SOME SUPPORT PROPERTIES FOR A CLASS OF Λ-FLEMING-VIOT PROCESSES

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Abstract. For a class of Λ-Fleming-Viot processes with underlying Brownian motion whose associated Λ-coalescents come down from infinity, we prove a one-sided modulus of continuity result for their ancestry processes recovered from the lookdown construction of Donnelly and Kurtz. As applications, we first show that such a Λ-Fleming-Viot support process has one-sided modulus of continuity (with modulus function $C\sqrt{t\log(1/t)}$) at any fixed time. We also show that the support is compact simultaneously at all positive times, and given the initial compactness, its range is uniformly compact over any finite time interval. In addition, under a mild condition on the Λ-coalescence rates, we find a uniform upper bound on Hausdorff dimension of the support and an upper bound on Hausdorff dimension of the range.

1. Introduction

Fleming-Viot process arises as a probability-measure-valued stochastic process on the distribution of allelic frequencies in a selectively neutral population with mutation. We refer to Ethier and Kurtz [17] and Etheridge [18] for surveys on the Fleming-Viot process and related mathematical models from population genetics.

Moments of the classical Fleming-Viot process can be expressed in terms of a dual process involving Kingman's coalescent and semigroup for the mutation operator. The Λ-Fleming-Viot process generalizes the classical Fleming-Viot process by replacing Kingman's coalescent with the Λ-coalescent allowing multiple collisions. Formally, the Λ-Fleming-Viot process is a Fleming-Viot process with general reproduction mechanism so that the total number of children from a parent can be comparable to the size of population. We refer to Birkner et al. [5] for a connection between mutationless Λ-Fleming-Viot processes and continuous state branching processes. In this paper we only consider the Fleming-Viot process with Brownian mutation that can also be interpreted as underlying spatial Brownian motion.

The support properties are interesting in the study of measure-valued processes. For the Dawson-Watanabe superBrownian motion arising as high density limit of empirical measures for near critical branching Brownian motions, the modulus of continuity and the carrying dimensions have been studied systematically for its support process. We refer to Chapter 7 of Dawson [7], Chapter 9 of Dawson [8] and Chapter III of Perkins [21] and references therein for a collection of these results. The proofs involve the historically cluster representation, the Palm distribution for the canonical measure and estimates

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obtained from PDE associated with the Laplace functional. For a superBrownian motion with a general branching mechanism, Delmas [13] obtained the Hausdorff dimension for its range using Brownian snake representation with subordination.

However, the approaches for Dawson-Watanabe superBrownian motions do not always apply to Fleming-Viot processes which are not infinitely divisible. Consequently, there are only a few results available for Fleming-Viot support processes so far. The earliest work on the compact support property for classical Fleming-Viot processes is due to Dawson and Hochberg [9]. It was shown in [9] that at any fixed time $T > 0$ the classical Fleming-Viot process with underlying Brownian motion has a compact support with Hausdorff dimension not greater than two. Using non-standard techniques Reimers [23] improved the above result by showing that the carrying dimension of the support is at most two simultaneously for all positive times. Applying a generalized Perkins disintegration theorem, the support dimension was found in Ruscher [24] for a Fleming-Viot-like process obtained from mass normalization and time change of superBrownian motion with stable branching. The Λ-Fleming-Viot process does not have a compact support if the associated Λ-coalescent does not come down from infinity. Liu and Zhou [20] recently extended the results in [9] to a class of Λ-Fleming-Viot processes whose associated Λ-coalescents come down from infinity. We are not aware of any results on the modulus of continuity for Fleming-Viot support processes although the modulus of continuity for superBrownian motion support had been first recovered by Dawson et al. [11] more than twenty years ago and further studied in Dawson and Vinogradov [12] and in Dawson el al. [10].

The lookdown construction of Donnelly and Kurtz [14, 15, 16] is a powerful technique in the study of the Fleming-Viot processes. Loosely speaking, the idea of lookdown construction is a discrete representation that leads to a nice version of the corresponding measure-valued process. The lookdown construction naturally results in a genealogy process describing the genealogical structure of the particles involved. In a sense it plays the role of historical processes for Dawson-Watanabe superprocesses.

Donnelly and Kurtz [14] first proposed the lookdown construction of a system of countable particles embedded into the classical Fleming-Viot process. They showed the duality between classical Fleming-Viot process and Kingman's coalescent and recovered some previous results on the classical Fleming-Viot process using this explicit representation. This representation was later extended in Donnelly and Kurtz [16] via a modified lookdown construction to a larger class of measure-valued processes including the Λ-Fleming-Viot processes and the Dawson-Watanabe superprocesses. Donnelly and Kurtz [15] also found a discrete representation for the classical Fleming-Viot models with selection and recombination.

Birkner and Blath [4] further discussed the modified lookdown construction in [16] for the Λ-Fleming-Viot process with jump type mutation operator. They also described how to recover the Λ-coalescent from the modified lookdown construction.

For the Ξ-coalescent allowing simultaneous multiple collisions, a Poisson point process construction of the Ξ-lookdown model can be found in Birkner et al. [6] by extending the modified lookdown construction of Donnelly and Kurtz [16]. It was proved in [6] that the empirical measure of the exchangeable particles converges almost surely in the Skorohod space of measure-valued paths to the so called Ξ-Fleming-Viot process with jump type mutation.

Using the modified lookdown construction of Donnelly and Kurtz, Liu and Zhou [20] proved that a class of Λ-Fleming-Viot processes with underlying Brownian motion have compact supports at any fixed time $T > 0$ provided the associated Λ-coalescents come down from infinity fast enough. Further, both lower and upper bounds were found in [20] on Hausdorff dimension for support of the Λ-Fleming-Viot process at the time T, where the exact Hausdorff dimension was shown to be two whenever the associated Λ-coalescent has a nontrivial Kingman component. These results generalize the previous results of Dawson and Hochberg [9] on the classical Fleming-Viot processes.

In this paper, for the class of Λ-Fleming-Viot processes in [20], we refine the arguments in [20] to further study their support properties. Our first result is a one-sided modulus of continuity type result for the ancestry process defined via the lookdown construction. The second result is the one-sided modulus of continuity for the Λ-Fleming-Viot support process at any fixed time. The third is on the uniform compactness of the Λ-Fleming-Viot support and the associated range. Under an additional mild condition on the coalescence rates of the corresponding Λ-coalescent, we also obtain two results on support dimensions. One result is an uniform upper bound on Hausdorff dimension for the support at all positive times. The other is an upper bound on Hausdorff dimension for the range of the Λ-Fleming-Viot support process. Again, the lookdown construction plays a key role throughout our arguments.

The paper is arranged as follows. In Section 2 we introduce the Λ-coalescent and the corresponding coming down from infinity property. In Section 3 we briefly discuss the lookdown construction for Λ-Fleming-Viot process with underlying Brownian motion and the associated ancestry process recovered from the lookdown construction. In Section 4 we present the main results of this paper together with corollaries and propositions. Proofs of the main results are deferred to Section 5.

2. The Λ-coalescent

2.1. **The Λ-coalescent.** We first introduce some notation. Put $[n] \equiv \{1, \dots, n\}$ and $[\infty] \equiv \{1, 2, \dots\}$. An ordered *partition* of $D \subset [\infty]$ is a countable collection $\pi = \{\pi_i, i = 1, 2, \dots\}$ of disjoint *blocks* such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$. Then blocks in π are ordered by their least elements.

Denote by $\mathcal{P}_n$ the set of ordered partitions of $[n]$ and by $\mathcal{P}_\infty$ the set of ordered partitions of $[\infty]$. Write $\mathbf{0}_{[n]} \equiv \{\{1\}, \dots, \{n\}\}$ for the partition of $[n]$ consisting of singletons and $\mathbf{0}_{[\infty]}$ for the partition of $[\infty]$ consisting of singletons. Given $n \in [\infty]$ and $\pi \in \mathcal{P}_\infty$, let $R_n(\pi) \in \mathcal{P}_n$ be the restriction of π to $[n]$.

Kingman's coalescent is a $\mathcal{P}_\infty$-valued time homogeneous Markov process such that all different pairs of blocks independently merge at the same rate. Pitman [22] and Sagitov [25] generalized the Kingman's coalescent to the Λ-*coalescent* which allows *multiple collisions*, i.e., more than two blocks may merge at a time. The Λ-coalescent is defined as a $\mathcal{P}_\infty$-valued Markov process $\Pi \equiv (\Pi(t))_{t\geq 0}$ such that for each $n \in [\infty]$, its restriction to $[n]$, $\Pi_n \equiv (\Pi_n(t))_{t\geq 0}$ is a $\mathcal{P}_n$-valued Markov process whose transition rates are described as follows: if there are currently b blocks in the partition, then each k-tuple of blocks ($2 \leq k \leq b$) independently merges to form a single block at rate

$$\lambda_{b,k} = \int_{[0,1]} x^{k-2}(1-x)^{b-k}\Lambda(dx), \tag{1}$$

where Λ is a finite measure on $[0,1]$. It is easy to check that the rates $(\lambda_{b,k})$ are consistent so that for all $2 \le k \le b$,

$$\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1}. \tag{2}$$

Consequently, for any $1 \le m < n \le \infty$, the coalescent process $R_m(\Pi_n(t))$ given $\Pi_n(0) = \pi_n$ has the same distribution as $\Pi_m(t)$ given $\Pi_m(0) = R_m(\pi_n)$.

With the transition rates determined by (1), there exists a one to one correspondence between Λ-coalescents and finite measures Λ on $[0,1]$.

For $n = 2, 3, \ldots$, denote by

$$\lambda_n = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \tag{3}$$

the total coalescence rate starting with n blocks. It is clear that $(\lambda_n)_{n\ge 2}$ is an increasing sequence, i.e., $\lambda_n \le \lambda_{n+1}$ for any $n \ge 2$. In addition, denote by

$$\gamma_n = \sum_{k=2}^{n} (k-1) \binom{n}{k} \lambda_{n,k}$$

the rate at which the number of blocks decreases.

2.2. **Coming down from infinity.** Given any Λ-coalescent $\Pi \equiv (\Pi(t))_{t\ge 0}$ with $\Pi(0) = \mathbf{0}_{[\infty]}$, let $\#\Pi(t)$ be the number of blocks in the partition $\Pi(t)$. The Λ-coalescent Π *comes down from infinity* if

$$\mathbb{P}(\#\Pi(t) < \infty) = 1$$

for all $t > 0$ and it *stays infinite* if

$$\mathbb{P}(\#\Pi(t) = \infty) = 1$$

for all $t > 0$. Suppose that the measure Λ has no atom at 1. It is shown by Schweinsberg [26] that

- the Λ-coalescent comes down from infinity if and only if $\sum_{n=2}^{\infty} \gamma_n^{-1} < \infty$;
- the Λ-coalescent stays infinite if and only if $\sum_{n=2}^{\infty} \gamma_n^{-1} = \infty$.

It is pointed out in Bertoin and Le Gall [3] that for

$$\psi(q) = \int_{[0,1]} (e^{-qx} - 1 + qx) x^{-2} \Lambda(dx),$$

$$\sum_{n=2}^{\infty} \gamma_n^{-1} < \infty \text{ if and only if } \int_a^{\infty} \frac{1}{\psi(q)} dq < \infty,$$

where the integral is finite for some (and then for all) $a > 0$.

Example 2.1. *In case of $\Lambda = \delta_0$, the corresponding coalescent is Kingman's coalescent and comes down from infinity.*

Example 2.2. *If $\beta \in (0,2)$ and*

$$\Lambda(dx) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} (1-x)^{\beta-1} dx,$$

the corresponding coalescent is Beta$(2-\beta,\beta)$-coalescent.

- *In case of $\beta \in (0,1]$, it stays infinite.*

- *In case of $\beta \in (1,2)$, it comes down from infinity.*

3. The Λ-Fleming-Viot process and its lookdown construction

In this section, we first discuss the lookdown construction of Λ-Fleming-Viot process with underlying Brownian motion. Then we explain how to recover the Λ-coalescent from the lookdown construction. Finally, we introduce the ancestry process for the Λ-Fleming-Viot process from the lookdown construction.

3.1. **Lookdown construction of Λ-Fleming-Viot process with underlying Brownian motion.** Donnelly and Kurtz [16] introduced a modified lookdown construction with the empirical measure process converging to measure-valued stochastic process. A key advantage of the lookdown construction is its projective property. Intuitively, in the lookdown model each particle is attached a "level" from the set $\{1, 2, \ldots\}$. The evolution of a particle at level n only depends on the evolution of the finite particles at lower levels. This property allows us to construct approximating particle systems, and their limit as $n \to \infty$ in the same probability space.

Following Birkner and Blath [4], we now give a brief introduction on the modified lookdown construction of the Λ-Fleming-Viot process with underlying Brownian motion. Let

$$(X_1(t), X_2(t), X_3(t), \ldots)$$

be an $(\mathbb{R}^d)^\infty$-valued random variable, where for any $i \in [\infty]$, $X_i(t)$ represents the spatial location of the particle at level i. We require the initial values $\{X_i(0), i \in [\infty]\}$ to be exchangeable random variables so that the limiting empirical measure

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(0)}$$

exists almost surely by de Finetti's theorem.

Let Λ be the finite measure associated to the Λ-coalescent. The reproduction in the particle system consists of two kinds of birth events: the events of single birth determined by measure $\Lambda(\{0\})\delta_0$ and the events of multiple births determined by measure Λ restricted to $(0,1]$ that is denoted by Λ_0.

To describe the evolution of the system during events of single birth, let $\{\mathbf{N}_{ij}(t) : 1 \le i < j < \infty\}$ be independent Poisson processes with common rate $\Lambda(\{0\})$. At a jump time t of $\mathbf{N}_{ij}$, the particle at level j looks down at the particle at level i and assumes its location (therefore, particle at level i gives birth to a new particle). Values of particles at levels above j are shifted accordingly, i.e., for $\Delta\mathbf{N}_{ij}(t) = 1$, we have

$$X_k(t) = \begin{cases} X_k(t-), & \text{if } k < j, \\ X_i(t-), & \text{if } k = j, \\ X_{k-1}(t-), & \text{if } k > j. \end{cases} \tag{4}$$

For those events of multiple births we can construct an independent Poisson point process $\tilde{\mathbf{N}}$ on $\mathbb{R}^+ \times (0,1]$ with intensity measure $dt \otimes x^{-2}\Lambda_0(dx)$. Let $\{U_{ij},\ i,j \in [\infty]\}$ be i.i.d. uniform $[0,1]$ random variables. Jump points $\{(t_i, x_i)\}$ for $\tilde{\mathbf{N}}$ correspond to the multiple birth events. For $t \ge 0$ and $J \subset [n]$ with $|J| \ge 2$, define

$$\mathbf{N}_J^n(t) \equiv \sum_{i:t_i \le t} \prod_{j\in J} \mathbf{1}_{\{U_{ij} \le x_i\}} \prod_{j\in[n]\setminus J} \mathbf{1}_{\{U_{ij} > x_i\}}. \tag{5}$$

Then $\mathbf{N}_J^n(t)$ counts the number of birth events among the particles from levels $\{1, 2, \dots, n\}$ such that exactly those at levels in J are involved up to time t. Intuitively, at a jump time t_i, a uniform coin is tossed independently for each level. All the particles at levels j with $U_{ij} \le x_i$ participate in the lookdown event. More precisely, those particles involved jump to the location of the particle at the lowest level involved. The spatial locations of particles on the other levels, keeping their original order, are shifted upwards accordingly, i.e., if $t = t_i$ is the jump time and j is the lowest level involved, then

$$X_k(t) = \begin{cases} X_k(t-), \ \text{for } k \le j, \\ X_j(t-), \ \text{for } k > j \text{ with } U_{ik} \le x_i, \\ X_{k-J_t^k}(t-), \ \text{otherwise}, \end{cases}$$

where $J_{t_i}^k \equiv \#\{m < k, U_{im} \le x_i\} - 1$.

Between jump times of the Poisson processes, particles at different levels move independently according to Brownian motions in $\mathbb{R}^d$.

We assume that the above-mentioned lookdown construction is carried out in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For each $t > 0$, $X_1(t), X_2(t), \dots$ are known to be exchangeable random variables so that

$$X(t) \equiv \lim_{n\to\infty} X^{(n)}(t) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}$$

exists almost surely by de Finetti's theorem and follows the probability law of the Λ-Fleming-Viot process with underlying Brownian motion. Further, we have that $X^{(n)}$ converges to X in the path space $D_{M_1(\mathbb{R}^d)}([0,\infty))$ equipped with the Skorohod topology, where $M_1(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. See Theorem 3.2 of [16].

In the sequel we always write X for such a Λ-Fleming-Viot process. Write $\operatorname{supp}\mu$ for the closed support of measure μ.

Lemma 3.1. *For any $t \ge 0$, $\mathbb{P}$-a.s. the spatial locations of the countably many particles in the lookdown construction satisfy*

$$\{X_1(t), X_2(t), X_3(t), \dots\} \subseteq \mathit{supp}\, X(t).$$

Proof. In the lookdown construction, $(X_n(t))_{n\ge1}$ are exchangeable at any time $t \ge 0$. By de Finetti's theorem (cf. Aldous [1]) such a system is a mixture of i.i.d. sequence, i.e., given the empirical measure

$$X(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)},$$

the random variables $\{X_i(t), i = 1, 2, \dots\}$ are jointly distributed as i.i.d. samples from the directing measure $X(t)$. Therefore, $X_n(t) \in \operatorname{supp} X(t)$ for any $n \in [\infty]$. □

3.2. **The Λ-coalescent in the lookdown construction.** The birth events induce a family structure to the particle system so we can present the *genealogy process* first introduced in Donnelly and Kurtz [16]. For any $0 \le t \le s$ and $n \in [\infty]$, denote by $L_n^s(t)$ the ancestor's level at time t for the particle with level n at time s. Given s and

n, $L_n^s(t)$ is nondecreasing and left continuous in t. Moreover, the genealogy processes $(L_n^s)_{s\geq 0}, n = 1, 2, \ldots$ satisfy the equations

$$\begin{aligned} L_n^s(t) = n &- \sum_{1\leq i<j<n} \int_{t-}^{s} \mathbf{1}_{\{L_n^s(u)>j\}} d\mathbf{N}_{ij}(u) \\ &- \sum_{1\leq i<j\leq n} \int_{t-}^{s} (j-i)\, \mathbf{1}_{\{L_n^s(u)=j\}} d\mathbf{N}_{ij}(u) \\ &- \sum_{J\subset[n]} \int_{t-}^{s} (L_n^s(u) - \min J)\, \mathbf{1}_{\{L_n^s(u)\in J\}} d\mathbf{N}_J^n(u) \\ &- \sum_{J\subset[n]} \int_{t-}^{s} (|J\cap\{1,\ldots,L_n^s(u)\}| - 1) \times \mathbf{1}_{\{L_n^s(u)>\min J, L_n^s(u)\notin J\}} d\mathbf{N}_J^n(u)\,. \end{aligned}$$

Given $T > 0$, for any $0 \leq t \leq T$ and $i \in [\infty]$, $L_i^T(T-t)$ represents the ancestor's level at time $T - t$ of the particle with level i at time T and $X_{L_i^T(T-t)}((T-t)-)$ represents that ancestor's location.

Write $\left(\Pi^T(t)\right)_{0\leq t\leq T}$ for the $\mathcal{P}_\infty$-valued process such that i and j belong to the same block of $\Pi^T(t)$ if and only if $L_i^T(T-t) = L_j^T(T-t)$, i.e., i and j belong to the same block if and only if the two particles with levels i and j, respectively, at time T share a common ancestor at time $T-t$. The process $\left(\Pi^T(t)\right)_{0\leq t\leq T}$ turns out to have the same law as the Λ-coalescent running up to time T. See Donnelly and Kurtz [16] and Birkner and Blath [4].

The next property of the genealogy process can be found in Lemma 3.1 of [20].

Lemma 3.2. *For any fixed $T > 0$, let $\left(\Pi^T(t)\right)_{0\leq t\leq T}$ be the Λ-coalescent recovered from the lookdown construction. Then given $t \in [0, T]$ and the ordered random partition $\Pi^T(t) = \left\{\pi_l(t) : l = 1, \ldots, \#\Pi^T(t)\right\}$, we have*

$$L_j^T(T-t) = l \text{ for any } j \in \pi_l(t).$$

3.3. **Ancestry process.** For any $T > 0$, denote by

$$(X_{1,s}, X_{2,s}, X_{3,s}, \ldots)_{0\leq s\leq T}$$

the *ancestry process* with $X_{i,s}$ defined by

$$X_{i,s}(t) \equiv X_{L_i^s(t)}(t-) \text{ for } 0 \leq t \leq s. \tag{6}$$

Intuitively $X_{i,s}$ keeps track of locations for all the ancestors of the particle with level i at time s.

For any $s \geq 0$, we can recover the Λ-coalescent $(\Pi^s(t))_{0\leq t\leq s}$ from the lookdown construction. For any $0 \leq r < s$, set

$$N^{r,s} \equiv \#\Pi^s(s-r)$$

and

$$\Pi^s(s-r) \equiv \{\pi_l : 1 \leq l \leq N^{r,s}\},$$

where $\pi_l \equiv \pi_l(r,s),\ 1 \leq l \leq N^{r,s}$ are all the disjoint blocks of $\Pi^s(s-r)$ ordered by their least elements. Let $H(r,s)$ be the maximal dislocation between the countably many particles at time s and their respective ancestors at time r. Applying Lemma 3.2, we have

$$\begin{aligned} H(r,s) &\equiv \max_{1\le l\le N^{r,s}} \max_{j\in\pi_l} \left| X_j(s) - X_{L_j^s(r)}(r-) \right| \\ &= \max_{1\le l\le N^{r,s}} \max_{j\in\pi_l} \left| X_j(s) - X_l(r-) \right|. \end{aligned}$$

4. Some properties of the Λ-Fleming-Viot process

4.1. **Main results.** For any $T>0$, let $(\Pi^T(t))_{0\le t\le T}$ be the Λ-coalescent recovered from the lookdown construction with $\Pi^T(0)=\mathbf{0}_{[\infty]}$. Write $\Pi\equiv(\Pi(t))_{t\ge 0}$ for the unique (in law) Λ-coalescent such that $(\Pi(t))_{0\le t\le T}$ has the same distribution as $(\Pi^T(t))_{0\le t\le T}$. We call Π the Λ-coalescent associated to the Λ-Fleming-Viot process X.

For any positive integer m, set

$$T_m \equiv \inf\left\{t\ge 0 : \#\Pi(t)\le m\right\} \tag{7}$$

with the convention $\inf\emptyset=\infty$.

Given $\eta>0$, for any Borel set $A\subset\mathbb{R}^d$, let $\mathbb{B}(A,\eta)$ be its *closed η-neighborhood* such that

$$\mathbb{B}(A,\eta)\equiv\overline{\bigcup_{x\in A}\mathbb{B}(x,\eta)},$$

where $\mathbb{B}(x,\eta)$ denotes the closed ball centered at x with radius η.

We now recall the definition of *Hausdorff dimension.* Given $A\subset\mathbb{R}^d$ and $\beta>0$, $\eta>0$, let

$$\mathcal{H}^\beta_\eta(A)\equiv\inf_{\{S_l\}\in\varphi_\eta}\sum_l d(S_l)^\beta,$$

where $d(S_l)$ denotes the diameter of ball S_l in $\mathbb{R}^d$ and φ_η denotes the collection of η-*covers* of set A by balls with diameters at most η. The Hausdorff β-measure of A is defined by

$$\mathcal{H}^\beta(A)=\lim_{\eta\to 0}\mathcal{H}^\beta_\eta(A).$$

The Hausdorff dimension of A is defined by

$$\dim A\equiv\inf\left\{\beta>0:\mathcal{H}^\beta(A)=0\right\}=\sup\left\{\beta>0:\mathcal{H}^\beta(A)=\infty\right\}.$$

Recall that X is the Λ-Fleming-Viot process with underlying Brownian motion. For any subset $\mathcal{I}\subset\mathbb{R}\cap[0,\infty)$, let

$$\mathcal{R}(\mathcal{I})\equiv\overline{\cup_{t\in\mathcal{I}}\operatorname{supp}X(t)}$$

be the range of $\operatorname{supp}X$ on the time interval $\mathcal{I}$.

Throughout the paper, we always write C or C with subscript for a positive constant and write $C(x)$ for a constant depending on x whose values might vary from place to place. The main results of this paper are the following theorems. We defer the proofs to Section 5.

Assumption I: There exists a constant $\alpha>0$ such that the associated Λ-coalescent Π satisfies

$$\limsup_{m\to\infty} m^\alpha\mathbb{E}T_m<\infty.$$

Theorem 4.1. *Under Assumption I and for any $T > 0$, there exist a positive random variable $\theta \equiv \theta(T, d, \alpha) < 1$ and a constant $C \equiv C(d, \alpha)$ such that $\mathbb{P}$-a.s. for all $r, s \in [0, T]$ satisfying $0 < s - r \leq \theta$, we have*

$$H(r, s) \leq C\sqrt{(s-r)\log(1/(s-r))}. \tag{8}$$

Theorem 4.2. *Under Assumption I and given any fixed $t \geq 0$, there exist a positive random variable $\theta \equiv \theta(t, d, \alpha) < 1$ and a constant $C \equiv C(d, \alpha)$ such that for any Δt with $0 < \Delta t \leq \theta$ we have $\mathbb{P}$-a.s.*

$$supp\, X(t + \Delta t) \subseteq \mathbb{B}\left(supp\, X(t), C\sqrt{\Delta t \log(1/\Delta t)}\right). \tag{9}$$

Theorem 4.3. *Under Assumption I, $supp\, X(t)$ is compact for all $t > 0$ $\mathbb{P}$-a.s.. Further, if $supp\, X(0)$ is compact, then $\mathcal{R}([0, t))$ is compact for all $t > 0$ $\mathbb{P}$-a.s..*

Condition A: There exists a constant $\alpha > 0$ such that the associated Λ-coalescent Π satisfies

$$\limsup_{m\to\infty} m^{\alpha} \sum_{b=m+1}^{\infty} {\lambda_b}^{-1} < \infty.$$

Remark 4.4. *The Kingman's coalescent satisfies Condition A with $\alpha = 1$. In case of $\beta \in (1, 2)$, the Beta$(2-\beta, \beta)$-coalescent satisfies Condition A with $\alpha = \beta - 1$.*

Theorem 4.5. *Suppose that Condition A holds. Then*

$$dim\, supp\, X(t) \leq 2/\alpha$$

for all $t > 0$ $\mathbb{P}$-a.s..

Theorem 4.6. *Suppose that Condition A holds. Then for any $0 < \delta < T$,*

$$dim\, \mathcal{R}([\delta, T)) \leq 2 + 2/\alpha \quad \mathbb{P}\text{-a.s..}$$

4.2. **A sufficient condition.** Recall the Markov chain introduced in [20]. For any n, $(\Pi_n(t))_{t\geq 0}$ is the Λ-coalescent Π restricted to $[n]$ with $\Pi_n(0) = \mathbf{0}_{[n]}$. For any $n > m$, the block counting process $(\#\Pi_n(t) \vee m)_{t\geq 0}$ is a Markov chain with initial value n and absorbing state m. For any $n \geq b > m$, let $(\mu_{b,k})_{m\leq k\leq b-1}$ be its transition rates such that

$$\begin{cases} \mu_{b,b-1} = \binom{b}{2}\lambda_{b,2}, \\ \mu_{b,b-2} = \binom{b}{3}\lambda_{b,3}, \\ \cdots\cdots \\ \mu_{b,m+1} = \binom{b}{b-m}\lambda_{b,b-m}, \\ \mu_{b,m} = \sum_{k=b-m+1}^{b}\binom{b}{k}\lambda_{b,k}. \end{cases} \tag{10}$$

The total transition rate is

$$\mu_b = \sum_{k=m}^{b-1}\mu_{b,k} = \sum_{k=2}^{b}\binom{b}{k}\lambda_{b,k} = \lambda_b.$$

For $b > m$, let $\gamma_{b,m}$ be the total rate at which the block counting Markov chain starting at b is decreasing, i.e.,

$$\gamma_{b,m} = \begin{cases} \sum_{k=2}^{b-m}(k-1)\binom{b}{k}\lambda_{b,k} + \sum_{k=b-m+1}^{b}(b-m)\binom{b}{k}\lambda_{b,k}, & \text{if } b \geq m+2, \\ \sum_{k=2}^{b}\binom{b}{k}\lambda_{b,k}, & \text{if } b = m+1. \end{cases} \tag{11}$$

Condition B: There exists a constant $\alpha > 0$ such that

$$\limsup_{m\to\infty} m^{\alpha} \sum_{b=m+1}^{\infty} {\gamma_{b,m}}^{-1} < \infty.$$

Remark 4.7. *It follows from the proof of Lemma 4.4 in* [20] *that*

$$\mathbb{E}T_m \leq \sum_{b=m+1}^{\infty} {\gamma_{b,m}}^{-1}.$$

Recalling the definitions of $\gamma_{b,m}$ *by (11) and* λ_b *by (3), we have* $\lambda_b \leq \gamma_{b,m}$ *for any* $b > m$*. Then for any* $\alpha > 0$*, we have*

$$m^{\alpha}\mathbb{E}T_m \leq m^{\alpha} \sum_{b=m+1}^{\infty} {\gamma_{b,m}}^{-1} \leq m^{\alpha} \sum_{b=m+1}^{\infty} {\lambda_b}^{-1}.$$

Therefore, Condition A implies Condition B which is sufficient for Assumption I.

Condition A is not a strong requirement since for the Beta coalescents Condition A is sufficient and necessary for coming down from infinity.

The speed of coming down from infinity for Λ*-coalescent is discussed in Berestycki et al.* [2]*. It is shown that there exists a deterministic function* $\nu : (0,\infty)\to(0,\infty)$ *such that* $\#\Pi(t)/\nu(t)\to 1$ *as* $t\to 0$ *both almost surely and in* L^p *for* $p \geq 1$*. For our purpose, it is possible to replace Assumption I with an assumption on the behavior of* $\nu(t)$ *for* t *close to* 0*.*

4.3. **Some Corollaries and Propositions.** For $t > 0$, let

$$r(t) \equiv \inf\left\{R \geq 0 : \operatorname{supp} X\left(t\right) \subseteq \mathbb{B}\left(0,R\right)\right\}.$$

The next result is similar to Theorem 2.1 of Tribe [27] on the support process of superBrownian motion; also see Theorem 9.3.2.3 of Dawson [8]. It follows immediately from Theorem 4.2.

Corollary 4.8. *Under Assumption I, there exists a constant* $C > 0$ *such that*

$$\mathbb{P}_{\delta_0}\left(\limsup_{t\downarrow 0} \frac{\sup_{0\leq u\leq t} r(u)}{\sqrt{t\log\left(1/t\right)}} \leq C\right) = 1,$$

where $\mathbb{P}_{\delta_0}$ *denotes the law of* X *with* $X\left(0\right) = \delta_0$*.*

Corollary 4.9. *Suppose that Condition A holds. For any* $T > 0$*, we have*

$$\mathbb{P}_{\delta_0}\left(\mathit{dim}\,\mathcal{R}\left([0,T)\right) \leq 2 + 2/\alpha\right) = 1.$$

We defer the proof of Corollary 4.9 to Section 5.

The next result follows from the proof of Theorem 4.5 and a standard result of Hausdorff measure; see Lemma 6.3 of Falconer [19].

Proposition 4.10. *Suppose that Condition A holds. Then* $\mathbb{P}$*-a.s. for all* $t > 0$ *and* $\epsilon > 0$ *we have*

$$\limsup_{r\to 0+} \frac{X(t)(\mathbb{B}(x,r))}{r^{2/\alpha+\epsilon}} > 0$$

for $X(t)$ *almost all* x*.*

For any $0 < t < 1$, let

$$h(t) \equiv \sqrt{t \log (1/t)}. \tag{12}$$

Proposition 4.11. *Let X be any Λ-Fleming-Viot process with $\Lambda(\{0\}) > 0$ and underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. Then given any fixed $t \geq 0$, with probability one the process $supp\, X(t)$ has the one-sided modulus of continuity with respect to Ch, where $C \equiv C(d)$ is the constant determined in Theorem 4.2. Further, with probability one $supp\, X(t)$ is compact for all $t > 0$ and if $supp\, X(0)$ is compact, then $\mathcal{R}([0,t))$ is also compact for all $t > 0$. In addition, with probability one*

$$dim\, supp\, X(t) \leq 2$$

for all $t > 0$. Finally, given any $0 < \delta < T$, with probability one

$$dim\, \mathcal{R}([\delta, T)) \leq 4.$$

Proof. Since $\Lambda(\{0\}) > 0$, the Λ-coalescent has a nontrivial Kingman component. Then

$$\lambda_b \geq \frac{1}{2}\Lambda(\{0\})b(b-1)$$

and

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \sum_{b=m+1}^{\infty} \frac{2}{\Lambda(\{0\})b(b-1)} = \frac{2}{\Lambda(\{0\})m},$$

i.e., Condition A holds with $\alpha = 1$. Therefore, the results follow from Remark 4.7 and Theorems 4.2-4.6. □

Remark 4.12. *The uniform upper bound on the Hausdorff dimension of classical Fleming-Viot support process was first proved by Reimers* [23]*, where a non-standard construction of the classical Fleming-Viot process is used to establish this result.*

Recall the (c, ϵ, γ)*-property* introduced in [20]. We say that a Λ-coalescent has the (c, ϵ, γ)-property, if there exist constants $c > 0$ and ϵ, $\gamma \in (0,1)$ such that the measure Λ restricted to $[0, \epsilon]$ is absolutely continuous with respect to Lebesgue measure and

$$\Lambda(dx) \geq cx^{-\gamma}dx \text{ for all } x \in [0, \epsilon].$$

The Λ-coalescents with the (c, ϵ, γ)-property come down from infinity.

Proposition 4.13. *Let X be any Λ-Fleming-Viot process with underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. If the associated Λ-coalescent has the (c, ϵ, γ)-property, then given any fixed $t \geq 0$, with probability one the process $supp\, X(t)$ has the one-sided modulus of continuity with respect to Ch, where $C \equiv C(d, \gamma)$ is the constant determined in Theorem 4.2. Further, with probability one $supp\, X(t)$ is compact for all $t > 0$ and if $supp\, X(0)$ is compact, then $\mathcal{R}([0,t))$ is also compact for all $t > 0$. In addition, with probability one*

$$dim\, supp\, X(t) \leq 2/\gamma$$

for all $t > 0$. Finally, given any $0 < \delta < T$, with probability one

$$dim\, \mathcal{R}([\delta, T)) \leq 2 + 2/\gamma.$$

Proof. It has been proved by Lemma 4.13 of [20] that for any $n \geq 2$, there exists a positive constant $C(c, \epsilon, \gamma)$ such that the total coalescence rate of the Λ-coalescent with the (c, ϵ, γ)-property satisfies

$$\lambda_n \geq C(c, \epsilon, \gamma) n^{1+\gamma}.$$

Then

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \frac{1}{C(c, \epsilon, \gamma)} \int_m^{\infty} \frac{1}{x^{1+\gamma}} dx \leq \frac{1}{\gamma C(c, \epsilon, \gamma) m^{\gamma}},$$

i.e., Condition A holds with $\alpha = \gamma$. Consequently, the results follow from Remark 4.7 and Theorems 4.2-4.6. □

Now we discuss the support properties for *Beta*$(2-\beta, \beta)$*-Fleming-Viot process* with underlying Brownian motion. It is known that the Beta$(2-\beta, \beta)$-coalescent stays infinite if $\beta \in (0, 1]$ and comes down from infinity if $\beta \in (1, 2)$. For $\beta \in (1, 2)$, given any $\epsilon \in (0, 1)$, the Beta$(2-\beta, \beta)$-coalescent has the $(c, \epsilon, \beta - 1)$-property. Therefore, the conclusions of Proposition 4.13 hold with $\gamma = \beta - 1$.

For $t \geq 0$ put

$$S_t \equiv \cap_{n=1}^{\infty} \mathcal{R}([t, t + 1/n)).$$

Proposition 4.14. *Under Assumption I and for any $T > 0$, there exist a positive random variable $\theta \equiv \theta(T, d, \alpha) < 1$ and a constant $C \equiv C(d, \alpha)$ such that $\mathbb{P}$-a.s.*

$$suppX(t + \Delta t) \subseteq \mathbb{B}(S_t, Ch(\Delta t))$$

for all $0 \leq t < t + \Delta t \leq T$ and $0 < \Delta t \leq \theta$.

We also defer the proof of Proposition 4.14 to Section 5.

5. PROOFS OF THEOREMS 4.1-4.6, COROLLARY 4.9 AND PROPOSITION 4.14

5.1. **Modulus of continuity for the ancestry process.** In this subsection we first obtain some estimates on the Λ-coalescent and on the maximal dislocation of the particles from their respective ancestors.

Denote by $\lfloor x \rfloor$ the integer part of x for any $x \in \mathbb{R}$. Given $T > 0$ and $\Delta > 0$, we can divide the interval $[0, T]$ into subintervals as follows:

$$[0, \Delta], [\Delta, 2\Delta], \ldots, [\lfloor T/\Delta - 1 \rfloor \Delta, \lfloor T/\Delta \rfloor \Delta], [\lfloor T/\Delta \rfloor \Delta, T].$$

Set $\Delta \equiv \Delta_n = 2^{-n}$. Let S_n^T be the collection of the endpoints of the first $\lfloor 2^n T \rfloor$ subintervals, i.e.,

$$S_n^T \equiv \left\{ k2^{-n} : 0 \leq k \leq 2^n T \right\}.$$

Put

$$S^T \equiv \bigcup_{n \geq 1} S_n^T = \bigcup_{n \geq 1} \left\{ k2^{-n} : 0 \leq k \leq 2^n T \right\}.$$

Clearly, given any $T > 0$, S^T is the collection of all the dyadic rationals in $[0, T]$. So S^T is a dense subset of $[0, T]$.

For any $n \in [\infty]$, let $\{\mathbb{A}_{n,k} : 1 \leq k \leq 2^n T\}$ be the collection of the first $\lfloor 2^n T \rfloor$ subintervals in the partition so that

$$\mathbb{A}_{n,k} \equiv \left[(k-1)2^{-n}, k2^{-n} \right].$$

For simplicity, we denote

$$N_{n,k} \equiv N^{(k-1)2^{-n},k2^{-n}}.$$

Also denote by $H_{n,k}$ the maximal dislocation over interval $\mathbb{A}_{n,k}$ of all the Brownian motions followed by the countably many particles alive at time $k2^{-n}$ and their respective ancestors at time $(k-1)\,2^{-n}$, i.e.,

$$H_{n,k} \equiv H\left((k-1)\,2^{-n}, k2^{-n}\right).$$

For any positive integer m, let

$$T_m^{n,k} \equiv \inf\left\{t \in [0, 2^{-n}] : \#\Pi^{k2^{-n}}(t) \le m\right\}$$

with the convention $\inf\emptyset = 2^{-n}$. Notice that for any fixed $n \in [\infty]$ and m, the random times $\{T_m^{n,k} : 1 \le k \le 2^n T\}$ follow the same distribution. Write $T_x^{n,k} \equiv T_{\lfloor x \rfloor}^{n,k}$ for any $x > 0$.

We need a standard estimate on Brownian motion.

Lemma 5.1. *Given any $x > 0$ and d-dimensional standard Brownian motion $(\mathbf{B}(s))_{s\ge 0}$, we have*

$$\mathbb{P}\left(\sup_{0\le s\le t} |\mathbf{B}(s)| > x\right) \le \sqrt{\frac{8d^3 t}{\pi}}\frac{1}{x}\exp\left(-\frac{x^2}{2dt}\right).$$

Lemma 5.2. *Under Assumption I and for any $T > 0$, there exists a positive constant $C_4(d,\alpha)$ such that $\mathbb{P}$-a.s.*

$$\max_{1\le k\le 2^n T} H_{n,k} \le C_4(d,\alpha) h\left(2^{-n}\right)$$

for n large enough, where h is defined by (12).

Proof. Given any n and $1 \le k \le 2^n T$, we first divide each interval $\mathbb{A}_{n,k}$ into countably many subintervals as follows:

$$J_0^{n,k} \equiv \left[(k-1)2^{-n}, k2^{-n} - T_{8^{(n+1)/\alpha}}^{n,k}\right]$$

and

$$J_l^{n,k} \equiv \left[k2^{-n} - T_{8^{(n+l)/\alpha}}^{n,k}, k2^{-n} - T_{8^{(n+l+1)/\alpha}}^{n,k}\right]$$

for $l = 1, 2, 3, \ldots$. Consequently, the lengths of these countably many subintervals satisfy that

$$\left|J_0^{n,k}\right| \le 2^{-n} \text{ and } \left|J_l^{n,k}\right| \le T_{8^{(n+l)/\alpha}}^{n,k} = T_{2^{(3n+3l)/\alpha}}^{n,k} \text{ for } l = 1, 2, 3, \ldots.$$

The right endpoints of these subintervals $\left(b_l^{n,k}\right)_{l\ge 1} \equiv \left(k2^{-n} - T_{2^{(3n+3l)/\alpha}}^{n,k}\right)_{l\ge 1}$ consist of a sequence of random times converging increasingly to $k2^{-n}$. Set $b_0^{n,k} \equiv (k-1)\,2^{-n}$ for convenience.

For $l = 0, 1, 2, \ldots$, let $D_l^{n,k}$ be the maximal dislocation of the ancestors (for those countably many particles alive at time $k2^{-n}$) at time $b_{l+1}^{n,k}$ from their respective ancestors at time $b_l^{n,k}$, i.e.,

$$D_l^{n,k} \equiv \max_{1\le i\le N^{b_l^{n,k},k2^{-n}}} \max_{j\in\pi_i} \left|X_{L_j^{k2^{-n}}\left(b_{l+1}^{n,k}\right)}\left(b_{l+1}^{n,k}-\right) - X_i\left(b_l^{n,k}-\right)\right|, \tag{13}$$

where $\left\{\pi_i : 1 \leq i \leq N^{b_l^{n,k}, k2^{-n}}\right\}$ denotes the collection of all the disjoint blocks of partition $\Pi^{k2^{-n}}\left(k2^{-n} - b_l^{n,k}\right)$ ordered by their least elements.

In the case of $b_{l+1}^{n,k} = b_l^{n,k}$, i.e., $\left|J_l^{n,k}\right| = 0$, which corresponds to the situation of either $T_{2^{(3n+3l+3)/\alpha}}^{n,k} = 2^{-n}$ or $T_{2^{(3n+3l+3)/\alpha}}^{n,k} = T_{2^{(3n+3l)/\alpha}}^{n,k}$, it follows from Lemma 3.2 that

$$L_j^{k2^{-n}}\left(b_{l+1}^{n,k}\right) = L_j^{k2^{-n}}\left(b_l^{n,k}\right) = i$$

for any $j \in \pi_i$ with $1 \leq i \leq N^{b_l^{n,k}, k2^{-n}}$. Hence we have $D_l^{n,k} = 0$ in (13).

By the lookdown construction and the coming down from infinity property, there exists a finite number of ancestors at each time $b_l^{n,k}$, $l = 0, 1, 2, \ldots$ for those countably many particles alive at time $k2^{-n}$, i.e.,

$$\#\left\{L_j^{k2^{-n}}\left(b_l^{n,k}\right) : j \in [\infty]\right\} < \infty.$$

So both maximums in (13) are in fact taken over finite sets. Put

$$D^{n,k} \equiv \sum_{l=0}^{\infty} D_l^{n,k}.$$

For dimension d and constant α in Assumption I, let $C_1(d, \alpha)$ be a positive constant satisfying

$$C_1(d, \alpha) > \sqrt{2d(3/\alpha + 1)}.$$

Now we estimate the total maximal dislocation $D^{n,k}$ as follows. Let

$$I_n \equiv \mathbb{P}\left(\max_{1 \leq k \leq 2^n T} D^{n,k} > \sum_{l=0}^{\infty} C_1(d, \alpha) h\left(2^{-(n+2l)}\right)\right).$$

Since $D^{n,k} = \sum_{l=0}^{\infty} D_l^{n,k}$, we have

$$\left\{D^{n,k} > \sum_{l=0}^{\infty} C_1(d, \alpha) h\left(2^{-(n+2l)}\right)\right\} \subseteq \bigcup_{l=0}^{\infty}\left\{D_l^{n,k} > C_1(d, \alpha) h\left(2^{-(n+2l)}\right)\right\}.$$

Therefore,

$$I_n \leq \sum_{k=1}^{2^n T} \sum_{l=0}^{\infty} \mathbb{P}\left(D_l^{n,k} > C_1(d, \alpha) h\left(2^{-(n+2l)}\right)\right).$$

Under Assumption I, there exists a positive constant C such that for $\mathbf{N}$ large enough and for all $n > \mathbf{N}$, $\mathbb{E}T_{8^{n/\alpha}} \leq C 8^{-n}$. For all those $n > \mathbf{N}$, since $D_l^{n,k} = 0$ for those l with interval length $\left|J_l^{n,k}\right| = 0$, we only need to consider the case of $\left|J_l^{n,k}\right| > 0$.

Observe that for $l = 0, 1, 2, \ldots$, the total number of Brownian motion paths connecting the ancestors (of the countably many particles alive at $k2^{-n}$) at time $b_{l+1}^{n,k}$ to their respective ancestors at earlier time $b_l^{n,k}$ is at most $8^{(n+l+1)/\alpha}$. Since $|J_0^{n,k}| = b_1^{n,k} - b_0^{n,k} \leq 2^{-n}$, we have

$$\mathbb{P}\left(D_0^{n,k} > C_1(d, \alpha) h\left(2^{-n}\right)\right) \leq 8^{\frac{n+1}{\alpha}} \mathbb{P}\left(\sup_{0 \leq s \leq 2^{-n}} |\mathbf{B}(s)| > C_1(d, \alpha) h\left(2^{-n}\right)\right).$$

For $l = 1, 2, \ldots$, we have

$$\begin{aligned}
&\mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right)\\
\leq&\mathbb{P}\left(\left|J_l^{n,k}\right| > 2^{-(n+2l)}\right) + \mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right), 0 < \left|J_l^{n,k}\right| \leq 2^{-(n+2l)}\right).
\end{aligned}$$

Since $|J_l^{n,k}| \leq T^{n,k}_{2^{(3n+3l)/\alpha}}$, for any $n > \mathbf{N}$ the length of interval $J_l^{n,k}$ satisfies

$$\begin{aligned}
\mathbb{P}\left(\left|J_l^{n,k}\right| > 2^{-(n+2l)}\right) \leq&\mathbb{P}\left(T^{n,k}_{2^{(3n+3l)/\alpha}} > 2^{-(n+2l)}\right)\\
\leq&2^{n+2l}\mathbb{E}T^{n,k}_{2^{(3n+3l)/\alpha}} \leq C2^{-(2n+l)}.
\end{aligned}$$

We further have

$$\begin{aligned}
&\mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right)\\
\leq&C2^{-(2n+l)} + 8^{\frac{n+l+1}{\alpha}}\mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_n \leq&2^nT8^{\frac{n+1}{\alpha}}\mathbb{P}\left(\sup_{0\leq s\leq 2^{-n}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-n}\right)\right)\\
&+ 2^nT\sum_{l=1}^{\infty}\left(C2^{-(2n+l)} + 8^{\frac{n+l+1}{\alpha}}\mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right)\right)\\
=&\sum_{l=1}^{\infty} CT2^{-(n+l)} + 2^nT\sum_{l=0}^{\infty} 8^{\frac{n+l+1}{\alpha}}\mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right).
\end{aligned}$$

It follows from Lemma 5.1 that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right)\\
\leq&\frac{1}{C_1(d,\alpha)}\sqrt{\frac{8d^3}{\pi(n+2l)\log 2}}\exp\left(-\frac{C_1^2(d,\alpha)(n+2l)\log 2}{2d}\right)\\
\leq&\frac{1}{C_1(d,\alpha)}\sqrt{\frac{8d^3}{\pi\log 2}}2^{-\frac{C_1^2(d,\alpha)(n+2l)}{2d}}\\
\equiv&C_2(d,\alpha)2^{-\frac{C_1^2(d,\alpha)(n+2l)}{2d}}.
\end{aligned}$$

Therefore, for any $n > \mathbf{N}$ we have

$$\begin{aligned}
I_n &\leq CT2^{-n} + 2^nT\sum_{l=0}^{\infty} 8^{\frac{n+l+1}{\alpha}}C_2(d,\alpha)2^{-\frac{C_1^2(d,\alpha)(n+2l)}{2d}}\\
&\leq CT2^{-n} + \sum_{l=0}^{\infty} TC_2(d,\alpha)2^{-\left(\frac{C_1^2(d,\alpha)}{2d}-\frac{3}{\alpha}-1\right)n-\left(\frac{C_1^2(d,\alpha)}{d}-\frac{3}{\alpha}\right)l+\frac{3}{\alpha}}.
\end{aligned}$$

Since $C_1(d,\alpha) > \sqrt{2d(3/\alpha+1)}$, it follows that

$$I_n \leq CT2^{-n} + TC_3(d,\alpha)2^{-\left(\frac{C_1^2(d,\alpha)}{2d}-\frac{3}{\alpha}-1\right)n}, \tag{14}$$

where

$$C_3(d,\alpha) \equiv \sum_{l=0}^{\infty} C_2(d,\alpha)2^{-\left(\frac{C_1^2(d,\alpha)}{d}-\frac{3}{\alpha}\right)l+\frac{3}{\alpha}}.$$

Both terms on the right hand side of (14) are summable with respect to n. Thus, $\sum_n I_n < \infty$, and it follows from the Borel-Cantelli lemma that $\mathbb{P}$-a.s.

$$\begin{aligned}\max_{1\leq k\leq 2^n T} D^{n,k} \leq& \sum_{l=0}^{\infty} C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\\ \leq& C_1(d,\alpha)\sqrt{2^{-n} n \log 2}\left(1+\sum_{l=1}^{\infty}\sqrt{2^{-2l+1} l}\right)\\ \equiv& C_4(d,\alpha)\sqrt{2^{-n} n\log 2}\end{aligned}$$

for n large enough.

By the lookdown construction and the arguments in Lemmas 4.6-4.7 of [20] we have $H_{n,k} \leq D^{n,k}$. Thus, $\mathbb{P}$-a.s.

$$\max_{1\leq k\leq 2^n T} H_{n,k} \leq \max_{1\leq k\leq 2^n T} D^{n,k} \leq C_4(d,\alpha) h\left(2^{-n}\right)$$

for n large enough. □

Lemma 5.3 follows from the lookdown construction.

Lemma 5.3. *For any r,t,s with $0\leq r\leq t\leq s$ we have*

$$H(r,s) \leq H(r,t) + H(t,s)$$

with the convention $H(r,r) = H(s,s) \equiv 0$.

We are ready to prove the one-sided modulus of continuity for the ancestry process.

Proof of Theorem 4.1. We first show that $\mathbb{P}$-a.s. for all $r,s \in S^T$ satisfying $0 < s-r \leq \theta$,

$$H(r,s) \leq Ch(s-r).$$

The following argument is similar to that in Section III.1 of Perkins [21].

By Lemma 5.2, given $T>0$, there exist an event $\Omega_{T,d,\alpha}$ of probability one, and an integer-valued random variable $\mathbf{N}(T,d,\alpha)$ big enough such that $2^{-\mathbf{N}(T,d,\alpha)} \leq e^{-1}$ and

$$\max_{1\leq k\leq 2^n T} H_{n,k} \leq C_4(d,\alpha)h\left(2^{-n}\right), \quad n > \mathbf{N}(\omega,T,d,\alpha),\ \omega\in\Omega_{T,d,\alpha}. \tag{15}$$

Let $\theta \equiv \theta(\omega,T,d,\alpha) = 2^{-\mathbf{N}(\omega,T,d,\alpha)}$. For any $r,s\in S^T$ with $0 < s-r \leq 2^{-\mathbf{N}(\omega,T,d,\alpha)} = \theta$, there exists an $n \geq \mathbf{N}(\omega,T,d,\alpha)$ such that $2^{-(n+1)} < s-r \leq 2^{-n}$. Recall that

$$S_k^T = \left\{l2^{-k} : 0 \leq l \leq 2^k T\right\} \text{ and } \overline{S^T} = \overline{\cup_{k\geq 1} S_k^T} = [0,T].$$

For any $k>n$, choose $s_k \in S_k^T$ such that $s_k \leq s$ and s_k is the largest such value. Then

$$s_k \uparrow s, \quad s_{k+1} = s_k + j_{k+1}2^{-(k+1)} \text{ with } j_{k+1}\in\{0,1\}.$$

Since $s \in S^T$, then $(s_k)_{k>n}$ is a sequence with at most finite terms that are not equal to s. Applying (15), we have

$$H(s_k, s_{k+1}) \le C_4(d,\alpha) j_{k+1} h\left(2^{-(k+1)}\right). \tag{16}$$

By Lemma 5.3,

$$\begin{aligned} H(s_{n+1}, s) &\le \sum_{k=n+1}^{\infty} H(s_k, s_{k+1}) \\ &\le \sum_{k=n+1}^{\infty} C_4(d,\alpha) j_{k+1} h\left(2^{-(k+1)}\right) \\ &\le C_4(d,\alpha) \sum_{k=n+1}^{\infty} \sqrt{2^{-(k+1)}(k+1)\log 2} \\ &\le C_4(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2} \sum_{k=1}^{\infty} \sqrt{2^{-k+1}k} \\ &\equiv C_5(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2}, \end{aligned} \tag{17}$$

where observe that only finitely many terms are nonzero in the summation on the right hand side of the first inequality.

Similarly, for any $k > n$, choose $r_k \in S_k^T$ such that $r_k \ge r$ and r_k is the smallest such value. Then

$$r_k \downarrow r, \quad r_{k+1} = r_k - j'_{k+1} 2^{-(k+1)} \text{ with } j'_{k+1} \in \{0,1\}.$$

Applying (15), we have

$$H(r_{k+1}, r_k) \le C_4(d,\alpha) j'_{k+1} h\left(2^{-(k+1)}\right).$$

Similar to (17), by Lemma 5.3 we have

$$\begin{aligned} H(r, r_{n+1}) &\le \sum_{k=n+1}^{\infty} H(r_{k+1}, r_k) \\ &\le \sum_{k=n+1}^{\infty} C_4(d,\alpha) j'_{k+1} h\left(2^{-(k+1)}\right) \\ &\le C_5(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2}. \end{aligned} \tag{18}$$

Since $2^{-(n+1)} < s - r \le 2^{-n}$, we have $0 \le s_{n+1} - r_{n+1} \le i_{n+1} 2^{-(n+1)}$ with $i_{n+1} \in \{0,1,2\}$. It comes from (16) and Lemma 5.3 that

$$\begin{aligned} H(r_{n+1}, s_{n+1}) &\le 2C_4(d,\alpha) h\left(2^{-(n+1)}\right) \\ &= 2C_4(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2}. \end{aligned} \tag{19}$$

Combining (17), (18) and (19), we have $\mathbb{P}$-a.s. for all $r, s \in S^T$ with $0 < s - r \leq \theta$

$$
\begin{aligned}
H(r,s) \leq & H(r, r_{n+1}) + H(r_{n+1}, s_{n+1}) + H(s_{n+1}, s) \\
\leq & 2C_4(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2} + 2C_5(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2} \\
\leq & C(d,\alpha)\sqrt{2^{-(n+1)}(n+1)\log 2},
\end{aligned}
$$

where $C(d, \alpha) \equiv 2C_4(d, \alpha) + 2C_5(d, \alpha)$.

Function h is increasing on $(0, e^{-1}]$. Since

$$2^{-(n+1)} < s - r \leq \theta \leq e^{-1},$$

we have

$$H(r,s) \leq C(d,\alpha) h\left(2^{-(n+1)}\right) \leq C(d,\alpha) h(s-r) \tag{20}$$

for all $r, s \in S^T$ satisfying $0 < s - r \leq \theta$.

Finally, for any $0 < r < s < T$ with $s - r < \theta/2$, find sequences $(r_m) \subseteq S^T$ and $(s_n) \subseteq S^T$ with $r_m \uparrow r$ and $s_n \downarrow s$. By the lookdown construction, for any $j \in [\infty]$,

$$
\begin{aligned}
& |X_j(s) - X_{L_j^s(r)}(r-)| \\
\leq & |X_j(s) - X_j(s_n)| + |X_j(s_n) - X_{L_j^{s_n}(r_m)}(r_m-)| \\
& + |X_{L_j^{s_n}(r_m)}(r_m-) - X_{L_j^{s_n}(r)}(r-)| + |X_{L_j^{s_n}(r)}(r-) - X_{L_j^s(r)}(r-)|.
\end{aligned}
\tag{21}
$$

Let both n and m be big enough such that $0 < s_n - r_m \leq \theta$. It follows from (20) that the second term on the right hand side of (21) is bounded from above by $C(d, \alpha)\, h(s_n - r_m)$. First fix n and let $m \to \infty$. The third term tends to 0 because $X_{L_j^{s_n}(\cdot)}(\cdot-)$ is continuous for any $j \in [\infty]$. Then letting $n \to \infty$, the first term tends to 0 because $X_j(\cdot)$ is right continuous for any $j \in [\infty]$. The last term is equal to 0 for large n since s_n is then so close to s that there is no lookdown event involving levels $\{1, 2, \ldots, j\}$ during time interval $(s, s_n]$. Consequently,

$$
\begin{aligned}
& |X_j(s) - X_{L_j^s(r)}(r-)| \\
\leq & \lim_{n\to\infty} |X_j(s) - X_j(s_n)| + \lim_{n\to\infty}\lim_{m\to\infty} C(d,\alpha)\, h(s_n - r_m) \\
& + \lim_{n\to\infty}\lim_{m\to\infty} |X_{L_j^{s_n}(r_m)}(r_m-) - X_{L_j^{s_n}(r)}(r-)| + \lim_{n\to\infty} |X_{L_j^{s_n}(r)}(r-) - X_{L_j^s(r)}(r-)| \\
= & C(d,\alpha)\, h(s-r).
\end{aligned}
$$

Then (8) follows. □

Remark 5.4. *It follows from estimate (14) that there exist positive constants $C_6 \equiv C_6(T, d, \alpha)$ and $C_7 \equiv C_7(d, \alpha)$ such that for $\epsilon > 0$ small enough*

$$\mathbb{P}(\theta \leq \epsilon) \leq C_6 \epsilon^{C_7}.$$

5.2. **Modulus of continuity for the Λ-Fleming-Viot support process and uniform compactness for the support and range.** We will need the following observation on weak convergence.

Lemma 5.5. *If $\left\{(\nu_n)_{n\geq 1}, \nu\right\} \subseteq M_1\left(\mathbb{R}^d\right)$ and ν_n weakly converges to ν, then we have*

$$supp\, \nu \subseteq \cap_{m\geq 1} \overline{\cup_{n\geq m} supp\, \nu_n}.$$

Proof. Suppose that there exists an $x \in \mathbb{R}^d$ such that

$$x \in \operatorname{supp} \nu \cap \overline{\cup_{n \geq m} \operatorname{supp} \nu_n}^{\,\mathrm{c}}$$

for some m. Since $\overline{\cup_{n \geq m} \operatorname{supp} \nu_n}^{\,\mathrm{c}}$ is an open set, there exists a positive value δ such that $\{y : |y - x| < \delta\} \subseteq \overline{\cup_{n \geq m} \operatorname{supp} \nu_n}^{\,\mathrm{c}}$. We can define a nonnegative and continuous function g satisfying $g > 0$ on $\{y : |y - x| < \delta/2\}$ and $g = 0$ on $\{y : |y - x| \geq \delta\}$. Then $\langle \nu_n, g \rangle = 0$ for any $n \geq m$ but $\langle \nu, g \rangle > 0$. Consequently, $\langle \nu_n, g \rangle \not\to \langle \nu, g \rangle$, which contradicts the fact that ν_n weakly converges to ν. □

Proof of Theorem 4.2. Applying Theorem 4.1, there exist a positive random variable $\theta \equiv \theta(T, d, \alpha)$ and a constant $C \equiv C(d, \alpha)$ such that given any fixed $t \in [0, T)$, $\mathbb{P}$-a.s. for all $r \in S^T \cap (t, t + \theta]$, we have

$$H(t, r) \leq Ch(r - t),$$

which gives the upper bound for the maximal dislocation between the countably many particles at time r and their corresponding ancestors at time t. By Lemma 3.2, the ancestors at time t are exactly $\{X_1(t-), X_2(t-), \ldots, X_{N^{t,r}}(t-)\}$, so we have $\mathbb{P}$ a.s.

$$\{X_1(r), X_2(r), \ldots\} \subseteq \bigcup_{1 \leq i \leq N^{t,r}} \mathbb{B}(X_i(t-), Ch(r - t)).$$

For the given $t \in [0, T)$, $\mathbb{P}$ a.s.

$$X_i(t) = X_i(t-) \text{ for any } i \in [\infty],$$

where $X_i(0-) \equiv X_i(0)$, so for any $r \in S^T \cap (t, t + \theta]$, we have $\mathbb{P}$ a.s.

$$\{X_1(r), X_2(r), \ldots\} \subseteq \bigcup_{1 \leq i \leq N^{t,r}} \mathbb{B}(X_i(t), Ch(r - t)). \tag{22}$$

Apply Lemma 3.1, for the given $t \in [0, T)$, $\mathbb{P}$ a.s.

$$\{X_1(t), X_2(t), \ldots, X_{N^{t,r}}(t)\} \subseteq \operatorname{supp} X(t).$$

It follows from (22) that

$$\{X_1(r), X_2(r), \ldots\} \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(r - t)).$$

For all $r \in S^T \cap (t, t + \theta]$, we have $\mathbb{P}$-a.s.

$$X^{(n)}(r) \equiv \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(r)} \to X(r).$$

Clearly,

$$\operatorname{supp} X^{(n)}(r) \subseteq \{X_1(r), X_2(r), \ldots\} \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(r - t))$$

for all n, which implies

$$\operatorname{supp} X(r) \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(r - t)). \tag{23}$$

Then for any s satisfying $t < s \leq (t + \theta/2) \wedge T$, we can choose a sequence $(s_l)_{l \geq 1} \subseteq S^T \cap (t, t + \theta]$ such that $s_l \downarrow s$. It follows from the right continuity of X and Lemma 5.5 that

$$\operatorname{supp} X(s) \subseteq \bigcap_{m \geq 1} \overline{\bigcup_{l \geq m} \operatorname{supp} X(s_l)}.$$

By (23), we have

$$\operatorname{supp} X(s_l) \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(s_l - t))$$

for all l. Consequently, for any $t < s \le (t + \theta/2) \wedge T$,

$$\begin{aligned}\operatorname{supp} X(s) &\subseteq \bigcap_{m\ge 1} \overline{\bigcup_{l\ge m} \mathbb{B}(\operatorname{supp} X(t), Ch(s_l - t))} \\ &= \bigcap_{m\ge 1} \mathbb{B}(\operatorname{supp} X(t), Ch(s_m - t)) \\ &= \mathbb{B}(\operatorname{supp} X(t), Ch(s - t)).\end{aligned}$$

Therefore, given any fixed $t \ge 0$, there exist a positive random variable $\theta \equiv \theta(t, d, \alpha)$ and a constant $C \equiv C(d, \alpha)$ such that for any Δt with $0 < \Delta t \le \theta$, $\mathbb{P}$-a.s.

$$\operatorname{supp} X(t + \Delta t) \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(\Delta t)) = \mathbb{B}\left(\operatorname{supp} X(t), C\sqrt{\Delta t \log(1/\Delta t)}\right).$$

□

Remark 5.6. *The constants $C \equiv C(d, \alpha)$ in Theorems 4.1 and 4.2 are the same. From the proofs of Lemma 5.2 and Theorems 4.1-4.2, it is clear that*

$$\begin{aligned}C(d, \alpha) &= 2C_4(d, \alpha) + 2C_5(d, \alpha) \\ &= 2C_4(d, \alpha) + 2C_4(d, \alpha) \sum_{k=1}^{\infty} \sqrt{2^{-k+1}k} \\ &= 2C_1(d, \alpha)\left(1 + \sum_{l=1}^{\infty} \sqrt{2^{-2l+1}l}\right)\left(1 + \sum_{k=1}^{\infty} \sqrt{2^{-k+1}k}\right),\end{aligned}$$

where $C_1(d, \alpha)$ is any constant satisfying $C_1(d, \alpha) > \sqrt{2d(3/\alpha + 1)}$.

Lemma 5.7. *Under Assumption I, we have $\mathbb{P}$-a.s.*

$$\max_{1\le k\le 2^n T} N_{n,k} < 4^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}$$

for n large enough.

Proof. Under Assumption I, there exists a positive constant C such that

$$\mathbb{E}T_m \le Cm^{-\alpha} \tag{24}$$

for m large enough.

Given n, $T^{n,k}_{4^{n/\alpha}n^{2/\alpha}}, 1 \le k \le 2^n T$ are i.i.d. random variables following the same distribution as $T_{4^{n/\alpha}n^{2/\alpha}} \wedge 2^{-n}$. Consequently, $N_{n,k}, 1 \le k \le 2^n T$ are also i.i.d. random variables. Choosing $4^{n/\alpha}n^{2/\alpha}$ large enough, by (24) we have

$$\begin{aligned}\mathbb{P}\left(\max_{1\le k\le 2^n T} N_{n,k} \ge 4^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) &= 1 - \prod_{1\le k\le 2^n T} \left(1 - \mathbb{P}(N_{n,k} \ge 4^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}})\right) \\ &\le 2^n T\mathbb{P}\left(N_{n,1} \ge 4^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) \\ &= 2^n T\mathbb{P}\left(T_{4^{n/\alpha}n^{2/\alpha}} \ge 2^{-n}\right) \\ &\le 2^n T\mathbb{E}T^{n,k}_{4^{n/\alpha}n^{2/\alpha}}/2^{-n} \\ &\le CTn^{-2},\end{aligned}$$

which is summable with respect to n. Applying Borel-Cantelli lemma, we then have $\mathbb{P}$-a.s.

$$\max_{1\le k\le 2^n T} N_{n,k} < 4^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}$$

for n large enough. □

Proof of Theorem 4.3. Under Assumption I, by Lemma 5.7 we have $\mathbb{P}$-a.s.

$$\max_{1\le k\le 2^n T} N_{n,k} < 4^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \tag{25}$$

for n large enough.

Given any positive constants σ and T with $0<\sigma<T$, we first show that $\mathcal{R}([\sigma,T))$ is a.s. compact. Applying Theorem 4.1, there exist a positive random variable $\theta \equiv \theta(T,d,\alpha)>0$ and a constant $C\equiv C(d,\alpha)$ such that $\mathbb{P}$-a.s. for all $r,s\in S^T$ satisfying $0<s-r\le\theta$,

$$H(r,s) \le Ch(s-r).$$

For the given σ, choose n big enough so that $2^{-n}\le\theta\wedge\sigma$ and (25) holds. For any $1\le k\le 2^nT$ and $t\in S^T\cap[k2^{-n},(k+1)2^{-n}\wedge T)$, we have

$$\begin{aligned} H\left((k-1)2^{-n},t\right) &\le H\left((k-1)2^{-n},k2^{-n}\right)+H\left(k2^{-n},t\right)\\ &\le 2Ch\left(2^{-n}\right). \end{aligned}$$

It follows from the lookdown construction and Lemma 3.2 that

$$\operatorname{supp} X(t) \subseteq \bigcup_{1\le i\le N^{(k-1)2^{-n},t}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right),2Ch\left(2^{-n}\right)\right).$$

By (25) we have

$$N^{(k-1)2^{-n},t} \le N^{(k-1)2^{-n},k2^{-n}} = N_{n,k} < 4^{n/\alpha}n^{2/\alpha}.$$

Consequently,

$$\operatorname{supp} X(t) \subseteq \bigcup_{1\le i<4^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right),2Ch\left(2^{-n}\right)\right). \tag{26}$$

For general $t\in[k2^{-n},(k+1)2^{-n}\wedge T)$. We can select a decreasing sequence

$$\left(t_l^{n,k}\right)_{l\ge1} \subseteq S^T\cap[k2^{-n},(k+1)2^{-n}\wedge T) \text{ satisfying } t_l^{n,k}\downarrow t \text{ as } l\to\infty.$$

Since the Λ-Fleming-Viot process X is right continuous, it follows from Lemma 5.5 that

$$\operatorname{supp} X(t) \subseteq \bigcap_{m\ge1} \overline{\bigcup_{l\ge m} \operatorname{supp} X\left(t_l^{n,k}\right)}.$$

By (26), we have

$$\operatorname{supp} X\left(t_l^{n,k}\right) \subseteq \bigcup_{1\le i<4^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right),2Ch\left(2^{-n}\right)\right).$$

Therefore, for any $t\in[k2^{-n},(k+1)2^{-n}\wedge T)$, we also have

$$\operatorname{supp} X(t) \subseteq \bigcup_{1\le i<4^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right),2Ch\left(2^{-n}\right)\right), \tag{27}$$

i.e., $\mathcal{R}([k2^{-n},(k+1)2^{-n}\wedge T))$ is contained in at most $\lfloor 4^{n/\alpha}n^{2/\alpha}\rfloor$ closed balls each of which has radius bounded from above by $2Ch(2^{-n})$. Then

$$
\begin{aligned}
\mathcal{R}([\sigma,T)) &\subseteq \mathcal{R}\left(\left[2^{-n},T\right)\right)\\
&\subseteq \bigcup_{1\le k\le 2^nT} \mathcal{R}\left(\left[k2^{-n},(k+1)\,2^{-n}\wedge T\right)\right)\\
&\subseteq \bigcup_{1\le k\le 2^nT}\ \bigcup_{1\le i<4^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right),2Ch\left(2^{-n}\right)\right),
\end{aligned}\tag{28}
$$

where the right hand side is the union of at most $\lfloor 2^nT\rfloor\times\lfloor 4^{n/\alpha}n^{2/\alpha}\rfloor$ closed and bounded balls. So $\mathcal{R}([\sigma,T))$ is compact.

Consequently, the random measure $X(t)$ has compact support for all times $t\in[\sigma,T)$ simultaneously. Let $\sigma=1/T$ and $T\to\infty$. Then the random measure $X(t)$ has compact support for all times $t\in(0,\infty)$ simultaneously.

Further, given that $\operatorname{supp}X(0)$ is compact, we can adapt the above-mentioned strategy to find a finite cover for $\mathcal{R}([0,T))$. Applying Theorem 4.2, for n large enough, we have

$$
\mathcal{R}\left([0,2^{-n})\right)=\overline{\bigcup_{t\in[0,2^{-n})}\operatorname{supp}X(t)}\subseteq\mathbb{B}\left(\operatorname{supp}X(0),Ch\left(2^{-n}\right)\right).
$$

Then

$$
\begin{aligned}
&\mathcal{R}([0,T))\\
&\subseteq \bigcup_{0\le k\le 2^nT}\mathcal{R}\left(\left[k2^{-n},(k+1)\,2^{-n}\wedge T\right)\right)\\
&\subseteq \mathbb{B}\left(\operatorname{supp}X(0),Ch\left(2^{-n}\right)\right)\bigcup\left(\bigcup_{1\le k\le 2^nT}\ \bigcup_{1\le i<4^{n/\alpha}n^{2/\alpha}}\mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right),2Ch\left(2^{-n}\right)\right)\right),
\end{aligned}
$$

where the right hand side is compact given the compactness of $\operatorname{supp}X(0)$. So, $\mathcal{R}([0,T))$ is compact.

Note that $\mathcal{R}([0,T))$ is increasing with respect to T. Let $T\to\infty$. It is clear that $\mathcal{R}([0,t))$ is compact for all $t>0$ $\mathbb{P}$-a.s.. □

5.3. **Upper bounds on Hausdorff dimensions for the support and range.** Given any Λ-coalescent $(\Pi(t))_{t\ge 0}$ with $\Pi(0)=\mathbf{0}_{[\infty]}$, recall that

$$
T_m\equiv\inf\left\{t\ge 0:\#\Pi(t)\le m\right\}
$$

with the convention $\inf\emptyset=\infty$. $(\Pi_n(t))_{t\ge 0}$ is its restriction to $[n]$ with $\Pi_n(0)=\mathbf{0}_{[n]}$. For any $n\ge m$, let

$$
T_m^n\equiv\inf\left\{t\ge 0:\#\Pi_n(t)\le m\right\}
$$

with the convention $\inf\emptyset=\infty$.

For any $x>0$, write $T_x^n\equiv T_{\lfloor x\rfloor}^n$ and $T_x\equiv T_{\lfloor x\rfloor}$.

Let $(\hat{T}_n)_{n\ge 2}$ be independent random variables such that $\hat{T}_n$ has the same distribution as T_{n-1}^n.

Lemma 5.8. *For any $n > m$, T_m^n is stochastically less than $\sum_{i=m+1}^n \hat{T}_i$, i.e., for any $t > 0$,*

$$\mathbb{P}\left(T_m^n \geq t\right) \leq \mathbb{P}\left(\sum_{i=m+1}^{n} \hat{T}_i \geq t\right). \tag{29}$$

Proof. We use a coupling argument by defining an auxiliary $[n] \times [n]$-valued continuous time Markov chain (Y_1, Y_2) describing the following urn model. Intuitively, there are balls in an urn of color either white or black. Let $Y_1(t)$ and $Y_2(t)$ represent the number of white and black balls at time t, respectively.

After each independent exponential sampling time a random number of balls are taken out of the urn and then immediately replaced with certain white or black colored balls so that the total number of balls in the urn decreases exactly by one overall afterwards. More precisely, given that there are w white balls and b black balls in the urn, at rate $\lambda_{w+b,k}$ each group of k balls with $k \leq w + b$ is independently removed. Suppose that w' white balls and $k - w'$ black balls have been chosen and removed at time t, we then immediately return $k - 1$ balls to the urn so that among the returned balls, either one is white and all the others are black if $w' > 0$ or all of them are black if $w' = 0$. At such a sampling time t we define

$$\begin{cases} Y_1(t) = w - w' + 1 \;\text{ and }\; Y_2(t) = b + w' - 2 = w + b - 1 - Y_1(t), \text{ if } w' > 0; \\ Y_1(t) = w \text{ and } Y_2(t) = b - 1, \text{ if } w' = 0, \end{cases}$$

and the value of (Y_1, Y_2) keeps unchanged between the sampling times. The above-mentioned procedure continues until there is one white ball left in the urn. Suppose that there are n white balls and no black balls in the urn initially, i.e., $(Y_1(0), Y_2(0)) = (n, 0)$.

Observe that Y_1 follows the law of the Λ-coalescent starting with n-blocks and $(\hat{T}_i)_{i \leq n}$ has the same distribution as the inter-decreasing times for process $Y_1 + Y_2$. Plainly,

$$\inf\{t : Y_1(t) \leq m\} \leq \inf\{t : Y_1(t) + Y_2(t) \leq m\}.$$

Inequality (29) thus follows. $\square$

The estimate in Lemma 5.7 is not enough for the proofs of Theorems 4.5-4.6. A sharper estimate is obtained in the following result under a stronger condition.

Lemma 5.9. *Suppose that Condition A holds. We have $\mathbb{P}$-a.s.*

$$\max_{1 \leq k \leq 2^n T} N_{n,k} < 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \tag{30}$$

for n large enough.

Proof. Under Condition A, there exists a positive constant C such that for n large enough and for any $b > 2^{n/\alpha} n^{2/\alpha}$,

$$\lambda_b \geq (C \lfloor 2^{n/\alpha} n^{2/\alpha} \rfloor^{-\alpha})^{-1} > 2^{n+1} n. \tag{31}$$

Letting $n \to \infty$ in (29), for any $t > 0$ and $m \in [\infty]$ we have

$$\mathbb{P}\left(T_m \geq t\right) \leq \mathbb{P}\left(\sum_{i > m} \hat{T}_i \geq t\right). \tag{32}$$

With estimate (32) we can find a sharper uniform upper bound for the maximal number of ancestors as follows:

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le 2^nT} N_{n,k} \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) &= 1 - \prod_{1\le k\le 2^nT}\left(1-\mathbb{P}(N_{n,k}\ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}})\right)\\
&\le 2^nT\mathbb{P}\left(N_{n,1}\ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right)\\
&\le 2^nT\mathbb{P}\left(T_{2^{n/\alpha}n^{2/\alpha}}\ge 2^{-n}\right)\\
&\le 2^nT\mathbb{P}\left(\sum_{i>2^{n/\alpha}n^{2/\alpha}}\hat{T}_i\ge 2^{-n}\right)\\
&\le 2^nTe^{-n}\mathbb{E}\exp\left(\sum_{i>2^{n/\alpha}n^{2/\alpha}}2^nn\hat{T}_i\right)\\
&= 2^nTe^{-n}\prod_{i>2^{n/\alpha}n^{2/\alpha}}\mathbb{E}\exp\left(2^nn\hat{T}_i\right),
\end{aligned}$$

where $\hat{T}_i$ follows an exponential distribution with parameter λ_i. It follows from (31) that when n is large enough, $\lambda_i > 2^n n$ for any $i > 2^{n/\alpha}n^{2/\alpha}$, which guarantees the existence of moment generating function for $\hat{T}_i$. As a result,

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le 2^nT} N_{n,k} \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) \le& 2^nTe^{-n}\prod_{i>2^{n/\alpha}n^{2/\alpha}}\frac{\lambda_i}{\lambda_i-n2^n}\\
\equiv& 2^nTe^{-n}Q.
\end{aligned}$$

Then

$$\begin{aligned}
\ln Q =& \sum_{i>2^{n/\alpha}n^{2/\alpha}}\ln\left(1+\frac{n2^n}{\lambda_i-n2^n}\right)\\
\le& \sum_{i>2^{n/\alpha}n^{2/\alpha}}\frac{n2^n}{\lambda_i-n2^n}\\
\le& n2^n\sum_{i>2^{n/\alpha}n^{2/\alpha}}\frac{1}{\lambda_i-\lambda_i/2}\\
\le& n2^{n+1}\sum_{i>2^{n/\alpha}n^{2/\alpha}}\frac{1}{\lambda_i}.
\end{aligned}$$

We have by Condition A for n large enough,

$$\ln Q \le n2^{n+1}C\left(\lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\rfloor\right)^{-\alpha} \le n2^{n+1}C\left(2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}/2\right)^{-\alpha} = 2^{\alpha+1}Cn^{-1}.$$

Then

$$\sum_n\mathbb{P}\left(\max_{1\le k\le 2^nT} N_{n,k} \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) < \infty,$$

which, by the Borel-Cantelli lemma, implies that $\mathbb{P}$-a.s.

$$\max_{1\leq k\leq 2^n T} N_{n,k} < 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}$$

for n large enough. □

Proof of Theorem 4.5. Given any $0 < \sigma < T$, we first consider the uniform upper bound on Hausdorff dimensions for supp $X(t)$ at all times $t \in [\sigma, T)$. We adapt the same idea as the proof of Theorem 4.3 to find a cover for the support at any time $t \in [\sigma, T)$. Since we have a sharper estimate for $N_{n,k}$ under Condition A, for n large enough, (27) in the proof of Theorem 4.3 can be replaced by

$$\operatorname{supp} X(t) \subseteq \bigcup_{1\leq i<2^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right), 2Ch\left(2^{-n}\right)\right)$$

for any $t \in [k2^{-n}, (k+1)2^{-n} \wedge T)$ and $1 \leq k \leq 2^n T$, i.e., for any $t \in [\sigma, T) \subseteq [2^{-n}, T)$, $\operatorname{supp} X(t)$ is contained in at most $\lfloor 2^{n/\alpha} n^{2/\alpha} \rfloor$ closed balls each of which has a radius bounded from above by $2Ch\,(2^{-n})$.

For any $\epsilon > 0$ we have

$$\begin{aligned}
\lim_{n\to\infty} \lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \rfloor \left(2Ch\left(2^{-n}\right)\right)^{\frac{2+\epsilon}{\alpha}} &\leq \lim_{n\to\infty} (2C)^{\frac{2+\epsilon}{\alpha}} 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \left(h\left(2^{-n}\right)\right)^{\frac{2+\epsilon}{\alpha}} \\
&= \lim_{n\to\infty} (2C)^{\frac{2+\epsilon}{\alpha}} (\log 2)^{\frac{2+\epsilon}{2\alpha}} 2^{-\frac{n\epsilon}{2\alpha}} n^{\frac{6+\epsilon}{2\alpha}} \\
&= 0,
\end{aligned}$$

which implies $\mathcal{H}^{\frac{2+\epsilon}{\alpha}}(\operatorname{supp} X(t)) = 0$. Since ϵ is arbitrary, the Hausdorff dimensions for $\operatorname{supp} X(t)$ at all times $t \in [\sigma, T)$ are uniformly bounded from above by $2/\alpha$.

Finally, let $\sigma \equiv 1/T$ and $T \to \infty$. The Hausdorff dimension for $\operatorname{supp} X(t)$ has uniform upper bound $2/\alpha$ at all positive times simultaneously. □

Proof of Theorem 4.6. Given any $0 < \delta < T$, we also follow the proof of Theorem 4.3 to find a finite cover for $\mathcal{R}\,([\delta, T))$. Choose n large enough such that $2^{-n} \leq \theta \wedge \delta$ and (30) holds. Similarly as (28) in the proof of Theorem 4.3, we have

$$\begin{aligned}
\mathcal{R}\,([\delta, T)) &\subseteq \mathcal{R}\left(\left[2^{-n}, T\right)\right) \\
&\subseteq \bigcup_{1\leq k\leq 2^n T} \mathcal{R}\left(\left[k2^{-n}, (k+1)\,2^{-n} \wedge T\right)\right) \\
&\subseteq \bigcup_{1\leq k\leq 2^n T} \bigcup_{1\leq i<2^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right), 2Ch\left(2^{-n}\right)\right),
\end{aligned}$$

which implies that $\mathcal{R}\,([\delta, T))$ is contained in at most $\lfloor 2^n T \rfloor \times \lfloor 2^{n/\alpha} n^{2/\alpha} \rfloor$ closed balls, each of which has radius bounded from above by $2Ch(2^{-n})$.

For any $\epsilon > 0$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \lfloor 2^n T \rfloor \times \left\lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \right\rfloor \left(2Ch(2^{-n})\right)^{\frac{2}{\alpha}+2+\epsilon} \\
\leq& C(T, d, \alpha, \epsilon) \lim_{n\to\infty} 2^{-\frac{n\epsilon}{2}} n^{\frac{3}{\alpha}+1+\frac{\epsilon}{2}} = 0.
\end{aligned}$$

Since ϵ is arbitrary, the Hausdorff dimension for the range $\mathcal{R}\,([\delta, T))$ is bounded from above by $2/\alpha + 2$. □

Proof of Corollary 4.9. With initial value δ_0, applying Theorem 4.2, it is clear that almost surely

$$\mathcal{R}\left([0,2^{-n})\right)\subseteq\mathbb{B}\left(0,Ch(2^{-n})\right)$$

for n large enough. From the proof of Theorem 4.6, we have

$$\begin{aligned}\mathcal{R}\left([0,T)\right)\subseteq&\mathcal{R}\left([0,2^{-n})\right)\bigcup\mathcal{R}\left([2^{-n},T)\right)\\ \subseteq&\mathbb{B}\left(0,Ch(2^{-n})\right)\bigcup\bigcup_{1\le k\le 2^nT}\bigcup_{1\le i<2^{n/\alpha}n^{2/\alpha}}\mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right),2Ch\left(2^{-n}\right)\right)\end{aligned}$$

for n large enough.

Therefore, $\mathcal{R}\left([0,T)\right)$ is contained in at most $\lfloor 2^nT\rfloor\times\lfloor 2^{n/\alpha}n^{2/\alpha}\rfloor+1$ closed balls, each of which has radius bounded from above by $2Ch(2^{-n})$.

For any $\epsilon>0$, we have

$$\lim_{n\to\infty}\left(\lfloor 2^nT\rfloor\times\left\lfloor 2^{\frac{n}{\alpha}}n^{\frac{2}{\alpha}}\right\rfloor+1\right)\left(2Ch(2^{-n})\right)^{\frac{2}{\alpha}+2+\epsilon}=0.$$

Since ϵ is arbitrary, the Hausdorff dimension for the range $\mathcal{R}\left([0,T)\right)$ is bounded from above by $2/\alpha+2$. □

Proof of Proposition 4.14. Let $\{t_i\}$ be any dense subset of $[0,T]$. Combining the proofs for Theorem 4.1 and Theorem 4.2, there exist $\theta\equiv\theta(T,d,\alpha)<e^{-1}$ and $C\equiv C(d,\alpha)$ such that $\mathbb{P}$-a.s.

$$\mathrm{supp}X(t_i+\Delta t)\subseteq\mathbb{B}(\mathrm{supp}X(t_i),Ch(\Delta t))$$

for all i and $0<\Delta t\le\theta\wedge(T-t_i)$. Then for any $t\in[0,T)$, there exists a subsequence (t_{i_j}) with $t_{i_j}\downarrow t$ such that given any $n>0$,

$$\begin{aligned}\mathrm{supp}X(t+\Delta t)&=\mathrm{supp}X\left(t_{i_j}+\Delta t-\left(t_{i_j}-t\right)\right)\\ &\subseteq\mathbb{B}(\mathrm{supp}X(t_{i_j}),Ch(\Delta t))\\ &\subseteq\mathbb{B}(\mathcal{R}\left([t,t+1/n)\right),Ch(\Delta t))\end{aligned}$$

for $0<\Delta t\le\theta\wedge(T-t)$ and j large enough. So,

$$\mathrm{supp}\,X(t+\Delta t)\subseteq\mathbb{B}(S_t,Ch(\Delta t))$$

since n is arbitrary. □

References

[1] D. J. Aldous. Exchangeability and related topics, Ecole dete de Probabilites de Sanit-Flour XIII-1983, Lecture Notes in Mathematics 1117, Springer, Berlin, 1985, 1-198.

[2] J. Berestycki, N. Berestycki and V. Limic. The Λ-coalescent speed of coming down from infinity, Annals of Probability, 38, No. 1, 207-233.

[3] J. Bertoin and J.-F. Le Gall. Stochastic flows associated to coalescent processes. III. Limit theorems. Illinois Journal of Mathematics, 50, (2006), 147-181.

[4] M. Birkner and J. Blath. Measure-valued diffusions, general coalescents and population genetic inference, Trends in Stochastic Analysis, LMS 353, Cambridge University Press, (2009), 329-363.

[5] M. Birkner, J. Blath, M. Capaldo, A. Etheridge, M. Möhle, J. Schweinsberg and A. Wakolbinger. α-stable branching and β-coalescents, Electronic Journal of Probability, 10, Paper no. 9, (2005), 303-325.

[6] M. Birkner, J. Blath, M. Möhle, M. Steinrücken and J. Tams. A modified lookdown construction for the Xi-Fleming-Viot process with mutation and populations with recurrent bottlenecks, Alea, 6, (2009), 25-61.

[7] D. A. Dawson. Infinitely Divisible Random Measures and Superprocesses, Stochastic analysis and related topics, Progress in Probability, 31, 1992, 1-129.
[8] D. A. Dawson. Measure-valued Markov Processes, Ecole dete de Probabilites de Sanit-Flour XIII-1991, Lecture Notes in Mathematics 1541, Springer, Berlin, 1993, 1-260.
[9] D. A. Dawson and K. J. Hochberg. Wandering random measures in the Fleming-Viot model, The Annals of Probability, 10, No. 3, (1982), 554-580.
[10] D. A. Dawson, K. J. Hochberg and V. Vinogradov. High-density limits of hierarchically structured branching-diffusing populations. Stochastic Processes and their Applications 62 (1996), 191-222.
[11] D. A. Dawson, I. Iscoe and E.A. Perkins. Super-Brownian motion: path properties and hitting probabilities. Probability Theory and related Fields, 83, (1989), 135-205.
[12] D. A. Dawson and V. Vinogradov. Almost-sure path properties of $(2, d, \beta)$-superprocesses, Stochastic Processes and their Application, 51, (1994), 221-258.
[13] J.-F. Delmas. Path properties of superprocesses with a general branching mechanism, The Annals of Probability, 27, No. 3, (1999), 1099-1134.
[14] P. Donnelly and T. G. Kurtz. A countable representation of the Fleming-Viot measure-valued diffusion, The Annals of Probability, 24, No. 2, (1996), 698-742.
[15] P. Donnelly and T. G. Kurtz. Genealogical processes for Fleming-Viot models with selection and recombination, The Annals of Applied Probability, 9, No. 4, (1999), 1091-1148.
[16] P. Donnelly and T. G. Kurtz. Particle representations for measure-valued population models, The Annals of Probability, 27, No. 1, (1999), 166-205.
[17] S. N. Ethier and T. G. Kurtz. Fleming-Viot processes in population genetics, SIAM Journal on Control and Optimization, 31, No. 2, (1993), 345-386.
[18] A. Etheridge. Some mathematical models from population genetics. Lecture Notes in Mathematics 2012, Springer, 2011.
[19] K. J. Falconer. The geometry of fractal sets, Cambridge University Press, 1985.
[20] H. L. Liu and X. W. Zhou. Compact support property of the Λ-Fleming-Viot process with underlying Brownian motion, Electronic Journal of Probability, 17, No. 73, (2012), 1-20.
[21] E. Perkins. Dawson-watanabe Superprocesses and Measure-valued Diffusions, Lecture Notes in Mathematics, Springer, 1781, 1999, 132-318.
[22] J. Pitman. Coalescents with multiple collisions, The Annals of Probability, 27, No. 4, (1999), 1870-1902.
[23] M. Reimers. A new result on the support of the Fleming-Viot process, proved by nonstandard construction, Stochastics Stochastics Rep., 44, 3-4, (1993), 213-223.
[24] J. G. Ruscher. Properties of superprocesses and interacting particle systems, Diploma Thesis, Technische Universitä Berlin, 2009.
[25] S. Sagitov. The general coalescent with asynchronous mergers of ancestral lines, Journal of Applied Probability, 36, No. 4, (1999), 1116-1125.
[26] J. Schweinsberg. A necessary and sufficient condition for the Λ-coalescent to come down from infinity, Electronic Communications in Probability, 5, (2000), 1-11.
[27] R. Tribe. Path properties of superprocesses. Ph.D. Thesis, University of British Columbia, 1989.

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