuous,  $\lambda \gg 1$  is a large parameter, and  $\xi$  denotes space-time white noise on  $\mathbf{R}_+ \times G$ .

The main result of this paper contains a near-dichotomy for the [expected squared] energy  $E(\|u_t\|_{L^2(G)}^2)$  of the solution. Roughly speaking, that dichotomy says that, in all known cases where  $u$  is intermittent, the energy of the solution behaves generically as  $\exp\{\text{const} \cdot \lambda^2\}$  when  $G$  is discrete and  $\geq \exp\{\text{const} \cdot \lambda^4\}$  when  $G$  is connected.

*Keywords:* The stochastic heat equation, intermittency, nonlinear noise excitation, Lévy processes, locally compact abelian groups.

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## 1 An informal introduction

Consider a stochastic heat equation of the form

$$\frac{\partial}{\partial t} u = \mathcal{L}u + \lambda\sigma(u)\xi. \tag{SHE}$$

Here,  $\sigma : \mathbf{R} \rightarrow \mathbf{R}$  is a Lipschitz continuous function,  $t > 0$  denotes the time variable,  $x \in G$  is the space variable, for a locally compact Hausdorff abelian group  $G$ —such as  $\mathbf{R}$ ,  $\mathbf{Z}^d$ , or  $[0, 1]$ —and the initial value  $u_0 : G \rightarrow \mathbf{R}$  is nonrandom and well behaved. The operator  $\mathcal{L}$  acts on the variable  $x$  only, and denotes the generator of a Lévy process on  $G$ , and  $\xi$  denotes space-time white noise on

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$(0, \infty) \times G$  whose control measure is the restriction of the Haar measure on  $\mathbf{R} \times G$  to  $(0, \infty) \times G$ . The number  $\lambda$  is a positive parameter; this is the so called *level of the noise*.

In this paper, we study the “noisy case.” That is when  $\lambda$  is a large quantity. The case that  $\lambda$  is small is also interesting; see for example the deep theory of Freidlin and Wentzel [24].

We will consider only examples of (SHE) that are *intermittent*. Intuitively speaking, “intermittency” is the property that the solution  $u_t(x)$  develops extreme oscillations at some values of  $x$ , typically when  $t$  is large. Intermittency was announced first (1949) by Batchelor and Townsend in a WHO conference in Vienna [1], and slightly later by Emmons [21] in the context of boundary-layer turbulence. Ever since that time, intermittency has been observed in an enormous number of scientific disciplines. Shortly we will point to concrete instances in theoretical physics. In the mean time, let us also mention that, in neuroscience, intermittency is observed as “spikes” in neural activity. [Tuckwell [42] contains a gentle introduction to SPDEs in neuroscience.] And in finance, intermittency is usually associated with financial “shocks.”

The standard mathematical definition of intermittency (see Molchanov [35] and Zeldovich et al [44]) is that

$$\frac{\gamma(k)}{k} < \frac{\gamma(k')}{k'} \quad \text{whenever } 2 \leq k < k' < \infty, \quad (1.1)$$

where  $\gamma$  denotes any reasonable choice of a so-called Lyapunov exponent of the moments of the energy of the solution: We may use either

$$\gamma(k) := \limsup_{t \rightarrow \infty} t^{-1} \log E \left( \|u_t\|_{L^2(G)}^k \right), \text{ or } \gamma(k) := \liminf_{t \rightarrow \infty} t^{-1} \log E \left( \|u_t\|_{L^2(G)}^k \right).$$

Other essentially-equivalent choices are also possible. One can justify this definition either by making informal analogies with finite-dimensional nonrandom dynamical systems [34], or by making a somewhat informal appeal to the Borel–Cantelli lemma [3]. Gibbon and Titi [26] contains an exciting modern account of mathematical intermittency and its role in our present-day understanding of physical intermittency.

In the case that  $G = \mathbf{R}$ ,  $G = [0, 1]$ , or  $G = \mathbf{Z}^d$ , there is a huge literature that is devoted to the intermittency properties of (SHE) when  $\sigma(x) = \text{const} \cdot x$ ; this particular model—the so-called *parabolic Anderson model*—is interesting in its own right, as it is connected deeply with a large number of diverse questions in probability theory and mathematical physics. See, for example, the ample bibliographies of Ref.s [3, 8, 10, 11, 17, 19, 22, 25, 29, 30, 35, 44].

The parabolic Anderson model arises in a surprisingly large number of diverse scientific problems; see Carmona and Molchanov [8, Introduction]. We mention quickly a few such instances: If  $\sigma(0) = 0$ ,  $u_0(x) > 0$  for all  $x \in G$ , and  $G$  is either  $\mathbf{R}$  or  $[0, 1]$  then Mueller’s comparison principle [37] shows that  $u_t(x) > 0$  almost surely for all  $t > 0$  and  $x \in G$ ; see also [14, p. 130]. In that case,  $h_t(x) := \log u_t(x)$  is well defined and is the so-called Cole–Hopf solution

to the KPZ equation of statistical mechanics [29, 30]. The parabolic Anderson model has many connections also with the stochastic Burger's equation [8] and Majda's model of shear-layer flow in turbulent diffusion [33].

Foondun and Khoshnevisan [22] have shown that the solution to (SHE) is fairly generically intermittent even when  $\sigma$  is nonlinear, as long as  $\sigma$  behaves as a line in one form or another.

It was noticed early on, in NMR spectroscopy, that intermittency can be associated strongly to nonlinear noise excitation. See for example Blümich [5]; Lindner et al [32] contains a survey of many related ideas in the physics literature. In the present context, this informal observation is equivalent to the existence of a nonlinear relationship between the energy  $\|u_t\|_{L^2(G)}$  of the solution at time  $t$  and the level  $\lambda$  of the noise. A precise form of such a relationship will follow as a ready consequence of our present work in all cases where the solution is known [and/or expected] to be intermittent. In fact, the main findings of this paper will imply that typically, when the solution is intermittent, there is a near-dichotomy:

- On one hand, if  $G$  is discrete then the energy of the solution behaves roughly as  $\exp\{\text{const} \cdot \lambda^2\}$ ;
- on the other hand, if  $G$  has a connected locally compact Hausdorff abelian subgroup, then the said energy behaves at least as badly as  $\exp\{\text{const} \cdot \lambda^4\}$ .

And quite remarkably, these properties do not depend in an essential way on the operator  $\mathcal{L}$ ; they depend only on the connectivity properties of the underlying state space  $G$ .

Every standard numerical method for solving (SHE) that is known to us begins by first discretizing  $G$  and  $\mathcal{L}$ . Our results suggest that when  $\lambda$  is modestly large, then nearly all such methods will generically underestimate by a vast margin when we use them to predict the size of the biggest intermittency islands [or shocks, or spikes] of the solution to (SHE).

Other SPDE models are analyzed in a companion paper [31] which should ideally be read before the present paper. That paper is less abstract than this one and, as such, has fewer mathematical prerequisites. We present in that paper the surprising result that the stochastic heat equation on an interval is typically significantly more noise excitable than the stochastic wave equation on the real line.

**Remark 1.1.** The referees of the paper have unanimously suggested that we describe, in words, an intuitive explanation for this near dichotomy. We agree that such an exposition will add value to the presentation of the paper, and would like to say a few things in this direction. Therefore, let us briefly consider the case that  $G$  is a very nice LCA group [such as a finite group,  $\mathbf{Z}^d$ , or  $\mathbf{R}$ ] and  $\sigma(u) = cu$  for some constant  $c > 0$  [the parabolic Anderson model]. First, one can see that when  $G$  is finite, (SHE) is another way to write a finite-dimensional stochastic differential equation; see Examples 4.1 and 4.2. In this case, it is not hard to verify directly, using only SDE technology, that the energy

of the solution to (SHE) typically grows as  $\exp\{\text{const} \cdot \lambda^2\}$  as  $\lambda \rightarrow \infty$ .<sup>1</sup> In some sense,  $G = \mathbf{Z}^d$  can be thought of as a limit of the finite case: Since most of the mass of the solution  $u_t$  is concentrated on compacts [because  $u_t \in L^2(G)$ ], this suggests that the case that  $G = \mathbf{Z}^d$  should behave as does the finite case. And it does. On the other hand, when  $\mathcal{L}$  is the generator of a nice Lévy process—say an isotropic  $\alpha$ -stable process—on  $G = \mathbf{R}$ , then  $\alpha$  is necessarily in  $(1, 2]$  [see Dalang [13]], and a simple scaling argument shows that the large- $\lambda$  behavior of the solution to (SHE) is the same as the large-time behavior of the solution to (SHE) with  $\lambda = 1$ , provided that we rescale time as  $T := \lambda^{2\alpha/(\alpha-1)}t$ . The existing literature on the parabolic Anderson model suggests that the energy at large time  $T$  of the solution to (SHE) with  $\lambda = 1$  should behave as  $\exp\{\text{const} \cdot T\}$ . Set  $T = \lambda^{2\alpha/(\alpha-1)}t$  in order to see that the energy to (SHE) with variable  $\lambda \gg 1$  ought to behave as  $\exp\{\text{const} \cdot \lambda^{2\alpha/(\alpha-1)}t\}$  as  $\lambda \rightarrow \infty$ , for all  $t > 0$  fixed. In other words, when  $\sigma(u) = cu$  and the underlying Lévy process is isotropic stable, the energy behaves as  $\exp\{\text{const} \cdot \lambda^q\}$  as  $\lambda \rightarrow \infty$  for  $q = 2\alpha/(\alpha - 1) \geq 4$ , where the time variable  $t$  is fixed.

## 2 Main results

The main goal of this article is to describe the behavior of (SHE) for a locally compact Hausdorff abelian group  $G$ , where the initial value  $u_0$  is nonrandom and is in the group algebra  $L^2(G)$ .<sup>2</sup> Compelling, as well as easy to understand, examples can be found in Section 4 below.

We assume throughout that the operator  $\mathcal{L}$  acts on the space variable only and denotes the generator of a Lévy process  $X := \{X_t\}_{t \geq 0}$  on  $G$ —see Section 3 for analysis on LCA groups and Section 5 for Lévy processes on LCA groups— $\sigma : \mathbf{R} \rightarrow \mathbf{R}$  is Lipschitz continuous and nonrandom, and  $\xi$  denotes space-time white noise on  $(0, \infty) \times G$ . That is,  $\xi$  is a generalized centered Gaussian process that is indexed by  $(0, \infty) \times G$  and whose covariance measure is described via

$$\text{Cov} \left( \int \varphi \, d\xi, \int \psi \, d\xi \right) = \int_0^\infty dt \int_G m_G(dx) \varphi_t(x) \psi_t(x), \quad (2.1)$$

for all  $\varphi, \psi \in L^2(dt \times dm_G)$ , where  $m_G$  denotes the Haar measure on  $G$ , and  $\int \varphi \, d\xi$  and  $\int \psi \, d\xi$  are defined as Wiener integrals. Finally,  $\lambda > 0$  designates a fixed parameter that is generally referred to as the *level of the noise*.

One can adapt the method of Dalang [13] in order to show that, in the linear case—that is when  $\sigma \equiv \text{constant}$ —(SHE) has a function solution iff

$$\int_{G^*} \left( \frac{1}{\beta + \text{Re}\Psi(\chi)} \right) m_{G^*}(d\chi) < \infty \quad \text{for one, hence all, } \beta > 0, \quad (\text{D})$$

<sup>1</sup>For an example, the reader is encouraged to consider the exponential martingale of Brownian motion. In that case, the  $\exp\{\text{const} \cdot \lambda^2\}$  behavior of the solution is more or less immediate.

<sup>2</sup>This is the usual space of all measurable functions  $f : G \rightarrow \mathbf{R}$  that are square integrable with respect to the Haar measure on  $G$ .

where  $\Psi$  denotes the characteristic exponent of our Lévy process  $\{X_t\}_{t \geq 0}$  and  $m_{G^*}$  denotes the Haar measure on the dual  $G^*$  to our group  $G$ . See also Brzeźniak and Jan van Neerven [6] and Peszat and Zabczyk [38]. Because we want (SHE) to have a function solution, at the very least in the linear case, we have no choice but to assume Dalang’s condition (D) from now on. Henceforth, we assume (D) without further mention.

In some cases, Condition (D) always holds. For example, suppose  $G$  is discrete. Because  $G^*$  is compact, thanks to Pontryagin–van Kampen duality [36, 41], continuity of the function  $\Psi$  implies its uniform boundedness, whence we find that the *Dalang Condition (D) always holds when  $G$  is discrete*. This simple observation is characteristic of many interesting results about the heat equation (SHE) in the sense that a purely-topological property of the group  $G$  governs important aspects of (SHE): In this case, we deduce the existence of a solution generically when  $G$  is discrete. For a probabilistic proof of this particular fact see Lemma 10.1 below.

We wish to establish that “noise excitation” properties of (SHE) are “intrinsic to the group  $G$ .” This goal forces us to try and produce solutions that take values in the group algebra  $L^2(G)$ . The following summarizes the resulting existence and regularity theorem that is good enough to help us begin our discussion of noise excitation. We note that an exact definition of a mild solution will be given in (7.2). That definition will imply that our solution is in  $L^2(G)$  at all times, and hence is a bona fide function on  $G$  at all times.

**Theorem 2.1.** *Suppose that  $\sigma$  is Lipschitz continuous and, in addition, that either  $G$  is compact or  $\sigma(0) = 0$ . Then for every nonrandom initial value  $u_0 \in L^2(G)$  and  $\lambda > 0$ , the stochastic heat equation (SHE) has a mild solution  $\{u_t\}_{t \geq 0}$ , with values in  $L^2(G)$ , that satisfies the following: There exist finite constants  $c_1 > 0$  and  $c_2 > 0$  that yield the energy inequality*

$$\mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right) \leq c_1 e^{c_2 t} \quad \text{for every } t \geq 0. \quad (2.2)$$

*Moreover, if  $v$  is an arbitrary mild solution that satisfies (2.2) subject to  $v_0 = u_0$ , then  $\mathbb{P}\{\|u_t - v_t\|_{L^2(G)} = 0\} = 1$  for all  $t \geq 0$ .*

**Remark 2.2.** For more explicit bounds on the constants  $c_1$  and  $c_2$  see the inequality (7.12) below. That inequality describes carefully how  $c_1$  and  $c_2$  depend on the various parameters of (SHE)—in particular it states abstractly how  $c_1$  and  $c_2$  depend on  $\lambda$ —and will be used several times in the sequel.

The proof of Theorem 2.1 will be given in Sections 7 and 8; see also Section 6, in which we develop the requisite machinery for Theorem 2.1 and the other main results in this paper. However, the preceding result is well-known for many euclidean examples; see in particular Dalang and Mueller [15].

Thus, we assume from now on, and without further mention, that

$$\text{either } G \text{ is compact, or } \sigma(0) = 0, \quad (2.3)$$

in order to know *a priori* that (SHE) has an  $L^2(G)$ -valued solution.<sup>3</sup>

The principal aim of this paper is to study the energy of the solution when  $\lambda$  is large. In order to simplify the exposition, let us denote the *energy of the solution at time  $t$*  by

$$\mathcal{E}_t(\lambda) := \sqrt{\mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right)}. \quad (2.4)$$

To be more precise,  $\mathcal{E}_t(\lambda)$  denotes the  $L^2(\mathbb{P})$ -norm of the energy of the solution. But we refer to it as the energy in order to save on the typography.

We begin our analysis of noise excitation by first noting the following fact: If  $\sigma$  is essentially bounded and  $G$  is compact, then the solution to (SHE) is *at most linearly noise excitable*. The following is the precise formulation of this statement (see Section 9 for the proof).

**Proposition 2.3** (Linear noise excitation). *If  $\sigma \in L^\infty(\mathbf{R})$  and  $G$  is compact, then*

$$\limsup_{\lambda \uparrow \infty} \frac{\mathcal{E}_t(\lambda)}{\lambda} < \infty \quad \text{for all } t > 0. \quad (2.5)$$

*This bound can be reversed in the following sense: If also  $\inf_{x \in G} |u_0(x)| > 0$  and  $\inf_{z \in \mathbf{R}} |\sigma(z)| > 0$ , then*

$$\liminf_{\lambda \uparrow \infty} \frac{\mathcal{E}_t(\lambda)}{\lambda} > 0 \quad \text{for all } t > 0. \quad (2.6)$$

We do not know what happens, at this level of generality, when  $\sigma \in L^\infty(\mathbf{R})$  and  $G$  is non compact.

The bulk of this paper is concerned with the behavior of (SHE) when the energy  $\mathcal{E}_t(\lambda)$  behaves as  $\exp(\text{const} \cdot \lambda^q)$ , for a fixed positive constant  $q$ , as  $\lambda \uparrow \infty$ . With this in mind, let us define for all  $t > 0$ ,

$$\underline{\mathfrak{e}}(t) := \liminf_{\lambda \uparrow \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda}, \quad \bar{\mathfrak{e}}(t) := \limsup_{\lambda \uparrow \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda}. \quad (2.7)$$

If  $\underline{\mathfrak{e}}(t) > 0$  for all  $t > 0$ , then the solution to (SHE) is expected to be also “intermittent,” not only in the usual mathematical sense [8], but also in a physical sense [that is, in cases where the solution to (SHE) represents the density of a particle system].

**Definition 2.4.** We refer to  $\bar{\mathfrak{e}}(t)$  and  $\underline{\mathfrak{e}}(t)$  respectively as the *upper* and the *lower excitation indices* of  $u$  at time  $t$ . In many cases of interest,  $\underline{\mathfrak{e}}(t)$  and  $\bar{\mathfrak{e}}(t)$  are equal and do not depend on the time variable  $t > 0$  [N.B. *not* to be confused with  $t \geq 0$ ]. In such cases, we tacitly write  $\mathfrak{e}$  for that common value, and we think of  $\mathfrak{e}$  as the *index of nonlinear noise excitation* of the solution to (SHE).

<sup>3</sup>In other words, we do not need to assume that  $\sigma(0) = 0$  when  $G$  is compact. However, we do need this condition in general when  $G$  is non-compact. There are examples of  $\sigma$  such that  $\sigma(0) \neq 0$ , non-compact LCA groups  $G$ , and Lévy process generators  $\mathcal{L}$  for which (SHE) does not have an  $L^2(G)$ -valued solution for all time.

Thus, Proposition (2.3) implies that  $\epsilon = 0$  when  $\sigma$  is essentially bounded and  $G$  is compact.

As a central part of our analysis, we will prove that both of these indices are natural quantities, as they are “group invariants” in a sense that will be made clear in Section 11. Moreover, one can deduce from our work that when  $G$  is unimodular [see Definition 11.2] the law of the solution to (SHE) is itself a “group invariant.” A careful explanation of the quoted terms will appear later on in Theorem 11.10. For now, we content ourselves by stating the main three results of this paper.

**Theorem 2.5** (Discrete case). *If  $G$  is discrete, then  $\bar{\epsilon}(t) \leq 2$  for all  $t > 0$ . In fact,  $\epsilon = 2$ , provided additionally that*

$$\ell_\sigma := \inf_{z \in \mathbf{R} \setminus \{0\}} |\sigma(z)/z| > 0. \quad (2.8)$$

Recall that the nonlinearity  $\sigma : \mathbf{R} \rightarrow \mathbf{R}$  is assumed to be Lipschitz continuous, and hence  $\sup_{z \in \mathbf{R} \setminus \{0\}} |\sigma(z)/z| < \infty$ . Thus, (2.8) is the assertion that the graph of  $\sigma$  lies globally in some cone.

**Theorem 2.6** (Connected case). *Suppose that  $G$  is connected and (2.8) holds. Then  $\underline{\epsilon}(t) \geq 4$  for all  $t > 0$ , provided that in addition either  $G$  is non compact or  $G$  is compact, metrizable, and has more than one element.*

**Remark 2.7.** The proofs will show a slightly more general statement, thanks to projection. Namely [see Proposition 12.1] that if  $G$  contains a non-compact connected LCA subgroup, or if  $G$  contains a compact metrizable connected LCA subgroup of more than one element, then  $\underline{\epsilon}(t) \geq 4$  as long as (2.8) holds.

**Theorem 2.8** (Connected case). *For every  $\theta \geq 4$  there are models of the triple  $(G, \mathcal{L}, u_0)$  for which  $\epsilon = \theta$ .*

The proofs of the above theorems are presented in Section 14 below, and use the results in §12 and 13. In particular, §12 enables us to obtain the lower bound of the lower excitation index in Theorem 2.6 “by projection.”

We now see that if (2.8) holds, in addition to the preceding conditions, then Theorems 2.5, 2.6, and 2.8 together imply the following: Either the energy of the solution behaves as  $\exp(\text{const} \cdot \lambda^2)$  or it is greater than  $\exp(\text{const} \cdot \lambda^4)$  for large noise levels, and this lower bound cannot be improved upon in general. Moreover, the connectivity properties of  $G$ —and not the operator  $\mathcal{L}$ —alone determine the first-order strength of the growth of the energy, viewed as a function of the noise level  $\lambda$ .

Finally, we will soon see that when the energy behaves as  $\exp(\text{const} \cdot \lambda^2)$ , this means that (SHE) is only as noise excitable as a classical Itô stochastic differential equation. Martin Hairer has asked [private communication] whether intermittency properties of (SHE) are always related to those of the McKean exponential martingale for Brownian motion. A glance at Example 4.1 below shows in some sense that, as far as nonlinear noise excitation is concerned,

intermittent examples of (SHE) behave as the exponential martingale if and only if  $G$  is essentially discrete.

Throughout,  $L_\sigma$  designates the optimal Lipschitz constant of the function  $\sigma$ . In more succinct terms, we have

$$L_\sigma := \sup_{-\infty < x < y < \infty} \left| \frac{\sigma(x) - \sigma(y)}{x - y} \right| < \infty. \quad (2.9)$$

### 3 Analysis on LCA groups

We follow the usual terminology of the literature and refer to a *locally compact Hausdorff abelian group* as an *LCA group*. Morris [36] and Rudin [41] are two standard references for the theory of LCA groups.

If  $G$  is an LCA group, then we let  $m_G$  denote the Haar measure on  $G$ .<sup>4</sup> The dual, or character, group to  $G$  denoted by  $G^*$ .<sup>5</sup> In addition, the *Fourier transform* on  $L^1(G)$  is defined via the following normalization:

$$\hat{f}(\chi) := \int_G (x, \chi) f(x) m_G(dx) \quad \text{for all } \chi \in G^* \text{ and } f \in L^1(G), \quad (3.1)$$

where  $(x, \chi) := \chi(x) := x(\chi)$  are interchangeable notations that all describe the natural pairing between  $x \in G$  and  $\chi \in G^*$ .<sup>6</sup>

Of course,  $m_G$  is defined uniquely only up to a multiplicative factor. Therefore, we always assume the *standard normalization* of Haar measures; that is any normalization that ensures that the Fourier transform has a continuous isometric extension to  $L^2(G) = L^2(G^*)$ . Analytically speaking, this means that our normalization of Haar measure ensures that the following formulation of the Plancherel identity is valid:

$$\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(G^*)} \quad \text{for all } f \in L^2(G). \quad (3.2)$$

Our normalization of Haar measure translates to well-known normalizations of Haar measures via Pontryagin–van Kampen duality [36, 41]:

**Case 1.** If  $G$  is compact, then  $G^*$  is discrete;  $m_G(G) = 1$ ; and  $m_{G^*}$  denotes the counting measure on subsets of  $\Gamma^*$ .

**Case 2.** If  $G$  is discrete, then  $G^*$  is compact,  $m_{G^*}(G^*) = 1$ , and  $m_G$  coincides with the counting measure on  $G$ .

<sup>4</sup>That is,  $m_G$  is a non-zero Radon measure on  $G$  that is translation invariant under group multiplication.

<sup>5</sup>That is,  $\chi \in G^*$  if and only if  $\chi : G \rightarrow \mathbf{C}$  is a group homomorphism from  $G$  to the circle group; that is,  $\chi$  is homeomorphic and satisfies  $\chi(xy) = \chi(x)\chi(y)$  for all  $x, y \in G$ . Every  $\chi \in G^*$  is called a *character* on  $G$ . Thus, for instance, if  $G = \mathbf{R}^d$ , then  $G^* = \mathbf{R}^d$  and  $\chi(x) = \exp(ix \cdot \chi)$ . Also, when  $G = \mathbf{Z}^d$ , then  $G^* = [0, 2\pi)^d$  and  $\chi(x) = \exp(ix \cdot \chi)$ .

<sup>6</sup>This notation is justified by the Pontryagin–van Kampen duality theorem [36, 41]: The dual of  $G^*$  is  $G$ . Consequently,  $x \in G$  acts on  $\chi \in G^*$  in the same way as  $\chi \in G^*$  acts on  $x \in G$ , whence  $x(\chi)$  can be identified with  $\chi(x)$ , as asserted. We emphasize that different authors use slightly different normalizations of Fourier transforms from us; see, for example Rudin [41].

**Case 3.** If  $G = \mathbf{R}^n$  for some integer  $n \geq 1$ , then  $G^* = \mathbf{R}^n$ ; we may choose  $m_G$  and  $m_{G^*}$ , in terms of  $n$ -dimensional Lebesgue measure, as  $m_G(dx) = a dx$  and  $m_{G^*}(dx) = b dx$  for any two positive reals  $a$  and  $b$  that satisfy the relation  $ab = (2\pi)^{-n}$ .

## 4 Some examples

The stochastic PDEs introduced here are quite natural; in many cases, they are in fact well-established equations. In this section, we identify some examples to highlight the preceding claims. Of course, one can begin with the most obvious examples of stochastic PDEs; for instance, where  $G = \mathbf{R}$ ,  $\mathcal{L} = \Delta$ , etc. But we prefer to have a different viewpoint: As far as interesting examples are concerned, it is helpful to sometimes think about concrete examples of LCA groups  $G$ ; then try to understand the Lévy processes on  $G$  [a kind of Lévy–Khintchine formula] in order to know which operators  $\mathcal{L}$  are relevant. And only then one can think about the actual resulting stochastic partial differential equation. This slightly-different viewpoint produces interesting examples.

**Example 4.1** (The trivial group). For our first example let us consider the trivial group  $G$  with only one element  $g$ . The only Lévy process on this group is  $X_t := g$ . All functions on the group  $G$  are, by default, constants. Therefore,  $\mathcal{L}f = 0$  for all  $f : G \rightarrow \mathbf{R}$ , and hence  $U_t := u_t(g)$  solves the Itô SDE

$$dU_t = \lambda\sigma(U_t) dB_t \quad \text{with } U_0 = u_0(g), \quad (4.1)$$

where  $B_t := \int_{[0,t] \times G} d\xi$  defines a Brownian motion. In other words, when  $G$  is the trivial group, (SHE) characterizes all drift-free 1-dimensional Itô diffusions!

**Example 4.2** (Cyclic groups, part I). For a slightly more interesting example consider the cyclic group  $G := \mathbf{Z}_2$  on two elements. We may think of  $G$  as  $\mathbf{Z}/2\mathbf{Z}$ ; i.e., the set  $\{0, 1\}$  endowed with binary addition [addition mod 1] and discrete topology. It is an elementary fact that the group  $G$  admits only one 1-parameter family of Lévy processes. Indeed, we can apply the strong Markov property to the first jump time of  $X$  to see that if  $X$  is a Lévy process on  $\mathbf{Z}_2$ , then there necessarily exists a number  $\kappa \geq 0$  such that, at independent exponential times, the process  $X$  changes its state at rate  $\kappa$ : From 0 to 1 if  $X$  is at 0 at the jump time, and from 1 to 0 when  $X$  is at 1 at the jump time [ $\kappa = 0$  yields the constant process]. In this way we find that (SHE) is an encoding of the coupled two-dimensional SDE

$$\begin{aligned} du_t(0) &= \kappa [u_t(1) - u_t(0)] dt + \lambda\sigma(u_t(0)) dB_t(0), \\ du_t(1) &= \kappa [u_t(0) - u_t(1)] dt + \lambda\sigma(u_t(1)) dB_t(1), \end{aligned} \quad (4.2)$$

where  $B(0)$  and  $B(1)$  are two independent 1-dimensional Brownian motions. In other words, when  $G = \mathbf{Z}_2$ , (SHE) describes a 2-dimensional Itô diffusion with local diffusion coefficients where the particles [coordinate processes] feel an attractive linear drift toward their neighbors [unless  $\kappa = 0$ , which corresponds to two decoupled diffusions].

**Example 4.3** (Cyclic groups, part II). Let us consider the case that  $G := \mathbf{Z}_n$  is the cyclic group on  $n$  elements when  $n \geq 3$ . We may think of  $G$  as  $\mathbf{Z}/n\mathbf{Z}$ ; that is, the set  $\{0, \dots, n-1\}$  endowed with addition  $(\bmod n)$  and discrete topology. If  $X$  is a Lévy process on  $G$ , then it is easy to see that there exist  $n-1$  parameters  $\kappa_1, \dots, \kappa_{n-1} \geq 0$  such that  $X$  jumps [at iid exponential times] from  $i \in \mathbf{Z}/n\mathbf{Z}$  to  $i+j \pmod n$  at rate  $\kappa_j$  for every  $i \in \{0, \dots, n-1\}$  and  $j \in \{1, \dots, n-1\}$ . In this case, our stochastic heat equation (SHE) is another way to describe the evolution of the  $n$ -dimensional Itô diffusion  $(u(1), \dots, u(n))$ , where for all  $i = 0, \dots, n-1$ ,

$$du_t(i) = \sum_{j=1}^{n-1} \kappa_j [u_t(i+j \pmod n) - u_t(i)] dt + \lambda \sigma(u_t(i)) dB_t(i), \quad (4.3)$$

for an independent system  $B(0), \dots, B(n-1)$  of one-dimensional Brownian motions. Thus, in this example, (SHE) encodes all possible  $n$ -dimensional diffusions with local diffusion coefficients and Ornstein–Uhlenbeck type attractive drifts. Perhaps the most familiar example of this type is the simple symmetric case in which  $\kappa_1 = \kappa_{n-1} := \kappa > 0$  and  $\kappa_j = 0$  for  $j \notin \{1, n-1\}$ . In that case, (4.3) simplifies to

$$du_t(i) = \kappa(\Delta u_t)(i) + \lambda \sigma(u_t(i)) dB_t(i), \quad (4.4)$$

where  $(\Delta f)(i) := f(i \boxplus 1) + f(i \boxminus 1) - 2f(i)$  denotes the “group Laplacian” of  $f : \mathbf{Z}_n \rightarrow \mathbf{R}$ ,  $a \boxplus b := a + b \pmod n$ , and  $a \boxminus b := a - b \pmod n$ .

**Example 4.4** (Lattice groups). In this example,  $G$  denotes a lattice subgroup of  $\mathbf{R}^d$ . This basically means that  $G = \delta \mathbf{Z}^d$  for some  $\delta > 0$  and  $d = 1, 2, \dots$ . The class of all Lévy processes on  $G$  coincides with the class of all continuous-time random walks on  $G$ . Thus, standard random walk theory tells us that there exists a constant  $\kappa \geq 0$ —the rate—and a probability function  $\{J(y)\}_{y \in \delta \mathbf{Z}^d}$ —the so-called jump measure—such that  $(\mathcal{L}f)(x) = \kappa \sum_{y \in \delta \mathbf{Z}^d} \{f(y) - f(x)\} J(y)$ , and hence (SHE) is an encoding of the following infinite system of interacting Itô-type stochastic differential equations:

$$du_t(x) = \kappa \sum_{y \in \delta \mathbf{Z}^d} [u_t(y) - u_t(x)] J(y) + \lambda \sigma(u_t(x)) dB_t(x), \quad (4.5)$$

for iid one-dimensional Brownian motions  $\{B(z)\}_{z \in \delta \mathbf{Z}^d}$  and all  $x \in \delta \mathbf{Z}^d$ . A particularly well-known case is when  $J(y)$  puts equal mass on the neighbors of the origin in  $\delta \mathbf{Z}^d$ . In that case,

$$du_t(x) = \frac{\kappa}{2d} (\Delta u_t)(x) + \lambda \sigma(u_t(x)) dB_t(x), \quad (4.6)$$

where  $(\Delta f)(x) := \sum_{|y-x|=1} \{f(y) - f(x)\}$  denotes the graph Laplacian of  $f : \delta \mathbf{Z}^d \rightarrow \mathbf{R}$  with  $|y-x| := \sum_{i=1}^d |y_i - x_i|$ .

**Example 4.5** (The real line). As an example, let us choose  $G := \mathbf{R}$  and  $X :=$  1-dimensional Brownian motion on  $\mathbf{R}$ . Then,  $\mathcal{L}f = f''$  and (SHE) becomes the usual stochastic heat equation

$$\frac{\partial u_t(x)}{\partial t} = \kappa \frac{\partial^2 u_t(x)}{\partial x^2} + \lambda \sigma(u_t(x)) \xi, \quad (4.7)$$

driven by space-time white noise on  $(0, \infty) \times \mathbf{R}$ .

**Example 4.6** (The torus). Next, we may consider  $G := [0, 1)$ ; as usual we identify the ends of  $[0, 1)$  in order to obtain the torus  $G := \mathbf{T}$ , endowed with addition mod 1. Let  $X :=$  Brownian motion on  $\mathbf{T}$ . Its generator is easily seen to be the Laplacian on  $[0, 1)$  with periodic boundary conditions. Hence, (SHE) encodes

$$\begin{cases} \frac{\partial u_t(x)}{\partial t} = \kappa \frac{\partial^2 u_t(x)}{\partial x^2} + \lambda \sigma(u_t(x)) \xi & \text{for all } 0 \leq x < 1, \\ \text{subject to } u_t(0) = u_t(1-), \end{cases} \quad (4.8)$$

in this case.

**Example 4.7** (Totally disconnected examples). Examples 4.1 through 4.6 are concerned with more or less standard SDE/SPDE models. Here we mention one among many examples where (SHE) is more exotic. Consider  $G := \mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots$  to be a countable direct product of the cyclic group on two elements. Then,  $G$  is a compact abelian group; this is a group that acts transitively on binary trees and is related to problems in fractal percolation. A Lévy process on  $G$  is simply a process that has the form  $X_t^1 \times X_t^2 \times \cdots$  at time  $t \geq 0$ , where  $X^1 \times \cdots \times X^k$  is a Lévy process on  $\prod_{i=1}^k \mathbf{Z}_2$  for every  $k \geq 1$  [see Example 4.1]. It is easy to see then that if  $f : G \rightarrow \mathbf{R}$  is a function that is constant in every coordinate except for the coordinates in some finite set  $F$ , then the generator of  $X$  acts on  $f$  as  $\prod_{j \in F} \mathcal{L}^j f$ , where  $\mathcal{L}^j$  denotes the generator of  $X^j$  [see Example 4.1] and  $\mathcal{A}\mathcal{B}$  denotes the compositions of operators  $\mathcal{A}$  and  $\mathcal{B}$ . The resulting stochastic heat equation (SHE) is not the subject of our analysis here *per se*. Thus, we mention only in passing that, in this case, (SHE) appears to have connections to interacting random walks on a random environment on a binary tree.

**Example 4.8** (Positive multiplicative reals). Our next, and last, example, requires a slightly longer discussion than its predecessors. But we feel that this is an illuminating example and thus worth the effort.

Let

$$h(x) := e^x \quad (x \in \mathbf{R}). \quad (4.9)$$

The range  $G := h(\mathbf{R})$  of the function  $h$  is the multiplicative positive reals. Frequently, one writes  $G$  as  $\mathbf{R}_{>0}^\times$ ; this is an LCA group, and  $h$  is an isomorphism between  $\mathbf{R}$  and  $\mathbf{R}_{>0}^\times$ . [There are of course other topological isomorphisms from  $\mathbf{R}$  to  $\mathbf{R}_{>0}^\times$ ; in fact,  $\mathbf{R} \ni x \mapsto \exp(qx) \in \mathbf{R}_{>0}^\times$  works for every real number  $q \neq 0$ .] As  $h$  also maps  $G^*$  to  $\mathbf{R}^* = \mathbf{R}$  homomorphically as well, it follows that the dual

of  $\mathbf{R}_{>0}^\times$  is  $\mathbf{R}$ , and that the Fourier transform on  $\mathbf{R}_{>0}^\times$  is none other than the classical Mellin transform.

Since  $h(x) = e^x$  is a topological isomorphism from  $\mathbf{R}$  onto  $\mathbf{R}_{>0}^\times$ , every Lévy process  $X := \{X_t\}_{t \geq 0}$  on  $\mathbf{R}_{>0}^\times$  can be written as  $X_t = \exp(Y_t)$ , where  $Y := \{Y_t\}_{t \geq 0}$  is a Lévy process on  $\mathbf{R}$ . An interesting special case is  $Y_t = B_t + \delta t$ , where  $B := \{B_t\}_{t \geq 0}$  denotes 1-dimensional Brownian motion on  $\mathbf{R}$  and  $\delta \in \mathbf{R}$  is a parameter. Thus,

$$t \mapsto X_t := e^{B_t + \delta t} \quad (4.10)$$

defines a continuous Lévy process on  $\mathbf{R}_{>0}^\times$ . The best-known example is the case that  $\delta = -1/2$ , in which case  $X$  is the exponential martingale.

An application of Itô's formula [or an appeal to classical generator computations] shows that if  $f \in C^\infty(\mathbf{R})$ , then for all  $x > 0$ ,

$$Ef(xX_t) = f(x) + \frac{t}{2} x^2 f''(x) + \frac{t(1+2\delta)}{2} x f'(x) + o(t) \quad \text{as } t \downarrow 0. \quad (4.11)$$

Thus, we can summarize the preceding as follows: The exponential martingale is a Lévy process on  $\mathbf{R}_{>0}^\times$  with generator  $(\mathcal{L}f)(x) = \frac{1}{2} x^2 f''(x) + (\delta + \frac{1}{2}) x f'(x)$ . Thus, we can understand our stochastic heat equation (SHE), in this context, as the following euclidean SPDE:

$$\frac{\partial u_t(x)}{\partial t} = \frac{x^2}{2} \frac{\partial^2 u_t(x)}{\partial x^2} + \left( \delta + \frac{1}{2} \right) x \frac{\partial u_t(x)}{\partial x} + \lambda \sigma(u_t(x)) \xi_h; \quad (4.12)$$

for  $t, x > 0$ . Moreover,  $\xi_h$  denotes a space-time white noise on  $(0, \infty) \times (0, \infty)$  whose control measure is proportional to  $x^{-1} dt dx \mathbf{1}_{(0, \infty)^2}(t, x)$  [the restriction of the Haar measure on  $\mathbf{R}_+ \times \mathbf{R}_{>0}^\times$  to  $(0, \infty) \times \mathbf{R}_{>0}^\times$ ]. We expend a few lines and make the following amusing observation as an aside: From the perspective of these SPDEs, the most natural case is the drift-free case where  $\delta = -1/2$ . In that case, the underlying Lévy process  $X$  is the exponential martingale, as was noted earlier. The exponential martingale is one of the archetypal classical examples of an intermittent process [44]. Moreover,  $X$  is centered when  $\delta = -1/2$  in the sense that  $EX_t$  is the group identity. Interestingly enough, the exponential martingale is natural in other sense as well: (1) The process  $X$  is a natural candidate for being a ‘‘Gaussian’’ process with values in the group  $\mathbf{R}_{>0}^\times$  in the sense that  $X$  is the image of a real-valued Gaussian process under the exponential map; and (2)  $X$  has quadratic variation  $t$ ; that is,

$$\lim_{n \rightarrow \infty} \sum_{0 \leq k \leq 2^n t} \left[ X_{(k+1)/2^n} X_{k/2^n}^{-1} \right]^2 = t \quad \text{almost surely for all } t \geq 0. \quad (4.13)$$

This property can be verified by standard methods.

## 5 Lévy processes

Let us recall some basic facts about Lévy processes on LCA groups. For more details, see Berg and Forst [2] and Port and Stone [39, 40]. Bertoin [4], and

Jacob [28] are masterly accounts of the probabilistic and analytic aspects of the theory of Lévy processes on  $\mathbf{R}^n$  and  $\mathbf{Z}^n$ .

Throughout,  $(\Omega, \mathcal{F}, \mathbb{P})$  is a fixed probability space.

Let  $G$  denote an LCA group, and suppose  $Y := \{Y_t\}_{t \geq 0}$  is a stochastic process on  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $G$ . [We always opt to write  $Y_t$  in place of  $Y(t)$ , as is customary in the theory of stochastic processes.] We say that  $Y$  is a *Lévy process* on  $G$  if:

1.  $Y_0 = e_G$ , the identity element of  $G$ ;
2.  $Y_{t+s}Y_s^{-1}$  is independent of  $\{Y_u\}_{u \in [0, s]}$  and has the same distribution as  $Y_t$ , for all  $s, t \geq 0$ ; and
3. The random function  $t \mapsto Y_t$  is right continuous and has left limits everywhere with probability one.

Our definition might appear to be slightly more stringent than the standard definition, but turns out to be equivalent to the standard definition, for instance, when  $G$  is metrizable.

Let  $\mu_t := \mathbb{P} \circ Y_t^{-1}$  denote the distribution of the random variable  $Y_t$ . Then,  $\{P_t\}_{t \geq 0}$  is a convolution semigroup, where

$$(P_t f)(x) := \mathbb{E}f(xY_t) := \int_G f(xy) \mu_t(dy). \quad (5.1)$$

We can always write the Fourier transform of the probability measure  $\mu_t$  as follows:

$$\hat{\mu}_t(\chi) = \mathbb{E}(Y_t, \chi) = e^{-t\Psi(\chi)} \quad \text{for all } t \geq 0 \text{ and } \chi \in G^*, \quad (5.2)$$

where  $\Psi : G^* \rightarrow \mathbf{C}$  is continuous and  $\Psi(e_{G^*}) = 0$ . It is easy to see that Dalang's Condition (D) always implies the following:

$$\int_{G^*} e^{-t\operatorname{Re}\Psi(\chi)} m_{G^*}(d\chi) < \infty \quad \text{for all } t > 0. \quad (5.3)$$

See, for example, [23, Lemma 8.1]. In this case, the following is well defined

$$p_t(x) = \int_{G^*} (x^{-1}, \chi) e^{-t\Psi(\chi)} m_{G^*}(d\chi) \quad \text{for all } t > 0 \text{ and } x \in G. \quad (5.4)$$

The following is a consequence of Fubini's theorem.

**Lemma 5.1.** *The function  $(t, x) \mapsto p_t(x)$  is well defined and bounded as well as uniformly continuous for  $(t, x) \in [\delta, \infty) \times G$  for every fixed  $\delta > 0$ . Moreover, we can describe the semigroup via*

$$(P_t f)(x) = \int f(xy) p_t(y) m(dy) \quad \text{for all } t > 0, x \in G, f \in L^1(G). \quad (5.5)$$

Consequently,  $p_t(x) \geq 0$  for all  $t > 0$  and  $x \in G$ .

We omit the proof, as it is elementary. Let us mention, however, that the preceding lemma guarantees that the Chapman–Kolmogorov equation holds pointwise. That is,

$$p_{t+s}(x) = (p_t * p_s)(x) \quad \text{for all } s, t > 0 \text{ and } x \in G, \quad (5.6)$$

where “ $*$ ” denotes the usual convolution on  $L^1(G)$ ; that is,

$$(f * g)(x) := \int_G f(y)g(xy^{-1}) m_G(dy). \quad (5.7)$$

Define, for all  $t > 0$  and  $x \in G$ ,

$$\bar{p}_t(x) := (P_t p_t)(x) = \int_G p_t(xy)p_t(y) m_G(dy). \quad (5.8)$$

In particular, we apply the preceding with  $x := e_G$  in order to see that

$$\bar{p}_t(e_G) = \|p_t\|_{L^2(G)}^2 \quad \text{for all } t > 0. \quad (5.9)$$

Furthermore, it can be shown that the following inversion theorem holds for all  $t > 0$  and  $x \in G$ :

$$\bar{p}_t(x) = \int_{G^*} (x^{-1}, \chi) e^{-2t \operatorname{Re} \Psi(\chi)} m_{G^*}(d\chi). \quad (5.10)$$

Thus, we find that

$$\Upsilon(\beta) := \int_0^\infty e^{-\beta t} \|p_t\|_{L^2(G)}^2 dt \quad (5.11)$$

satisfies

$$\Upsilon(\beta) := \int_0^\infty e^{-\beta t} \|p_t\|_{L^2(G)}^2 dt = \int_{G^*} \frac{m_{G^*}(d\chi)}{\beta + 2 \operatorname{Re} \Psi(\chi)}. \quad (5.12)$$

Consequently, Dalang’s Condition (D) can be recast equivalently and succinctly as the condition that  $\Upsilon : [0, \infty) \rightarrow [0, \infty)$  is finite on  $(0, \infty)$ .

Since  $t \mapsto \int_0^t \bar{p}_s(e_G) ds$  is nondecreasing, Lemma 3.3 of [23] implies the following abelian/tauberian bound:

$$e^{-1} \Upsilon(1/t) \leq \int_0^t \bar{p}_s(e_G) ds \leq e \Upsilon(1/t) \quad \text{for all } t > 0. \quad (5.13)$$

Finally, by *the generator* of  $\{X_t\}_{t \geq 0}$  we mean the linear operator  $\mathcal{L}$  with domain

$$\operatorname{Dom}[\mathcal{L}] := \left\{ f \in L^2(G) : \mathcal{L}f := \lim_{t \downarrow 0} t^{-1}(P_t f - f) \text{ in } L^2(G) \right\}. \quad (5.14)$$

This defines  $\mathcal{L}$  as an  $L^2$ -generator, which is a slightly different operator than the one that is usually obtained from the Hille–Yosida theorem. The  $L^2$ -theory makes good sense here for a number of reasons; chief among them is the fact

that  $G$  need not be second countable and hence the standard form of the Hille–Yosida theorem is not applicable. The  $L^2$ -theory has the added advantage that the domain is more or less explicit, as will be seen shortly.

Recall that each  $P_t$  is a contraction on  $L^2(G)$ , and observe that

$$\widehat{P_t f}(\chi) = \hat{f}(\chi) \exp \left\{ -t \overline{\Psi(\chi)} \right\} \quad \text{for all } t \geq 0 \text{ and } \chi \in G^*. \quad (5.15)$$

Therefore, for all  $f, g \in L^2(G)$ ,

$$\int_G g(P_t f - f) \, dm_G = - \int_{G^*} \hat{f}(\chi) \overline{\hat{g}(\chi)} \left( 1 - e^{-t \overline{\Psi(\chi)}} \right) m_{G^*}(\mathrm{d}\chi). \quad (5.16)$$

It follows fairly readily from this relation that  $\mathcal{L} : \text{Dom}[\mathcal{L}] \rightarrow L^2(G)$ ,

$$\text{Dom}[\mathcal{L}] = \left\{ f \in L^2(G) : \int_{G^*} |\hat{f}(\chi)|^2 |\Psi(\chi)|^2 m_{G^*}(\mathrm{d}\chi) < \infty \right\}, \quad (5.17)$$

and for all  $f \in \text{Dom}[\mathcal{L}]$  and  $g \in L^2(G)$ ,

$$\int_G g \mathcal{L} f \, dm_G = - \int_{G^*} \hat{f}(\chi) \overline{\hat{g}(\chi) \Psi(\chi)} m_{G^*}(\mathrm{d}\chi). \quad (5.18)$$

The latter identity is another way to write

$$\widehat{\mathcal{L} f}(\chi) = -\hat{f}(\chi) \overline{\Psi(\chi)} \quad \text{for all } f \in \text{Dom}[\mathcal{L}] \text{ and } \chi \in G^*. \quad (5.19)$$

In other words,  $\mathcal{L}$  is a pseudo-differential operator on  $L^2(G)$  with Fourier multiplier [“symbol”]  $-\overline{\Psi}$ .

## 6 Stochastic convolutions

Throughout this paper,  $\xi$  will denote space-time white noise on  $\mathbf{R}_+ \times G$ . That is,  $\xi$  is a set-indexed Gaussian random field, indexed by Borel subsets of  $\mathbf{R}_+ \times G$  that have finite measure  $\text{Leb} \times m_G$  [product of Lebesgue and Haar measures, respectively on  $\mathbf{R}_+$  and  $G$ ]. Moreover,  $E\xi(A \times T) = 0$  for all measurable  $A \subset \mathbf{R}_+$  and  $T \subset G$  of finite measure [resp. Lebesgue and Haar], and

$$\text{Cov}(\xi(B \times T), \xi(A \times S)) = \text{Leb}(B \cap A) \cdot m_G(T \cap S), \quad (6.1)$$

for all Borel sets  $A, B \subset \mathbf{R}_+$  that have finite Lebesgue measure and all Borel sets  $S, T \subseteq G$  that have finite Haar measure. It is easy to see that  $\xi$  is then a vector-valued measure with values in  $L^2(\mathbf{P})$ .

The principal goal of this section is to introduce and study stochastic convolutions of the form

$$(K \circledast Z)_t(x) := \int_{(0,t) \times G} K_{t-s}(yx^{-1}) Z_s(y) \xi(\mathrm{d}s \, \mathrm{d}y), \quad (6.2)$$

where  $Z$  is a suitable space-time random field and  $K$  is a nice nonrandom space-time function from  $(0, \infty) \times G$  to  $\mathbf{R}$ ; Lemma 6.5 below will make precise the meaning of “suitable” in this context.

If  $Z$  is a predictable random field, in the sense of Walsh [43] and Dalang [13], and satisfies

$$\sup_{t \in [0, T]} \sup_{x \in G} \mathbf{E} (|Z_t(x)|^2) < \infty, \quad \int_0^T ds \int_G m_G(dy) [K_s(y)]^2 < \infty, \quad (6.3)$$

for all  $T > 0$ , then the stochastic convolution  $K \circledast Z$  is the same stochastic integral that has been obtained in Walsh [43] and, in particular, Dalang [13]. One of the essential properties of the resulting stochastic integral is the following  $L^2$  isometry:

$$\mathbf{E} \left( |(K \circledast Z)_t(x)|^2 \right) = \int_0^t ds \int_G m_G(dy) [K_{t-s}(yx^{-1})]^2 \mathbf{E} (|Z_s(y)|^2). \quad (6.4)$$

In this section we briefly describe an extension of the Walsh–Dalang stochastic integral that has the property that  $t \mapsto (K \circledast Z)_t$  is a stochastic process with values in the group algebra  $L^2(G)$ . Thus, the resulting stochastic convolution need not be, and in general is not, a random field in the modern sense of the word. Rather, we can realize the stochastic convolution process  $t \mapsto (K \circledast Z)_t$  as a Hilbert-space-valued stochastic process, where the Hilbert space is  $L^2(G)$ .

Our construction has a similar flavor as some other recent constructions; see, in particular, Da Prato and Zabczyk [12] and Dalang and Quer–Sardanyons [16]. However, our construction also has some novel aspects.

Let us set forth some notation first. As always, let  $(\Omega, \mathcal{F}, \mathbf{P})$  denote a probability space.

**Definition 6.1.** Let  $Z := \{Z_t(x)\}_{t \in I, x \in G}$  be a two-parameter [space-time] real-valued stochastic process indexed by  $I \times G$ , where  $I$  is a measurable subset of  $\mathbf{R}_+$ . We say that  $Z$  is a *random field* when the function  $Z : (\omega, t, x) \mapsto Z_t(x)(\omega)$  is product measurable from  $\Omega \times I \times G$  to  $\mathbf{R}$ .

The preceding definition is somewhat unconventional; our random fields are frequently referred to as “universally measurable random fields.” Because we will never have need for any other random fields than universally measurable ones, we feel justified in abbreviating the terminology.

**Definition 6.2.** For every random field  $Z := \{Z_t(x)\}_{t \geq 0, x \in G}$  and  $\beta \geq 0$ , let us define

$$\mathcal{N}_\beta(Z; G) := \sup_{t \geq 0} \left\{ e^{-2\beta t} \mathbf{E} \left( \|Z_t\|_{L^2(G)}^2 \right) \right\}^{1/2}. \quad (6.5)$$

We may sometimes only write  $\mathcal{N}_\beta(Z)$  when it is clear which underlying group we are referring to.

Each  $\mathcal{N}_\beta$  defines a norm on space-time random fields, provided that we identify a random field with all of its versions.

**Definition 6.3.** For every  $\beta \geq 0$ , we define  $\mathcal{L}_\beta^2(G)$  be the  $L^2$ -space of all measurable functions  $\Phi : (0, \infty) \times G \rightarrow \mathbf{R}$  with  $\|\Phi\|_{\mathcal{L}_\beta^2(G)} < \infty$ , where

$$\|\Phi\|_{\mathcal{L}_\beta^2(G)}^2 := \int_0^\infty e^{-2\beta s} \|\Phi_s\|_{L^2(G)}^2 ds. \quad (6.6)$$

We emphasize that the elements of  $\mathcal{L}_\beta^2(G)$  are nonrandom. Define, for every  $\varphi \in L^2(G)$  and  $t \geq 0$ ,

$$B_t(\varphi) := \int_{(0,t) \times G} \varphi(y) \xi(ds dy). \quad (6.7)$$

The preceding is understood as a Wiener integral, and it is easy to see that  $\{B_t(\varphi)\}_{t \geq 0}$  is Brownian motion scaled to have variance  $\|\varphi\|_{L^2(G)}$  at time one. Let  $\mathcal{F}_t$  denote the  $\sigma$ -algebra generated by all random variables of the form  $B_s(\varphi)$ , as  $s$  ranges within  $[0, t]$  and  $\varphi$  ranges within  $L^2(G)$ . Then,  $\{\mathcal{F}_t\}_{t \geq 0}$  is the [raw] filtration of the white noise  $\xi$ . Without changing the notation, we will complete [P] every  $\sigma$ -algebra  $\mathcal{F}_t$  and also make  $\{\mathcal{F}_t\}_{t \geq 0}$  right continuous in the usual way. In this way, we may apply the martingale-measure machinery of Walsh [43] whenever we need to.

A space-time stochastic process  $Z := \{Z_t(x)\}_{t \geq 0, x \in G}$  is called an *elementary random field* [43] if we can write  $Z_t(x) = X \mathbf{1}_{[a,b)}(t) \psi(x)$ , where  $0 < a < b$ ,  $\psi \in C_c(G)$  [the usual space of real-valued continuous functions with compact support on  $G$ ], and  $X \in L^2(\mathbf{P})$  is  $\mathcal{F}_a$ -measurable. Clearly, elementary random fields are random fields in the sense mentioned earlier.

A space-time stochastic process is a *simple random field* [43] if it is a finite nonrandom sum of elementary random fields.

**Definition 6.4.** For every  $\beta \geq 0$ , we define  $\mathcal{P}_\beta^2(G)$  to be the completion of the collection of simple random fields in the norm  $\mathcal{N}_\beta$ . We may observe that: (i) Every  $\mathcal{P}_\beta^2(G)$  is a Banach space, once endowed with norm  $\mathcal{N}_\beta$ ; and (ii) If  $\alpha < \beta$ , then  $\mathcal{P}_\alpha^2(G) \subseteq \mathcal{P}_\beta^2(G)$ .

We can think of an element of  $\mathcal{P}_\beta^2(G)$  as a “predictable random field” in some extended sense.

Let us observe that if  $K \in \mathcal{L}_\beta^2(G)$ , then  $\int_0^T ds \int_G m_G(dy) [K_s(y)]^2 < \infty$  all  $T > 0$ . Indeed,

$$\int_0^T ds \int_G m_G(dy) [K_s(y)]^2 \leq e^{2\beta T} \|K\|_{\mathcal{L}_\beta^2(G)}^2. \quad (6.8)$$

Therefore, we can define the stochastic convolution  $K \otimes Z$  for all simple random fields  $Z$  and all  $K \in \mathcal{L}_\beta^2(G)$  as in Walsh [43]. The following yields further information on this stochastic convolution. For other versions of such stochastic Young inequalities see Foondun and Khoshnevisan [22], and especially Conus and Khoshnevisan [9].

**Lemma 6.5** (Stochastic Young inequality). *Suppose that  $Z$  is a simple random field and  $K \in \mathcal{L}_\beta^2(G)$  for some  $\beta \geq 0$ . Then,  $K \circledast Z \in \mathcal{P}_\beta^2(G)$ , and*

$$\mathcal{N}_\beta(K \circledast Z) \leq \mathcal{N}_\beta(Z) \cdot \|K\|_{\mathcal{L}_\beta^2(G)}. \quad (6.9)$$

If  $K \in \mathcal{L}_\beta^2(G)$ , then Walsh's theory [43] produces a space-time stochastic process  $(t, x) \mapsto (K \circledast Z)_t(x)$ ; that is, a collection of random variables  $(K \circledast Z)_t(x)$ , one for every  $(t, x) \in (0, \infty) \times G$ . Thus, the stochastic convolution in Lemma 6.5 is well defined.

Lemma 6.5 implies that the stochastic convolution operator  $K \circledast \bullet$  is a bounded linear map from  $Z \in \mathcal{P}_\beta^2(G)$  to  $K \circledast Z \in \mathcal{P}_\beta^2(G)$  with operator norm being at most  $\|K\|_{\mathcal{L}_\beta^2(G)}$ . In particular, it follows readily from this lemma that  $K \circledast Z$  is a random field, since it is an element of  $\mathcal{P}_\beta^2(G)$ .

*Proof.* It suffices to consider the case that  $Z$  is an elementary random field.

Let us say that a function  $K : (0, \infty) \times G \rightarrow \mathbf{R}$  is *elementary* [in the sense of Lebesgue] if we can write  $K_s(y) = A \mathbf{1}_{[c,d]}(s) \phi(y)$  where  $A \in \mathbf{R}$ ,  $0 \leq c < d$ , and  $\phi \in C_c(G)$  [the usual space of continuous real-valued functions on  $G$  that have compact support]. Let us say also that  $K$  is a *simple* function [also in the sense of Lebesgue] if it is a finite sum of elementary functions. These are small variations on the usual definitions of the Lebesgue theory of integration. But they produce the same theory as that of Lebesgue. Here, these variations are particularly handy.

From now on, let us choose and fix some constant  $\beta \geq 0$ , and let us observe that if  $K$  were an elementary function, then  $K \in \mathcal{L}_\beta^2(G)$  for every  $\beta \geq 0$ .

Suppose we could establish (6.9) in the case that  $K$  is an elementary function. Then of course (6.9) also holds when  $K$  is a simple function. Because  $C_c(G)$  is dense in  $L^1(m_G)$  [41, E8, p. 267], the usual form of Lebesgue's theory ensures that simple functions are dense in  $\mathcal{L}_\beta^2(G)$ . Therefore by density, if we could prove that " $K \circledast Z \in \mathcal{P}_\beta^2(G)$ " and (6.9) both hold in the case that  $K$  is elementary, then we can deduce " $K \circledast Z \in \mathcal{P}_\beta^2(G)$ " and (6.9) for all  $K \in \mathcal{L}_\beta^2(G)$ . This reduces our entire problem to the case where  $Z$  is an elementary random field and  $K$  is an elementary function, properties that we assume to be valid throughout the remainder of this proof. Thus, from now on we consider

$$K_s(y) = A \cdot \mathbf{1}_{[c,d]}(s) \phi(y) \quad \text{and} \quad Z_t(x) = X \cdot \mathbf{1}_{[a,b]}(t) \psi(x), \quad (6.10)$$

where  $A \in \mathbf{R}$ ,  $0 \leq c < d$ ,  $0 < a < b$ ,  $X \in L^2(\mathbf{P})$  is  $\mathcal{F}_a$ -measurable,  $\psi \in C_c(G)$ , and  $\phi \in C_c(G)$ . The remainder of the proof works is divided naturally into three steps.

**Step 1 (measurability).** We first show that  $K \circledast Z$  is a random field in the sense of this paper.

Choose and fix some  $T > 0$ . According to the Walsh theory [43],

$$(K \circledast Z)_t(x) = AX \cdot \int_{\mathcal{T}(t) \times G} \phi(yx^{-1}) \psi(y) \xi(ds dy), \quad (6.11)$$

where  $\mathcal{T}(t) := (0, t) \cap [a, b) \cap [t - d, t - c)$ , and the stochastic integral can be understood as a Wiener integral, since the integrand is nonrandom and square integrable  $[ds \times m_G(dy)]$ . In particular, we may observe that for all  $x, w \in G$  and  $t \in [0, T]$ ,

$$\begin{aligned} & \mathbb{E} \left( |(K \otimes Z)_t(x) - (K \otimes Z)_t(w)|^2 \right) \\ &= A^2 \mathbb{E}(X^2) |\mathcal{T}(t)| \cdot \int_G m_G(dy) [\psi(y)]^2 |\phi(yx^{-1}) - \phi(yw^{-1})|^2 \\ &\leq \text{const} \cdot \int_G |\phi(yw^{-1}x) - \phi(y)|^2 m_G(dy), \end{aligned} \quad (6.12)$$

where  $|\mathcal{T}(t)| = t(b - a)(d - c)$  denotes the Lebesgue measure of  $\mathcal{T}(t)$ , and the implied constant does not depend on  $(t, x, w) \in [0, T] \times G \times G$ . Similarly, for every  $0 \leq t \leq \tau \leq T$  and  $x \in G$ ,

$$\mathbb{E} \left( |(K \otimes Z)_t(x) - (K \otimes Z)_\tau(x)|^2 \right) \leq \text{const} \cdot (\tau - t), \quad (6.13)$$

where the implied constant does not depend on  $(t, x, w) \in [0, T] \times G \times G$ . Consequently,

$$\lim_{\substack{x \rightarrow w \\ t \rightarrow \tau}} \mathbb{E} \left( |(K \otimes Z)_t(x) - (K \otimes Z)_\tau(w)|^2 \right) = 0, \quad (6.14)$$

uniformly for all  $\tau \in [0, T]$  and  $w \in G$ . In light of a separability theorem of Doob [18, Ch. 2], the preceding implies that  $(\Omega, (0, \infty), G) \ni (\omega, t, x) \mapsto (K \otimes Z)_t(x)(\omega)$  has a product-measurable version.<sup>7</sup>

**Step 2 (extended predictability).** Next we prove that  $K \otimes Z \in \mathcal{P}_\beta^2(G)$ .

Let us define another elementary function  $\bar{K}_s(y) := A \mathbf{1}_{[c, d)}(s) \bar{\phi}(y)$  where  $A$  and  $(c, d)$  are the same as they were in the construction of  $K$ , but  $\bar{\phi} \in L^2(G)$  is not necessarily the same as  $\phi$ . It is easy to see that

$$\begin{aligned} & \mathbb{E} \left( |(K \otimes Z)_t(x) - (\bar{K} \otimes Z)_t(x)|^2 \right) \\ &= A^2 \mathbb{E}(X^2) |\mathcal{T}(t)| \cdot \int_G [\psi(y)]^2 |\phi(yx^{-1}) - \bar{\phi}(yx^{-1})|^2 m_G(dy) \\ &\leq \text{const} \cdot \|\phi - \bar{\phi}\|_{L^2(G)}^2, \end{aligned} \quad (6.15)$$

where the implied constant does not depend on  $(t, x, \phi, \bar{\phi})$ . The definition of the stochastic convolution shows that

$$\text{supp}((K \otimes Z)_t) \subseteq \text{supp}(\psi) \oplus \text{supp}(\phi), \quad (6.16)$$

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<sup>7</sup>As written, Doob's theorem is applicable to the case of stochastic processes that are indexed by euclidean spaces. But the very same proof will work for processes that are indexed by  $\mathbf{R}_+ \times G$ .

almost surely for all  $t \geq 0$ , where ‘‘supp’’ denotes ‘‘support.’’ Since  $K \otimes Z$  and  $\bar{K} \otimes Z$  are both random fields (Step 1), we can integrate both sides of (6.15)  $[\exp(-2\beta t) dt \times m_G(dx)]$  in order to find that

$$[\mathcal{N}_\beta(K \otimes Z - \bar{K} \otimes Z)]^2 \leq \text{const} \cdot \|\phi - \bar{\phi}\|_{L^2(G)}^2 \cdot m_G(\text{supp}(\psi) \oplus S), \quad (6.17)$$

where  $S$  is any compact set that contains both the supports of both  $\phi$  and  $\bar{\phi}$ . Of course,  $\text{supp}(\psi) \oplus S$  has finite  $m_G$ -measure since it is a compact set.

We now use the preceding computations as follows: Let us choose in place of  $\bar{\phi}$  a sequence of functions  $\phi^1, \phi^2, \dots$ , all in  $L^2(G)$  and all supported in one fixed compact set  $S \supset \text{supp}(\phi)$ , such that: (i) Each  $\phi^j$  can be written as  $\phi^j(x) := \sum_{i=1}^{n_j} a_{i,j} \mathbf{1}_{E_i}(x)$  for some constants  $a_{i,j}$ 's and compact sets  $E_j \subset G$ ; and (ii)  $\|\phi - \phi^j\|_{L^2(G)} \rightarrow 0$  as  $j \rightarrow \infty$ . The resulting kernel can be written as  $K^j$  [in place of  $\bar{K}$ ]. Thanks to (6.17),

$$\lim_{j \rightarrow \infty} \mathcal{N}_\beta(K \otimes Z - K^j \otimes Z) = 0. \quad (6.18)$$

A direct computation shows that  $K^j \otimes Z$  is an elementary random field, and hence it is in  $\mathcal{P}_\beta^2$ . Thanks to the preceding display,  $K \otimes Z$  is also in  $\mathcal{P}_\beta^2$ . This completes the proof of Step 2.

**Step 3 (proof of (6.9)).** Since

$$\mathbb{E} \left( |(K \otimes Z)_t(x)|^2 \right) = \int_0^t ds \int_G m_G(dy) [K_{t-s}(yx^{-1})]^2 \mathbb{E}(|Z_s(y)|^2), \quad (6.19)$$

we integrate both sides  $[dm]$  in order to obtain

$$\begin{aligned} \mathbb{E} \left( \|(K \otimes Z)_t\|_{L^2(G)}^2 \right) &= \int_0^t \|K_{t-s}\|_{L^2(G)}^2 \mathbb{E} \left( \|Z_s\|_{L^2(G)}^2 \right) ds \\ &\leq e^{2\beta t} [\mathcal{N}_\beta(Z)]^2 \int_0^t e^{-2\beta(t-s)} \|K_{t-s}\|_{L^2(G)}^2 ds \\ &\leq e^{2\beta t} [\mathcal{N}_\beta(Z)]^2 \|K\|_{\mathcal{L}_\beta^2}^2. \end{aligned} \quad (6.20)$$

The interchange of integrals and expectation is justified by Tonelli's theorem, thanks to Step 1. Divide by  $\exp(-2\beta t)$  and optimize over  $t \geq 0$  to deduce (6.9) whence the lemma.  $\square$

Now we extend the definition of the stochastic convolution as follows: Suppose  $K \in \mathcal{L}_\beta^2$  and  $Z \in \mathcal{P}_\beta^2$  for some  $\beta \geq 0$ . Then we can find simple random fields  $Z^1, Z^2, \dots$  such that  $\lim_{n \rightarrow \infty} \mathcal{N}_\beta(Z^n - Z) = 0$ . Lemma 6.5 ensures that

$$\lim_{n \rightarrow \infty} \mathcal{N}_\beta(K^n \otimes Z - K \otimes Z) = 0, \quad (6.21)$$

and hence the following result holds.

**Theorem 6.6.** *If  $K \in \mathcal{L}_\beta^2(G)$  and  $Z \in \mathcal{P}_\beta^2(G)$  for some  $\beta \geq 0$ , then there exists  $K \circledast Z \in \mathcal{P}_\beta^2(G)$  such that  $(K, Z) \mapsto K \circledast Z$  is a.s. a bilinear map that satisfies (6.9). This stochastic convolution  $K \circledast Z$  agrees with the Walsh stochastic convolution when  $Z$  is a simple random field.*

The random field  $K \circledast Z$  is the *stochastic convolution* of  $K$  and  $Z$ . Let us emphasize, however, that this construction of  $K \circledast Z$  produces a stochastic process  $t \mapsto (K \circledast Z)_t$  with values in  $L^2(G)$ .

## 7 Proof of Theorem 2.1: Part 1

The proof of Theorem 2.1 is divided naturally in two parts: First we study the case that  $\sigma(0) = 0$ ; after that we visit the case that  $G$  is compact. The two cases are handled by different methods. Throughout this section, we address only the first case, and hence we assume that

$$\sigma(0) = 0, \quad \text{whence} \quad |\sigma(z)| \leq L_\sigma |z| \quad \text{for all } z \in \mathbf{R}; \quad (7.1)$$

see (2.9).

Our derivation follows ideas of Walsh [43] and Dalang [13], but has novel features as well, since our stochastic convolutions are not defined as classical [everywhere defined] random fields but rather as elements of the space  $\cup_{\beta \geq 0} \mathcal{P}_\beta^2(G)$ . Therefore, we hash out some of the details of the proof of Theorem 2.1. Throughout, we write  $u_t(x)$  in place of  $u(t, x)$ , as is customary in the theory of stochastic processes. Thus, let us emphasize that we *never* write  $u_t$  in place of  $\partial u / \partial t$ .

Let us follow [essentially] the treatment of Walsh [43], and say that a stochastic process  $u := \{u_t\}_{t \geq 0}$  with values in  $L^2(G)$  is a *mild solution* to (SHE) with initial function  $u_0 \in L^2(G)$ , when  $u$  satisfies

$$u_t = P_t u_0 + \lambda (p \circledast \sigma(u))_t \quad \text{a.s. for all } t > 0, \quad (7.2)$$

viewed as a random dynamical system on  $L^2(G)$ .<sup>8</sup> Somewhat more precisely, we wish to find a  $\beta \geq 0$ , sufficiently large, and solve the preceding as a stochastic integration equation for processes in  $\mathcal{P}_\beta^2(G)$ , using that value of  $\beta$ . Since the spaces  $\{\mathcal{P}_\beta^2(G)\}_{\beta \geq 0}$  are nested, there is no unique choice. But as it turns out there is a minimal acceptable choice for  $\beta$ , which we also will identify for later purposes.

The proof proceeds, as usual, by an appeal to Picard iteration. Let  $u_t^{(0)}(x) := u_0(x)$  and define iteratively

$$u_t^{(n+1)} := P_t u_0 + \lambda \left( p \circledast \sigma \left( u^{(n)} \right) \right)_t, \quad (7.3)$$

for all  $n \geq 1$ . Since

$$\mathcal{N}_\beta(P_t u_0) \leq \sup_{t \geq 0} \|P_t u_0\|_{L^2(G)} = \|u_0\|_{L^2(G)} \quad \text{for all } \beta \geq 0, \quad (7.4)$$

<sup>8</sup>In statements such as this, we sometimes omit writing ‘‘a.s.’’ particularly when the ‘‘almost sure’’ assertion is implied clearly.

and because  $\|p\|_{\mathcal{L}_\beta^2}^2 = \Upsilon(2\beta)$ , it follows from Lemma 6.5 that

$$\begin{aligned} \mathcal{N}_\beta \left( u^{(n+1)} \right) &\leq \|u_0\|_{L^2(G)} + \lambda \mathcal{N}_\beta \left( \sigma \circ u^{(n)} \right) \left( \int_0^\infty e^{-2\beta s} \|p_s\|_{L^2(G)}^2 ds \right)^{1/2} \\ &= \|u_0\|_{L^2(G)} + \lambda \mathcal{N}_\beta \left( \sigma \circ u^{(n)} \right) \sqrt{\Upsilon(2\beta)}, \end{aligned} \quad (7.5)$$

for all  $n \geq 1$  and  $\beta \geq 0$ . Next we apply the Lipschitz condition of  $\sigma$  together with the fact that  $\sigma(0) = 0$  in order to deduce the iterative bound

$$\mathcal{N}_\beta \left( u^{(n+1)} \right) \leq \|u_0\|_{L^2(G)} + \mathcal{N}_\beta \left( u^{(n)} \right) \lambda L_\sigma \sqrt{\Upsilon(2\beta)}. \quad (7.6)$$

Now we choose  $\beta$  somewhat carefully. Let us choose and fix some  $\varepsilon \in (0, 1)$ , and then define

$$\beta := \frac{1}{2} \Upsilon^{-1} \left( \frac{1}{(1+\varepsilon)^2 \lambda^2 L_\sigma^2} \right), \quad (7.7)$$

which leads to the identity  $\lambda L_\sigma \sqrt{\Upsilon(2\beta)} = (1+\varepsilon)^{-1}$ , whence

$$\mathcal{N}_\beta \left( u^{(n+1)} \right) \leq \|u_0\|_{L^2(G)} + \frac{1}{(1+\varepsilon)} \mathcal{N}_\beta \left( u^{(n)} \right). \quad (7.8)$$

Since  $\mathcal{N}_\beta(u_0) = \|u_0\|_{L^2(G)}$ , it follows that

$$\sup_{n \geq 0} \mathcal{N}_\beta \left( u^{(n)} \right) \leq \frac{1+\varepsilon}{\varepsilon} \|u_0\|_{L^2(G)}. \quad (7.9)$$

The same value of  $\beta$  can be applied in a similar way in order to deduce that

$$\mathcal{N}_\beta \left( u^{(n+1)} - u^{(n)} \right) \leq \frac{1}{1+\varepsilon} \mathcal{N}_\beta \left( u^{(n)} - u^{(n-1)} \right). \quad (7.10)$$

This shows, in particular, that  $\sum_{n=0}^\infty \mathcal{N}_\beta(u^{(n+1)} - u^{(n)}) < \infty$ , whence there exists  $u$  such that  $\lim_{n \rightarrow \infty} \mathcal{N}_\beta(u^{(n)} - u) = 0$ . Since

$$\begin{aligned} &\mathcal{N}_\beta \left( p \circledast \left[ \sigma \left( u^{(n)} \right) - \sigma \left( u \right) \right] \right) \\ &\leq \lambda \mathcal{N}_\beta \left( \sigma \left( u^{(n)} \right) - \sigma \left( u \right) \right) \cdot \left( \int_0^\infty e^{-2\beta s} \|p_s\|_{L^2(G)}^2 ds \right)^{1/2} \\ &\leq \lambda L_\sigma \cdot \mathcal{N}_\beta \left( u^{(n)} - u \right) \sqrt{\Upsilon(2\beta)}, \end{aligned} \quad (7.11)$$

it follows that the stochastic convolution  $p \circledast \sigma(u^{(n)})$  converges in norm  $\mathcal{N}_\beta$  to the stochastic convolution  $p \circledast \sigma(u)$ . Thus, it follows that  $u$  solves the stochastic heat equation and the  $L^2$  moment bound on  $u$  is a consequence of the fact that  $\mathcal{N}_\beta(u) \leq (1+\varepsilon)\varepsilon^{-1} \|u_0\|_{L^2(G)}$ , for the present choice of  $\beta$ . The preceding can be unscrambled as follows:

$$\mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right) \leq \frac{(1+\varepsilon)^2}{\varepsilon^2} \|u_0\|_{L^2(G)}^2 \exp \left\{ \frac{t}{2} \Upsilon^{-1} \left( \frac{1}{(1+\varepsilon)^2 \lambda^2 L_\sigma^2} \right) \right\}, \quad (7.12)$$

for all  $\varepsilon \in (0, 1)$  and  $t \geq 0$ . Of course, (2.2) is a ready consequence. This proves the existence of the right sort of mild solution to (SHE).

The proof of uniqueness follows the ideas of Dalang [13] but computes norms in  $L^2(G)$  rather than pointwise norms. To be more specific, suppose  $v$  is another solution that satisfies (2.2) for some finite constant  $c \geq 0$ . Then of course  $v$  satisfies (2.2) also when  $c$  is replaced by any other larger constant. Therefore, there exists  $\beta \geq c \geq 0$  such that  $u, v \in \mathcal{P}_\beta^2$  [for the same  $\beta$ ]. A calculation, very much similar to those we made earlier for Picard's iteration, shows that

$$\mathcal{N}_\beta(u - v) \leq \lambda L_\sigma \cdot \mathcal{N}_\beta(u - v) \cdot \sqrt{\Upsilon(2\beta)}, \quad (7.13)$$

whence it follows that the  $L^2(G)$ -valued stochastic processes  $\{u_t\}_{t \geq 0}$  and  $\{v_t\}_{t \geq 0}$  are modifications of one another. This completes the proof.  $\square$

## 8 Proof of Theorem 2.1: Part 2

It remains to prove Theorem in the case that  $G$  is compact. If, additionally,  $\sigma(0) = 0$ , then the existence and uniqueness of a solution follows from the proof of the non-compact case. That proof states, in an *a priori* sense, that if  $u_0 \in L^2(G)$  and  $\sigma(0) = 0$ , then  $u_t \in L^2(G)$  for all  $t > 0$  as well. This property is not in general true. Therefore, we need to proceed otherwise. Our approach is to reduce the problem to the case that  $u_0 \in C_c(G)$ , by approximation. Then we show that, in the case that  $u_0 \in C_c(G)$ , (SHE) has a pointwise [random field] solution that has the property that  $C_T := \sup_{t \in [0, T]} \sup_{x \in G} \mathbb{E}(|u_t(x)|^2) < \infty$  for all  $T > 0$ . It then follows from Tonelli's theorem that  $\sup_{t \in [0, T]} \mathcal{E}_t(\lambda) \leq C_T < \infty$ , since  $m_G(G) = 1$  in the compact case.

The actual proof requires a number of small technical steps.

Recall the norms  $\mathcal{N}_\beta$ . We now introduce a slightly different family of norms that were introduced earlier in Foondun and Khoshnevisan [22].

**Definition 8.1.** For every  $\beta \geq 0$  and for all everywhere-defined random fields  $Z := \{Z_t(x)\}_{t \geq 0, x \in G}$  we define

$$\mathcal{M}_\beta(Z) := \sup_{t \geq 0} \sup_{x \in G} \left\{ e^{-2\beta t} \mathbb{E}(|Z_t(x)|^2) \right\}^{1/2}. \quad (8.1)$$

We can define predictable random fields  $\mathcal{P}_\beta^\infty(G)$  with respect to the preceding norms, just as we defined spaces  $\mathcal{P}_\beta^2(G)$  of predictable random fields for  $\mathcal{N}_\beta$  in Definition 6.4.

**Definition 8.2.** For every  $\beta \geq 0$ , we define  $\mathcal{P}_\beta^\infty(G)$  to be the completion of the collection of simple random fields in the norm  $\mathcal{M}_\beta$ . We may observe that: (i) Every  $\mathcal{P}_\beta^\infty(G)$  is a Banach space, once endowed with norm  $\mathcal{M}_\beta$ ; and (ii) If  $\alpha < \beta$ , then  $\mathcal{P}_\alpha^\infty(G) \subseteq \mathcal{P}_\beta^\infty(G)$ .

Note that  $\mathcal{M}_\beta$  is a larger norm than  $\mathcal{N}_\beta$  on  $\mathcal{P}_\beta^\infty(G)$ , since  $G$  is compact. Indeed, because  $m_G(G) = 1$  it follows that  $\mathcal{N}_\beta(Z) \leq \mathcal{M}_\beta(Z)$  for all  $Z \in \mathcal{P}_\beta^\infty(G)$ .

The stochastic convolution  $K \circledast Z$  can be defined for  $Z \in \mathcal{P}_\beta^\infty(G)$  as well, just as one does it for  $Z \in \mathcal{P}_\beta^2(G)$  [Theorem 6.6]. The end result is the following.

**Theorem 8.3.** *If  $K \in \mathcal{L}_\beta^2(G)$  and  $Z \in \mathcal{P}_\beta^\infty(G)$  for some  $\beta \geq 0$ , then there exists  $K \circledast Z \in \mathcal{P}_\beta^\infty(G)$  such that  $(K, Z) \mapsto K \circledast Z$  is a.s. a bilinear map that satisfies the stochastic Young inequality,*

$$\mathcal{M}_\beta(K \circledast Z) \leq \mathcal{M}_\beta(Z) \cdot \|K\|_{\mathcal{L}_\beta^2(G)}. \quad (8.2)$$

*This stochastic convolution  $K \circledast Z$  agrees with the Walsh stochastic convolution when  $Z$  is a simple random field.*

The proof of Theorem 8.3 follows the same general pattern of the proof of Theorem 6.6 but one has to make a few adjustments that, we feel, are routine. Therefore, we omit the details. However, we would like to emphasize that this stochastic convolution is not always the same as the one that was constructed in the previous sections. In particular, let us note that if  $K \in \mathcal{L}_\beta^2(G)$  and  $Z \in \mathcal{P}_\beta^\infty(G)$  for some  $\beta \geq 0$ , then  $(K \circledast Z)_t(x)$  is a well-defined uniquely-defined random variable for all  $t > 0$  and  $x \in G$ . This should be compared to the fact that  $(K \circledast Z)_t$  is defined only as an element of  $L^2(G)$  when  $Z \in \mathcal{P}_\beta^2(G)$ .

The next result shows that (SHE) has a a.s.-unique mild pointwise solution  $u$  whenever  $u_0 \in L^\infty(G)$ , in the sense that  $u$  is the a.s.-unique solution to the equation

$$u_t(x) = (P_t u_0)(x) + (p \circledast \sigma(u))_t(x), \quad (8.3)$$

valid a.s. for every  $x \in G$  and  $t > 0$ . The preceding stochastic convolution is understood to be the one that we just constructed in this section. Among other things, the following tacitly ensures that the said stochastic convolution is well defined.

**Theorem 8.4.** *Let  $G$  be an LCA group, and  $\{X_t\}_{t \geq 0}$  be a Lévy process on  $G$ . If  $u_0 \in L^\infty(G)$ , then for every  $\lambda > 0$ , the stochastic heat equation (SHE) has a mild pointwise solution  $u$  that satisfies the following: There exists a finite constant  $b \geq 1$  that yields the energy inequality*

$$\sup_{x \in G} \mathbb{E}(|u_t(x)|^2) \leq b e^{bt} \quad \text{for every } t \geq 0. \quad (8.4)$$

*Moreover, if  $v$  is any mild solution that satisfies (2.2) as well as  $v_0 = u_0$ , then  $\mathbb{P}\{u_t(x) = v_t(x)\} = 1$  for all  $t \geq 0$  and  $x \in G$ .*

One can model a proof of Theorem 8.4 after the already-proved portion of Theorem 2.1 [that is, in the case that  $\sigma(0) = 0$ ], but use the norm  $\mathcal{M}_\beta$  in place of  $\mathcal{N}_\beta$ . In fact, such a proof will imply that (8.4) has a solution that is in  $L^\infty(G)$  at all times as long as  $u_0 \in L^\infty(G)$ , even if  $G$  is not compact and  $\sigma(0)$  is not 0. When  $G = \mathbf{R}$ , the latter facts are also contained within the theory of Dalang [13]. For these reasons, we omit the proof of Theorem 8.4. But let us emphasize that since  $u$  is a random field in the sense of the present paper, (8.4) and Fubini's theorem together imply that if  $u_0 \in L^\infty(G)$ , then

$$\mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right) \leq b e^{bt} m_G(G). \quad (8.5)$$

Now let us recall that for our present purposes  $G$  is compact, and hence  $m_G(G) = 1$ . It follows from these conditions that the solution  $u_t$  is also in  $L^2(G)$ , for all  $t > 0$ , as long as  $u_0 \in L^\infty(G)$ .<sup>9</sup>

Now we begin our proof of Theorem 2.1 in the case that  $G$  is compact, an assumption which we assume for the remainder of the section.

Our normalization of Haar measure ensures that  $m_G(G) = 1$  in the present compact case. Consequently,  $L^\infty(G) \subset L^2(G)$ , and hence if  $u_0 \in L^\infty(G)$ , then (SHE) has a random field solution, with values in  $L^2(G) \cap L^\infty(G)$  at all times, such that

$$\mathbf{E} \left( \|u_t\|_{L^2(G)}^2 \right) \leq b e^{bt}. \quad (8.6)$$

We also find, *a priori*, that  $u \in \mathcal{P}_\beta^2(G)$  for all sufficiently large  $\beta$ . This proves the theorem when  $G$  is compact and  $u_0 \in L^\infty(G)$ .

In fact, we can now use the *a priori* existence bounds that we just developed in order to argue, somewhat as in the Walsh theory, and see that [in this case where  $u_0 \in L^\infty(G)$ ]

$$\begin{aligned} & \mathbf{E} (|u_t(x)|^2) \\ &= |(P_t u_0)(x)|^2 + \lambda^2 \int_0^t ds \int_G m_G(dy) [p_{t-s}(yx^{-1})]^2 \mathbf{E} (|\sigma(u_s(y))|^2), \end{aligned} \quad (8.7)$$

for all  $t > 0$  and  $x \in G$ . But we will not need this formula at this time. Instead, let us observe the following variation: If  $v$  solves (SHE)—for the same white noise  $\xi$ —with  $v_0 \in L^\infty(G)$ , then

$$\begin{aligned} & \mathbf{E} (|u_t(x) - v_t(x)|^2) \\ &= |(P_t u_0)(x) - (P_t v_0)(x)|^2 \\ &\quad + \lambda^2 \int_0^t ds \int_G m_G(dy) [p_{t-s}(yx^{-1})]^2 \mathbf{E} (|\sigma(u_s(y)) - \sigma(v_s(y))|^2) \\ &\leq |(P_t u_0)(x) - (P_t v_0)(x)|^2 \\ &\quad + \lambda^2 \mathbf{L}_\sigma^2 \cdot \int_0^t ds \int_G m_G(dy) [p_{t-s}(yx^{-1})]^2 \mathbf{E} (|u_s(y) - v_s(y)|^2). \end{aligned} \quad (8.8)$$

Since each  $P_t$  is a linear contraction on  $L^2(G)$ , we may integrate both sides of the preceding inequality in order to deduce the following from Fubini's theorem: For every  $\beta \geq 0$ ,

$$\begin{aligned} & \mathbf{E} \left( \|u_t - v_t\|_{L^2(G)}^2 \right) \\ &\leq \|u_0 - v_0\|_{L^2(G)}^2 + \lambda^2 \mathbf{L}_\sigma^2 \cdot \int_0^t \|p_{t-s}\|_{L^2(G)}^2 \mathbf{E} \left( \|u_s - v_s\|_{L^2(G)}^2 \right) \\ &\leq \|u_0 - v_0\|_{L^2(G)}^2 + \lambda^2 \mathbf{L}_\sigma^2 e^{2\beta t} [\mathcal{N}_\beta(u - v)]^2 \cdot \Upsilon(2\beta). \end{aligned} \quad (8.9)$$

<sup>9</sup>This property can fail when  $G$  is not compact and  $\sigma(0)$  is not zero. For example, if  $u_0 = 0$ ,  $G = \mathbf{R}$ , and  $\sigma \equiv 1$  (the linear stochastic heat equation), then there is a unique solution that is in  $L^\infty(\mathbf{R})$  at all times but there is no solution that is in  $L^2(\mathbf{R})$  at any time  $t > 0$ .

In particular,

$$[\mathcal{N}_\beta(u - v)]^2 \leq \|u_0 - v_0\|_{L^2(G)}^2 + \lambda^2 \mathbf{L}_\sigma^2 [\mathcal{N}_\beta(u - v)]^2 \Upsilon(2\beta). \quad (8.10)$$

Owing to (8.6), we know that  $\mathcal{N}_\beta(u - v) < \infty$  if  $\beta$  is sufficiently large. By the dominated convergence theorem,  $\lim_{\beta \uparrow \infty} \Upsilon(2\beta) = 0$ , whence we have

$$\lambda^2 \mathbf{L}_\sigma^2 \Upsilon(2\beta) \leq 1/2 \quad \text{for all } \beta \text{ large enough.} \quad (8.11)$$

This shows that

$$\mathcal{N}_\beta(u - v) \leq \text{const} \cdot \|u_0 - v_0\|_{L^2(G)}, \quad (8.12)$$

for all  $u_0, v_0 \in L^\infty(G)$  and an implied constant that is finite and depends only on  $(\lambda, \mathbf{L}_\sigma, \Upsilon)$ .

Now that we have proved (8.12), we can complete the proof of Theorem 2.1 [in the case that  $G$  is compact] as follows: Suppose  $u_0 \in L^2(G)$ . Since  $C_c(G)$  is dense in  $L^2(G)$ , we can find  $u_0^{(1)}, u_0^{(2)}, \dots \in C_c(G)$  such that  $u_0^{(n)} \rightarrow u_0$  in  $L^2(G)$  as  $n \rightarrow \infty$ . Let  $u^{(n)} := \{u_t^{(n)}(x)\}_{t \geq 0, x \in G}$  denote the solution to (SHE) starting at  $u_0^{(n)}$ . Eq. (8.12) shows that  $\{u^{(n)}\}_{n=1}^\infty$  is a Cauchy sequence in  $\mathcal{P}_\beta^2(G)$  provided that  $\beta$  is chosen to be sufficiently large [but fixed!]. Therefore,  $w := \lim_{n \rightarrow \infty} u^{(n)}$  exists in  $\mathcal{P}_\beta^2(G)$ . Lemma 6.5 ensures that  $p \otimes u^{(n)}$  converges to  $p \otimes w$ , and hence  $w$  solves (SHE) starting at  $u_0$ . This proves existence. Uniqueness is proved by similar approximation arguments.

## 9 Proof of Proposition 2.3

First, consider the case that  $u_0 \in L^\infty(G)$ . In that case, we may apply (8.7) in order to see that the solution  $u$  is defined pointwise and satisfies

$$\mathbb{E}(|u_t(x)|^2) \leq |(P_t u_0)(x)|^2 + \lambda^2 \|\sigma\|_{L^\infty(\mathbf{R})}^2 \int_0^t \|p_s\|_{L^2(G)}^2 ds. \quad (9.1)$$

Since  $\int_0^t \|p_s\|_{L^2(G)}^2 ds = \int_0^t \bar{p}_s(e_G) ds \leq e\Upsilon(1/t) < \infty$  [(5.13)] and  $G$  is compact, the  $L^2(G)$ -contractive property of  $P_t$  yields

$$[\mathcal{E}_t(\lambda)]^2 = \mathbb{E}\left(\|u_t\|_{L^2(G)}^2\right) \leq \|u_0\|_{L^2(G)}^2 + e\lambda^2 \|\sigma\|_{L^\infty(\mathbf{R})}^2 \Upsilon(1/t). \quad (9.2)$$

If  $u$  is known to be only in  $L^2(G)$ , then by density we can find for every  $\varepsilon > 0$  a function  $v \in L^\infty(G)$  such that  $\|u_0 - v_0\|_{L^2(G)} \leq \varepsilon$ . The preceding paragraph and (8.12) together yield

$$\begin{aligned} & [\mathcal{E}_t(\lambda)]^2 \\ & \leq 2e^{2\beta t} [\mathcal{N}_\beta(u - v)]^2 + 2\left(\|v_0\|_{L^2(G)}^2 + e\lambda^2 \|\sigma\|_{L^\infty(\mathbf{R})}^2 \Upsilon(1/t)\right) \\ & \leq \text{const} \cdot 2e^{2\beta t} \varepsilon^2 + 2\left(2\|u_0\|_{L^2(G)}^2 + 2\varepsilon^2 + e\lambda^2 \|\sigma\|_{L^\infty(\mathbf{R})}^2 \Upsilon(1/t)\right). \end{aligned} \quad (9.3)$$

This is more than enough to show that  $\mathcal{E}_t(\lambda) = O(\lambda)$  for all  $t > 0$ . In fact, it yields also the quantitative bound,

$$\mathcal{E}_t(\lambda) \leq \text{const} \cdot \left( \|u_0\|_{L^2(G)} + \lambda \|\sigma\|_{L^\infty(\mathbf{R})} \sqrt{\Upsilon(1/t)} \right), \quad (9.4)$$

for a finite universal constant. This completes the first portion of the proof.

If  $|\sigma|$  is bounded uniformly from below, then we reduce the problem to the case that  $u_0 \in L^\infty(G)$  just as we did in the first half, using (8.12), and then apply (8.7) in order to see that [in the case that  $u_0 \in L^\infty(G)$ ],

$$\mathbb{E}(|u_t(x)|^2) \geq \inf_{z \in G} |u_0(z)|^2 + \lambda^2 \inf_{z \in \mathbf{R}} |\sigma(z)|^2 \cdot \int_0^t \|p_s\|_{L^2(G)}^2 ds. \quad (9.5)$$

We will skip the remaining details on how one makes the transition from considerations of initial values  $u_0 \in L^\infty(G)$  to initial values  $u_0 \in L^2(G)$ : This issue has been dealt with already in the first half of the proof. Instead, let us conclude the proof by observing that the preceding is consistent, since  $\int_0^t \|p_s\|_{L^2(G)}^2 ds > 0$ , for if this integral were zero for all  $t$  then the proof would fail. But because  $G$  is compact and  $m_G$  is a probability measure on  $G$ , Jensen's inequality reveals that  $\|p_s\|_{L^2(G)}^2 \geq \|p_s\|_{L^1(G)}^2 = 1$ . Therefore,  $\int_0^t \|p_s\|_{L^2(G)}^2 ds \geq t$  is positive when  $t$  is positive, as was advertized.

## 10 Condition (D) and local times

Dalang's Condition (D) is connected intimately to the theory of local times for Lévy processes. This connection was pointed out in Foondun, Khoshnevisan, and Nualart [23] when  $G = \mathbf{R}$ ; see also Eisenbaum et al [20]. Here we describe how one can extend that connection to the present, more general, setting where  $G$  is an LCA group.

Let  $Y := \{Y_t\}_{t \geq 0}$  be an independent copy of  $X$ , and consider the stochastic process

$$S_t := X_t Y_t^{-1} \quad (t \geq 0). \quad (10.1)$$

It is easy to see that  $S := \{S_t\}_{t \geq 0}$  is a Lévy process with characteristic function

$$\mathbb{E}(S_t, \chi) = e^{-2t \text{Re} \Psi(\chi)} \quad \text{for all } t \geq 0 \text{ and } \chi \in G^*, \quad (10.2)$$

where  $\Psi$  denote the Lévy–Khintchine exponent, or characteristic exponent, of the Lévy process  $\{X_t\}_{t \geq 0}$ . The process  $S$  is called the Lévy symmetrization of  $X$ ; the nomenclature is motivated by the fact that each  $S_t$  is a *symmetric* random variable in the sense that  $S_t$  and  $S_t^{-1}$  have the same distribution for all  $t \geq 0$ .

Let  $J$  denote the *weighted occupation measure* of  $S$ ; that is,

$$J(A) := \int_0^\infty \mathbf{1}_A(S_s) e^{-s} ds, \quad (10.3)$$

for all Borel sets  $A \subset G$ . It is easy to see that

$$\hat{J}(\chi) := \int_G (x, \chi) J(dx) = \int_0^\infty (S_s, \chi) e^{-s} ds \quad (\chi \in G^*), \quad (10.4)$$

whence

$$\mathbb{E} \left( |\hat{J}(\chi)|^2 \right) = 2 \int_0^\infty e^{-t} dt \int_0^t e^{-s} ds \mathbb{E} \left[ (S_s, \chi) \overline{(S_t, \chi)} \right]. \quad (10.5)$$

For every  $s, t \geq 0$  and for all characters  $\chi \in G^*$ ,

$$(S_s, \chi) \overline{(S_t, \chi)} = \chi(S_s) \chi(S_t^{-1}) = \chi(S_s S_t^{-1}) = (S_s S_t^{-1})(\chi). \quad (10.6)$$

Note that  $S_s S_t^{-1} = (S_t S_s^{-1})^{-1}$ , and that the distribution of  $S_t S_s^{-1}$  is the same as the distribution of  $S_{t-s}$  for  $t \geq s \geq 0$ . Since  $S_{t-s}$  has the same distribution as that of  $S_{t-s}^{-1}$ , by the symmetry of  $S$ , it follows that

$$\begin{aligned} \mathbb{E} \left( |\hat{J}(\chi)|^2 \right) &= 2 \int_0^\infty e^{-t} dt \int_0^t e^{-s} ds \mathbb{E} [(S_{t-s}, \chi)] \\ &= 2 \int_0^\infty e^{-s} ds \int_s^\infty e^{-t} dt e^{-(t-s)\text{Re}\Psi(\chi)} \\ &= \frac{1}{1 + 2\text{Re}\Psi(\chi)}, \end{aligned} \quad (10.7)$$

for every  $\chi \in G^*$ . Therefore,

$$\mathbb{E} \left( \|\hat{J}\|_{L^2(G^*)}^2 \right) = \int_{G^*} \left( \frac{1}{1 + 2\text{Re}\Psi(\chi)} \right) m_{G^*}(d\chi) = \Upsilon(1). \quad (10.8)$$

In particular, we have proved that Dalang's Condition (D) is equivalent to the condition that

$$\ell(x) := \frac{dJ}{dm_G}(x) \text{ exists and is in } L^2(\mathbb{P} \times m_G), \quad (10.9)$$

and in this case,

$$\mathbb{E} \left( \|\ell\|_{L^2(G)}^2 \right) = \mathbb{E} \left( \|\hat{J}\|_{L^2(G^*)}^2 \right) = \Upsilon(1), \quad (10.10)$$

thanks to Plancherel's theorem. For real-valued Lévy processes, this observation is due essentially to Hawkes [27].

The random field  $\ell$  is called the *local times* of  $\{S_t\}_{t \geq 0}$ ;  $\ell$  has, by its very definition, the property that it is a random probability function on  $G$  such that

$$\int_G f \ell dm_G = \int_0^\infty f(S_t) e^{-t} dt \quad \text{a.s.}, \quad (10.11)$$

for all nonrandom functions  $f \in L^2(G)$ .

Let us now return to the following remark that was made in the Introduction.

**Lemma 10.1.** *Dalang’s Condition (D) holds whenever  $G$  is discrete.*

This lemma was shown to hold as a consequence of Pontryagin–van Kampen duality. We can now understand this lemma, probabilistically.

*A probabilistic proof of Lemma 10.1.* When  $G$  is discrete, local times always exist and are described via

$$\ell(x) := \int_0^\infty \mathbf{1}_{\{x\}}(S_t) e^{-t} dt \quad (x \in G). \quad (10.12)$$

In light of (10.10), it remains to check only that  $\ell \in L^2(\mathbb{P} \times m_G)$ , since it is evident that  $\ell = dJ/dm_G$  in this case. But since  $m_G$  is the counting measure on  $G$ ,

$$\begin{aligned} \Upsilon(1) &= \|\ell\|_{L^2(\mathbb{P} \times m_G)}^2 \\ &= 2 \sum_{x \in G} \int_0^\infty e^{-s} ds \int_s^\infty e^{-t} dt \mathbb{P}\{S_s = x, S_t = x\} \\ &= 2 \int_0^\infty e^{-s} ds \int_s^\infty e^{-t} dt \mathbb{P}\{S_{t-s} = e_G\}, \end{aligned} \quad (10.13)$$

where  $e_G$  denotes the identity element in  $G$ . Since  $\mathbb{P}\{S_{t-s} = e_G\} \leq 1$ , it follows readily that  $\Upsilon(1) < \infty$ , whence follows Condition (D).  $\square$

## 11 Group invariance of the excitation indices

The principal aim of this section is to prove that the noise excitation indices  $\bar{\epsilon}(t)$  and  $\underline{\epsilon}(t)$  are “group invariants.” In order to do this we need to apply some care, but it is easy to describe informally what group invariance means: If we apply a topological isomorphism to  $G$  then we do not change the values of  $\bar{\epsilon}(t)$  and  $\underline{\epsilon}(t)$ .

**Definition 11.1.** Recall that two LCA groups  $G$  and  $\Gamma$  are *isomorphic* [as topological groups] if there exists a homeomorphic homomorphism  $h : G \rightarrow \Gamma$ . We will denote by  $\text{Iso}(G, \Gamma)$  the collection of all such topological isomorphisms, and write “ $G \simeq \Gamma$ ” when  $\text{Iso}(G, \Gamma) \neq \emptyset$ ; that is precisely when  $G$  and  $\Gamma$  are isomorphic to one another.

Throughout this section, we consider two LCA groups  $G \simeq \Gamma$ .

It is easy to see that if  $h \in \text{Iso}(G, \Gamma)$ , then  $m_\Gamma \circ h$  is a translation-invariant Borel measure on  $G$  whose total mass agrees with the total mass of  $m_G$ . Therefore, we can find a constant  $\mu(h) \in (0, \infty)$  such that

$$m_\Gamma \circ h = \mu(h) m_G \quad \text{for all } h \in \text{Iso}(G, \Gamma). \quad (11.1)$$

**Definition 11.2.** We refer to  $\mu : \text{Iso}(G, \Gamma) \rightarrow (0, \infty)$  as the *modulus function*, and  $\mu(h)$  as the *modulus of an isomorphism*  $h \in \text{Iso}(G, \Gamma)$ . In particular, we say that  $G$  is *unimodular* when  $\mu(h) = 1$ .

This definition is motivated by the following: Since  $G \simeq G$ , the collection  $\text{Aut}(G) := \text{Iso}(G, G)$  of all automorphisms of  $G$  is never empty. Recall that  $\text{Aut}(G)$  is in general a non-abelian group endowed with group product  $h \circ g$  [composition] and group inversion  $h^{-1}$  [functional inversion]. It is then easy to see that  $\mu$  is a homomorphism from  $\text{Aut}(G)$  into the multiplicative positive reals  $\mathbf{R}_{>0}^\times$ ; i.e., that  $\mu(h \circ g) = \mu(h)\mu(g)$  and  $\mu(h^{-1}) = 1/\mu(h)$  for every  $h, g \in \text{Aut}(G)$ . Thus, the Definition 11.2 of a unimodular group agrees with the usual one when  $\Gamma = G$ .

The following simple lemma is an immediate consequence of our standard normalization of Haar measures and states that compact and/or discrete LCA groups are unimodular. But it is worth recording.

**Lemma 11.3.** *Every element of  $\text{Iso}(G, \Gamma)$  is measure preserving when  $G$  is either compact or discrete. In other words, if  $G$  is compact or discrete, then so is  $\Gamma$ , and  $\mu(h) = 1$  for every  $h \in \text{Iso}(G, \Gamma)$ .*

Next, let  $\xi$  denote a space time white noise on  $\mathbf{R}_+ \times G$ . Given a function  $h \in \text{Iso}(G, \Gamma)$ , we may define a random set function  $\xi_h$  on  $\Gamma$  as follows:

$$\xi_h(A \times B) := \sqrt{\mu(h)} \xi(A \times h^{-1}(B)), \quad (11.2)$$

for all Borel sets  $A \subset \mathbf{R}_+$  and  $B \subset \Gamma$  with finite respective measures  $\text{Leb}(A)$  and  $m_G(B)$ . In this way, we find that  $\xi_h$  is a totally scattered Gaussian random measure on  $\mathbf{R}_+ \times \Gamma$  with control measure  $\text{Leb} \times m_\Gamma$ . Moreover,

$$\begin{aligned} \mathbb{E} \left( |\xi_h(A \times B)|^2 \right) &= \mu(h) \text{Leb}(A) (m_G \circ h^{-1})(B) \\ &= \text{Leb}(A) m_\Gamma(B). \end{aligned} \quad (11.3)$$

In other words, we have verified the following simple fact.

**Lemma 11.4.** *Let  $\xi$  denote a space-time white noise on  $\mathbf{R}_+ \times G$ . Then,  $\xi_h$  is a white noise on  $\mathbf{R}_+ \times \Gamma$  for every  $h \in \text{Iso}(G, \Gamma)$ .*

Note, in particular, that we can solve SPDEs on  $(0, \infty) \times \Gamma$  using the space-time white noise  $\xi_h$ . We will return to this matter shortly.

If  $f \in L^2(G)$  and  $h \in \text{Iso}(G, \Gamma)$ , then  $f \circ h^{-1}$  can be defined uniquely as an element of  $L^2(\Gamma)$  as well as pointwise. Here is how: First let us consider  $f \in C_c(G)$ , in which case  $f \circ h^{-1} : \Gamma \rightarrow \mathbf{R}$  is defined pointwise and is in  $C_c(\Gamma)$ . Next we observe that

$$\begin{aligned} \|f \circ h^{-1}\|_{L^2(\Gamma)}^2 &= \int_\Gamma |f(h^{-1}(x))|^2 m_\Gamma(dx) \\ &= \int_G |f(y)|^2 (m_\Gamma \circ h)(dy) \\ &= \mu(h) \|f\|_{L^2(G)}^2. \end{aligned} \quad (11.4)$$

Since  $C_c(G)$  is dense in  $L^2(G)$ , the preceding constructs uniquely  $f \circ h^{-1} \in L^2(\Gamma)$  for every topological isomorphism  $h : G \rightarrow \Gamma$ . Moreover, it follows that (11.4)

is valid for all  $f \in L^2(G)$ . This construction has a handy consequence which we describe next.

For the sake of notational simplicity, if  $Z$  is a random field, then we write  $Z \circ h^{-1}$  for the random field  $Z_t(h^{-1}(x))$ , whenever  $h$  is such that this definition makes sense. Of course, if  $Z$  is nonrandom, then we may use the very same notation; thus,  $K \circ h^{-1}$  makes sense equally well in what follows.

**Lemma 11.5.** *Let  $\beta \geq 0$  and  $h \in \text{Iso}(G, \Gamma)$ . If  $Z \in \mathcal{P}_\beta^2(G)$ , then  $Z \circ h^{-1} \in \mathcal{P}_\beta^2(\Gamma)$ , where*

$$(Z \circ h^{-1})_t(x) := Z_t(h^{-1}(x)) \quad \text{for all } t > 0 \text{ and } x \in \Gamma. \quad (11.5)$$

Moreover,

$$\mathcal{N}_\beta(Z \circ h^{-1}; \Gamma) = \sqrt{\mu(h)} \mathcal{N}_\beta(Z; G). \quad (11.6)$$

*Proof.* It suffices to prove the lemma when  $Z$  is an elementary random field. But then the result follows immediately from first principles, thanks to (11.4).  $\square$

Our next result is a change of variables formula for Wiener integrals.

**Lemma 11.6.** *If  $F \in L^2(\mathbf{R}_+ \times \Gamma)$  and  $h \in \text{Iso}(G, \Gamma)$ , then*

$$\int_{\mathbf{R}_+ \times G} (F \circ h) \, d\xi = \frac{1}{\sqrt{\mu(h)}} \int_{\mathbf{R}_+ \times \Gamma} F \, d\xi_h \quad \text{a.s.} \quad (11.7)$$

*Proof.* Thanks to the very construction of Wiener integrals, it suffices to prove the lemma in the case that  $F_t(x) = A \mathbf{1}_{[c,d]}(t) \mathbf{1}_Q(x)$  for some  $A \in \mathbf{R}$ ,  $0 \leq c < d$ , and Borel-measurable set  $Q \subset \Gamma$  with  $m_\Gamma(Q) < \infty$ . In this special case,  $(F \circ h)_t(x) = A \mathbf{1}_{[c,d]}(t) \mathbf{1}_{h^{-1}(Q)}(x)$ , whence we have:

$$\int_{\mathbf{R}_+ \times G} (F \circ h) \, d\xi = A \xi([c, d] \times h^{-1}(Q)) \quad (11.8)$$

which is  $[\mu(h)]^{-1/2}$  times  $A \xi_h([c, d] \times Q) = \int_{\mathbf{R}_+ \times \Gamma} F \, d\xi_h$ , by default.  $\square$

**Lemma 11.7.** *Let  $\otimes$  denote stochastic convolution with respect to the white noise  $\xi$  on  $\mathbf{R}_+ \times G$ , as before. For every  $h \in \text{Iso}(G, \Gamma)$ , let  $\otimes_h$  denote stochastic convolution with respect to the white noise  $\xi_h$  on  $\mathbf{R}_+ \times \Gamma$ . Choose and fix some  $\beta \geq 0$ . Then, for all  $K \in \mathcal{L}_\beta^2(\Gamma)$  and  $Z \in \mathcal{P}_\beta^2(\Gamma)$ ,*

$$(K \circ h) \otimes (Z \circ h) = \frac{1}{\sqrt{\mu(h)}} (K \otimes_h Z) \circ h, \quad (11.9)$$

*almost surely.*

*Proof.* Lemma 11.4 shows that  $\xi_h$  is indeed a white noise on  $\mathbf{R}_+ \times \Gamma$ ; and Lemma 11.5 guarantees that  $Z \circ h \in \mathcal{P}_\beta^2(G)$ . In order for  $(K \circ h) \otimes (Z \circ h)$  to be a well-defined stochastic convolution, we need  $K \circ h$  to be in  $\mathcal{L}_\beta^2(G)$  [Theorem 6.6]. But (11.4) tells us that

$$\|K_t \circ h\|_{L^2(G)}^2 = \frac{1}{\mu(h)} \|K_t\|_{L^2(\Gamma)}^2 \quad \text{for all } t > 0, \quad (11.10)$$

and hence

$$\|K \circ h\|_{\mathcal{L}_\beta^2(G)}^2 = \frac{1}{\mu(h)} \|K\|_{\mathcal{L}_\beta^2(\Gamma)}^2 < \infty. \quad (11.11)$$

This shows that  $(K \circ h) \circledast (Z \circ h)$  is a properly-defined stochastic convolution.

In order to verify (11.9), which is the main content of the lemma, it suffices to consider the case that  $K$  and  $Z$  are both elementary; see Lemma 6.5 and our construction of stochastic convolutions. In other words, it remains to consider the case that  $K$  and  $Z$  have the form described in (6.10): That is, in the present context: (i)  $K_s(y) = A \mathbf{1}_{(c,d]}(s) \phi(y)$  where  $A \in \mathbf{R}$ ,  $0 \leq c < d$ , and  $\phi \in C_c(\Gamma)$ ; and (ii)  $Z_t(x) = X \mathbf{1}_{(a,b)}(t) \psi(x)$  for  $0 < a < b$ ,  $X \in L^2(\mathbf{P})$  is  $\mathcal{F}_a$ -measurable, and  $\psi \in C_c(\Gamma)$ . In this case,

$$(K \circ h)_s(y) = A \mathbf{1}_{(c,d]}(s) \phi(h(y)), \quad (Z \circ h)_t(x) = X \mathbf{1}_{(a,b)}(t) \psi(h(x)). \quad (11.12)$$

Therefore,

$$\begin{aligned} & [(K \circ h) \circledast (Z \circ h)]_t(x) \\ &= AX \int_{(0,t) \times G} \mathbf{1}_{(c,d]}(s) \mathbf{1}_{(a,b]}(t-s) \phi(h(yx^{-1})) \psi(h(y)) \xi(ds dy). \end{aligned} \quad (11.13)$$

The preceding integral is a Wiener integral, and the above quantity is almost surely equal to

$$\begin{aligned} & \frac{AX}{\sqrt{\mu(h)}} \int_{(0,t) \times \Gamma} \mathbf{1}_{(c,d]}(s) \mathbf{1}_{(a,b]}(t-s) \phi(y(h(x))^{-1}) \psi(y) \xi_h(ds dy) \\ &= \frac{1}{\sqrt{\mu(h)}} (K \circledast_h Z)_t(h(x)), \end{aligned} \quad (11.14)$$

thanks to Lemma 11.6.  $\square$

Finally, if  $X := \{X_t\}_{t \geq 0}$  is a Lévy process on  $G$ , then  $Y_t := h(X_t)$  defines a Lévy process  $Y := h \circ X$  on  $\Gamma$ . In order to identify better the process  $Y := h \circ X$ , let us first recall [36, Ch. 4] that since  $\Gamma = h(G)$ , every character  $\zeta \in \Gamma^*$  is of the form  $\chi \circ h^{-1}$  for some  $\chi \in G^*$  and vice versa. In particular, we can understand the dynamics of  $Y = h \circ X$  via the following computation:

$$\begin{aligned} \mathbf{E}(\zeta, Y_t) &= \mathbf{E}(\chi \circ h^{-1}, Y_t) = \mathbf{E}[\chi(h^{-1}(Y_t))] \\ &= \mathbf{E}[\chi(X_t)] = \mathbf{E}(\chi, X_t) = \mathbf{E}(\zeta \circ h, X_t), \end{aligned} \quad (11.15)$$

for every  $t \geq 0$  and  $\zeta = \chi \circ h^{-1} \in \Gamma^*$ . Let  $\Psi_W$  denote the characteristic exponent of every Lévy process  $W$ . Then, it follows that

$$\Psi_{h \circ X}(\zeta) = \Psi_X(\zeta \circ h) \quad \text{for all } \zeta \in \Gamma^*. \quad (11.16)$$

In particular, we can evaluate the  $\Upsilon$ -function for  $Y := h \circ X$  as follows:

$$\begin{aligned} & \int_{\Gamma^*} \left( \frac{1}{1 + \operatorname{Re} \Psi_{h \circ X}(\zeta)} \right) m_{\Gamma^*}(d\zeta) \\ &= \int_{\Gamma^*} \left( \frac{1}{1 + \operatorname{Re} \Psi_X(\zeta \circ h)} \right) m_{\Gamma^*}(d\zeta). \end{aligned} \quad (11.17)$$

Since  $\zeta \circ h$  is identified with  $\chi$  through the Pontryagin-van Kampen duality pairing, we find the familiar fact that  $\Gamma^* \simeq G^*$  [36, Ch. 4], whence we may deduce the following:

$$\begin{aligned} & \int_{\Gamma^*} \left( \frac{1}{1 + \operatorname{Re} \Psi_{h \circ X}(\zeta)} \right) m_{\Gamma^*}(d\zeta) \\ &= \int_{G^*} \left( \frac{1}{1 + \operatorname{Re} \Psi_X(\chi)} \right) (m_{\Gamma^*} \circ h^{-1})(d\chi) \quad (11.18) \\ &= \mu(h) \cdot \int_{G^*} \left( \frac{1}{1 + \operatorname{Re} \Psi_X(\chi)} \right) m_{G^*}(d\chi) \end{aligned}$$

This  $\mu(h)$  is the same as the constant in (11.1), because our normalization of Haar measures makes the Fourier transform an  $L^2$ -isometry.

In other words, we have established the following.

**Lemma 11.8.** *Let  $X := \{X_t\}_{t \geq 0}$  denote a Lévy process on  $G$ , and choose and fix  $h \in \operatorname{Iso}(G, \Gamma)$ . Then, the  $G$ -valued process  $X$  satisfies Dalang's Condition (D) if and only if the  $\Gamma$ -valued process  $Y := h \circ X$  satisfies Dalang's Condition (D).*

Let us make another simple computation, this time about the invariance properties of semigroups and their  $L^2$ -generators.

**Lemma 11.9.** *Let  $X := \{X_t\}_{t \geq 0}$  denote a Lévy process on  $G$ , with semigroup  $\{P_t^X\}_{t \geq 0}$  and generator  $\mathcal{L}^X$ , and choose and fix  $h \in \operatorname{Iso}(G, \Gamma)$ . Then, the semigroup and generator of  $Y := h \circ X$  are*

$$(P_t^{h \circ X} f)(y) = (P_t^X (f \circ h))(h^{-1}(y)), \quad (11.19)$$

and

$$(\mathcal{L}^{h \circ X} f)(y) = (\mathcal{L}^X (f \circ h))(h^{-1}(y)), \quad (11.20)$$

respectively, where  $t \geq 0$ ,  $y \in \Gamma$ , and  $f \in L^2(\Gamma)$ .

*Proof.* If  $t \geq 0$  and  $y \in \Gamma$ , then  $yh(X_t) = h(h^{-1}(y)X_t)$ , whence it follows that for all  $f \in C_c(\Gamma)$ ,

$$(P_t^{h \circ X} f)(y) = \mathbb{E}[f(yh(X_t))] = \mathbb{E}[(f \circ h)(h^{-1}(y)X_t)]. \quad (11.21)$$

This yields the semigroup of  $h \circ X$  by the density of  $C_c(G)$  in  $L^2(G)$ . Differentiate with respect to  $t$  to compute the generator.  $\square$

As a ready consequence of Lemma 11.9, we find that if  $X := \{X_t\}_{t \geq 0}$  denotes a Lévy process on  $G$  with transition densities  $p^X$  [with respect to  $m_G$ ], and if  $h \in \operatorname{Iso}(G, \Gamma)$ , then  $h \circ X$  is a Lévy process on  $\Gamma$  with transition densities  $p^{h \circ X}$  [with respect to  $m_\Gamma$ ] that are given by

$$p^{h \circ X} := \frac{p^X \circ h^{-1}}{\mu(h)}. \quad (11.22)$$

Indeed, Lemma 11.9 and the definition of  $\mu(h)$  together imply that

$$\int \psi p_t^{h \circ X} dm_\Gamma = \mathbb{E}[\psi(h(X_t))], \quad (11.23)$$

for all  $t > 0$  and  $\psi \in C_c(G)$ . Therefore,  $p^{h \circ X}$  is a version of the transition density of  $h \circ X$ . Lemma 5.1 ensures that  $p^{h \circ X}$  is in fact the unique continuous version of any such transition density.

We are ready to present and prove the main result of this section. Throughout,  $X := \{X_t\}_{t \geq 0}$  denotes a Lévy process on  $G$  that satisfies Dalang's Condition (D), and recall our convention that either  $G$  is compact or  $\sigma(0) = 0$ . In this way, we see that (SHE) has a unique solution for every nonrandom initial function in  $L^2(G)$ .

**Theorem 11.10** (Group invariance of SPDEs). *Suppose  $u_0 \in L^2(G)$  is nonrandom, and let  $u$  denote the solution to (SHE)—viewed as an SPDE on  $(0, \infty) \times G$ —whose existence and uniqueness is guaranteed by Theorem 2.1. Choose and fix  $h \in \text{Iso}(G, \Gamma)$ . Then,  $v_t := u_t \circ h^{-1}$  defines the unique solution to the stochastic heat equation*

$$\begin{cases} \frac{\partial v_t(x)}{\partial t} = (\mathcal{L}^{h \circ X} v_t)(x) + \lambda \sqrt{\mu(h)} \sigma(v_t(x)) \xi_h, \\ v_0 = u_0 \circ h^{-1}, \end{cases} \quad (11.24)$$

viewed as an SPDE on  $\Gamma = h(G)$ , for  $x \in \Gamma$  and  $t > 0$ .

*Proof.* With the groundwork under way, the proof is quite simple. Let  $v$  be the solution to (11.24); its existence is guaranteed thanks to Lemma 11.8 and Theorem 2.1.

Let  $v^{(n)}$  and  $u^{(n)}$  respectively denote the Picard iterates of (11.24) and  $u$ . That is,  $u^{(n)}$ 's are defined iteratively by (7.3), and  $v$ 's are defined similarly as

$$v_t^{(n+1)} := P_t^{h \circ X} v_0 + \lambda \sqrt{\mu(h)} \left( p^{h \circ X} \otimes_h \sigma \left( v^{(n)} \right) \right)_t. \quad (11.25)$$

We first claim that for all  $t > 0$ ,

$$v_t^{(n)} = u_t^{(n)} \circ h^{-1} \quad \text{a.s. for all } n \geq 0. \quad (11.26)$$

This is a tautology when  $n = 0$ , by construction. Suppose  $v_t^{(n)} = u_t^{(n)} \circ h^{-1}$  a.s. for every  $t > 0$ , where  $n \geq 0$  is an arbitrary fixed integer. We next verify that  $v_t^{(n+1)} = u_t^{(n+1)} \circ h^{-1}$  a.s. for all  $t > 0$ , as well. This and a relabeling  $[n \leftrightarrow n+1]$  will establish (11.26).

Thanks to the induction hypothesis, Lemma 11.9, and (11.22),

$$v_t^{(n+1)} := (P_t^X u_0) \circ h^{-1} + \frac{\lambda}{\sqrt{\mu(h)}} \left( (p^X \circ h^{-1}) \otimes_h \sigma \left( u^{(n)} \circ h^{-1} \right) \right)_t, \quad (11.27)$$

almost surely. Therefore, Lemma 11.7 implies that

$$v_t^{(n+1)} := (P_t^X u_0) \circ h^{-1} + \lambda \left( p^X \otimes \sigma \left( u^{(n)} \right) \right)_t \circ h^{-1}, \quad (11.28)$$

almost surely. We now merely recognize the right-hand side as  $u_t^{(n+1)} \circ h^{-1}$ ; see (7.3). In this way we have proved (11.26).

Since we now know that  $v^{(n)} = u^{(n)} \circ h^{-1}$ , two appeals to Theorem 2.1 [via Lemma 11.5] show that if  $\beta$  is sufficiently large, then  $v^{(n)}$  converges in  $\mathcal{P}_\beta^2(\Gamma)$  to  $v$  and  $u^{(n)} \rightarrow u$  in  $\mathcal{P}_\beta^2(G)$ , as  $n \rightarrow \infty$ . Thus it follows from a second application of Lemma 11.5 that  $v = u \circ h^{-1}$ .  $\square$

The following is a ready corollary of Theorem 11.10; its main content is in the last line where it shows that our noise excitation indices are “invariant under group isomorphisms.”

**Corollary 11.11.** *In the context of Theorem 11.10, we have the following energy identity*

$$\mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right) = \frac{1}{\mu(h)} \mathbb{E} \left( \|v_t\|_{L^2(\Gamma)}^2 \right), \quad (11.29)$$

valid for all  $t \geq 0$ . In particular,  $u$  and  $v$  have the same noise excitation indices.

*Proof.* Since  $v_t(x) = u_t(h^{-1}(x))$ , it follows from Theorem 11.10 and (11.4) that

$$\|u_t\|_{L^2(G)}^2 = \frac{1}{\mu(h)} \|v_t\|_{L^2(\Gamma)}^2 \quad \text{a.s.}, \quad (11.30)$$

which is more than enough to imply (11.29). The upper noise-excitation index of  $u$  at time  $t \geq 0$  is

$$\bar{\epsilon}(t) = \limsup_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \log \sqrt{\mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right)}, \quad (11.31)$$

whereas the upper noise excitation index of  $v$  at time  $t$  is

$$\limsup_{\lambda \uparrow \infty} \frac{1}{\log \left[ \lambda \sqrt{\mu(h)} \right]} \log \log \sqrt{\mathbb{E} \left( \|v_t\|_{L^2(\Gamma)}^2 \right)}, \quad (11.32)$$

which is equal to  $\bar{\epsilon}(t)$ , thanks to (11.29) and the fact that  $\log[\lambda \sqrt{\mu(h)}] \sim \log \lambda$  as  $\lambda \uparrow \infty$ . This proves that the upper excitation indices of  $u$  and  $v$  are the same. The very same proof shows also that the lower excitation indices are shared as well.  $\square$

## 12 Projections

Consider our stochastic heat equation (SHE) in the case that the underlying LCA group  $G$  is non compact, metrizable, and has more than one element; that is, consider the general setting of Theorem 2.6. According to the structure theory of LCA groups, which we will recall in due time, we can write  $G \simeq \mathbf{R}^n \times K$  for a nonnegative integer  $n$  and a compact LCA group  $K$ . It is easy to see that the underlying Lévy process on  $G$  can then be written—coordinatewise—

$X \times Y := \{X_t \times Y_t\}_{t \geq 0}$ , where  $\{X_t\}_{t \geq 0}$  is a Lévy process on  $\mathbf{R}^n$  and  $\{Y_t\}_{t \geq 0}$  a Lévy process on  $K$ . The results of this section will allow us to compare the energy of our stochastic PDE to the energy of another version of (SHE), whose  $x$ -variable now ranges in  $\mathbf{R}^n$ , and whose operator  $\mathcal{L}$  is the generator of  $\{X_t\}_{t \geq 0}$ . This comparison principle is a kind of parallel to the classical energy inequality of potential theory. In the present setting it states that the energy of (SHE) on  $G \simeq \mathbf{R}^n \times K$ —using the Lévy process  $X \times Y$ —is greater than or equal to the energy of (SHE) on  $\mathbf{R}^n$ —using the Lévy process  $X$ . Moreover, if (SHE) has a solution—that is, if  $X \times Y$  satisfies Dalang’s condition (D)—then (SHE) on  $\mathbf{R}^n$  must have a solution—that is,  $X$  must satisfy Dalang’s condition (D)—and hence  $n = 1$ . The structure theory of Lévy processes on  $\mathbf{R}$  will then show us that the lowest energy we can expect is from the case that  $X$  is Brownian motion. In that case, a simple scaling argument can yield the desired  $\exp\{\text{const} \cdot \lambda^4\}$  lower bound, which will ultimately verify Theorem 2.6.

In this section we study the natural projection  $G$  of a [larger] LCA group  $G \times K$ , where  $K$  is a compact abelian group. It is easy to see from first principles that such a projection maps a Lévy process on  $G \times K$  to a Lévy process on  $G$ . One of the main results of this section is that if the original process on  $G$  satisfied Dalang’s Condition (D)—on  $G \times K$ —then the new process on  $G$  will satisfy Condition (D) on  $G$ . Thanks to the structure theory of LCA groups, this fact and its ensuing “energy inequality” will be instrumental in the proof of Theorem 2.6 (see Section 14).

We will prove Theorem 2.6 in §14. Presently, we satisfy ourselves by stating and proving a general form of the mentioned projection theorem/energy inequality.

Throughout this section, we let  $G$  denote an LCA group and  $K$  a compact abelian group. Then, it is well known, and easy to see directly, that  $G \times K$  is an LCA group with dual group  $(G \times K)^* = G^* \times K^*$  [36, Ch. 4]. [For purposes of comparison, let us state that the  $G \times K$  of this section is going to play the role of  $G \simeq \mathbf{R}^n \times K$  of the preceding paragraphs.]

Let  $\pi : G \times K \rightarrow G$  denote the canonical projection map. Since  $\pi$  is a [continuous] group homomorphism, it follows that if  $X := \{X_t\}_{t \geq 0}$  is a Lévy process on  $G \times K$ , then  $(\pi \circ X)_t := \pi(X_t)$  defines a Lévy process on  $G$ . If  $\chi \in G^*$ , then  $\chi \circ \pi \in (G \times K)^*$ , and hence

$$\mathbb{E}(\chi, \pi(X_t)) = \mathbb{E}[(\chi \circ \pi, X_t)] = e^{-t\Psi_X(\chi \circ \pi)}, \quad (12.1)$$

for all  $t \geq 0$  and  $\chi \in G^*$ . In other words, we can write the characteristic exponent of  $\pi \circ X$  in terms of the characteristic exponent of  $X$  as follows:

$$\Psi_{\pi \circ X}(\chi) = \Psi_X(\chi \circ \pi) \quad \text{for all } \chi \in G^*. \quad (12.2)$$

**Proposition 12.1.** *If  $X$  satisfies Dalang’s Condition (D) on  $G \times K$ , then the Lévy process  $\pi \circ X$  satisfies Condition (D) on  $G$ . In fact, we have the following “energy inequality,”*

$$\Upsilon_{\pi \circ X}(\beta) \leq \Upsilon_X(\beta) \quad \text{for all } \beta \geq 0, \quad (12.3)$$

where  $\Upsilon_W$  is the function defined in (5.11) and/or (5.12) for every Lévy process  $W$  that has transition densities.

*Proof.* First of all, note that the product measure  $m_G \times m_K$  is a translation-invariant Borel measure on  $G \times K$ , whence  $m_{G \times K} = cm_G \times m_K$  for some constant  $c$ . It is easy to see that  $c \in (0, \infty)$ ; let us argue next that  $c = 1$ . If  $f \in L^2(G)$  and  $g \in L^2(K)$  satisfy  $m_G\{f > 0\} > 0$  and  $m_K\{g > 0\} > 0$ , then  $(f \otimes g)(x \times y) := f(x)g(y)$  satisfies  $f \otimes g \in L^2(G \times K)$ , and

$$\|f \otimes g\|_{L^2(G \times K)} = \|f\|_{L^2(G)} \|g\|_{L^2(K)} = \|f \otimes g\|_{L^2(m_G \times m_K)}. \quad (12.4)$$

Since the left-most term is equal to  $c$  times the right-most term, it follows that  $c = 1$ .

Let  $p^W$  denote the transition densities of  $W$  for every Lévy process  $W$  that possesses transition densities. It is a simple fact about “marginal probability densities” that since  $X$  has nice transition densities  $p^X$  [see Lemma 5.1], so does  $\pi \circ X$ . In fact, because  $m_{G \times K} = m_G \times m_K$ —as was proved in the previous paragraph—we may deduce that

$$p_t^{\pi \circ X}(x) = \int_K p_t^X(x \times y) m_K(dy) \quad \text{for all } t > 0 \text{ and } x \in G. \quad (12.5)$$

Now we simply compute: Because  $K$  is compact,  $m_K$  is a probability measure, and hence

$$\begin{aligned} \|p_t^X\|_{L^2(G \times K)}^2 &= \int_G m_G(dx) \int_K m_K(dy) |p_t^X(x \times y)|^2 \\ &\geq \int_G m_G(dx) \left| \int_K m_K(dy) p_t^X(x \times y) \right|^2 \\ &= \|p_t^{\pi \circ X}\|_{L^2(G)}^2, \end{aligned} \quad (12.6)$$

$$(12.7)$$

for all  $t > 0$ , owing to the Cauchy–Schwarz inequality. We can integrate both sides of the preceding  $[\exp(-\beta t) dt]$  in order to see that

$$\int_0^\infty e^{-\beta s} \|p_s^{\pi \circ X}\|_{L^2(G)}^2 ds \leq \int_0^\infty e^{-\beta s} \|p_s^X\|_{L^2(G \times K)}^2 ds, \quad (12.8)$$

for all  $\beta \geq 0$ , and the result follows.  $\square$

### 13 An abstract lower bound

The main result of this section is an abstract lower estimate for the energy of the solution in terms of the function  $\Upsilon$  that was defined in (5.11); see also (5.12).

**Proposition 13.1.** *If  $u_0 \in L^2(G)$ ,  $\|u_0\|_{L^2(G)} > 0$ , and (2.8) holds, then there exists a finite constant  $c \geq 1$  such that*

$$\mathcal{E}_t(\lambda) \geq c^{-1} \exp(-ct) \cdot \sqrt{1 + \sum_{j=1}^{\infty} \left( \frac{\ell_\sigma^2 \lambda^2}{e} \cdot \Upsilon(j/t) \right)^j}, \quad (13.1)$$

for all  $t \geq 0$ . The constant  $c$  depends on  $u_0$  as well as the underlying Lévy process  $X$ .

*Proof.* Consider first the case that

$$u_0 \in L^\infty(G) \cap L^2(G). \quad (13.2)$$

Thanks to (13.2), we may apply (8.7); upon integration  $[m_G(dx)]$ , this and Fubini's theorem together yield the following formula:

$$\begin{aligned} \mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right) &= \|P_t u_0\|_{L^2(G)}^2 + \lambda^2 \int_0^t \|p_{t-s}\|_{L^2(G)}^2 \mathbb{E} \left( \|\sigma \circ u_s\|_{L^2(G)}^2 \right) ds \\ &\geq \|P_t u_0\|_{L^2(G)}^2 + \ell_\sigma^2 \lambda^2 \int_0^t \|p_{t-s}\|_{L^2(G)}^2 \mathbb{E} \left( \|u_s\|_{L^2(G)}^2 \right) ds \quad (13.3) \\ &= \|P_t u_0\|_{L^2(G)}^2 + \ell_\sigma^2 \lambda^2 \int_0^t \bar{p}_{t-s}(e_G) \mathbb{E} \left( \|u_s\|_{L^2(G)}^2 \right) ds. \end{aligned}$$

Appeals to Fubini's theorem are indeed justified, since Theorem 2.1 contains implicitly the desired measurability statements about  $u$ .

Next we prove that (13.3) holds for every  $u_0 \in L^2(G)$  and not just those that satisfy (13.2). With this aim in mind, let us appeal to density in order to find  $u_0^{(1)}, u_0^{(2)}, \dots \in L^\infty(G) \cap L^2(G)$  such that

$$\lim_{n \rightarrow \infty} \|u_0^{(n)} - u_0\|_{L^2(G)} = 0. \quad (13.4)$$

Then, (8.12) assures us that there exists  $\beta > 0$ , sufficiently large, such that

$$\lim_{n \rightarrow \infty} \mathcal{N}_\beta \left( u^{(n)} - u \right) = 0, \quad (13.5)$$

where  $u_t^{(n)}(x)$  denotes the solution to (SHE) with initial value  $u_0^{(n)}$ . Eq. (13.5) implies readily that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \left\| u_t^{(n)} \right\|_{L^2(G)}^2 \right) = \mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right) \quad \text{for all } t \geq 0. \quad (13.6)$$

And because  $P_t$  is contractive on  $L^2(G)$ ,

$$\lim_{n \rightarrow \infty} \|P_t u_0^{(n)}\|_{L^2(G)} = \|P_t u_0\|_{L^2(G)} \quad \text{for all } t \geq 0. \quad (13.7)$$

Therefore, our claim that (13.3) holds is verified once we show that

$$\lim_{n \rightarrow \infty} \int_0^t \bar{p}_{t-s}(e_G) \mathbb{E} \left( \left\| u_s^{(n)} - u_s \right\|_{L^2(G)}^2 \right) ds = 0 \quad (13.8)$$

for every  $t > 0$ . This is so because of (13.5) and the fact that the preceding integral is bounded above by

$$\begin{aligned} \left[ \mathcal{N}_\beta \left( u^{(n)} - u \right) \right]^2 \cdot \int_0^t e^{-2\beta(t-s)} \bar{p}_{t-s}(e_G) ds \\ \leq \left[ \mathcal{N}_\beta \left( u^{(n)} - u \right) \right]^2 \cdot \Upsilon(2\beta); \end{aligned} \quad (13.9)$$

see also (5.12). Thus, we have established (13.3) in all cases of interest. We can now proceed to prove the main part of the proposition.

Let us define, for all  $t > 0$ ,<sup>10</sup>

$$\mathcal{P}(t) := \ell_\sigma^2 \lambda^2 \bar{p}_t(e_G), \quad \mathcal{I}(t) := \|P_t u_0\|_{L^2(G)}^2, \quad \mathcal{E}(t) := \mathbb{E} \left( \|u_t\|_{L^2(G)}^2 \right). \quad (13.10)$$

Thanks to (13.3), we obtain the pointwise convolution inequality,

$$\begin{aligned} \mathcal{E} &\geq \mathcal{I} + (\mathcal{P} * \mathcal{E}) \\ &\geq \mathcal{I} + (\mathcal{P} * \mathcal{I}) + (\mathcal{P} * \mathcal{P} * \mathcal{E}) \\ &\vdots \\ &\geq \mathcal{I} + (\mathcal{P} * \mathcal{I}) + (\mathcal{P} * \mathcal{P} * \mathcal{I}) + (\mathcal{P} * \mathcal{P} * \mathcal{P} * \mathcal{I}) + \dots, \end{aligned} \quad (13.11)$$

where  $(\psi * \phi)(t) := \int_0^t \psi(s)\phi(t-s) ds$  defines the usual [temporal] convolution operator “\*.” In particular, we may note that the final quantity in (13.11) depends only on the function  $\mathcal{I}$ , which is related only to the initial function  $u_0$ .

A direct computation shows us that the Fourier transform of  $P_t u_0$ , evaluated at  $\chi \in G^*$ , is  $\exp\{-t\Psi(\chi^{-1})\}\hat{u}_0(\chi)$ ; see (5.4). Therefore, we may apply the Plancherel’s theorem to see that

$$\mathcal{I}(t) = \int_{G^*} e^{-2t\operatorname{Re}\Psi(\chi)} |\hat{u}_0(\chi)|^2 m_{G^*}(d\chi) \quad \text{for all } t > 0. \quad (13.12)$$

Since  $u_0 \in L^2(G)$ , we can find a compact neighborhood  $K$  of the identity of  $G^*$  such that

$$\int_K |\hat{u}_0(\chi)|^2 m_{G^*}(d\chi) \geq \frac{1}{2} \int_{G^*} |\hat{u}_0(\chi)|^2 m_{G^*}(d\chi) = \frac{1}{2} \|u_0\|_{L^2(G)}^2, \quad (13.13)$$

thanks to Plancherel’s theorem [as well as the monotone convergence theorem, of course]. In this way, we find that

$$\mathcal{I}(t) \geq \frac{\|u_0\|_{L^2(G)}^2}{2} e^{-c_0 t} \quad \text{for all } t > 0. \quad (13.14)$$

where

$$c_0 := 2 \sup_{\chi \in K} \operatorname{Re}\Psi(\chi). \quad (13.15)$$

We will require the fact that  $0 \leq c_0 < \infty$ ; this fact holds simply because  $\Psi$  is continuous and  $\operatorname{Re}\Psi$  is non negative. In this way, (13.14) yields an estimate for the first term on the right-hand side of (13.11).

As for the other terms, let us write  $\mathcal{P}^{*(n)}$  in place of the  $n$ -fold convolution,  $\mathcal{P} * \dots * \mathcal{P}$ , where  $\mathcal{P}^{*(1)} := \mathcal{P}$ . Then, it is easy to deduce from (13.14) that

$$\left( \mathcal{P}^{*(n)} * \mathcal{I} \right) (t) \geq \frac{\|u_0\|_{L^2(G)}^2}{2} e^{-c_0 t} \left( \mathcal{P}^{*(n)} * \mathbf{1} \right) (t) \quad \text{for all } t > 0, \quad (13.16)$$

<sup>10</sup>It is easy to write  $\mathcal{E}$  in terms of the energy of the solution. Indeed,  $\mathcal{E}(t) = [\mathcal{E}_t(\lambda)]^2$ .

where  $\mathbf{1}(t) := 1$  for all  $t > 0$ . Thus, we conclude from (13.11) that

$$\mathcal{E}(t) \geq \frac{\|u_0\|_{L^2(G)}^2}{2} e^{-c_0 t} \cdot \sum_{n=0}^{\infty} \left( \mathcal{P}^{*(n)} * \mathbf{1} \right) (t), \quad (13.17)$$

where  $\mathcal{P}^{*(0)} * \mathbf{1} := 1$ .

Now,

$$(\mathcal{P} * \mathbf{1})(t) = \ell_\sigma^2 \lambda^2 \cdot \int_0^t \bar{p}_s(e_G) ds. \quad (13.18)$$

Consequently,

$$\begin{aligned} (\mathcal{P} * \mathcal{P} * \mathbf{1})(t) &= \ell_\sigma^4 \lambda^4 \cdot \int_0^t \bar{p}_{s_2}(e_G) ds_2 \int_0^{t-s_2} \bar{p}_{s_1}(e_G) ds_1, & (13.19) \\ (\mathcal{P} * \mathcal{P} * \mathcal{P} * \mathbf{1})(t) &= \ell_\sigma^8 \lambda^8 \cdot \int_0^t \bar{p}_{s_3}(e_G) ds_3 \int_0^{t-s_3} \bar{p}_{s_2}(e_G) ds_2 \int_0^{t-s_3-s_2} \bar{p}_{s_1}(e_G) ds_1, \\ &\vdots \end{aligned}$$

For all real  $t \geq 0$  and integers  $n \geq 1$ ,

$$\begin{aligned} \left( \mathcal{P}^{*(n)} * \mathbf{1} \right) (t) &\geq \ell_\sigma^{2n} \lambda^{2n} \left( \int_0^{t/n} \bar{p}_s(e_G) ds \right)^n \\ &\geq \left( \frac{\ell_\sigma^2 \lambda^2}{e} \cdot \Upsilon(n/t) \right)^n. \end{aligned} \quad (13.20)$$

The first bound follows from an application of induction to the variable  $n$ , and the second follows from (5.13). Since  $(\mathcal{P}^{*(0)} * \mathbf{1})(t) = 1$ , the proposition follows from (13.17).  $\square$

## 14 Proofs of the main theorems

We have set in place all but one essential ingredients of our proofs. The remaining part is the following simple real-variable result. We prove the result in detail, since we will need the following quantitative form of the ensuing estimates.

**Lemma 14.1.** *For all integers  $a \geq 0$  and real numbers  $\rho > 0$ , there exists a positive and finite constant  $c_{a,\rho} > 1$  such that*

$$\sum_{j=a}^{\infty} \left( \frac{b}{j^\rho} \right)^j \geq c_{a,\rho}^{-1} \exp\left((\rho/e)b^{1/\rho}\right) \quad \text{for all } b \geq c_{a,\rho}. \quad (14.1)$$

*Proof.* It is an elementary fact that  $(j/e)^j \leq j!$  for every integer  $j \geq 1$ . Therefore, whenever  $n$ ,  $m$ , and  $jm/n$  are positive integers,

$$\left( \frac{jm}{en} \right)^{jm/n} \leq \left( \frac{jm}{n} \right)!. \quad (14.2)$$

In particular, for all  $b > 0$ ,

$$\sum_{j=a}^{\infty} \left( \frac{b}{j^{m/n}} \right)^j \geq \sum_{\substack{j \geq a \\ jm \in n\mathbf{Z}_+}} \frac{b^j (m/en)^{jm/n}}{(jm/n)!} \geq \sum_{\substack{k \geq am/n \\ k \in \mathbf{Z}_+}} \frac{c^k}{k!}, \quad (14.3)$$

where  $c := b^{n/m} m/(en)$ . Since

$$\sum_{\substack{k < am/n \\ k \in \mathbf{Z}_+}} \frac{c^k}{k!} \leq \max(b^a, 1) \sum_{k=0}^{\infty} \frac{(m/en)^k}{k!} = \exp\left\{\frac{m}{en}\right\} \cdot \max(b^a, 1), \quad (14.4)$$

we immediately obtain the inequality

$$\begin{aligned} \sum_{j=a}^{\infty} \left( \frac{b}{j^{m/n}} \right)^j &\geq e^c - \exp\left\{\frac{m}{en}\right\} \cdot \max(b^a, 1) \\ &= \exp\left\{\frac{b^{n/m} m}{en}\right\} - \exp\left\{\frac{m}{en}\right\} \cdot \max(b^a, 1). \end{aligned} \quad (14.5)$$

The preceding bound is valid for all integers  $n$  and  $m$  that are strictly positive. We can choose now a sequence  $n_k$  and  $m_k$  of positive integers such that  $\lim_{k \rightarrow \infty} (m_k/n_k) = \rho$ . Apply the preceding with  $(m, n)$  replaced by  $(m_k, n_k)$  and then let  $k \rightarrow \infty$  to deduce the following bound:

$$\sum_{j=a}^{\infty} \left( \frac{b}{j^\rho} \right)^j \geq \exp\left((\rho/e)b^{1/\rho}\right) - \exp(\rho/e) \cdot \max(b^a, 1). \quad (14.6)$$

Since the preceding is valid for all  $b > 0$ , the lemma follows readily.  $\square$

With the preceding under way, we conclude the paper by proving Theorems 2.5, 2.6, and 2.8 in this order.

*Proof of Theorem 2.5.* We plan to appeal to (7.12) in order to verify the stated energy upper bound.

Since  $\text{Re}\Psi$  is non negative,

$$\Upsilon(\beta) \leq \beta^{-1} \quad \text{for all } \beta > 0, \quad (14.7)$$

and hence for every  $\varepsilon \in (0, 1)$ ,

$$\Upsilon^{-1}\left(\frac{1}{(1+\varepsilon)^2 \lambda^2 \Gamma_\sigma^2}\right) \leq \text{const} \cdot \lambda^2 \quad \text{for all } \lambda > 1, \quad (14.8)$$

where the implied constant is independent of  $\lambda$ . Now we merely apply (7.12) in order to see that there exist finite constants  $a$  and  $b$  such that  $\mathcal{E}_t(\lambda) \leq a \exp(b\lambda^2)$  for all  $\lambda > 1$ . This proves that  $\bar{\epsilon}(t) \leq 2$ .

For the converse bound we recall that  $m_{G^*}$  has total mass one because  $G^*$  is compact. Since  $\Psi$  is continuous, it follows that  $\operatorname{Re}\Psi$  is bounded uniformly on  $G^*$ , and hence for all  $\beta_0 > 0$  there exists a positive constant such that

$$\Upsilon(\beta) = \int_{G^*} \left( \frac{1}{\beta + \operatorname{Re}\Psi(\chi)} \right) m_{G^*}(d\chi) \geq \frac{\operatorname{const}}{\beta} \quad \text{for all } \beta > \beta_0. \quad (14.9)$$

Proposition 13.1 then ensures that

$$\mathcal{E}_t(\lambda) \geq \operatorname{const} \cdot \sqrt{1 + \sum_{j=1}^{\infty} \left( \frac{t\ell_\sigma^2 \lambda^2}{e^j} \right)^j} \geq a \exp(b\lambda^2), \quad (14.10)$$

for some finite  $a$  and  $b$  that depend only on  $t$ , and in particular are independent of  $\lambda > c_{1,1}$ . [We have appealed to Lemma 14.1—with  $\rho := 1$  and  $a := 1$ —in order to see that  $c_{1,1}$  is strictly greater than one; we have also used the assumption that  $\ell_\sigma > 0$ .] This proves that  $\underline{\mathfrak{c}}(t) \geq 2$  when  $\ell_\sigma > 0$ , and concludes our proof of Theorem 2.5.  $\square$

*Proof of Theorem 2.6.* First we consider the case that  $G$  is non compact.

According to the structure theory of LCA groups [36, Ch. 6], since  $G$  is connected we can find an integer  $n \geq 0$  and a compact abelian group  $K$  such that

$$G \simeq \mathbf{R}^n \times K. \quad (14.11)$$

Because  $G$  is not compact, we must have  $n \geq 1$ . Now we put forth the following claim:

$$n = 1. \quad (14.12)$$

In order to prove (14.12), let  $\pi$  denote the canonical projection from  $G \simeq \mathbf{R}^n \times K$  to  $\mathbf{R}^n$ . Because Condition (D) holds for the Lévy process  $X$  on  $G \simeq \mathbf{R}^n \times K$ , Proposition 12.1 assures us that the Lévy process  $\pi \circ X$  on  $\mathbf{R}^n$  also satisfies Condition (D). That is,  $\Upsilon_{\pi \circ X}(\beta) < \infty$  for one, hence all,  $\beta > 0$ . Recall from (5.12) that

$$\Upsilon_{\pi \circ X}(\beta) = \operatorname{const} \cdot \int_{\mathbf{R}^n} \left( \frac{1}{\beta + \operatorname{Re}\Psi_{\pi \circ X}(z)} \right) dz \quad \text{for all } \beta > 0, \quad (14.13)$$

where “const” accounts for a suitable normalization of Haar measure on  $\mathbf{R}^n$ . Since  $\pi \circ X$  is a Lévy process on  $\mathbf{R}^n$ , a theorem of Bochner [7, see (3.4.14) on page 67] ensures that there exists  $A \in (0, \infty)$  such that

$$\operatorname{Re}\Psi_{\pi \circ X}(z) \leq A(1 + \|z\|^2) \quad \text{for all } z \in \mathbf{R}^n. \quad (14.14)$$

Because  $\Upsilon_{\pi \circ X}(\beta) < \infty$ , by assumption, it follows that  $\int_{\mathbf{R}^n} (\beta + \|z\|^2)^{-1} dz < \infty$  and hence  $n = 1$ .<sup>11</sup> This proves our earlier assertion (14.12).

<sup>11</sup>This illustrates, in the present setting, the well-known folklore fact that the SHE does not have a mild solution as a function on  $\mathbf{R}^n$  when  $n \geq 2$ ; see Dalang [13] and Peszat and Zabczyk [38].

Now that we have (14.12), we know that  $G \simeq \mathbf{R} \times K$  for a compact abelian group  $K$ . Because of Theorem 11.10 we may assume, without loss of generality, that our LCA group  $G$  is in fact equal to  $\mathbf{R} \times K$ . Thus, thanks to Propositions 12.1 and 13.1,

$$\begin{aligned} \mathbb{E} \left( \|u_t\|_{L^2(\mathbf{R} \times K)}^2 \right) &\geq \text{const} \cdot \left\{ 1 + \sum_{j=1}^{\infty} \left( \frac{\ell_{\sigma}^2 \lambda^2}{e} \cdot \Upsilon_X(j/t) \right)^j \right\} \\ &\geq \text{const} \cdot \left\{ 1 + \sum_{j=1}^{\infty} \left( \frac{\ell_{\sigma}^2 \lambda^2}{e} \cdot \Upsilon_{\pi \circ X}(j/t) \right)^j \right\}. \end{aligned} \quad (14.15)$$

According to Bochner's estimate (14.14),

$$\Upsilon_{\pi \circ X}(\beta) \geq \text{const} \cdot \int_0^{\infty} \frac{dx}{\beta + x^2} \geq \frac{\text{const}}{\sqrt{\beta}}, \quad (14.16)$$

uniformly for all  $\beta \geq \beta_0$ , for every fixed  $\beta_0 > 0$ . Thus, we may appeal to Lemma 14.1—with  $\rho := 1/2$  and  $a = 1$ —in order to see that  $\mathbb{E}(\|u_t\|_{L^2(\mathbf{R} \times K)}^2) \geq a \exp(b\lambda^4)$ , simultaneously for all  $\lambda > c_{1,1/2}$ , where  $c_{1,1/2}$  is a finite constant that is independent of  $\lambda$ . This proves that  $\mathfrak{g}(t) \geq 4$  when  $G$  is non compact [as well as connected].

We complete the proof of the theorem by proving it when  $G$  is compact, connected, metrizable, and has at least 2 elements.

A theorem of Pontryagin [36, Theorem 33, p. 106] states that if  $G$  is a locally connected LCA group that is also metrizable then

$$G \simeq \mathbf{R}^n \times \mathbf{T}^m \times D, \quad (14.17)$$

where  $0 \leq n < \infty$ ,  $0 \leq m \leq \infty$ , and  $D$  is discrete. Of course,  $\mathbf{T}^{\infty} := \mathbf{T} \times \mathbf{T} \times \cdots$  denotes the countable direct product of the torus  $\mathbf{T}$  with itself, as is customary.

Since  $G$  is compact and connected, we can deduce readily that  $n = 0$  and  $D$  is trivial; that is,  $G \simeq \mathbf{T}^m$  for some  $0 \leq m \leq \infty$ . Because, in addition,  $G$  contains at least 2 elements, we can see that  $m \neq 0$ ; thus,

$$G \simeq \mathbf{T}^m \quad \text{for some } 1 \leq m \leq \infty. \quad (14.18)$$

As a matter of fact, the forthcoming argument can be refined to prove that  $m = 1$ ; see our earlier proof of (14.12) for a model of such a proof. But since we will not need this fact, we will not prove explicitly that  $m = 1$ . Suffice it to say that, since  $m \geq 1$ , an application of Tychonoff's theorem yields

$$G \simeq \mathbf{T} \times K, \quad (14.19)$$

for a *compact* Hausdorff abelian group  $K$ . Theorem 11.10 reduces our problem to the case that  $G = \mathbf{T} \times K$ , owing to projection.

Let now  $\pi$  denote the canonical projection from  $\mathbf{T} \times K$  to  $\mathbf{T}$ , and argue as in the non-compact case to see that

$$\mathbb{E} \left( \|u_t\|_{L^2(\mathbf{T} \times K)}^2 \right) \geq \text{const} \cdot \left\{ 1 + \sum_{j=1}^{\infty} \left( \frac{\ell_\sigma^2 \lambda^2}{e} \cdot \Upsilon_{\pi \circ X}(j/t) \right)^j \right\}. \quad (14.20)$$

Bochner's estimate (14.14) has the following analogue for the Lévy process  $\pi \circ X$  on  $\mathbf{T}$ : There exists  $A \in (0, \infty)$  such that

$$\text{Re} \Psi_{\pi \circ X}(n) \leq A(1 + n^2) \quad \text{for all } n \in \mathbf{Z}. \quad (14.21)$$

[The proof of this bound is essentially the same as the proof of (14.14).] Since the dual to  $\mathbf{T}$  is  $\mathbf{Z}$ , it follows that

$$\Upsilon_{\pi \circ X}(\beta) = \text{const} \cdot \sum_{n=-\infty}^{\infty} \frac{1}{\beta + \text{Re} \Psi(n)} \geq \frac{\text{const}}{\sqrt{\beta}}, \quad (14.22)$$

uniformly for all  $\beta \geq \beta_0$ , for every fixed  $\beta_0 > 0$ . A final appeal to Lemma 14.1—with  $\rho := 1/2$ —finishes the proof.  $\square$

*Proof of Theorem 2.8.* Consider the special case that  $G = \mathbf{R}$  and  $X$  is a symmetric stable Lévy process with index  $\alpha \in (0, 2]$ ; that is,  $\Psi(\xi) = |\xi|^\alpha$ . Condition (D) holds if and only if  $\alpha \in (1, 2]$ , a condition which we now assume. The generator of  $X$  is the fractional Laplacian  $\mathcal{L} := -(-\Delta)^{\alpha/2}$  on  $\mathbf{R}$ . A direct computation reveals that

$$\Upsilon(\beta) = \text{const} \cdot \int_0^\infty \frac{dx}{\beta + x^\alpha} = \text{const} \cdot \beta^{-(\alpha-1)/\alpha}. \quad (14.23)$$

In particular, for every  $\varepsilon \in (0, 1)$ ,

$$\Upsilon^{-1} \left( \frac{1}{(1 + \varepsilon)^2 \lambda^2 L_\sigma^2} \right) \leq \text{const} \cdot \lambda^{2\alpha/(\alpha-1)} \quad \text{for all } \lambda > 1, \quad (14.24)$$

This yields

$$\bar{\mathbf{e}}(t) \leq \frac{2\alpha}{\alpha - 1}, \quad (14.25)$$

in this case; see the proof of the first portion of Theorem 2.5, for more details. And an appeal to Lemma 14.1 yields

$$\underline{\mathbf{e}}(t) \geq \frac{2\alpha}{\alpha - 1}. \quad (14.26)$$

See the proof of Theorem 2.6 for some details.

Thus, for every  $\alpha \in (1, 2]$ , we have found a model whose noise excitation index is

$$\mathbf{e} = \frac{2\alpha}{\alpha - 1}. \quad (14.27)$$

Since  $\theta := 2\alpha/(\alpha - 1)$  can take any value in  $[4, \infty)$ , as  $\alpha$  varies in  $(1, 2]$ , Eq. (14.27) proves the theorem.  $\square$

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