

BAYESIAN METHODS FOR GENETIC ASSOCIATION ANALYSIS WITH HETEROGENEOUS SUBGROUPS: FROM META-ANALYSES TO GENE-ENVIRONMENT INTERACTIONS

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Genetic association analyses often involve data from multiple potentially-heterogeneous subgroups. The expected amount of heterogeneity can vary from modest (e.g. a typical meta-analysis), to large (e.g. a strong gene-environment interaction). However, existing statistical tools are limited in their ability to address such heterogeneity. Indeed, most genetic association meta-analyses use a “fixed effects” analysis, which assumes no heterogeneity. Here we develop and apply Bayesian association methods to address this problem. These methods are easy to apply (in the simplest case, requiring only a point estimate for the genetic effect, and its standard error, from each subgroup), and effectively include standard frequentist meta-analysis methods, including the usual “fixed effects” analysis, as special cases. We apply these tools to two large genetic association studies: one a meta-analysis of genome-wide association studies from the Global Lipids consortium, and the second a cross-population analysis for expression quantitative trait loci (eQTLs). In the Global Lipids data we find, perhaps surprisingly, that effects are generally quite homogeneous across studies. In the eQTL study we find that eQTLs are generally shared among different continental groups, and discuss consequences of this for study design.

1. Introduction. We consider the following problem, which arises frequently in genetic association analysis: how to test for association while allowing for heterogeneity of effects among subgroups. We are motivated particularly by the following applications:

Motivating Application 1: The Global Lipids Genome-wide Association Study. The Global Lipids consortium (Teslovich et al. (2010)) conducted a large meta-analysis of genome-wide genetic association studies of blood lipids phenotypes (total cholesterol (TC), low-density lipoprotein cholesterol (LDL-C), high-density lipoprotein cholesterol (HDL-C) and triglycerides (TG)). This study, like most meta-analyses, aimed to increase power by combining information across studies. The consortium amassed a total of more than 100,000 individuals, through 46 separate studies. These studies involve different investigators, at different centers, with different enrollment

criteria. Consequently one would expect genetic effect sizes to differ among studies. However, Teslovich et al. (2010), following standard practice in this field, analyzed the data assuming no heterogeneity. This analysis appeared highly successful, identifying genetic associations at a total of 95 different genetic loci, 53 of them novel. Our work here was motivated by a desire to analyze these data, and others like them, *taking account of potential heterogeneity among studies*, and to see whether this would identify additional genetic associations.

Motivating Application 2: assessing heterogeneity of genetic effects on gene expression (eQTLs) among populations. An expression quantitative trait locus (eQTL) is a genetic variant that is associated with expression (activity) of a gene. Identifying eQTLs is important because such variants are candidates for being functional (i.e. actually causing changes in gene expression), and hence candidates for having other, perhaps medically important, consequences. See Gilad, Rifkin and Pritchard (2008) for more on insights to be gained from eQTL studies.

Here we analyze data from Stranger et al. (2007), who measured gene expression on lymphoblastoid cell lines derived from unrelated individuals sampled from three major continental groups: Europeans (CEU), Asians (ASN), and Africans (YRI). The main aim of our analysis is to assess heterogeneity of eQTL effects across groups: for example, do eQTL effects tend to vary among groups, and do some eQTLs appear to be active in only some groups? Understanding heterogeneity in this context could yield insights into differences in the gene-regulatory mechanisms acting in each group, and also has important implications for generalizability of studies performed in one subgroup to other subgroups. Similar questions also arise frequently in eQTL studies involving different tissue or cell types (Dimas et al. (2009); Brown, Mangravite and Engelhardt (2012); Flutre et al. (2013)).

Motivating Application 3: Identifying biological interactions with environment. Our third example is more generic, but nonetheless important: identifying genetic associations in the presence of environmental interactions. Strong environmental interactions can result in genetic effects varying among subgroups, and in extreme cases could even cause effects to have different signs in different subgroups. In such cases ignoring heterogeneity would substantially reduce power. For example, by separately analyzing male and females, Kong et al. (2008) identified a strong genetic association with recombination rate that is missed by a standard analysis that ignores heterogeneity. However, separately analyzing subgroups is less attractive than

analyzing them jointly and allowing for potential heterogeneity; this motivated development of methods described here.

These three applications differ in their expected heterogeneity. For example, whereas interactions could cause genetic effects to differ in sign among subgroups, differences in sign seem less likely in the meta-analysis setting. However, they also share an important element in common: the vast majority of genetic variants are unassociated with any given phenotype within *all* subgroups. Consequently, it is of considerable interest to identify genetic variants that show association in *any* subgroup, or in other words to reject the “global” null hypothesis of *no association within any subgroup*. This focus on rejecting the global null hypothesis distinguishes genetic association analyses from other settings, and calls for analysis approaches tailored to this goal; see Lebec, Stijnen and Houwelingen (2010) for relevant discussion. Thus, although there has been previous work on Bayesian methods for meta-analysis (e.g. Sutton and Abrams (2001); Stangl and Berry (2000); DuMouchel and Harris (1983); Whitehead and Whitehead (1991); Li and Begg (1994); Eddy, Hasselblad and Schachter (1990); Givens, Smith and Tweedie (1997); Verzilli et al. (2008); De Iorio et al. (2011); Burgess et al. (2010); Mila and Ngugi (2011)), the nature of our applications calls for a different focus. Specifically our applications need tools for association testing and model comparison, via Bayes Factors (BFs), rather than for estimation. In addition, because genetic association studies often involve very many association tests, computational speed is important, so we focus on obtaining fast numerical approximations to BFs (rather than using MCMC say). Finally, because in many cases, including the Global Lipids data above, only summary data (e.g. effect size estimates and standard errors in the different subgroups) are easily available, we need methods that can work with summary data.

Although our methods are Bayesian, they have close connections with related frequentist methods. Indeed, while this work was in progress, Lebec, Stijnen and Houwelingen (2010) (and, later, Han and Eskin (2011)) published frequentist approaches to association testing based on models very similar to those used here. Further, as we show in Section 4, some standard frequentist meta-analysis tests correspond closely to BFs obtained under certain priors. Consequently, ranking SNP associations by these standard test statistics is equivalent to making specific (and not necessarily realistic) prior assumptions. Thus, although our primary goal is to provide a practical solution to a common applied problem, we also provide theoretical results linking our methods to widely-used existing methods.

2. Models and Methods. The problems outlined above have two key goals:

1. To test whether a genetic variant is associated with phenotype in any subgroup, allowing for potential heterogeneity of effects among subgroups.
2. Given that an association exists, assess the support for different levels of heterogeneity.

We tackle these problems by specifying a family of alternative models with varying levels of heterogeneity, and developing computational tools to calculate the support in the data (the BF) for each alternative model compared with the null model of no association. Within this framework the goal of testing the global null (1 above) is accomplished by assessing the overall support for any of the alternative hypotheses, whereas the goal of examining heterogeneity among groups is achieved by comparing the relative support for different alternative models.

All our motivating examples involve quantitative outcomes, and we focus on this case. However, in common to many meta-analysis methods, the simplest of our methods requires only an estimated effect size in each study, and its corresponding standard deviation, and thus can be applied to any setting where such estimates are available (e.g. generalized linear models). See appendix B for details.

2.1. Notation and Assumptions. Assume quantitative phenotype data and genotype data are available on S pre-defined subgroups. Like most association analyses, we analyze each genetic variant in turn, one at a time. Assume that the data within subgroup s come from n_s randomly-sampled unrelated individuals. Let the n_s -vectors \mathbf{y}_s and \mathbf{g}_s denote, respectively, the corresponding phenotype and genotype data, and $\mathbf{Y} := (\mathbf{y}_1, \dots, \mathbf{y}_S)$ and $\mathbf{G} := (\mathbf{g}_1, \dots, \mathbf{g}_S)$. Here each genotype is coded as 0, 1 or 2 copies of a reference allele, so $\mathbf{g}_s \in \{0, 1, 2\}^{n_s}$. (For imputed genotypes we replace each genotype with its posterior mean Guan and Stephens (2008), in which case $\mathbf{g}_s \in [0, 2]^{n_s}$.)

2.2. Models for Effect-size Heterogeneity. Within each subgroup, we model genotype-phenotype association using a standard linear model:

$$(2.1) \quad \mathbf{y}_s = \mu_s \mathbf{1} + \beta_s \mathbf{g}_s + \mathbf{e}_s, \quad \mathbf{e}_s \sim N(0, \sigma_s^2 I),$$

with residual errors \mathbf{e}_s assumed independent across subgroups. (Additional, possibly study-specific, covariates are easily added to the right hand side

of (2.1). If independent flat priors are used for the coefficients of these covariates within each study then our main results below still hold, effectively unchanged. This treatment is analogous to the frequentist mixed effects model, where such covariates are typically assumed to have study-specific effects.)

The “global” null hypothesis is no genotype-phenotype association within any subgroup; i.e. $\beta_s = 0$ for all s .

Under the alternative hypothesis the genetic effects are non-zero. To allow for heterogeneity among subgroups we assume that these effects are normally distributed about some unknown common mean. We consider two different definitions of genetic effects: the “standardized effects” $b_s := \beta_s/\sigma_s$, and the unstandardized effects, β_s , leading to the following models:

1. *Exchangeable Standardized Effects (ES model)*. The standardized effects b_s are normally distributed among subgroups, about some unknown mean \bar{b} , to which we assign a normal prior:

$$(2.2) \quad b_s | \bar{b}, \sigma_s \sim N(\bar{b}, \phi^2), \quad [\text{or equivalently, } \beta_s | \sigma_s \sim N(\sigma_s \bar{b}, \sigma_s^2 \phi^2),]$$

$$(2.3) \quad \bar{b} \sim N(0, \omega^2).$$

Alternatively, and equivalently, the vector $\mathbf{b} = (b_1, \dots, b_S)$ is multivariate normally distributed:

$$(2.4) \quad \mathbf{b} \sim N_S(0, \Sigma_b)$$

where Σ_b is an $S \times S$ matrix with diagonal elements $\text{Var}(b_s) = \phi^2 + \omega^2$ and off-diagonal elements $\text{Cov}(b_s, b_{s'}) = \omega^2$.

2. *Exchangeable Effects (EE model)*. The unstandardized effects β_s are normally distributed about some unknown mean $\bar{\beta}$, to which we assign a normal prior:

$$(2.5) \quad \beta_s | \bar{\beta} \sim N(\bar{\beta}, \psi^2),$$

$$(2.6) \quad \bar{\beta} \sim N(0, w^2).$$

Alternatively, and equivalently, the vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_S)$ is multivariate normally distributed:

$$(2.7) \quad \boldsymbol{\beta} \sim N_S(0, \Sigma_\beta)$$

where Σ_β is an $S \times S$ matrix with diagonal elements $\text{Var}(\beta_s) = \psi^2 + w^2$ and off-diagonal elements $\text{Cov}(\beta_s, \beta_{s'}) = w^2$.

In both ES and EE models we assume conjugate priors for (μ_s, σ_s) :

$$(2.8) \quad \text{ES} : \mu_s | \sigma_s \sim \text{N}(0, \sigma_s^2 u_s^2); \quad \sigma_s^{-2} \sim \Gamma(m_s/2, l_s/2).$$

$$(2.9) \quad \text{EE} : \mu_s \sim \text{N}(0, u_s^2); \quad \sigma_s^{-2} \sim \Gamma(m_s/2, l_s/2).$$

Specifically, we consider posteriors and BFs that arise in the limits $u_s^2 \rightarrow \infty$ and $l_s, m_s \rightarrow 0$. These limiting priors correspond to standard improper priors for normal regressions, and ensure that the BFs satisfy certain invariance properties (see Servin and Stephens (2008)).

Both ES and EE have two key hyper-parameters, one (ω in ES; w in EE) that controls the prior expected size of the *average effect* across subgroups, and another (ϕ in ES; ψ in EE) that controls the prior expected degree of *heterogeneity* among subgroups. A complimentary view is that $\omega^2 + \phi^2$ (respectively, $w^2 + \psi^2$) controls the expected (marginal) effect size in each study and ϕ/ω (respectively, ψ/w) controls the degree of heterogeneity. Thus one can allow for different levels of heterogeneity by considering different values of these hyperparameters (see below). Note that $\phi = 0$ (respectively, $\psi = 0$) corresponds to the assumption, commonly used in practice, of no heterogeneity among subgroups.

Of the two models, ES has the advantage that it results in analyses (e.g. BFs) that are invariant to the phenotype measurement scale used within each subgroup. This makes it robust to users accidentally specifying phenotype measurements in different subgroups on different scales (a non-trivial issue in complex analyses involving collaboration among many research groups). It also makes it applicable when measurement scales are difficult to harmonize across subgroups, for example due to use of different measurement technologies. For these reasons we prefer ES for general use. However, in some cases EE may be easier to apply. For example, if one has access only to published point estimates and standard errors for the effect size β_s in each study, then this suffices to approximate the BF under EE, but not under ES. Note that ES and EE will produce similar results if the residual error variances are similar in all subgroups.

2.2.1. *Limiting Heterogeneity: A Curved Exponential Family Normal Prior.*

The above priors assume independence of the mean (\bar{b}) and variance (ω^2) of the effects. In some settings this assumption may be unattractive. For example, in a typical meta-analysis we expect effects to show “limited heterogeneity” across studies, and typically to have the same sign (Owen (2009)), regardless whether \bar{b} is small or large. But the independence assumption implies that the probability that the effects have the same sign is much larger when \bar{b} is large than when it is small. To address this we can replace the

priors (2.2) and (2.5) with, respectively,

$$(2.10) \quad b_s \sim N(\bar{b}, k^2 \bar{b}^2),$$

$$(2.11) \quad \beta_s \sim N(\bar{\beta}, k^2 \bar{\beta}^2).$$

Here k determines the amount of heterogeneity, with smaller k indicating less heterogeneity, and $k = 0$ indicating no heterogeneity. Under these priors the probability of effects differing in sign depends only on k and not on \bar{b}

$$(2.12) \quad \Pr(b_s \text{ has a different sign from } \bar{b}) = \Phi\left(-\frac{1}{|k|}\right),$$

where Φ is the cumulative distribution function of a standard normal distribution. For example, when $k = 1/2$, (2.12) is approximately 2.3%.

We call these priors ‘‘Curved Exponential Family Normal’’ (CEFN) priors, reflecting their functional relationship between the mean and variance.

2.3. Bayes Factors for Testing the Global Null Hypothesis. For simplicity we focus on calculations for the ES model; details for EE, and modifications for CEFN, are given in appendices.

The ES model has two hyperparameters, ϕ and ω . The global null hypothesis, which is most naturally written as $\beta_s \equiv 0$ for all s , can also be written as

$$(2.13) \quad H_0 : \phi = \omega = 0.$$

The support in the data for a particular alternative model, specified by hyperparameters (ϕ, ω) , vs H_0 , is given by the Bayes Factor (BF):

$$(2.14) \quad \text{BF}^{\text{ES}}(\phi, \omega) = \frac{P(\mathbf{Y}|\mathbf{G}, \phi, \omega)}{P(\mathbf{Y}|\mathbf{G}, H_0)}.$$

Each value of (ω, ϕ) corresponds to a particular alternative model, with ω controlling the typical average effect size, and ϕ controlling the degree of heterogeneity among subgroups (or in a re-parametrization, $\omega^2 + \phi^2$ controls the expected marginal effect size in each subgroup and ϕ/ω controls the degree of heterogeneity). To allow for uncertainty about appropriate values for ϕ and ω , we use a (discrete) prior distribution on a set of plausible values $\{(\phi^{(i)}, \omega^{(i)}) : i = 1, \dots, M\}$ (see Applications for details). (This induces a prior on the effects that is a mixture of multivariate normals.) A discrete prior is more flexible than fixing ϕ, ω to specific values, while maintaining computational convenience. Indeed, if π_i denotes the prior weight on

$(\phi^{(i)}, \omega^{(i)})$ then the resulting BF against H_0 is the weighted average of the individual BFs:

$$(2.15) \quad \text{BF}_{\text{av}}^{\text{ES}} := \sum_{i=1}^M \pi_i \text{BF}^{\text{ES}}(\phi^{(i)}, \omega^{(i)}).$$

This average could be extended to include other models (e.g. some using the CEFN prior, others not). The fact that BFs under different assumptions for heterogeneity can be both averaged in this way (to assess evidence against the global null, allowing for heterogeneity), and compared with one another (to assess the evidence for different levels or types of heterogeneity), is one nice feature of the Bayesian framework.

We make two comments regarding the need to specify the prior weights π in (2.15). First, the usual practice of simply ignoring heterogeneity is implicitly making a particular, and rather strong, decision about π . In this sense, specifying weights for different amounts of heterogeneity is simply turning a usually-implicit decision into an explicit decision. (See Wakefield (2009) for analogous discussion regarding choice of priors on effect sizes.) Second, in some applications it will be possible, and desirable, to learn these weights from the data via a hierarchical model, reducing the subjectivity of the analysis. Indeed, the ability of BFs to be naturally incorporated into a hierarchical model is one advantage over analogous frequentist test statistics Lebec, Stijnen and Houwelingen (2010) that maximize over ϕ, ω . This idea is illustrated in Section 3.3.

2.3.1. Calculating Bayes Factors. Calculating $\text{BF}^{\text{ES}}(\phi, \omega)$ involves evaluating a multi-dimensional integral. In appendix A we present two different approximations, both based on applying Laplace's method and both having error terms that decay inversely with the average sample size across subgroups. The first of these, which effectively follows methods from Butler and Wood (2002) for computing confluent hyper-geometric functions, is very accurate, even for small sample sizes. Indeed, for the special case of a single subgroup ($S = 1$), the approximation becomes *exact*, and for small numbers of subgroups we have checked numerically (appendix D) that it provides almost identical results to an alternative approach based on adaptive quadrature (which is practical only for small S). However, it requires a numerical optimization step and has a somewhat complex form, which although not a practical barrier to its use does hinder intuitive interpretation. In what follows we use $\widehat{\text{BF}}^{\text{ES}}$ to denote this approximation.

The second approximation is less accurate for small samples sizes, but converges asymptotically (with average sample size) to the correct answer.

Further, a simple modification, described in appendix C, yields much greater accuracy. For the special case of $S = 1$ it yields an analogue of the approximate BFs from Wakefield (2009) and Johnson (2008), and in what follows we use ABF^{ES} to denote this approximation under the ES model. The nice feature of ABF^{ES} is that it has an intuitive analytic form, which is detailed after the Applications, in Section 4 (Proposition 4.1).

2.3.2. Special Cases. Our ES and EE models include, in the special cases $\phi = 0$ and $\psi = 0$, the case where there is no heterogeneity of effects across subgroups. The assumption of no heterogeneity also underlies standard “fixed effects” meta-analysis methods, and so we use $\text{ABF}_{\text{fix}}^{\text{ES}}$ and $\text{ABF}_{\text{fix}}^{\text{EE}}$ to denote ABFs computed under these assumptions. These ABFs are the Bayesian analogues of standard frequentist test statistics for “fixed effect” analyses; see Section 4.2.1 for details.

At the other extreme, the cases $\omega = 0$ (ES model) and $w = 0$ (EE model) maximize heterogeneity across subgroups, and we use $\text{ABF}_{\text{maxH}}^{\text{ES}}$, $\text{ABF}_{\text{maxH}}^{\text{EE}}$ to denote ABFs computed under these assumptions. These ABFs can be viewed as the Bayesian analogues of frequentist methods that combine information across subgroups ignoring the direction of the effect in each subgroup (e.g. Fisher’s method); see Section 4.2.2 for details.

2.3.3. Bayes Factors from summary statistics. It turns out that both the true BFs (BF^{ES} , BF^{EE}) and the approximations ($\widehat{\text{BF}}^{\text{ES}}$, $\widehat{\text{BF}}^{\text{EE}}$, ABF^{ES} , ABF^{EE}) depend on the observed data in each subgroup only through a set of summary statistics, a 6-tuple $(n_s, \mathbf{1}'\mathbf{y}_s, \mathbf{1}'\mathbf{g}_s, \mathbf{y}'_s\mathbf{y}_s, \mathbf{g}'_s\mathbf{g}_s, \mathbf{y}'_s\mathbf{g}_s)$. Furthermore, for the simplest approximations (the ABFs), the summary statistics needed from each subgroup are reduced to only $(\hat{b}_s, \text{se}(\hat{b}_s))$ for the ES model and $(\hat{\beta}_s, \text{se}(\hat{\beta}_s))$ for the EE model. These are exactly the quantities used in traditional meta-analysis applications.

These properties have important practical implications. First, they aid collaboration among groups, where sharing of raw data can be more difficult than sharing summary data. Second, and perhaps more importantly, it means that the methods, particularly the ABFs, are extremely flexible. Indeed, in any setting where an effect size estimate and its standard error can be obtained for each subgroup, these can be plugged in to compute an ABF. This can be viewed as an additional Laplace approximation (Appendix B). Thus the methods can easily handle study-specific covariates, and non-normal data (e.g. using a generalized linear model within each subgroup).

2.3.4. Software. We implemented these methods in a software package MeSH (Meta analysis with Subgroup Heterogeneity), available from [http:](http://)

[//www.github.com/xqwen/mesh](https://www.github.com/xqwen/mesh).

3. Applications.

3.1. *Illustrative Example: deCODE Recombination Study.* We first illustrate the methods on motivating application 3 where heterogeneity of effects is known to occur. The example involves three correlated genetic variants (SNPs) that were found, in a genetic association study of recombination rate by Kong et al. (2008), to be strongly associated with both male and female recombination rates, but with estimated effects in opposite directions (i.e. the allele associated with lower recombination rate in males is associated with higher recombination rate in females).¹

We applied our methods to these data, exploiting the ability to compute ABFs from summary data. Specifically, Table 1 of Kong et al. (2008) gives estimated effect sizes $\hat{\beta}_{\text{male}}$ and $\hat{\beta}_{\text{female}}$ for three relevant SNPs, as well as corresponding p values, from which we infer approximate values for the standard errors $\text{se}(\hat{\beta}_{\text{male}})$ and $\text{se}(\hat{\beta}_{\text{female}})$. These summaries suffice to compute ABFs under the EE model. We treat males and females as two subgroups and consider 4 levels of expected marginal overall effect sizes with $\sqrt{\psi^2 + w^2} = 5, 10, 20, 40$ (the phenotype scale being centi-Morgans) and 5 levels of heterogeneity with $\psi^2/w^2 = 0, 0.5, 1, 2, \infty$. This yields a grid of 4×5 different (ψ, w) combinations, and we treat every grid value as *a priori* equally likely when computing $\text{ABF}_{\text{av}}^{\text{EE}}$. (A denser grid could be used to obtain more precise estimates of ψ and w , but this coarse grid suffices for our purposes here.)

The resulting BFs are shown in Table 1. Notably, the association signal in the joint analysis of males and females, allowing for heterogeneity, $\text{ABF}_{\text{av}}^{\text{EE}}$, is many orders of magnitude larger than either of the subgroup-specific BFs, which are themselves larger than the BF under a fixed effects model, $\text{ABF}_{\text{fix}}^{\text{EE}}$. This analysis illustrates two simple but important points. First, a joint analysis can yield considerably stronger signal than subgroup-specific analyses. Second, a standard fixed-effects meta-analysis would be ineffective in this case. Of course, in general, the “right” level of heterogeneity is unknown; $\text{ABF}_{\text{av}}^{\text{EE}}$ deals with this by averaging over different levels of heterogeneity.

¹A subsequent study suggests that this genetic region may actually contain more than one genetic variant affecting recombination rates, some acting in males and others in females, rather than a single genetic variant with antagonistic effects in the two groups (Fledel-Alon et al. (2011)). This finding emphasizes the fact that apparent heterogeneity may differ from actual heterogeneity, particularly when examining genetic markers that are not the causal variant. The power benefits of accounting for heterogeneity in an association analysis largely depend on apparent, rather than actual, heterogeneity.

This ability to average over unknown quantities is an attractive feature of the Bayesian approach, although it can also be helpful to examine the components of this average separately (see the next application).

SNP	Male		Female		Meta BFs	
	Effect (p-value)	$ABF_{\text{single,male}}^{\text{EE}}$	Effect (p-value)	$ABF_{\text{single,female}}^{\text{EE}}$	$ABF_{\text{fix}}^{\text{EE}}$	$ABF_{\text{av}}^{\text{EE}}$
rs3796619	-67.9 (1.1×10^{-14})	$10^{11.12}$	67.6 (7.9×10^{-6})	$10^{2.81}$	$10^{3.07}$	$10^{13.91}$
rs1670533	-66.1 (1.8×10^{-11})	$10^{8.06}$	92.8 (4.1×10^{-8})	$10^{4.55}$	$10^{1.10}$	$10^{12.58}$
rs2045065	-66.2 (1.6×10^{-11})	$10^{8.11}$	92.2 (6.0×10^{-8})	$10^{4.40}$	$10^{1.18}$	$10^{12.49}$

TABLE 1

Bayesian analysis of genetic associations with recombination rate. The SNPs, estimated effect sizes, and p-values are taken directly from Kong et al. (2008) Table 1. We compute approximate BFs under the EE model using these summary statistics. $ABF_{\text{single,male}}^{\text{EE}}$ and $ABF_{\text{single,female}}^{\text{EE}}$ are approximate BFs computed using only male subgroup and female subgroup data respectively. $ABF_{\text{fix}}^{\text{EE}}$ and $ABF_{\text{av}}^{\text{EE}}$ are approximate BFs computed using both males and females jointly, either assuming no heterogeneity ($ABF_{\text{fix}}^{\text{EE}}$) or averaging over a range of values of heterogeneity ($ABF_{\text{av}}^{\text{EE}}$).

Our methods can also assess which associations show evidence for heterogeneity. This could help identify potentially-interesting interactions (as in this case), or potentially-suspect association signals (see next example). In these data $ABF_{\text{av}}^{\text{EE}}$ is substantially larger than $ABF_{\text{fix}}^{\text{EE}}$, indicating that the data are inconsistent with the fixed effects assumption of equal effects in both subgroups. This comparison is a Bayesian analogue of the frequentist test for heterogeneity in a random effects meta-analysis. To further investigate heterogeneity we compare BFs for different values of the heterogeneity parameter (ψ^2/w^2); in this case the data are consistent with infinite values for this parameter (i.e. $w^2 = 0$).

This example illustrates that an association analysis accounting for heterogeneity can identify associations that would be missed by a fixed effects analysis that ignores heterogeneity. Next we attempt to exploit this to identify novel associations in a large-scale genome-wide association study.

3.2. Global Lipids Genome-wide Association Study. We now return to motivating application 1, a large scale meta-analysis of genome-wide genetic association studies of blood lipids phenotypes conducted by the Global Lipids consortium (Teslovich et al. (2010)). In this study, more than 100,000 individuals of European ancestry were amassed through 46 separate studies (grouped into 25 studies in their final analysis). For each individual, measures of total cholesterol (TC), low-density lipoprotein cholesterol (LDL-C), high-density lipoprotein cholesterol (HDL-C) and triglycerides (TG) were

obtained. Genotypes at 2.7 million SNPs across the genome were also measured or imputed. In each study, the phenotypes were independently quantile normal transformed; single SNP association tests were performed for all SNPs and all phenotypes using the linear model (2.1) and estimated effects and their standard errors were computed. The meta-analysis combined these summary data using the software METAL (Willer, Li and Abecasis (2010)) to compute a weighted Z statistic that we discuss later (specifically they used equation (4.10) with weights $w_s = \sqrt{n_s}$). This can be viewed as an approximation to a fixed effects analysis under the ES model (see Section 4.2.1). Teslovich et al. (2010) reported 168 SNP-phenotype associations exceeding their “genome-wide significant” threshold (p -value $< 5 \times 10^{-8}$), and identified 95 genes, with 59 showing genome-wide significant association for the first time.

We hypothesized that taking account of heterogeneity among subgroups might help identify additional novel associations, so we reanalyzed the data using our Bayesian tools to allow for heterogeneity. We were able to obtain access to summary data from each study, in the form of an estimated effect size (computed from the quantile-transformed phenotype data) and its standard error for each SNP in each study. With these data we can perform analyses under the EE model for the quantile normalized data (rather than the ES model effectively used in the original analysis). We computed ABFs under increasing amounts of heterogeneity: the fixed effects model, the “limited heterogeneity” CEFN model, and the maximum heterogeneity model. In each case we assumed a discrete uniform prior on the overall genetic effect size (i.e., $(k^2 + 1)w^2$ in the CEFN model and $w^2 + \psi^2$ in the other models) on the set $\{0.1^2, 0.2^2, 0.4^2, 0.6^2, 0.8^2\}$. For the fixed effects model, $\psi = 0$; for the maximum heterogeneity model, $w = 0$; and for the CEFN model we set $k = 0.326$ which gives a prior probability of 1/1,000 that the genetic effect in each study has an opposite sign to $\bar{\beta}$. Although a fully automated Bayesian analysis would naturally average over the different models for heterogeneity, in practice, because of their different sensitivity to particular data features (see below) we found it helpful to examine each separately.

As an initial check on data handling we verified that our Bayesian fixed effects analysis produced similar results to the original fixed effects analysis. As expected, we found that $\text{ABF}_{\text{fix}}^{\text{EE}}$ ranked SNP associations very similarly to the original reported (ES model) p -values. However, there were some notable exceptions. In particular, a few SNPs showed much stronger association signal in $\text{ABF}_{\text{fix}}^{\text{EE}}$ than the original analysis. Further investigation suggested that these results likely reflected the EE analysis being less robust to data processing errors than the original ES analysis. For example,

SNP rs17061870 with LDL phenotype, had a huge signal in our EE analysis ($ABF_{\text{fix}}^{\text{EE}} > 10^{17}$), and a very modest signal in the original analysis (p -value = 0.028), but examination of the study-specific data for this SNP showed a suspicious pattern: the p -value was 2×10^{-31} in one study (the Family Heart Study, FHS), but no smaller than 0.1 in the 5 other studies for which this SNP had genotype data available. Furthermore, the very small p -value in FHS was driven primarily by a very small, probably erroneous, estimate of the residual error in that study (the sample size of this particular study is not large) which under the EE model results in a very high weight on that study, but in the ES model does not (see Section 4.2.1). We emphasize that we performed the EE analysis here because it was what we were able to do easily with the available summary data, rather than because we prefer it.

Next we assessed evidence for heterogeneity of effects in the 168 association signals reported by the original analysis. We did this by comparing the support for the limited heterogeneity model ($ABF_{\text{cefn}}^{\text{EE}}$) with the support for the no heterogeneity model ($ABF_{\text{fix}}^{\text{EE}}$). The majority of phenotype-SNP pairs (111/168) showed stronger support for the no heterogeneity model (Figure 1). This result was surprising to us: this meta-analysis involved a large number of different studies, encompassing a range of different enrollment criteria, so we expected to find much stronger evidence for heterogeneity among studies. We did find three strong associations that showed overwhelming support for heterogeneity ($ABF_{\text{cefn}}^{\text{EE}}/ABF_{\text{fix}}^{\text{EE}} > 10^{10}$; Figure 1). Forest plots for these SNPs (Figure 2) suggest that in all three cases this signal for heterogeneity comes from modest variation in effect size among all studies, rather than a strong difference in one or a few studies (although the B58C-WTCCC study is, arguably, something of an outlier at rs3764261).

Finally, we addressed the primary question of interest: whether allowing for heterogeneity across studies yields novel associations. To do this, we performed a genome-wide analysis for each phenotype, in each case excluding all SNPs within 1Mb of any SNPs originally reported as associated with that phenotype. We searched for SNPs that showed strong evidence for association under one of the heterogeneity models ($ABF_{\text{cefn}}^{\text{EE}}$ or $ABF_{\text{maxH}}^{\text{EE}} \geq 10^6$, where $ABF_{\text{maxH}}^{\text{EE}}$ denotes the approximate Bayes factor computed by restricting $w = 0$, see section 4.2.2 for precise definition) but not under the fixed-effects model ($ABF_{\text{fix}}^{\text{EE}} < 10^6$). This threshold (10^6) corresponds very roughly to, and is perhaps slightly more conservative than, the threshold effectively used in the initial analysis. (We used the same threshold for all three models for simplicity, but different thresholds might be more appropriate; for example, one might prefer a more stringent threshold for $ABF_{\text{maxH}}^{\text{EE}}$ because strong heterogeneity is unexpected in this context.)

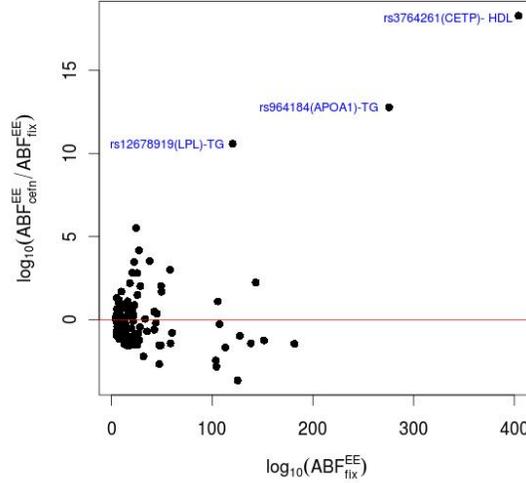


FIG 1. Assessment of evidence for heterogeneity in 168 reported phenotype-SNP association signals from Teslovich et al. (2010). Large values on the y axis indicate stronger support for the model allowing for limited heterogeneity ($\text{ABF}_{\text{ceiln}}^{\text{EE}}$) compared with the model with no heterogeneity ($\text{ABF}_{\text{fix}}^{\text{EE}}$). The three highlighted points correspond to associations with overwhelming evidence for heterogeneity, $\text{ABF}_{\text{ceiln}}^{\text{EE}} / \text{ABF}_{\text{fix}}^{\text{EE}} > 10^{10}$; forest plots for these are shown in Figure 2

Overall we found 42 SNP-phenotype associations satisfying this criteria (after removing SNPs in LD with one another), representing associations potentially missed by the original analysis. However, detailed investigation suggested that 36 of these were not genuine associations. Specifically, these 36 associations, which showed strong signals in $\text{ABF}_{\text{maxH}}^{\text{EE}}$ only, were driven by strong associations in the FHS study that are likely due to data processing errors (the FHS p -values at these SNPs for quantile transformed phenotypes were many orders of magnitude smaller than for the original phenotypes). We therefore dropped the FHS data and re-performed the association analysis.

After dropping FHS, all 6 remaining signals from the previous analysis still satisfy our association criteria (Table 2). Of those, the first two listed are almost certainly genuine: the genes ABCA1 and KLF14 are reported in Teslovich et al. (2010) as associated with other lipid phenotypes (ABCA1 with HDL and TC; KLF14 with HDL), but not with the phenotypes we listed in Table 2, and both reflect associations that just missed being significant in the original fixed effects analysis. The next two associations may also be real:

GENETIC ASSOCIATION ANALYSIS WITH HETEROGENEOUS SUBGROUPS15

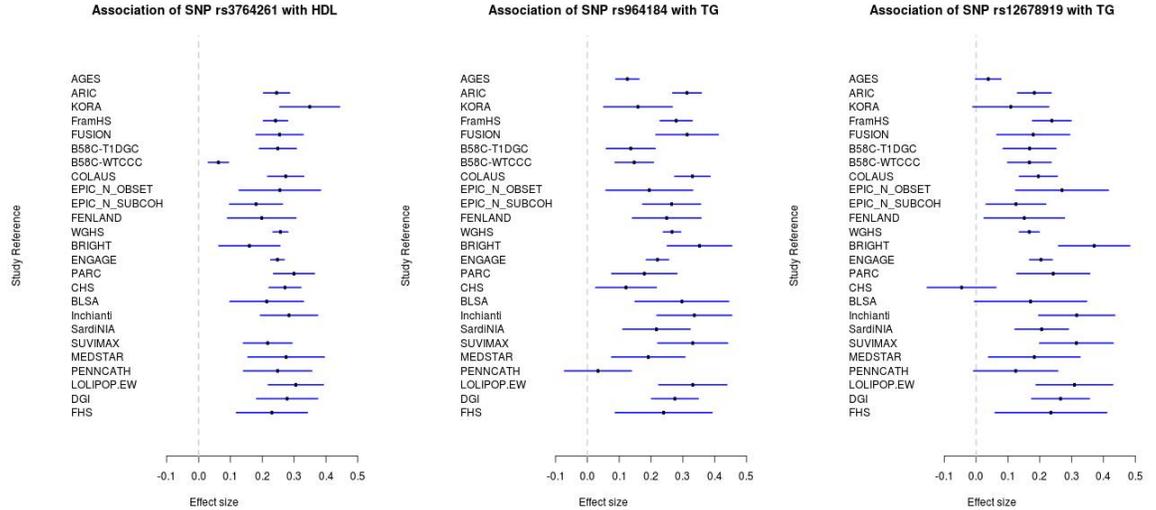


FIG 2. Forest plots for the three highlighted association signals in Figure 1, which showed overwhelming evidence for heterogeneity of (apparent) effects.

they map approximately 6Mb apart on chromosome 11, in a region that is densely populated with olfactory receptor genes, and this same genetic region is also identified in a multivariate association analysis of these same data (Stephens (2013)), although we know of no further independent evidence to support them. One slight cause for caution is that, in humans, SNPs this far apart would usually not be correlated with one another, but these two are slightly correlated ($r^2 \approx 0.07$ in the European 1000 Genomes data), raising the possibility of mapping errors. That is, the precise locations of these SNPs may be in question, and certainly it is difficult to say which genes they might implicate; the table simply lists the nearest gene for reference. Finally, based on examination of the raw data, we suspect that the last two associations are false positives, driven by apparent anomalies in a single study (this time B58C-WTCCC).

In summary, we find that the original fixed effects analysis in Teslovich et al. (2010) was highly effective. This may seem surprising, since these data seem to provide ample opportunities for heterogeneity of effects. Indeed, some of the associations identified by the original study do show a substantially stronger signal in analyses allowing for heterogeneity (Figure 1), and a genome wide association analysis allowing for heterogeneity identified at least two apparently real associations that just missed being significant under the original fixed effects analysis. Thus, despite the success of the

Phenotype	SNP	Gene Region	$\log_{10}(\text{BF}_{\text{fix}}^{\text{EE}})$	$\log_{10}(\text{BF}_{\text{maxH}}^{\text{EE}})$	$\log_{10}(\text{BF}_{\text{cefn}}^{\text{EE}})$
LDL	rs1800978	5'UTR of ABCA1	5.2	3.4	6.0
TG	rs1562398	Flanking KLF14	5.3	-0.2	6.5
HDL	rs11229165	Flanking OR4A16	4.6	4.9	6.4
HDL	rs7108164	Flanking OR4A42P	4.2	4.9	6.3
HDL	rs11984900	N.A.	-1.1	16.6	6.2
HDL	rs6995137	Flanking SFRP1	-0.4	6.9	4.8

TABLE 2

Association signals that show strong association under the models allowing for heterogeneity ($\text{ABF}_{\text{cefn}}^{\text{EE}}$ or $\text{ABF}_{\text{maxH}}^{\text{EE}} \geq 10^6$) but less strong under a model with no heterogeneity ($\text{ABF}_{\text{fix}}^{\text{EE}} < 10^6$). It seems likely that the last two of these represent false positive associations, but we include them in the table for completeness (see text for discussion).

fixed effects analysis, analyses allowing for heterogeneity could modestly increase in power for GWAS meta-analyses in general. On the other hand, our results also provide a cautionary tale: in the context of meta-analysis of genetic association studies, when associations appear only under models allowing for strong heterogeneity, and not under fixed effects models, the reasons for the discrepancy must be examined carefully, and the results interpreted critically. Indeed, we found that searching for SNPs showing strong heterogeneity is an effective way to identify data processing errors that may otherwise lurk undetected!

3.3. Heterogeneity of eQTLs among populations. Now we consider our second motivating application, examining heterogeneity in the effects of expression quantitative trait loci (eQTLs) among populations. An eQTL is a genetic variant (here, a SNP) that is associated with gene expression. Understanding heterogeneity of eQTL effects among population subgroups is important for several reasons. For example, it is important for designing and interpreting experiments, because it influences how generalizable results obtained in one subgroup are to other subgroups. In addition, identifying heterogeneous effects could yield insights into biological differences among subgroups: if an eQTL is more active in one subgroup than others it may indicate a difference in the regulatory mechanisms operating in that subgroup.

To assess heterogeneity of eQTLs among European, African and Asian subgroups we analyzed gene expression measurements from Stranger et al. (2007), obtained using the Illumina Sentrix Human-6 Expression BeadChip, on lymphoblastoid cell lines. Specifically we considered the subset of 141 cell lines (41 Europeans (CEU), 59 Asians (ASN) and 41 Africans (YRI))

ϕ^2/ω^2	posterior mean	95% credible interval
0	0.700	(0.640, 0.753)
1/4	0.265	(0.195, 0.330)
1/2	0.015	(0.002, 0.052)
1	0.008	(0.003, 0.016)
2/1	0.007	(0.002, 0.015)
4/1	0.004	(0.001, 0.012)
∞	0.003	(0.000, 0.010)

TABLE 3

Estimated heterogeneity of eQTL effects among Europeans, Africans and Asians, obtained by fitting a hierarchical model to combine information across genes.

$\sqrt{\omega^2 + \phi^2}$	posterior mean	95% credible interval
0.1	0.004	(0.001, 0.008)
0.2	0.004	(0.001, 0.007)
0.4	0.008	(0.003, 0.014)
0.8	0.976	(0.966, 0.983)
1.6	0.007	(0.002, 0.015)

TABLE 4

Estimated standard deviation of average eQTL effects ($\sqrt{\omega^2 + \phi^2}$), obtained by fitting a hierarchical model to combine information across genes.

that were fully sequenced in the pilot project of the 1000 Genomes project (Durbin et al. (2010)). We analyzed the 8,427 distinct autosomal genes that were confirmed to be expressed in the same African samples by an independent experiment (Pickrell et al. (2010)). We used SNP genotype data on 14.4 million SNPs from the final release (March, 2010) of the pilot SNP calls from the 1000 genomes project, with no additional allele frequency filtering. In addition to the original normalization Stranger et al. (2007), we performed quantile normal transformations to expression values for each gene, separately within each population group, to reduce the influence of outliers or other deviations from normality. Previous studies have shown that most eQTLs are located near to the gene whose expression they influence (so-called “*cis*-eQTLs”). Therefore for each gene we restricted our association analysis to the “*cis* SNPs” which lie within the region 500kb upstream of the transcription start site and 500kb downstream of the transcription end site.

Our analysis focuses on the question: how much do eQTL effects vary among continental groups? To assess this we applied the ES model with $\sqrt{\phi^2 + \omega^2} \in \{0.1, 0.2, 0.4, 0.8, 1.6\}$ and $\phi^2/\omega^2 \in \{0, 1/4, 1/2, 1, 2, 4, \infty\}$, producing a total grid of 35 different (ϕ, ω) combinations. These values were

chosen to cover a wide range of possible effect sizes and levels of heterogeneity. Since the amount of heterogeneity is our main interest, we estimate the weights (π) on the 35 combinations using a Bayesian hierarchical model that jointly analyzes all 8,427 genes. In brief, this model assumes: i) the data at each gene are independent; ii) each gene has at most one eQTL, with each SNP being equally likely; and iii) that each eQTL draws its (ϕ, ω) value from the grid of 35 different values, according to π . In addition, we assign a uniform prior to π . Under these assumptions, we implement a Markov Chain Monte Carlo (MCMC) algorithm to perform posterior inference on π . (See Wen (2011); Flutre et al. (2013) for full details of the computational methods and modeling assumptions.)

The results (Table 3) suggest that eQTL effects typically vary little among subgroups: the estimates from the hierarchical model put 97% of the total weight on the two smallest heterogeneity parameters, 0 and 1/4. (For completeness we also include estimated grid weights on $\phi^2 + \omega^2$, which control the average eQTL effect sizes, in Table 4.)

The above analysis effectively assumes that each eQTL is active in all three populations, and allows for heterogeneity by allowing that the effect size may vary among populations. That is, it is effectively a model for “quantitative heterogeneity”. A different model for heterogeneity is that some eQTLs may be active in only a subset of the populations, with no effect in others. That is that heterogeneity might be qualitative, with some eQTLs being “population-specific”.² Although our quantitative-heterogeneity analysis above suggests that heterogeneity is generally low, it does not preclude the existence of some population-specific effects, so we performed an additional analysis to assess how common such population-specific effects might be. To apply our methods to this situation, we introduce C to denote a binary string of indicators for whether an eQTL is active (i.e. has non-zero effect) in each population. For example $C = (110)$ would indicate that the eQTL is active only in the first two populations. For the three populations in our data, C has 2^3 possible values, which we refer to as “configurations”. The support (BF) for each configuration, relative to the null model $C = (000)$,

²It could be objected that the notion of a “population-specific” eQTL is too simplistic, and that apparent absence of effects in some populations more likely reflects very small, non-zero, effects. While sympathetic to this argument, we also find the simplicity has a certain appeal, and we view such models as potentially useful nonetheless.

Configuration	Estimate (posterior mean)	95% credible interval
CEU only	0.001	(0.000, 0.003)
ASN only	0.001	(0.000, 0.008)
YRI only	0.001	(0.000, 0.004)
CEU and YRI	0.001	(0.000, 0.013)
ASN and YRI	0.017	(0.010, 0.035)
CEU and ASN	0.026	(0.014, 0.041)
CEU and ASN and YRI	0.953	(0.934, 0.970)

TABLE 5

Estimated proportion of eQTLs that are shared among the three continental subgroups, represented by CEU (European), ASN (Asian) and YRI (African) samples. Estimates come from fitting a hierarchical model to combine information across genes; see text. The vast majority of eQTLs are estimates to be shared among all three subgroups.

is easily computed. For example, for $C = (110)$,

$$\begin{aligned}
 \text{BF}_{C=(110)} &= \frac{P(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 | \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, C = (110))}{P(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 | \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, C = (000))} \\
 (3.1) \qquad &= \frac{P(\mathbf{y}_1, \mathbf{y}_2 | \mathbf{g}_1, \mathbf{g}_2)}{P(\mathbf{y}_1, \mathbf{y}_2 | H_0)}.
 \end{aligned}$$

(The simplification is due to the assumption that the vectors of residual errors in (2.1) are independent across populations.)

To estimate the proportion of eQTLs that follow each (non-null) configuration, we introduce hyperparameters, $\eta_{001}, \dots, \eta_{111}$ to represent the frequency of each non-null configuration. So each eQTL draws its configuration C independently according to η . Furthermore, given C , we assume the standardized eQTL effect sizes follow the CEFN prior with $k = 0.314$ and ω drawn independently from a grid $\{0.1, 0.2, 0.4, 0.8, 1.6\}$ according to weights π' . Putting this all together into a single hierarchical model, with uniform priors on π' and η , we use MCMC to sample from the joint posterior distribution on all parameters. Full details are given in Wen (2011); Flutre et al. (2013).

The resulting estimates for η are given in Table 5. Consistent with the conclusions on low overall heterogeneity above, the vast majority of eQTLs behave consistently across populations: indeed, we estimate 95% of eQTLs to be active in all three populations. Nonetheless, we find some evidence for occasional deviations from this pattern, with approximately 2% of eQTLs being active only in European and Asian samples, and 2% being active only in Asian and African. Illustrative examples of potential exceptions to the general rule of sharing among populations are shown in Figures 3 and 4.

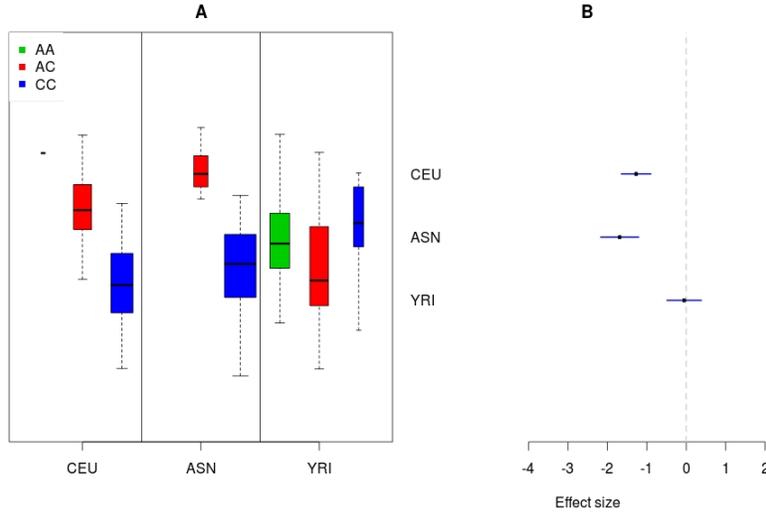


FIG 3. Example of a potential population specific eQTL presented only in CEU and ASN. A: Boxplots of the gene expression levels of gene CMAH (Ensemble ID ENSG00000168405) according to genotypes of SNP rs6906102 in the three Hapmap populations. B: forest plot of estimated effect sizes of this eQTL. Allele A of this SNP has allele frequencies 0.22, 0.08 and 0.67 in ASN, CEU and YRI, respectively.

4. Analytic Expressions for the Bayes Factors and Connections with Frequentist Statistics. We now provide analytic expressions for the ABFs mentioned above (Proposition 4.1 below). These expressions provide intuitive insights, and highlight connections with standard frequentist test statistics, effectively establishing the “implicit prior assumptions” underlying some standard frequentist procedures. We start by introducing necessary notation.

- *Association Testing in a Single Subgroup.* Consider analyzing a single subgroup, s . Let $\hat{\beta}_s$ and $\hat{\sigma}_s$ denote the least square estimates of β_s and σ_s from the linear regression model (2.1) using only data from subgroup s . The following expressions give an estimate for the standardized effect b_s (\hat{b}_s), its standard error under H_0 (δ_s), and a t statistic

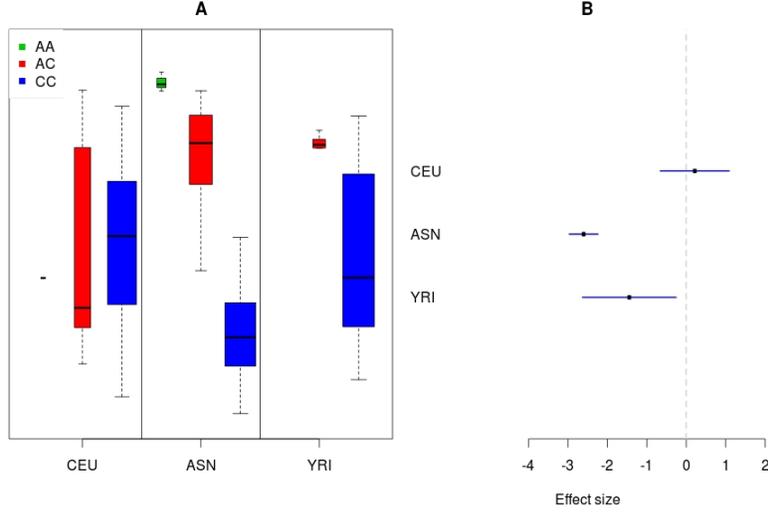


FIG 4. *Example of a potential population specific eQTL presented only in ASN and YRI. A: Boxplots of the gene expression levels of gene PAQR8 (Ensemble IDENSG00000170915) according to genotypes of SNP rs3180068 in the three Hapmap populations. B: forest plot of estimated effect sizes of this eQTL. Allele A of this SNP has sample allele frequencies 0.14, 0.22 and 0.07 in ASN, CEU and YRI, respectively.*

for testing $b_s = 0$ (T_s):

$$(4.1) \quad \hat{b}_s := \hat{\beta}_s / \hat{\sigma}_s,$$

$$(4.2) \quad \delta_s^2 := \frac{1}{\mathbf{g}'_s \mathbf{g}_s - n_s \bar{g}_s^2},$$

$$(4.3) \quad T_s^2 := \frac{\hat{b}_s^2}{\text{se}(\hat{b}_s)^2} = \frac{\hat{\beta}_s^2}{\hat{\sigma}_s^2 \delta_s^2}.$$

Note that T_s is also equal to $\hat{\beta}_s / \text{se}(\hat{\beta}_s)$, which is the usual t-statistic for testing $\beta_s = 0$.

Both Wakefield (2009) and Johnson (2008) derive the following approximate BF for testing $b_s \sim N(0, \phi^2)$ vs. $b_s = 0$:

$$(4.4) \quad \text{ABF}_{\text{single}}^{\text{ES}}(T_s, \delta_s; \phi) := \sqrt{\frac{\delta_s^2}{\delta_s^2 + \phi^2}} \exp\left(\frac{T_s^2}{2} \frac{\phi^2}{\delta_s^2 + \phi^2}\right).$$

As noted by Wakefield, if ϕ is chosen differently for each SNP, and proportional to the value of δ_s^2 for that SNP, then $\text{ABF}_{\text{single}}^{\text{ES}}$ ranks the

SNPs in the same way as the usual test statistic T_s . This result connects the standard frequentist analysis to a particular (approximate) Bayesian analysis in the case of a single subgroup. Proposition 4.1 below extends this to multiple subgroups, allowing for heterogeneity among subgroups.

- *Testing Average Effect in a Random-effect Meta-analysis Model.* Consider the standard frequentist test of $\bar{b} = 0$ in a random-effect meta-analysis of all subgroups, with $b_s \sim N(\bar{b}, \phi^2)$. If ϕ is considered known, then an estimate for \bar{b} , its standard error ζ , and a test statistic $\mathcal{T}_{\text{ES}}^2$ for testing $\bar{b} = 0$ are given by:

$$(4.5) \quad \hat{\bar{b}} := \frac{\sum_s (\delta_s^2 + \phi^2)^{-1} \hat{b}_s}{\sum_s (\delta_s^2 + \phi^2)^{-1}},$$

$$(4.6) \quad \zeta^2 := \frac{1}{\sum_s (\delta_s^2 + \phi^2)^{-1}},$$

$$(4.7) \quad \mathcal{T}_{\text{ES}}^2 := \frac{\hat{\bar{b}}^2}{\text{se}(\hat{\bar{b}})^2}.$$

Applying Johnson's idea (Johnson (2005, 2008)) we can “translate” this test statistic into an approximate BF for testing $\bar{b} \sim N(0, \omega^2)$ vs. $\bar{b} = 0$, which yields

$$(4.8) \quad \text{ABF}_{\text{single}}^{\text{ES}}(\mathcal{T}_{\text{ES}}^2, \zeta; \omega) := \sqrt{\frac{\zeta^2}{\zeta^2 + \omega^2}} \exp\left(\frac{\mathcal{T}_{\text{ES}}^2}{2} \frac{\omega^2}{\zeta^2 + \omega^2}\right).$$

Now $\text{ABF}^{\text{ES}}(\phi, \omega)$ can be written as a simple product of the ABFs (4.4) and (4.8).

PROPOSITION 4.1. *Under the ES model, applying a version of Laplace's method to approximate $\text{BF}^{\text{ES}}(\phi, \omega)$ yields the approximation*

$$(4.9) \quad \text{BF}^{\text{ES}}(\phi, \omega) \approx \text{ABF}^{\text{ES}}(\phi, \omega) := \text{ABF}_{\text{single}}^{\text{ES}}(\mathcal{T}_{\text{ES}}^2, \zeta; \omega) \cdot \prod_s \text{ABF}_{\text{single}}^{\text{ES}}(T_s^2, \delta_s; \phi),$$

and $\text{ABF}^{\text{ES}}(\phi, \omega)$ converges (almost surely) to $\text{BF}^{\text{ES}}(\phi, \omega)$ as $n_s \rightarrow \infty$ for all subgroups s .

PROOF. See appendix A.1. □

NOTE 4.1. *If the study-specific residual error variances, σ_s are considered known (rather than being assigned a prior distribution), and used*

in place of $\hat{\sigma}_s$ to compute ABF^{ES} , then the approximation is exact, and $\text{ABF}^{\text{ES}}(\phi, \omega) = \text{BF}^{\text{ES}}(\phi, \omega)$. This fact, together with the fact that the estimators $\hat{\sigma}_s$ are consistent for σ_s , explains, intuitively, why the proposition holds.

NOTE 4.2. *The numerical accuracy of ABF^{ES} as an approximation to BF^{ES} depends on sample sizes, and for small sample sizes it may be too inaccurate for routine application. However, a simple modification, described in appendix C, yields much greater accuracy.*

Proposition 4.1 partitions the evidence for association into two parts: one part reflects the evidence in each subgroup (the second term) and the other reflects consistency of effects among subgroups (the first term). In particular, if all subgroups show effects in the same direction, then the first term may be large ($\gg 1$) and “boost” the evidence for association. A similar result holds for the EE model (appendix A.2).

4.1. Properties of Bayes Factors.

4.1.1. *Induced Single Study Bayes Factors.* For the ES model, in the special case of one subgroup ($S = 1$), both the actual BF and our approximations reduce to results from previous work. Specifically, the approximation $\widehat{\text{BF}}^{\text{ES}}$ becomes exact in this case, and equal to the BF derived by Servin and Stephens (2008), whereas ABF^{ES} is equal to the ABF in Wakefield (2009) (see also Johnson (2005) and Johnson (2008)).

4.1.2. *Non-informative Subgroup Data.* Suppose that in one subgroup, s , sample genotypes vary very little. This might arise, for example, in cross-population genetic studies, if one SNP allele is very rare in one population. Intuitively, subgroup s then contains little information for testing H_0 . Indeed, this will be reflected in the standard error for the effect size, δ_s , being large, which will result in study s contributing little to the BF. Specifically, in the limit $\delta_s \rightarrow \infty$, the ABF (4.9) is unaffected by the association data in study s (T_s); a similar result holds for the exact BF (appendix A) and for both EE and ES. Thus, the BF correctly reflects the non-informativeness of the data from study s . Although one might expect every reasonable statistical procedure to possess this very intuitive property, many widely-used methods do not (e.g. Fisher’s combined probability test).

4.2. *Extreme Models and Connections with Frequentist Tests.* The proposed models are very flexible, covering a wide range of types and degrees of

heterogeneity by setting different values for (ϕ, ω) (or k in the CEFN prior). Here we discuss the two extremes of no heterogeneity (“fixed effects”) and maximum heterogeneity, and establish connections with frequentist testing approaches in these settings.

4.2.1. *The Fixed Effects Model.* The fixed effects model assumes genetic effects to be homogeneous across subgroups, and corresponds to $\phi = 0$ in ES, or $\psi = 0$ in EE. In these cases the test statistics \mathcal{T}_{ES} and \mathcal{T}_{EE} have particularly simple forms, being a weighted sum of individual T_s statistics from each study (often referred to as a weighted sum of Z scores when sample sizes are large). Specifically,

$$(4.10) \quad \mathcal{T} = \frac{\sum_s w_s T_s}{\sqrt{\sum_{s'} w_{s'}^2}}$$

where

1. For the ES model, $w_s = \text{se}(\hat{b}_s)^{-1} \approx \sqrt{2n_s f_s(1 - f_s)}$,
2. For the EE model, $w_s = \text{se}(\hat{\beta}_s)^{-1} \approx \hat{\sigma}_s^{-1} \sqrt{2n_s f_s(1 - f_s)}$,

and f_s denotes the allele frequency of the target SNP in subgroup s . (The approximations come from assuming Hardy–Weinberg equilibrium in each subgroup.) These representations clarify a key practical difference between the ES and EE models: EE up-weights studies with small residual error variance. Note also that T_s is the same for both EE and ES, and independent of measurement scale, but σ_s depends on measurement scale, so \mathcal{T}_{ES} is robust to studies using different measurement scales (or different transformations of the phenotypes) but \mathcal{T}_{EE} is not. In addition, these representations clarify the connection between these statistics and the methods used in the meta-analysis software METAL (Willer, Li and Abecasis (2010)). Specifically, METAL implements tests using the weighted statistic (4.10) with two different weighting schemes, one corresponding to the EE model weights above, and the other with the weights equal to $\sqrt{n_s}$. This latter scheme corresponds to the ES model only if f_s is equal across studies. (Where f_s varies across studies the weighting in the ES model seems, to us, preferable to the METAL scheme since studies with small $f_s(1 - f_s)$ provide less information.)

Returning now to the BFs, when $\phi = 0$ the ABF (4.9) simplifies to

$$(4.11) \quad \text{ABF}_{\text{fix}}^{\text{ES}}(\omega) := \text{ABF}^{\text{ES}}(\phi = 0, \omega) = \sqrt{\frac{\zeta^2}{\zeta^2 + \omega^2}} \cdot \exp\left(\frac{\mathcal{T}_{\text{ES}}^2}{2} \frac{\omega^2}{\zeta^2 + \omega^2}\right),$$

where

$$(4.12) \quad \zeta^2 = \frac{1}{\sum_s \delta_s^{-2}}.$$

A similar expression holds for $\text{ABF}_{\text{fix}}^{\text{EE}}(w) := \text{ABF}^{\text{EE}}(\psi = 0, w)$.

We now answer the following question: under what prior assumptions will $\text{ABF}_{\text{fix}}^{\text{ES}}$ produce the same SNP rankings as the frequentist test statistic \mathcal{T}_{ES} ? Wakefield (2009) names this kind of prior the “implicit p -value prior”, as it identifies the implicit prior assumptions being made when one ranks SNPs by their p -value computed from \mathcal{T}_{ES} .

Although for a *given* SNP $\text{ABF}_{\text{fix}}^{\text{ES}}(\omega)$ is a monotone function of \mathcal{T}_{ES} , for a fixed value of ω the two statistics will not generally rank SNPs in the same way because ζ varies among SNPs. If however ω is assumed to vary among SNPs in a particular way, then the two statistics produce the same ranking. (A similar result holds for $\text{ABF}_{\text{fix}}^{\text{EE}}$.)

PROPOSITION 4.2. (Implicit p -value prior, fixed effects) *In the ES model, if the prior hyperparameter ω is allowed to vary among SNPs, with*

$$(4.13) \quad \omega_p = K\zeta_p,$$

where p indexes SNPs, and K is any positive constant, then $\text{ABF}_{\text{fix}}^{\text{ES}}$ and \mathcal{T}_{ES} will produce the same ranking of SNPs.

PROOF. This follows directly from substituting (4.13) into (4.11). \square

NOTE 4.3. *Recall that ζ_p is the standard error of \hat{b} for SNP p , so large ζ_p corresponds to less information about \bar{b} (which could occur, for example, due to the SNP having small minor allele frequency, or being typed in only a few studies). Recall also that large values of ω_p correspond to a prior assumption that the effect size \bar{b} at SNP p is likely to be large (in absolute value). Thus the implicit p -value makes the curious assumption that SNPs with less information have larger effects (see also Guan and Stephens (2008)).*

NOTE 4.4. *When data on all SNPs are available on all subgroups, and the subgroups also have similar allele frequencies at every SNP (as might happen if the subgroups come from a single random mating population) then the sample genotype variance of SNP p in subgroup s can be well approximated by $2n_s f_p(1 - f_p)$, where f_p is the population allele frequency of SNP p . (Note the slight abuse of notation, since we previously indexed f by subgroup,*

whereas here it is indexed by SNP.) Consequently, the implicit frequentist prior (4.13) can be written as

$$(4.14) \quad \omega_p = K \sqrt{\frac{1}{f_p(1-f_p)} \frac{1}{\sum_s n_s}},$$

which is effectively the same as the single subgroup case discussed by Wakefield (2009).

4.2.2. *Maximum Heterogeneity Model.* Now consider the other end of the spectrum: models with very high heterogeneity. Specifically, within the class of ES models with some fixed prior expected marginal effect size $(\phi^2 + \omega^2)$, the model with $\omega = 0$ has maximum heterogeneity. In this case, the average effect \bar{b} is identically 0, and the effects $b_s|\phi$ are independent, $\sim N(0, \phi^2)$.

It can be shown from (A.13) and (A.27) that for both EE and ES, the exact BF under this setting, $\text{BF}_{\max\text{H}}$, is the product of the individual BFs,

$$(4.15) \quad \text{BF}_{\max\text{H}} = \prod_s \text{BF}_{\text{single}, s},$$

where $\text{BF}_{\text{single}, s}$ is the exact BF calculated using data only from subgroup s . This relationship also holds for the ABF, i.e.

$$(4.16) \quad \text{ABF}_{\max\text{H}}^{\text{ES}}(\phi) := \text{ABF}^{\text{ES}}(\phi, \omega = 0) = \prod_s \sqrt{\frac{\delta_s^2}{\delta_s^2 + \phi^2}} \exp\left(\frac{T_s^2}{2} \frac{\phi^2}{\delta_s^2 + \phi^2}\right).$$

The frequentist test that corresponds to this “maximum heterogeneity” BF turns out to be the likelihood ratio test of $H_0 : b_s = 0$ (for all s) vs the general unconstrained alternative, which can be written

$$(4.17) \quad \text{LR}_{\max\text{H}} = \prod_s \text{LR}_s,$$

where LR_s is the likelihood ratio test statistic for $H_0 : b_s = 0$ vs $H_1 : b_s$ unconstrained. For sufficiently large samples, LR_s is well approximated by

$$(4.18) \quad \lim_{n_s \rightarrow \infty} \text{LR}_s \approx \exp\left(-\frac{T_s^2}{2}\right),$$

so

$$(4.19) \quad \text{LR}_{\max\text{H}} \approx \exp\left(-\frac{\sum_s T_s^2}{2}\right).$$

Thus the likelihood ratio test is approximately the same as a test based on $\sum_s T_s^2$, which (again assuming large sample sizes) is the sum of the squared Z values, and p -values can be obtained by noting that under the global null hypothesis this sum will be $\sim \chi_S^2$. This is very similar to Fisher’s approach to combining test statistics from multiple studies.

Under what prior assumptions will $\text{ABF}_{\text{maxH}}^{\text{ES}}$ give the same SNP ranking as $\sum_s T_s^2$? Under the ES model no single ϕ value will give this result. However, we have the following:

PROPOSITION 4.3. (Implicit p -value prior, maximum heterogeneity) *In the ES model, if the prior hyperparameter ϕ is allowed to vary among subgroups, with*

$$(4.20) \quad \phi_s^2 = K \delta_s^2,$$

where K is a constant for all subgroups and all SNPs tested, then $\text{ABF}_{\text{maxH}}^{\text{ES}}$ yields the same SNP ranking as $\sum_s T_s^2$.

PROOF. This follows directly from substituting (4.20) into (4.16). \square

NOTE 4.5. *Recalling that δ_s is the standard error for b_s , the implicit p -value prior (4.20) assumes bigger effects in subgroups with less information. There seems to be no good justification, in general, for this prior assumption.*

5. Discussion. Motivated by the need to allow for heterogeneity of effects in genetic association studies, we developed and applied a flexible toolbox of Bayesian methods for this problem. Our applications demonstrate how these tools can i) identify associations allowing for different amounts and types of heterogeneity; and ii) assess the amount and type of heterogeneity. The tools are sufficiently flexible to tackle a wide range of applications, from those involving limited heterogeneity (e.g. a typical meta-analysis), to the more extreme heterogeneity that might be encountered in Gene-Environment interaction studies. We presented computational methods that are practical for large studies, and highlighted connections between BFs and standard frequentist test statistics in this context (Propositions 4.2 and 4.3).

The models and priors considered here are closely connected to other models employed in meta-analysis. In particular, they are similar to mixed effect meta-analysis models in standard frequentist approaches for quantitative phenotypes, where the subgroup-specific intercept terms μ_s in (2.1) are regarded as fixed effects terms and genetic effect β_s (or b_s) are regarded as random effect terms. Our models are also connected with, but differ in an

important way from, models used in gene-environment ($G \times E$) interaction studies:

$$(5.1) \quad y_i = \mu + \beta_{e,s_i} + \beta_g g_i + \beta_{[g:e],s_i} g_i + e_i, \quad e_i \sim N(0, \sigma^2),$$

where s_i denotes the subgroup membership of individual i , and $\beta_{[g:e]}$ denotes the subgroup-genotype interaction terms. This linear model can be re-written as

$$(5.2) \quad y_i = (\mu + \beta_{e,s_i}) + (\beta_g + \beta_{[g:e],s_i}) g_i + e_i, \quad e_i \sim N(0, \sigma^2),$$

to emphasize that each subgroup has its own intercept, $\mu + \beta_{e,s_i}$, and its own genetic effect, $\beta_g + \beta_{[g:e],s_i}$. (If no marginal effect of subgroup is included, the model makes the stronger assumption of equal intercepts for different subgroups, which can be dangerous in practice and may lead to Simpson's paradox (Bravata and Olkin (2001))). The key difference between this model (5.2) and ours (2.1) is their assumptions on the error variances: our model allows for a different variance in each subgroup, whereas (5.1) assumes them to be equal. Allowing for subgroup-specific variances improves robustness, and can improve power Flutre et al. (2013).

One important issue that we have largely ignored is the question of how to weigh evidence of heterogeneity in the data (e.g. large BFs for high heterogeneity models) against an *a priori* belief that, in general, strong heterogeneity might be rare. In principle this is straightforward: given a prior distribution on different types of heterogeneity, it is trivial to use the BFs to compute posterior distributions. However, there remains an issue of choice of appropriate priors (which also arises in a disguised form in frequentist approaches, for example in selecting appropriate p -value thresholds when testing for heterogeneity). Here we have often used discrete uniform distributions for convenience. In general, one might want to change this, and appropriate priors may be context-dependent. For example, in a meta-analysis one might upweight models with limited heterogeneity (the CEFN prior), whereas in a gene-environment interaction study one might allow more heterogeneity.

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APPENDIX A: COMPUTING BAYES FACTORS

In this section, we show the detailed calculations of various BFs.

A.1. Computation in ES Model. A particular ES model, describing an alternative hypothesis H_a , is fully specified by setting values for (ϕ, ω) and hyper-parameters $(v_1, \dots, v_S, l_1, m_1, \dots, l_S, m_S)$. Under the contrasting null model H_0 , we set $\phi = \omega = 0$ while keeping all other hyper-parameters the same.

Let $\boldsymbol{\beta}_s = (\mu_s, \boldsymbol{\beta}_s)$, $\tau_s = \sigma_s^{-2}$ and $\boldsymbol{\theta} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_S, \tau_1, \dots, \tau_S, \bar{b})$, the marginal likelihood under model H_a can be written as

$$\begin{aligned}
 (A.1) \quad P(\mathbf{Y}|\mathbf{G}, H_a) &= \int P(\mathbf{Y}|\mathbf{G}, \boldsymbol{\theta}, H_a) p(\boldsymbol{\theta}|H_a) d\boldsymbol{\theta} \\
 &= \int \left(\prod_s P(\mathbf{y}_s|\mathbf{g}_s, \boldsymbol{\beta}_s, \tau_s) \prod_s P(\boldsymbol{\beta}_s|\tau_s, \bar{b}, H_a) \prod_s P(\tau_s|H_a) P(\bar{b}|H_a) \right) d\boldsymbol{\beta}_1 \cdots d\boldsymbol{\beta}_S d\tau_1 \cdots d\tau_S d\bar{b} \\
 &= \int \left(\int \left(\prod_s \int P(\mathbf{y}_s|\mathbf{g}_s, \boldsymbol{\beta}_s, \tau_s) P(\boldsymbol{\beta}_s|\tau_s, \bar{b}, H_a) d\boldsymbol{\beta}_s \right) p(\bar{b}|H_a) d\bar{b} \right) \prod_s P(\tau_s|H_a) d\tau_1 \cdots d\tau_S
 \end{aligned}$$

Let $\mathbf{X}_s = (\mathbf{1} \ \mathbf{g}_s)$ denote the design matrix of regression model (2.1) for subgroup s , it follows that

$$\begin{aligned}
 (A.2) \quad P(\mathbf{y}_s|\mathbf{g}_s, \boldsymbol{\beta}_s, \tau_s) &= \left(\frac{2\pi}{\tau_s}\right)^{-n_s/2} \exp\left(-\frac{\tau_s}{2}(\mathbf{y}_s - \mathbf{X}_s\boldsymbol{\beta}_s)'(\mathbf{y}_s - \mathbf{X}_s\boldsymbol{\beta}_s)\right) \\
 &= \left(\frac{2\pi}{\tau_s}\right)^{-n_s/2} \exp\left(-\frac{1}{2}(\tilde{\mathbf{y}}_s - \mathbf{X}_s\mathbf{b}_s)'(\tilde{\mathbf{y}}_s - \mathbf{X}_s\mathbf{b}_s)\right),
 \end{aligned}$$

where $\tilde{\mathbf{y}}_s = \sqrt{\tau_s}\mathbf{y}_s$ and $\mathbf{b}_s = \sqrt{\tau_s}\boldsymbol{\beta}_s = (\sqrt{\tau_s}\mu_s, \mathbf{b}_s)$. We further denote

$$(A.3) \quad \bar{\mathbf{b}} = \begin{pmatrix} 0 \\ \bar{b} \end{pmatrix} \quad \text{and} \quad \Phi_s = \begin{pmatrix} v_s^2 & 0 \\ 0 & \phi^2 \end{pmatrix},$$

and write prior distribution $P(\mathbf{b}_s|\bar{\mathbf{b}}, H_a)$ in following matrix form,

$$(A.4) \quad \mathbf{b}_s|\bar{\mathbf{b}}, H_a \sim \mathbf{N}(\bar{\mathbf{b}}, \Phi_s).$$

We compute the marginal likelihood by sequentially evaluating the following integrals,

$$\begin{aligned}
 (A.5) \quad F_{H_a, s} &= \int P(\mathbf{y}_s|\mathbf{X}_s, \mathbf{b}_s, \tau_s) P(\mathbf{b}_s|\bar{\mathbf{b}}, H_a) d\mathbf{b}_s \\
 &= \left(\frac{2\pi}{\tau_s}\right)^{-n_s/2} |\Phi_s|^{-\frac{1}{2}} \cdot |\mathbf{X}_s'\mathbf{X}_s + \Phi_s^{-1}|^{-\frac{1}{2}} \\
 &\quad \cdot \exp\left(-\frac{1}{2} \left(\tilde{\mathbf{y}}_s'\tilde{\mathbf{y}}_s - (\mathbf{X}_s'\tilde{\mathbf{y}}_s + \Phi_s^{-1}\bar{\mathbf{b}})'(\mathbf{X}_s'\mathbf{X}_s + \Phi_s^{-1})^{-1}(\mathbf{X}_s'\tilde{\mathbf{y}}_s + \Phi_s^{-1}\bar{\mathbf{b}}) + \bar{\mathbf{b}}'\Phi_s^{-1}\bar{\mathbf{b}} \right)\right).
 \end{aligned}$$

Let $J_{H_a} = \int (\prod_s F_{H_a,s}) P(\bar{b}|H_a) d\bar{b}$; this quantity is also analytically computable by straightforward algebra.

To compute BF of H_a versus H_0 under the ES model, we take limits with respect to hyper-parameters $(v_1, \dots, v_S, l_1, m_1, \dots, l_S, m_S)$, such that $u_s^2 \rightarrow \infty$ and $l_s, m_s \rightarrow 0, \forall s$. This yields,

$$(A.6) \quad \begin{aligned} \text{BF}^{\text{ES}}(\phi, \omega) &= \lim \frac{\int J_{H_a} \prod_s P(\tau_s) d\tau_1 \cdots d\tau_S}{\int J_{H_0} \prod_s P(\tau_s) d\tau_1 \cdots d\tau_S} \\ &= \frac{\int K_{H_a} d\tau_1 \cdots d\tau_S}{\int K_{H_0} d\tau_1 \cdots d\tau_S}. \end{aligned}$$

Let us denote

$$(A.7) \quad \text{RSS}_{0,s} = \mathbf{y}'_s \mathbf{y}_s - n_s \bar{y}_s^2,$$

$$(A.8) \quad \text{RSS}_{1,s} = \mathbf{y}'_s \mathbf{y}_s - \mathbf{y}'_s \mathbf{X}_s (\mathbf{X}'_s \mathbf{X}_s)^{-1} \mathbf{X}'_s \mathbf{y}_s,$$

$$(A.9) \quad \delta_s^2 = \frac{1}{\mathbf{g}'_s \mathbf{g}_s - n_s \bar{g}_s^2},$$

$$(A.10) \quad \hat{\beta}_s = \frac{\mathbf{y}'_s \mathbf{g}_s - n_s \bar{y}_s \bar{g}_s}{\mathbf{g}'_s \mathbf{g}_s - n_s \bar{g}_s^2},$$

$$(A.11) \quad \zeta^2 = \frac{1}{\sum_s (\delta_s^2 + \phi^2)^{-1}},$$

where \bar{y}_s and \bar{g}_s are the sample means of phenotypes and genotypes in subgroup s . It can be shown that,

$$(A.12) \quad K_{H_0} = \prod_s \tau_s^{\frac{n_s}{2}-1} \exp\left(-\frac{1}{2} \sum_s \tau_s \cdot \text{RSS}_{0,s}\right),$$

and

$$(A.13) \quad \begin{aligned} K_{H_a} &= \sqrt{\frac{\zeta^2}{\zeta^2 + \omega^2}} \prod_s \sqrt{\frac{\delta_s^2}{\delta_s^2 + \phi^2}} \\ &\cdot \prod_s \tau_s^{\frac{n_s}{2}-1} \exp\left(-\frac{1}{2} \sum_s \tau_s \left(\frac{\phi^2}{\delta_s^2 + \phi^2} \cdot \text{RSS}_{1,s} + \frac{\delta_s^2}{\delta_s^2 + \phi^2} \cdot \text{RSS}_{0,s}\right)\right) \\ &\cdot \exp\left(\frac{1}{2} \frac{\omega^2 \zeta^2}{\zeta^2 + \omega^2} \left(\sum_s \frac{\hat{\beta}_s \sqrt{\tau_s}}{\delta_s^2 + \phi^2}\right)^2\right). \end{aligned}$$

The multidimensional integral $\int K_{H_a} d\tau_1 \cdots d\tau_S$ generally does not have a simple analytic form (although it can be represented as finite sums of complicated hypergeometric functions). Next, we show two different approximations, both based on Laplace's method, to evaluate this integral. The

first approximation is a direct application of Butler and Wood (2002) and the second one yields a simple analytic expression. Although the integral $\int K_{H_0} d\tau_1 \cdots d\tau_S$ can be analytically computed as a gamma function, for computing the BF, we also use Laplace’s method to numerically evaluate it (which essentially is applying Sterling’s formula) – we find this recipe yields more accurate result for the final BF: in particular, when there is only one subgroup ($S = 1$, where the BF can be analytically computed as in Servin and Stephens (2008)), we obtain the exact result by applying the first Laplace’s approximation.

Laplace’s method approximates a multivariate integral in the following way,

$$(A.14) \quad \int_D h(\boldsymbol{\tau}) e^{g(\boldsymbol{\tau})} d\boldsymbol{\tau} \approx (2\pi)^{S/2} |H_{\hat{\boldsymbol{\tau}}}|^{-1/2} h(\hat{\boldsymbol{\tau}}) e^{g(\hat{\boldsymbol{\tau}})}$$

where $\boldsymbol{\tau}$ is an S -vector,

$$(A.15) \quad \hat{\boldsymbol{\tau}} = \arg \max_{\boldsymbol{\tau}} g(\boldsymbol{\tau}),$$

and $|H_{\hat{\boldsymbol{\tau}}}|$ is the absolute value of the determinant of the Hessian matrix of the function g evaluated at $\hat{\boldsymbol{\tau}}$. Note that the factorization of the integrand is rather arbitrary, it only requires that function h is smooth and positively valued and the smooth function g has a unique maximum lying in the interior of D (for detailed discussion, see Butler (2007)).

Our first approach to apply Laplace’s method sets $h(\boldsymbol{\tau}) \equiv 1$. Except for some trivial situations (e.g. $S = 1$), the maximization of $\log K_{H_a}$ with respect to $\boldsymbol{\tau}$ is analytically intractable. In practice, we use the Broyden-Fletcher-Goldfarb-Shanno (BFGS2) algorithm, a gradient-based numerical optimization routine (implemented in the GNU Scientific Library), to perform numerical maximization. This procedure leads to $\widehat{\text{BF}}^{\text{ES}}(\phi, \omega)$.

Alternatively, we apply Laplace’s method by factoring the integrand in such a way that g can be analytically maximized. This approach results in a closed-form approximation. More specifically, we factor K_{H_a} into

$$(A.16) \quad K_{H_a} = h(\tau_1, \dots, \tau_S) e^{g(\tau_1, \dots, \tau_S)},$$

where

$$\begin{aligned}
 (A.17) \quad h(\tau_1, \dots, \tau_S) &= \sqrt{\frac{\zeta^2}{\zeta^2 + \omega^2}} \prod_s \sqrt{\frac{\delta_s^2}{\delta_s^2 + \phi^2}} \\
 &\cdot \prod_s \exp\left(-\frac{1}{2} \sum_s \frac{\delta_s^2}{\delta_s^2 + \phi^2} \cdot \tau_s \cdot (\text{RSS}_{0,s} - \text{RSS}_{1,s})\right) \\
 &\cdot \exp\left(\frac{1}{2} \frac{\omega^2 \zeta^2}{\zeta^2 + \omega^2} \left(\sum_s \frac{\hat{\beta}_s \sqrt{\tau_s}}{\delta_s^2 + \phi^2}\right)^2\right)
 \end{aligned}$$

and

$$(A.18) \quad e^{g(\tau_1, \dots, \tau_S)} = \prod_s \tau_s^{\frac{n_s}{2} - 1} \cdot \exp\left(-\frac{1}{2} \sum_s \tau_s \cdot \text{RSS}_{1,s}\right).$$

It is straightforward to show that the unique maximum of $g(\tau_1, \dots, \tau_S)$ is attained at

$$(A.19) \quad \hat{\tau}_s = \frac{n_s - 2}{\text{RSS}_{1,s}}, \quad s = 1, \dots, S,$$

which coincides with the REML estimate of τ_s in subgroup-level regression model (2.1). Similarly, we factor K_{H_0} into

$$(A.20) \quad K_{H_0} = \exp\left(-\frac{1}{2} \sum_s \tau_s (\text{RSS}_{0,s} - \text{RSS}_{1,s})\right) \cdot \prod_s \tau_s^{\frac{n_s}{2} - 1} \cdot \exp\left(-\frac{1}{2} \sum_s \tau_s \cdot \text{RSS}_{1,s}\right),$$

and expand it around $\hat{\tau}_s$ as well.

Following the notations in section 2.3 and noting the relationship between t and F statistics in the simple linear regression,

$$(A.21) \quad T_s^2 = \hat{\tau}_s \cdot (\text{RSS}_{0,s} - \text{RSS}_{1,s}),$$

Applying (A.14) results in

$$\begin{aligned}
 (A.22) \quad \text{BF}^{\text{ES}}(\phi, \omega) &= \sqrt{\frac{\zeta^2}{\zeta^2 + \omega^2}} \exp\left(\frac{T_{\text{ES}}^2}{2} \frac{\omega^2}{\zeta^2 + \omega^2}\right) \cdot \prod_s \left(\sqrt{\frac{\delta_s^2}{\delta_s^2 + \phi^2}} \exp\left(\frac{T_s^2}{2} \frac{\phi^2}{\delta_s^2 + \phi^2}\right)\right) \\
 &\cdot \left(1 + O\left(\frac{1}{n_s}\right)\right).
 \end{aligned}$$

In conclusion, we have obtained

$$(A.23) \quad \text{ABF}^{\text{ES}}(\phi, \omega) = \sqrt{\frac{\zeta^2}{\zeta^2 + \omega^2}} \exp\left(\frac{T_{\text{ES}}^2}{2} \frac{\omega^2}{\zeta^2 + \omega^2}\right) \prod_s \left(\sqrt{\frac{\delta_s^2}{\delta_s^2 + \phi^2}} \exp\left(\frac{T_s^2}{2} \frac{\phi^2}{\delta_s^2 + \phi^2}\right) \right).$$

Remarks. Note, in case τ_1, \dots, τ_S are known, we can directly compute the exact BF using

$$(A.24) \quad \text{BF}^{\text{ES}}(\phi, \omega) = \lim \frac{J_{H_a}}{J_{H_0}}$$

without evaluating the multi-dimensional integrals in (A.6). In this particular case, it is easy to show that the exact BF has the exact functional form as in (A.23), only with all the $\hat{\tau}_s$'s replaced by the corresponding true values of τ_s 's.

Finally, we give the proof for 4.1:

PROOF OF PROPOSITION 4.1. The derivation above serves as a proof. An alternative proof can be obtained by noting that the REML estimate of $\hat{\boldsymbol{\tau}}$ asymptotically converges to the true value of $\boldsymbol{\tau}$ with probability 1. From the remarks above, by applying continuous mapping theorem, we conclude that $\text{ABF}^{\text{ES}}(\phi, \omega)$ converges to $\text{BF}^{\text{ES}}(\phi, \omega)$ with probability 1. \square

A.2. Computation in EE Model. The procedure for computing BF assuming an EE model is essentially the same, we omit repeating the details but only show the final results of the BF of an EE model H_b , specified by (ψ, w) , versus the null model H_0 ,

$$(A.25) \quad \text{BF}^{\text{EE}}(\psi, w) = \frac{\int K_{H_b} d\tau_1 \cdots d\tau_S}{\int K_{H_0} d\tau_1 \cdots d\tau_S}.$$

The expression of K_{H_0} remains the same as (A.12). We denote

$$(A.26) \quad \eta^2 = \left(\sum_s \frac{\tau_s}{\delta_s^2 + \tau_s \psi^2} \right)^{-1}.$$

It can be shown

$$(A.27) \quad \begin{aligned} K_{H_b} = & \sqrt{\frac{\eta^2}{\eta^2 + w^2}} \prod_s \sqrt{\frac{\delta_s^2}{\delta_s^2 + \tau_s \psi^2}} \\ & \cdot \prod_s \tau_s^{\frac{n_s}{2}-1} \exp \left(-\frac{1}{2} \sum_s \tau_s \left(\frac{\tau_s \psi^2}{\delta_s^2 + \tau_s \psi^2} \cdot \text{RSS}_{1,s} + \frac{\delta_s^2}{\delta_s^2 + \tau_s \psi^2} \cdot \text{RSS}_{0,s} \right) \right) \\ & \cdot \exp \left(\frac{1}{2} \frac{w^2}{\eta^2 + w^2} \frac{\left(\sum_s \frac{\tau_s}{\delta_s^2 + \tau_s \psi^2} \cdot \hat{\beta}_s \right)^2}{\eta^2} \right). \end{aligned}$$

We use the similar numerical procedure to obtain $\widehat{\text{BF}}^{\text{EE}}(\psi, w)$ as in the ES model.

To derive ABF^{EE} , we factor K_{H_b} into

$$(A.28) \quad K_{H_b} = h(\tau_1, \dots, \tau_S) e^{g(\tau_1, \dots, \tau_S)},$$

where,

$$(A.29) \quad \begin{aligned} h(\tau_1, \dots, \tau_S) = & \sqrt{\frac{\eta^2}{\eta^2 + w^2}} \prod_s \sqrt{\frac{\delta_s^2}{\delta_s^2 + \tau_s \psi^2}} \\ & \cdot \prod_s \exp \left(-\frac{1}{2} \sum_s \frac{\tau_s \delta_s^2}{\delta_s^2 + \tau_s \psi^2} \cdot (\text{RSS}_{0,s} - \text{RSS}_{1,s}) \right) \\ & \cdot \exp \left(\frac{1}{2} \frac{w^2}{\eta^2 + w^2} \frac{\left(\sum_s \frac{\tau_s}{\delta_s^2 + \tau_s \psi^2} \cdot \hat{\beta}_s \right)^2}{\eta^2} \right) \end{aligned}$$

and

$$(A.30) \quad e^{g(\tau_1, \dots, \tau_S)} = \prod_s \tau_s^{\frac{n_s}{2}-1} \cdot \exp \left(-\frac{1}{2} \sum_s \tau_s \cdot \text{RSS}_{1,s} \right).$$

Again, function $g(\tau_1, \dots, \tau_S)$ is maximized at

$$(A.31) \quad \hat{\tau}_s = \frac{n_s - 2}{\text{RSS}_{1,s}}, \quad s = 1, \dots, S.$$

We denote

$$(A.32) \quad d_s^2 = \frac{1}{\hat{\tau}_s} \delta_s^2 = \frac{\hat{\sigma}_s^2}{\mathbf{g}'_s \mathbf{g}_s - n_s \bar{g}_s^2},$$

$$(A.33) \quad T_s^2 = \frac{\hat{\beta}_s}{d_s^2},$$

$$(A.34) \quad \xi^2 = \left(\sum_s \frac{\hat{\tau}_s}{\delta_s^2 + \hat{\tau}_s \psi^2} \right)^{-1} = \frac{1}{\sum_s (d_s^2 + \psi^2)^{-1}},$$

$$(A.35) \quad \hat{\beta} = \frac{\sum_s (d_s^2 + \psi^2)^{-1} \hat{\beta}_s}{\sum_s (d_s^2 + \psi^2)^{-1}},$$

and

$$(A.36) \quad \mathcal{T}_{EE}^2 = \frac{\hat{\beta}^2}{\xi^2} = \frac{\left(\sum_s \frac{\hat{\beta}}{d_s^2 + \psi^2} \right)^2}{\eta^2}.$$

Using a similar procedure as in the ES model, we obtain

$$(A.37) \quad \text{ABF}^{EE}(\psi, w) = \sqrt{\frac{\xi^2}{\xi^2 + w^2}} \exp\left(\frac{\mathcal{T}_{EE}^2}{2} \frac{w^2}{\xi^2 + w^2}\right) \prod_s \left(\sqrt{\frac{d_s^2}{d_s^2 + \psi^2}} \exp\left(\frac{T_s^2}{2} \frac{\psi^2}{d_s^2 + \psi^2}\right) \right).$$

As discussed in **Remarks** of section A.1, if τ_1, \dots, τ_S are known, the exact BF of the EE model has the same function form as in (A.37), with $\hat{\tau}_s$'s replaced by the corresponding τ_s 's.

A.3. Computation using CEFN Priors. Using curved exponential family normal prior, the computation of the BFs is slightly different than what we show in the previous sections. Here, we use the ES model as a demonstration, the procedure for the EE model is very similar.

To compute the BF of a CEFN-ES model defined by parameters (k, ω) vs. the null model, we can carry out the same and exact calculation up to (A.5). However, due to the nature of the CEFN prior, we can no longer perform analytic calculation to integrate out \bar{b} . Instead, we exchange the order of integrations by first analytically approximating the multi-dimensional integration with respect to τ_1, \dots, τ_S using the second procedure of Laplace's method described in previous sections. As a result, we obtain the approxi-

mate BF as a one-dimensional integral

(A.38)

$$\begin{aligned} \text{ABF}_{\text{CEFN}}^{\text{ES}}(k, \omega) &= \frac{1}{\sqrt{2\pi\omega}} \prod_s \left(\frac{\text{RSS}_{0,s}}{\text{RSS}_{1,s}} \right)^{n_s/2} \int_{-\infty}^{\infty} \prod_s \sqrt{\frac{\delta_s^2}{\delta_s^2 + kb^2}} \\ &\cdot \exp \left[-\frac{1}{2} \left(\left(\sum_s \frac{1}{\delta_s^2 + kb^2} + \frac{1}{\omega^2} \right) \bar{b}^2 - 2 \sum_s \left(\frac{\hat{b}_s}{\delta_s^2 + kb^2} \right) \bar{b} + \sum_s \frac{\delta_s^2}{\delta_s^2 + kb^2} T_s^2 \right) \right] d\bar{b}. \end{aligned}$$

We then apply an adaptive Gaussian quadrature method, QAGI (implemented in GNU scientific library), to numerically evaluate this integral. Essentially, this method first maps the integrand to the semi-open interval $[0, 1)$ using the transformation $y = (1 - \bar{b})/\bar{b}$, then apply the standard adaptive Gaussian quadrature routine for the finite interval integration.

For the EE model with CEFN prior, the final one-dimensional integral can be shown as

(A.39)

$$\begin{aligned} \text{ABF}_{\text{CEFN}}^{\text{EE}}(k, w) &= \frac{1}{\sqrt{2\pi w}} \prod_s \left(\frac{\text{RSS}_{0,s}}{\text{RSS}_{1,s}} \right)^{n_s/2} \int_{-\infty}^{\infty} \prod_s \sqrt{\frac{d_s^2}{d_s^2 + k\bar{\beta}^2}} \\ &\cdot \exp \left[-\frac{1}{2} \left(\left(\sum_s \frac{1}{d_s^2 + k\bar{\beta}^2} + \frac{1}{w^2} \right) \bar{\beta}^2 - 2 \sum_s \left(\frac{\hat{\beta}_s}{d_s^2 + k\bar{\beta}^2} \right) \bar{\beta} + \sum_s \frac{d_s^2}{d_s^2 + k\bar{\beta}^2} T_s^2 \right) \right] d\bar{\beta}. \end{aligned}$$

APPENDIX B: BF FOR GENERALIZED LINEAR MODELS

In this section, we show the computation of BF for generalized linear models. As a useful application in genetic association study, these results can be directly applied to case-control data where a logistic link function is typically used.

For an appropriate link function $g(\cdot)$, we modify the subgroup-level linear model (2.1) into

$$(B.1) \quad \mathbf{E}(\mathbf{y}_s) = g^{-1}(\mu_s \mathbf{1} + \beta_s \mathbf{g}_s).$$

Let us denote $\boldsymbol{\beta}_s = (\mu_s, \beta_s)$. The key component in our computation is to approximate subgroup-level log-likelihood function $l(\boldsymbol{\beta}_s)$ with a quadratic form expanding around its maximum likelihood estimate, i.e.

$$(B.2) \quad \log P(\mathbf{y}_s | \mathbf{g}_s, \boldsymbol{\beta}_s) = l(\boldsymbol{\beta}_s) \simeq l(\hat{\boldsymbol{\beta}}_s) - \frac{1}{2} (\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s)' I_s(\hat{\boldsymbol{\beta}}_s) (\boldsymbol{\beta}_s - \hat{\boldsymbol{\beta}}_s),$$

where $I_s(\hat{\boldsymbol{\beta}}_s) = \begin{pmatrix} i_{\hat{\mu}_s \hat{\mu}_s} & i_{\hat{\mu}_s \hat{\beta}_s} \\ i_{\hat{\beta}_s \hat{\mu}_s} & i_{\hat{\beta}_s \hat{\beta}_s} \end{pmatrix}$ is the expected Fisher information evalu-

ated at $\hat{\beta}_s$. We note that

$$(B.3) \quad \gamma_s^2 := \text{Var}(\hat{\beta}_s) = (i_{\hat{\beta}_s \hat{\beta}_s} - i_{\hat{\beta}_s \hat{\mu}_s} i_{\hat{\mu}_s \hat{\mu}_s}^{-1} i_{\hat{\mu}_s \hat{\beta}_s})^{-1}$$

is the estimated asymptotic variance of MLE $\hat{\beta}_s$.

Given approximate log-likelihood function (B.2) and a model H_c specified by (ψ, w) , the prior distribution for β_s is given by

$$(B.4) \quad \beta_s | \bar{\beta}, H_c \sim N(\bar{\beta}, \Psi_s),$$

where

$$(B.5) \quad \bar{\beta} = \begin{pmatrix} 0 \\ \bar{\beta} \end{pmatrix} \quad \text{and} \quad \Psi_s = \begin{pmatrix} v_s^2 & 0 \\ 0 & \psi^2 \end{pmatrix}.$$

It follows that

$$(B.6) \quad \begin{aligned} F_{H_c, s} &= \int P(\mathbf{y}_s | \mathbf{g}_s, \beta_s) P(\beta_s | \bar{\beta}, H_c) d\beta_s \\ &= \exp(l(\hat{\beta}_s)) \cdot |\Psi_s|^{-\frac{1}{2}} \cdot |I_s + \Psi_s^{-1}|^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2} \hat{\beta}_s' (I_s - I_s (I_s + \Psi_s^{-1})^{-1} I_s) \hat{\beta}_s\right) \\ &\quad \cdot \exp\left(-\frac{1}{2} \left(\bar{\beta}' (\Psi_s^{-1} - \Psi_s^{-1} (I_s + \Psi_s^{-1})^{-1} \Psi_s^{-1}) \bar{\beta} - \bar{\beta}' \eta_s - \eta_s' \bar{\beta}\right)\right), \end{aligned}$$

with $\eta_s = \Psi_s^{-1} (I_s + \Psi_s^{-1})^{-1} I_s \hat{\beta}_s$.

Under contrasting null model H_0 , the parameter space is restricted to $\beta_s = 0$, for β_s satisfies this restriction

$$(B.7) \quad (\beta_s - \hat{\beta}_s)' I_s (\hat{\beta}_s) (\beta_s - \hat{\beta}_s) = i_{\hat{\mu}_s \hat{\mu}_s} \cdot (\mu_s - \hat{m}_s)^2 + \frac{\hat{\beta}_s^2}{\gamma_s^2},$$

where $\hat{m}_s = \hat{\mu}_s + \frac{i_{\hat{\mu}_s \hat{\beta}_s}}{i_{\hat{\mu}_s \hat{\mu}_s}} \hat{\beta}_s$. It can be shown that

$$(B.8) \quad \begin{aligned} F_{H_0, s} &= \int P(\mathbf{y}_s | \mathbf{g}_s, \beta_s) P(\beta_s | \bar{\beta}, H_0) d\beta_s \\ &= \exp(l(\hat{\beta}_s)) \cdot v_s^{-1} (i_{\hat{\mu}\hat{\mu}} + v_s^{-2})^{-\frac{1}{2}} \cdot \exp\left(-\frac{\hat{\beta}_s^2}{2\gamma_s^2}\right) \\ &\quad \cdot \exp\left(-\frac{1}{2} \left(\hat{m}_s' i_{\hat{\mu}\hat{\mu}} \hat{m}_s - (i_{\hat{\mu}\hat{\mu}} \hat{m}_s)' (i_{\hat{\mu}\hat{\mu}} + v_s^{-2})^{-1} (i_{\hat{\mu}\hat{\mu}} \hat{m}_s)\right)\right). \end{aligned}$$

Finally, we compute

$$(B.9) \quad \text{ABF}(\psi, w) = \lim \frac{\int (\prod_s F_{H_c, s}) P(\bar{\beta} | H_c) d\bar{\beta}}{\prod_s F_{H_0, s}},$$

where the limit is taken as $v_s \rightarrow \infty$, $\forall s$. By straightforward algebra, we obtain the following final result

(B.10)

$$\text{BF}(\psi, w) \approx \text{ABF}(\psi, w) := \text{ABF}_{\text{single}}(\mathcal{Z}_{\text{cc}}^2, \xi; w) \cdot \prod_s \text{ABF}_{\text{single}}(Z_s^2, \gamma_s; \psi),$$

where

$$(B.11) \quad \text{ABF}_{\text{single}}(Z_s^2, \gamma_s; \psi) = \sqrt{\frac{\gamma_s^2}{\gamma_s^2 + \psi^2}} \exp\left(\frac{Z_s^2}{2} \frac{\psi^2}{\gamma_s^2 + \psi^2}\right),$$

$$(B.12) \quad \text{ABF}_{\text{single}}^{\text{CC}}(\mathcal{Z}_{\text{cc}}^2, \xi; w) = \sqrt{\frac{\xi^2}{\xi^2 + w^2}} \exp\left(\frac{\mathcal{Z}_{\text{cc}}^2}{2} \frac{w^2}{\xi^2 + w^2}\right),$$

(B.13)

and

$$(B.14) \quad \gamma_s^2 := \text{se}(\hat{\beta}_s)^2,$$

$$(B.15) \quad Z_s^2 = \frac{\hat{\beta}_s^2}{\gamma_s^2},$$

$$(B.16) \quad \hat{\beta} = \frac{\sum_s (\gamma_s^2 + \psi^2)^{-1} \hat{\beta}_s}{\sum_s (\gamma_s^2 + \psi^2)^{-1}},$$

$$(B.17) \quad \xi^2 := \text{se}(\hat{\beta})^2 = \frac{1}{\sum_s (\gamma_s^2 + \psi^2)^{-1}},$$

$$(B.18) \quad \mathcal{Z}_{\text{cc}}^2 = \frac{\hat{\beta}^2}{\xi^2}.$$

APPENDIX C: SMALL SAMPLE SIZE CORRECTION FOR APPROXIMATE BAYES FACTORS

The accuracy of ABF^{ES} relies on the sample sizes in subgroups: when sample sizes are small in some subgroups, the approximation may become inaccurate. In particular, we consider the behavior of the approximate BF when the null hypothesis is true. A valid BF has the property that

$$(C.1) \quad \text{E}(\text{BF}|H_0) = 1,$$

where the expectation is taken with respect to the data distribution (\mathbf{Y} in our settings) under the null model. This is because,

$$(C.2) \quad \text{E}(\text{BF}|H_0) = \int \frac{P(\mathbf{Y}|H_1)}{P(\mathbf{Y}|H_0)} \cdot P(\mathbf{Y}|H_0) d\mathbf{Y} = 1.$$

Unfortunately, when sample sizes are small, (C.1) can be violated when the approximate BF is used (as the expected value is strictly greater than 1) and this essentially indicates that the approximation becomes inaccurate.

To demonstrate a violation of (C.1), we consider the special case of one single subgroup. The approximate BF assuming the ES model with parameters (ϕ, ω) is given by

$$(C.3) \quad \text{ABF}_{\text{single}}^{\text{ES}}(\phi, \omega) = \sqrt{1 - \lambda} \exp\left(\frac{\lambda}{2} T_s^2\right),$$

and,

$$(C.4) \quad \log(\text{ABF}_{\text{single}}^{\text{ES}}(\phi, \omega)) = \frac{1}{2} \log(1 - \lambda) + \frac{\lambda}{2} T_s^2,$$

where $\lambda = \frac{\phi^2 + \omega^2}{\phi^2 + \omega^2 + \delta_s^2}$ and takes values from $[0, 1]$. Under H_0 , T_s follows t-distribution with $n_s - 2$ degree of freedom and

$$(C.5) \quad \text{E}(T_s^2 | H_0) = \frac{n_s - 2}{n_s - 4} > 1.$$

Now consider the continuous function

$$(C.6) \quad f(\lambda) = \frac{1}{\lambda} \log\left(\frac{1}{1 - \lambda}\right)$$

for $\lambda \in [0, 1]$, it can be shown that

$$(C.7) \quad \lim_{\lambda \rightarrow 0} f(\lambda) = 1$$

$$(C.8) \quad \lim_{\lambda \rightarrow 1} f(\lambda) = \infty.$$

Hence, there must exist values of $\lambda \in (0, 1)$, such that

$$(C.9) \quad 1 < f(\lambda) < \text{E}(T_s^2 | H_0).$$

Consequently, by Jensen's inequality, for those λ values

$$(C.10) \quad \log(\text{E}(\text{ABF}_{\text{single}}^{\text{ES}} | H_0)) \geq \text{E}(\log(\text{ABF}_{\text{single}}^{\text{ES}}) | H_0) > 0.$$

This shows property (C.1) does not generally hold for approximate BFs, and when sample size n_s is small, the inaccuracy may become severe.

We now propose a simple correction procedure for small sample sizes, which ensures the resulting approximation satisfies property (C.1). Specifically, we modify (4.9) into the following form

$$(C.11) \quad \text{A}^* \text{BF}^{\text{ES}}(\phi, \omega) = \text{ABF}_{\text{single}}^{\text{ES}}(q(\mathcal{T}_{\text{es}}^2), \zeta; \omega) \cdot \prod_s \text{ABF}_{\text{single}}^{\text{ES}}(q_s(T_s^2), \delta_s; \phi),$$

where the function q_s denotes a one-to-one quantile transformation from a t-distribution with $n_s - 2$ degree of freedom to a standard normal distribution, and the function q is defined as

$$(C.12) \quad q(\mathcal{T}_{\text{ES}})^2 = \frac{\hat{b}_{\text{cor}}^2}{\zeta^2},$$

where

$$(C.13) \quad \hat{b}_{\text{cor}} = \frac{\sum_s (\delta_s^2 + \phi^2)^{-1} \delta_s q_s(T_s)}{\sum_s (\delta_s^2 + \phi^2)^{-1}}.$$

Note, the quantile transformation functions q_s and q converge to the identity mappings as $n_s \rightarrow \infty$ and the asymptotic property of (4.9) is preserved. The numerical performance of this correction is demonstrated in appendix D.

To show the corrected version of approximate BF satisfying (C.1), we note that ABF^{ES} depends on data \mathbf{Y} only through T_s (δ_s depends on genotype data but not \mathbf{Y}). Further, from **Remark** in appendix A.1, we also notice the approximation becomes an exact BF (for which property (C.1) is guaranteed) if estimated error variance terms $\hat{\sigma}_s^2$'s are replaced by their corresponding true values. When the true error variances are plugged in, under the H_0 , T_s 's instead follow standard norm distributions. It is therefore sufficient to satisfy property (C.1) by quantile transforming each individual T_s in (4.9) from the t-distribution to standard normal distribution. In essence, the correction can be viewed as a general strategy of providing a better point estimate of residual errors, therefore the similar strategy also likely improves the accuracy of approximate BF when EE model or CEFN model is used.

APPENDIX D: NUMERICAL ACCURACY OF BF EVALUATIONS

In this section, we evaluate the numerical accuracy of various approximation methods for computing the BFs.

We use the dataset from population eQTL study (Stranger et al. (2007)) discussed in section 3.3 for this purpose. For each of the 8,427 genes examined, we select the top associated *cis*-SNP based on the values of $\widehat{\text{BF}}_{\text{av}}^{\text{ES}}$ and re-calculate the BF directly based on (A.6) using a general adaptive Gaussian quadrature procedure (Note, because of its high computational cost in numerically evaluating multi-dimensional integrals, this numerical recipe does not apply in general practice). We treat these results as the ‘‘truth’’ and make comparison with $\widehat{\text{BF}}_{\text{av}}^{\text{ES}}$ and $\text{ABF}_{\text{av}}^{\text{ES}}$ (with and without small sample corrections). Moreover, we convert various numerical results

of BFs to log 10 scale and compute Root Mean Squared Errors (RMSE) for each approximation.

The results of the numerical evaluation for the ES model shown in Table 6 and Figure 5. Although the sample sizes in each subgroup are quite small in this dataset (41 Europeans, 59 Asians and 41 Africans), the numerical results of $\widehat{\text{BF}}_{\text{av}}^{\text{ES}}$ are almost identical to the results obtained from the adaptive Gaussian quadrature procedure (RMSE = 1.2×10^{-4} in log 10 scale). As expected, the approximate BF, $\text{ABF}_{\text{av}}^{\text{ES}}$, has the worst numerical performance, mainly due to the small sample sizes in this dataset. Nevertheless, the ranking of the SNPs by $\text{ABF}_{\text{av}}^{\text{ES}}$ is quite consistent with true values (rank correlation = 0.99). Figure 5 suggests that under small sample situations, $\text{ABF}_{\text{av}}^{\text{ES}}$ tends to over-evaluate the true value and this over-evaluation can become quite severe when the true values are extremely large. On the other hand, the proposed small sample size correction method seems very effective: with this simple correction, the resulting $\text{A}^*\text{BF}_{\text{av}}^{\text{ES}}$ are quite accurate comparing with the true values.

We also perform a similar experiment for the EE model using the same dataset with five levels of $\sqrt{\psi^2 + w^2}$ values: 0.1, 0.2, 0.4, 0.8, 1.6, and seven degrees of heterogeneities characterized by ψ^2/w^2 values: 0, 1/4, 1/2, 1, 2, 4, ∞ , and we assign these 35 grid values equal prior weight. The results are similar with the case in the EE model and shown in Table 7.

	$\log_{10}(\widehat{\text{BF}}_{\text{av}}^{\text{ES}})$	$\log_{10}(\text{ABF}_{\text{av}}^{\text{ES}})$	$\log_{10}(\text{A}^*\text{BF}_{\text{av}}^{\text{ES}})$
RMSE	1.2×10^{-4}	4.95	0.14

TABLE 6

Numerical accuracy of three approximations for evaluating BFs under the ES model. $\widehat{\text{BF}}_{\text{av}}^{\text{ES}}$ is based on the first approximation of Laplace’s method discussed in appendix A, $\text{ABF}_{\text{av}}^{\text{ES}}$ is computed using (4.9) and $\text{A}^\text{BF}_{\text{av}}^{\text{ES}}$ is based on (C.11) which is corrected for small sample sizes.*

	$\log_{10}(\widehat{\text{BF}}_{\text{av}}^{\text{EE}})$	$\log_{10}(\text{ABF}_{\text{av}}^{\text{EE}})$	$\log_{10}(\text{A}^*\text{BF}_{\text{av}}^{\text{EE}})$
RMSE	4.1×10^{-4}	5.03	0.09

TABLE 7

Numerical accuracy of three approximations for evaluating BFs under the EE model.

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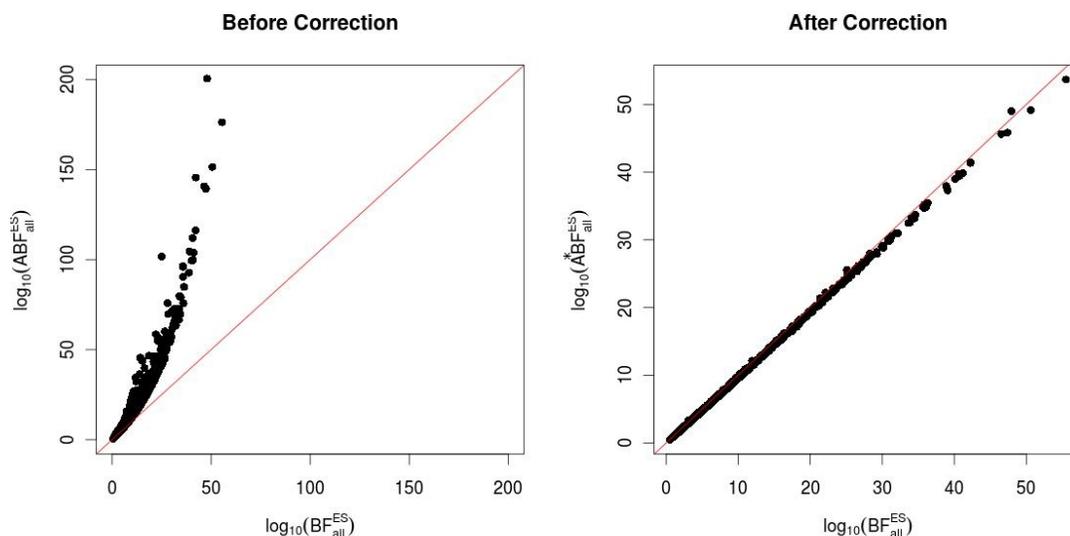


FIG 5. Comparison of approximate BF's before and after applying small sample size corrections.

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