A LOCAL LIMIT LAW FOR THE EMPIRICAL SPECTRAL DISTRIBUTION OF THE ANTICOMMUTATOR OF INDEPENDENT WIGNER MATRICES

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ABSTRACT. Our main result is a local limit law for the empirical spectral distribution of the anticommutator of independent Wigner matrices, modeled on the local semicircle law. Our approach is to adapt some techniques from recent papers of Erdös-Yau-Yin. We also use an algebraic description of the law of the anticommutator of free semicircular variables due to Nica-Speicher, the linearization trick due to Haagerup-Schultz-Thorbjørnsen in a self-adjointness-preserving variant and the Schwinger-Dyson equation. A byproduct of our work is a relatively simple deterministic version of the local semicircle law.

CONTENTS

1. INTRODUCTION AND FORMULATION OF THE MAIN RESULT

Our main result is a local limit law for the anticommutator of independent Wigner matrices, modeled on the local semicircle law. The latter has emerged from the recent great progress in universality for Wigner matrices. Concerning universality, without attempting to be comprehensive, we mention [6], [7], [8], [9], [20], [21] and [24]. The paper [9] has especially influenced us. We obtain our results by using on the one hand techniques derived from [9] and on the other hand techniques derived from [11] and [12], most notably the linearization trick.

The self-adjointness-preserving variant of the linearization trick used here was introduced in [1]. (See also [2] and [4] for slicker treatments.) It turns out to mesh well with "self-improving" estimates of the type characteristic of the paper [9].

1.1. **Setup for the main result.** We formulate our main result forthwith. See §2 below for a table of notation.

1.1.1. *Random matrices.* Fix constants $\alpha_0 > 0$ and $\alpha_1 \geq 1$. Let $N \geq 2$ be a integer. Let $U, V \in \mathrm{Mat}_N$ be random hermitian matrices with the following properties:

$$(1) \qquad \sup_{p \in [2,\infty)} p^{-\alpha_0} \left(\bigvee_{i,j=1}^N \|U(i,j)\|_p \vee \bigvee_{i,j=1}^N \|V(i,j)\|_p \right) \leq \sqrt{\frac{\alpha_1}{N}}.$$

(2)   The family $\{U(i,j), V(i,j)\}_{1 \leq i \leq j \leq N}$ is independent.

(3)   All entries of U and V have mean zero.

$$(4) \qquad \|U(i,j)\|_2 = \|V(i,j)\|_2 = \frac{1}{\sqrt{N}} \quad \text{for distinct } i, j = 1, \ldots, N.$$

(Here $U(i,j)$ is the (i,j)-entry of U and $\|U(i,j)\|_p = (\mathbf{E}|U(i,j)|^p)^{1/p}$. Also we write $x \vee y$ (resp., $x \wedge y$) for the maximum (resp., minimum) of x and y.) This is a class of Wigner matrices similar to that considered in [9]. Condition (1) is merely a technically convenient way of imposing uniformly a tail bound of exponential type. (See Proposition 8.3 below for the equivalence.)

1.1.2. *Apparatus from free probability.* (For background see [3, Chap. 5], [16], [22].) Let $\mathbf{u}$ and $\mathbf{v}$ be freely independent semicircular noncommutative random variables. Let $\mu_{\{\mathbf{uv}\}}$ denote the law of $\{\mathbf{uv}\} = \mathbf{uv} + \mathbf{vu}$ and let

$$(5) \qquad m_{\{\mathbf{uv}\}}(z) = \int \frac{\mu_{\{\mathbf{uv}\}}(dt)}{t - z} \quad \text{for } z \in \mathfrak{h} = \{z \in \mathbb{C} \mid \Im z > 0\}$$

denote the Stieltjes transform of that law. Context permitting (most of the time) we will write briefly $m = m_{\{\mathbf{uv}\}}(z)$. Although m depends on z the notation does not show it. It was shown in [15, Eq. (1.15)] as part of a general discussion of commutators of free random variables that m satisfies the equation

$$(6) \qquad zm^3 - m^2 - zm - 1 = 0.$$

(Caution: Our sign convention for the Stieltjes transform is opposed to that of [15].) From (6) it follows that the support of $\mu_{\{\mathbf{uv}\}}$ is the interval $[-\zeta, \zeta]$ where

$$(7) \qquad \zeta = \sqrt{\frac{11 + 5\sqrt{5}}{2}} \cong 3.33.$$

More precisely, it was shown in [15] that $\mu_{\{\mathbf{uv}\}}$ has a density with respect to Lebesgue measure, this density was calculated explicitly, and the support $[-\zeta, \zeta]$ was thus verified. (See [15, Eq. (1.17)].) The density will not be needed here.

See [5] for a recent discussion and application of the law $\mu_{\{\mathbf{uv}\}}$ in another context.

1.1.3. *The function h.* For $z \in \mathfrak{h}$ let

(8)
$$h = |z + \zeta| \wedge |z - \zeta| \wedge 1.$$

The number $0 < h \leq 1$ depends on z but the notation does not show it.

Here is our main result.

Theorem 1.2. *Notation and assumptions are as above. (Also see §2 for general notation.) There exists a random variable $\mathbf{K} \geq 1$ with the following two properties.*

(9)   *On the event $[\llbracket U \rrbracket \vee \llbracket V \rrbracket \leq 4]$ one has*

$$\bigvee_{i=1}^{N} \left| (\{UV\} - z\mathbf{I}_N)^{-1}(i,i) - m_{\{\mathbf{uv}\}}(z) \right| \leq \frac{\mathbf{K}}{\sqrt{Nh\Im z}}$$

for $z \in \mathfrak{h}$ such that $|\Re z| \vee \Im z \leq 64$ and $\mathbf{K}^2/N \leq h^2\Im z$.

(10)   *For every $t > 0$ one has $\Pr(\mathbf{K} > t^{2\alpha_0+1}) \leq \beta_0 N^{\beta_1} \exp(-\beta_2 t)$,*

for positive constants β_0 and β_2 depending only on α_0 and α_1

and a positive absolute constant β_1.

(In particular, β_0, β_1 and β_2 are independent of N.) The theorem is not so sharp as the sharpest available concerning the local semicircle law. The novelty here, rather, is to have made inroads on the general problem of proving local limit laws for polynomials in Wigner matrices. Looking forward, we have given some of our arguments in a general setting when this could be done without making the paper significantly longer. (See §4 and §6 below.) But some arguments are quite *ad hoc* (see §5 below) and implicitly pose the problem of finding conceptual general arguments with which to replace them.

One has delocalization of eigenvectors in our setup in the following sense.

Corollary 1.3. *Evaluate $\{UV\}$ and $\mathbf{K}$ at a sample point of the event $[\llbracket U \rrbracket \vee \llbracket V \rrbracket \leq 4]$. We still write $\{UV\}$ and $\mathbf{K}$ for these evaluations, respectively. Let λ be an eigenvalue of $\{UV\}$ and let v be a corresponding unit-length (right) eigenvector. Let $\rho = \mathbf{K}^2/N$ and for simplicity assume that $\rho < 1$. Let $\sigma \in [\rho, \rho^{1/3}]$ be defined by the equation $\rho = h^2\Im z|_{z=\lambda+i\sigma}$. Then we have*

(11)
$$\bigvee_{i=1}^{N} |v(i)| \leq \sqrt{2\sigma}.$$

This result is roughly comparable to [9, Cor. 3.2]. Figure 1 shows σ as a function of λ for $\rho = 0.2, 0.02, 0.002, 0.0002$. Note that in the bulk one simply has $\rho = \sigma$. However, the bound (11) is not optimal near the edge of the spectrum and it is an open problem to optimize it.

Proof. Let $32 \geq \lambda_1 \geq \cdots \geq \lambda_N \geq -32$ be the eigenvalues of $\{UV\}$ and let $v_1, \ldots, v_N$ be corresponding unit-length eigenvectors. We have for $i = 1, \ldots, N$ and $z \in \mathfrak{h}$ the standard formula

$$\frac{\Im(\{UV\} - z\mathbf{I}_N)^{-1}(i,i)}{\Im z} = \sum_{j=1}^{N} \frac{|v_j(i)|^2}{|z - \lambda_j|^2}$$

which we will apply presently. We may assume that $\lambda = \lambda_{i_0}$ and $v = v_{i_0}$ for a suitable index i_0. Let $z_0 = \lambda + i\sigma$ and $h_0 = h|_{z=z_0}$, noting that
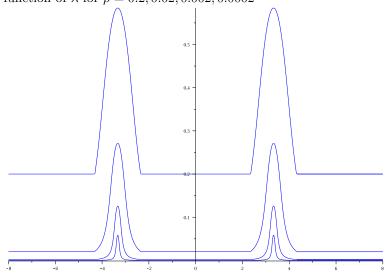
$$|\lambda| \vee \sigma = |\Re z_0| \vee \Im z_0 \leq 64 \ \text{ and} \frac{\mathbf{K}}{\sqrt{N h_0 \Im z_0}} = \sqrt{h_0} \leq 1$$

by our assumption that $[\![U]\!] \vee [\![V]\!] \leq 4$ and simplifying assumption that $\rho < 1$. Thus we have

$$\begin{aligned} 2 \ &\geq \ 1 + \frac{\mathbf{K}}{\sqrt{N h_0 \Im z_0}} \ \geq \ \Im(\{UV\} - z_0 \mathbf{I}_N)^{-1}(i,i) \\ &= \ \sum_{j=1}^{N} \frac{\sigma |v_j(i)|^2}{(\lambda_j - \lambda_{i_0})^2 + \sigma^2} \ \geq \ \frac{|v(i)|^2}{\sigma} \end{aligned}$$

by Theorem 1.2 and the uniform bound $|m| < 1$ from Proposition 5.2 below.   □

FIGURE 1. Closest permissible approach σ to the real axis as a function of λ for $\rho = 0.2, 0.02, 0.002, 0.0002$

1.4. **Decay of** $\Pr([\![U]\!] \vee [\![V]\!] > 4)$**.** The conditioning on the event $[\![U]\!] \vee [\![V]\!] \leq 4$ taking place in Theorem 1.2 is not costly. In the setup of the theorem, one has

$$\Pr([\![U]\!] \vee [\![V]\!] > 4) \leq c_0 \exp(-c_1 N^{c_2})$$

for some positive constants c_0, c_1 and c_2 depending only on α_0 and α_1. See, e.g., the argument presented immediately after [3, Lemma 2.1.23]. The lemma in question is a combinatorial lemma somewhat weaker than the classical result of [10] and weaker still than the more refined results of [23]. We will not deal further here with the rate of decay of $\Pr([\![U]\!] \vee [\![V]\!] > 4)$ as $N \to \infty$.

Our proof of Theorem 1.2 is structured overall by the following trivial remark.

Proposition 1.5. *Let $f_1, f_2, f_3 : \mathcal{X} \to [0, \infty)$ be continuous functions on a connected topological space $\mathcal{X}$. Make the following assumptions.*

(12) $$f_1(x_0) < f_2(x_0) \text{ for some } x_0 \in \mathcal{X}.$$

(13) $$f_1(x) \leq f_2(x) \Rightarrow f_1(x) \leq f_3(x) \text{ for all } x \in \mathcal{X}.$$

(14) $$f_3(x) < f_2(x) \text{ for all } x \in \mathcal{X}.$$

Then we have

(15) $$f_1(x) \leq f_3(x) \text{ for all } x \in \mathcal{X}.$$

The proposition is a less technically demanding way to think about estimates in the self-improving style of [9].

Proof. We have $\emptyset \neq \{f_1 < f_2\} \subset \{f_1 \leq f_3\} \subset \{f_1 < f_2\}$ by hypotheses (12), (13) and (14), respectively. Since $\{f_1 \leq f_3\}$ is open, closed and nonempty, in fact $\{f_1 \leq f_3\} = \mathcal{X}$ by connectedness of $\mathcal{X}$. $\square$

1.6. **Further comments on methods of proofs.**

1.6.1. An explicit if somewhat involved description of the random variable $\mathbf{K}$ will be given later. Given this description, the proof of property (10) turns out to be an exercise involving methods from the toolbox of [9]. Under more restrictive hypotheses it is likely one could obtain stronger results using the Hanson-Wright inequality. For an illuminating modern treatment of the latter see the recent preprint [17].

1.6.2. The main technical result of the paper by which means we prove (9) is a deterministic statement of a form perhaps not seen before in connection with local limit laws. (See Theorem 7.1 below.) Its proof is a reworking of the idea of a self-improving estimate—rather than marching by short steps toward the real axis, updating estimates at each step as in [9], we get our result at once by using Proposition 1.5.

1.6.3. We employ here generalized resolvent techniques from [1]. But we do so with significant simplifications, e.g., we do not use two-variable generalized resolvents and Stieltjes transforms—rather, we just use the classical parameter z.

1.7. **The deterministic local semicircle law.** To facilitate comparison of our results to the literature on the local semicircle law, as well as to rehearse main ideas in a simplified context, we include an appendix in which we state and prove a semi-circular analogue of Theorem 7.1, which we call the *deterministic local semicircle law*. (See Theorem 9.2 below.)

1.8. **Outline of the paper.** In §2 we provide a table of notation. In §3 we introduce the generalized resolvent formalism for anticommutators and we prove several identities and inequalities. In §4 we review the general Schwinger-Dyson equation and present key examples of solutions. (See Propositions 4.2 and 4.3.) Then we analyze stability of a general nondegenerate solution. (See Proposition 4.4.) In §5 we prove Proposition 4.3 and in passing pose a general problem for the free probability theorists. (See §5.3 below.) In §6 we analyze a general matrix-valued version of the self-consistent equation [9, Lemma 4.3]. (See Proposition 6.2 below.) In §7 we do the main work of proving (9). (See Theorem 7.1 below.) In §8 we finish the proof of Theorem 1.2 using methods of the type discussed in [9, Appendix B]. Finally, in the appendix provided in §9, we present the deterministic local semicircle law.

2. Table of notation

2.1. **Basic notation.** Let $\{xy\} = xy + yx$ denote the anticommutator of x and y. We write $\mathrm{i} = \sqrt{-1}$ (roman typeface). For real numbers x and y, let $x \vee y$ (resp., $x \wedge y$) denote the maximum (resp., minimum) of x and y. For $x \geq 0$, let $x_\bullet = x \vee 1$. Let $\Re z$ and $\Im z$ denote the real and imaginary parts of a complex number z, respectively, and let z^* denote the complex conjugate of z. Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \Im z > 0\}$ denote the upper half-plane. For a $\mathbb{C}$-valued random variable Z and $p \in [1, \infty)$, let $\|Z\|_p = (\mathbf{E}|Z|^p)^{1/p}$ and furthermore, let $\|Z\|_\infty$ denote the essential supremum of $|Z|$.

2.2. **Matrix notation.** Let $\mathrm{Mat}_{k \times \ell}$ denote the space of k-by-ℓ matrices with entries in $\mathbb{C}$. Let $\mathrm{Mat}_N = \mathrm{Mat}_{N \times N}$. Let $\mathbf{I}_N \in \mathrm{Mat}_N$ denote the N-by-N identity matrix. Context permitting, we may write 1 instead of $\mathbf{I}_N$. Given $A \in \mathrm{Mat}_{k \times \ell}$, let $[\![A]\!]$ denote the largest singular value of A and let $A^* \in \mathrm{Mat}_{\ell \times k}$ denote the transpose conjugate of A. For $A \in \mathrm{Mat}_N$, let $\Re A = \frac{A + A^*}{2}$ and $\Im A = \frac{A - A^*}{2\mathrm{i}}$. For $A \in \mathrm{Mat}_N$, we write $A > 0$ (resp., $A \geq 0$) if A is hermitian and positive definite (resp., positive semidefinite). Given for $\nu = 1, 2$ a matrix $A^{(\nu)} \in \mathrm{Mat}_{k_\nu \times \ell_\nu}$, recall that the *Kronecker product* $A^{(1)} \otimes A^{(2)} \in \mathrm{Mat}_{k_1 k_2 \times \ell_1 \ell_2}$ is defined by the rule

$$A^{(1)} \otimes A^{(2)} = \begin{bmatrix} & \vdots & \\ \cdots & A^{(1)}(i,j)A^{(2)} & \cdots \\ & \vdots & \end{bmatrix}.$$

2.3. **The matrix norms $[\![\cdot]\!]_p$.** Given a matrix $A \in \mathrm{Mat}_{k \times \ell}$ with singular values $\mu_1 \geq \mu_2 \geq \cdots$ and $p \in [1, \infty)$, let $[\![A]\!]_p = (\sum_i \mu_i^p)^{1/p}$. Also let $[\![A]\!] = [\![A]\!]_\infty$. Standard properties of the matrix norms $[\![\cdot]\!]_p$ are taken for granted, e.g., $[\![A]\!]_2^2 = \sum_{i,j} |A(i,j)|^2 = \mathrm{tr}\, AA^*$. Of particular importance is the *Hölder inequality* which asserts that $[\![AB]\!]_r \leq [\![A]\!]_p [\![B]\!]_q$ whenever $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$ and the matrix product AB is defined. See [14] or [19] for background. Actually only $p = 1, 2, \infty$ will be important.

2.4. **Stieltjes transforms.** In general, given a probability measure μ on the real line, we define its *Stieltjes transform* by the formula $S_\mu(z) = \int \frac{\mu(dt)}{t-z}$ for $z \in \mathfrak{h}$. Note that with this sign convention we have $\Im S_\mu(z) > 0$ for $\Im z > 0$. We also have a uniform bound $|S(z)| \leq 1/\Im z$.

2.5. **Banach spaces.** Banach spaces always have complex scalars. The norm in a Banach space $\mathcal{V}$ is denoted by $[\![\cdot]\!]_\mathcal{V}$ or simply by $[\![\cdot]\!]$ when (usually) context permits. A *unital* Banach algebra $\mathcal{A}$ is one equipped with a unit $1_\mathcal{A}$ satisfying $[\![1_\mathcal{A}]\!] = 1$. Other notation may be used for the unit, e.g., $\mathbf{I}_n = 1_{\mathrm{Mat}_n}$ or $1 = 1_\mathcal{A}$. We invariably equip Mat_n with unital Banach algebra structure by means of the largest-singular-value norm. Let $B(\mathcal{V})$ denote the space of bounded linear maps from $\mathcal{V}$ to itself normed by the rule $[\![T]\!]_{B(\mathcal{V})} = \sup_{v \in \mathrm{Ball}_\mathcal{V}(0,1)} [\![T(v)]\!]_\mathcal{V}$. Given $v_0 \in \mathcal{V}$ and $\epsilon \geq 0$, let $\mathrm{Ball}_\mathcal{V}(v_0, \epsilon) = \{v \in \mathcal{V} \mid [\![v - v_0]\!]_\mathcal{V} \leq \epsilon\}$ (a closed ball).

2.6. **Inexplicit constants.** These may be denoted by c, C, etc. and their values may change from context to context and even from line to line. When recalling a previously defined constant we sometimes do so by referencing as a subscript the theorem, proposition, corollary, or lemma in which the constant was defined, e.g., $c_{4.3}$ denotes the constant c from Proposition 4.3.

3. THE GENERALIZED RESOLVENT FORMALISM FOR ANTICOMMUTATORS

We enumerate the main objects of study and work out several relations among them. Our viewpoint and methods are deterministic except in §3.6, where we pause to discuss the probabilistic motivations.

3.1. The main objects of study.

3.1.1. *Data.* Arbitrarily fix hermitian matrices $U, V \in \mathrm{Mat}_N$ where $N \geq 2$ and a point $z \in \mathfrak{h}$. These data remain fixed throughout §3 and throughout calculations later to be undertaken in §7. We take (U, V, z) to be deterministic here and in §7, except in §3.6 where we temporarily identify U and V with the random matrices figuring in Theorem 1.2. The emphasis in §3 and §7 will be on deterministic estimates with constants independent of N, U, V and z.

3.1.2. *The generalized resolvent R.* Let

$$(16) \qquad \Lambda \;=\; \begin{bmatrix} z & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_3,$$

$$(17) \qquad X \;=\; \begin{bmatrix} 0 & \frac{U-V}{\sqrt{2}} & \frac{-U-V}{\sqrt{2}} \\ \frac{U-V}{\sqrt{2}} & 0 & 0 \\ \frac{-U-V}{\sqrt{2}} & 0 & 0 \end{bmatrix} \in \mathrm{Mat}_{3N} \text{ and}$$

$$(18) \qquad W \;=\; \begin{bmatrix} \mathbf{I}_N & 0 & 0 \\ \frac{-U+V}{\sqrt{2}} & \mathbf{I}_N & 0 \\ \frac{-U-V}{\sqrt{2}} & 0 & \mathbf{I}_N \end{bmatrix} \in \mathrm{Mat}_{3N}.$$

Note that Λ depends on z although the notation does not show it. Note that X is hermitian. Note that both X and W depend on U and V although the notation does not show it. Note that

$$(19) \quad 1 \leq [\![W]\!] = [\![W^{-1}]\!] = [\![W^*]\!] = [\![(W^*)^{-1}]\!] \quad \text{and} \quad [\![X]\!] \vee [\![W]\!] \leq 8([\![U]\!] \vee [\![V]\!] \vee 1).$$

We have a factorization

$$(20) \qquad W^*(X - \Lambda \otimes \mathbf{I}_N)W = \begin{bmatrix} UV + VU - z\mathbf{I}_N & 0 & 0 \\ 0 & \mathbf{I}_N & 0 \\ 0 & 0 & -\mathbf{I}_N \end{bmatrix}.$$

It follows that $X - \Lambda \otimes \mathbf{I}_N$ is invertible. Let

$$(21) \qquad R = (X - \Lambda \otimes \mathbf{I}_N)^{-1} = W \begin{bmatrix} (\{UV\} - z\mathbf{I}_N)^{-1} & 0 & 0 \\ 0 & \mathbf{I}_N & 0 \\ 0 & 0 & -\mathbf{I}_N \end{bmatrix} W^*,$$

which we call the *generalized resolvent* for anticommutators. The matrix R depends on (U, V, z) but the notation does not show it. Clearly, the dependence of R on (U, V, z) is continuous. Crucially, the resolvent of $\{UV\} = UV + VU$ appears as the upper left N-by-N block of R. For discussion of the self-adjoint linearization trick by which means generalized resolvents such as R are contrived, see [1], [2] or [4].

3.1.3. *The matrix M.* With $m = m_{\{\mathbf{uv}\}}(z)$ as on line (5) above, let

$$(22) \qquad M = \begin{bmatrix} m & 0 & 0 \\ 0 & -\frac{1}{m-1} & 0 \\ 0 & 0 & -\frac{1}{m+1} \end{bmatrix} \in \mathrm{Mat}_3.$$

Since $\Im m > 0$, in fact M is well-defined and moreover invertible. Although M depends on z, the notation does not show it. We remark that the function h defined on line (8) will be used often in conjunction with M.

3.1.4. *The linear map Φ.* Let $\Phi \in B(\mathrm{Mat}_3)$ be the (constant) linear map defined by the formula

$$(23) \qquad \Phi(A) = (e_{12} + e_{21})A(e_{12} + e_{21}) + (e_{13} + e_{31})A(e_{13} + e_{31})$$

where $\{e_{ij}\}_{i,j=1}^{3}$ is the standard basis for Mat_3 consisting of elementary matrices. A straightforward calculation shows that the definition (23) can be rewritten

$$(24) \qquad \Phi\left(\begin{bmatrix} x_1 & x_4 & x_6 \\ x_5 & x_2 & x_8 \\ x_7 & x_9 & x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + x_3 & x_5 & x_7 \\ x_4 & x_1 & 0 \\ x_6 & 0 & x_1 \end{bmatrix}.$$

The peculiar numbering of matrix entries will be useful later. See §3.6 for probabilistic motivation for the definition of Φ.

3.1.5. *Specialized matrix notation.* For $i = 1, \ldots, N$, let $e_i \in \mathrm{Mat}_{1 \times N}$ denote the ith row of $\mathbf{I}_N$, let $\mathbf{e}_i = \mathbf{I}_3 \otimes e_i \in \mathrm{Mat}_{3 \times 3N}$, let $\hat{e}_i \in \mathrm{Mat}_{(N-1) \times N}$ denote $\mathbf{I}_N$ with the ith row deleted and let $\hat{\mathbf{e}}_i = \mathbf{I}_3 \otimes \hat{e}_i \in \mathrm{Mat}_{3(N-1) \times 3N}$.

3.1.6. *Further objects associated with R.* For $i = 1, \ldots, N$ let

$$\begin{aligned} G_i &= \mathbf{e}_i R \mathbf{e}_i^* \in \mathrm{Mat}_3, \ \ G = \frac{1}{N} \sum_{i=1}^{N} G_i \in \mathrm{Mat}_3, \\ R_i &= (\hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^* - \Lambda \otimes \mathbf{I}_{N-1})^{-1} \in \mathrm{Mat}_{3(N-1)}, \\ \widehat{G}_i &= \frac{1}{N} \sum_{j=1}^{N} \mathbf{e}_j \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i \mathbf{e}_j^* \in \mathrm{Mat}_3, \\ Q_i &= \mathbf{e}_i X \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i X \mathbf{e}_i^* - \mathbf{e}_i X \mathbf{e}_i^* - \Phi(\widehat{G}_i) \in \mathrm{Mat}_3, \\ \mathfrak{K}_i &= 1 \vee \frac{[\![Q_i]\!]}{\frac{1}{\sqrt{N}}\left(1 \vee \frac{[\![R_i]\!]_2}{\sqrt{N}}\right)} \in [1, \infty) \ \text{ and } \ \mathfrak{K} = \bigvee_{i=1}^{N} \mathfrak{K}_i. \end{aligned}$$

All these objects depend on (U, V, z) but the notation does not show it. Clearly, dependence on (U, V, z) is continuous. All of these objects have counterparts in the study of single Wigner matrices, as we explain in an appendix. (See §9 below.) Theorem 7.1 below will explain the role of the most complicated object, namely $\mathfrak{K}$. Ultimately we will define the random variable $\mathbf{K}$ in Theorem 1.2 in terms of $\mathfrak{K}$.

3.2. **Basic relations.**

3.2.1. *The Schwinger-Dyson equation.* In §5.4.1 below it is proved that

$$(25) \qquad\qquad \mathbf{I}_3 + M(\Lambda + \Phi(M)) = 0.$$

This solution of the *Schwinger-Dyson equation* will be studied in §5 in great detail. The general equation will be studied in §4 and §6.

3.2.2. *The linearization bound.* The relation deserving emphasis as the starting point for the proof of Theorem 1.2 is the bound

(26) $$|(\{UV\} - z\mathbf{I}_N)^{-1}(i,i) - m| \leq [\![G_i - M]\!] \text{ for } i = 1, \ldots, N.$$

The latter holds because firstly, the resolvent of the anticommutator $\{UV\}$ appears as the upper left N-by-N block of the generalized resolvent R and secondly, we have $m = M(1,1)$ by definition of M.

3.2.3. *Finer relations between R and the resolvent of $\{UV\} = UV + VU$.* Let

(27) $$r = \begin{bmatrix} (\{UV\} - z\mathbf{I}_N)^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \text{Mat}_{3N},$$

which is just the resolvent of $\{UV\}$ bordered by some zeros. Let

(28) $$\Lambda^0 = \lim_{z \to 0} \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{Mat}_3.$$

Recall that by definition $\Im A = \frac{A - A^*}{2i}$ for $A \in \text{Mat}_n$. We have

(29) $$R + \Lambda^0 \otimes \mathbf{I}_N = WrW^*, \quad \frac{dR}{dz} = Wr^2W^* \text{ and } \frac{\Im R}{\Im z} = Wrr^*W^* = Wr^*rW^*$$

as one can straightforwardly deduce from (21).

3.2.4. *A finer* a priori *bound for G_i.* We have

(30) $$\bigvee_{i=1}^N [\![G_i + \Lambda_0]\!] \leq \frac{[\![W]\!]^2}{\Im z} \leq \frac{2^6([\![U]\!] \vee [\![V]\!] \vee 1)^2}{\Im z}$$

by combining (19), (29) and the standard resolvent bound $[\![r]\!] \leq 1/\Im z$.

We will apply the following well-known facts concerning Schur complements to derive further relations among the objects associated with R.

Proposition 3.3. *Let $\{1, \ldots, n\} = I \coprod J$ be a disjoint union decomposition. Let $\mathbf{e}$ (resp., $\hat{\mathbf{e}}$) be the result of striking rows indexed by J (resp., I) from $\mathbf{I}_n$. Let $A \in \text{Mat}_n$ be a matrix such that both A and $\hat{\mathbf{e}}A\hat{\mathbf{e}}^*$ are invertible. Then $\mathbf{e}A^{-1}\mathbf{e}^*$ is invertible and we have relations*

(31) $$(\mathbf{e}A^{-1}\mathbf{e}^*)^{-1} = \mathbf{e}A\mathbf{e}^* - \mathbf{e}A\hat{\mathbf{e}}^*(\hat{\mathbf{e}}A\hat{\mathbf{e}}^*)^{-1}\hat{\mathbf{e}}A\mathbf{e}^*,$$

(32) $$A^{-1} = \hat{\mathbf{e}}^*(\hat{\mathbf{e}}A\hat{\mathbf{e}}^*)^{-1}\hat{\mathbf{e}} + A^{-1}\mathbf{e}^*(\mathbf{e}A^{-1}\mathbf{e}^*)^{-1}\mathbf{e}A^{-1}.$$

Proof. Write

$$\begin{bmatrix} \mathbf{e} \\ \hat{\mathbf{e}} \end{bmatrix} A \begin{bmatrix} \mathbf{e}^* & \hat{\mathbf{e}}^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{e} \\ \hat{\mathbf{e}} \end{bmatrix} A^{-1} \begin{bmatrix} \mathbf{e}^* & \hat{\mathbf{e}}^* \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

noting that $\begin{bmatrix} \mathbf{e} \\ \hat{\mathbf{e}} \end{bmatrix}$ is a permutation matrix. By hypothesis $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and d are invertible. Thus we have a factorization

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix},$$

hence the Schur complement $a - bd^{-1}c$ is also invertible and we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & d^{-1} \end{bmatrix} + \begin{bmatrix} 1 \\ -d^{-1}c \end{bmatrix} (a - bd^{-1}c)^{-1} \begin{bmatrix} 1 & -bd^{-1} \end{bmatrix}.$$

This already proves invertibility of $\mathbf{e}A^{-1}\mathbf{e}^*$ and identity (31). It follows that

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & d^{-1} \end{bmatrix} + \begin{bmatrix} p \\ r \end{bmatrix} p^{-1} \begin{bmatrix} p & q \end{bmatrix}.$$

The latter identity after conjugation by $\begin{bmatrix} \mathbf{e}^* & \hat{\mathbf{e}}^* \end{bmatrix}$ on both sides becomes (32).   □

3.4. **Further relations.** We have seen that $X - \Lambda \otimes \mathbf{I}_N$ is invertible and for similar reasons $\hat{\mathbf{e}}_i^*(X - \Lambda \otimes \mathbf{I}_N)\hat{\mathbf{e}}_i$ is also invertible. Thus by Proposition 3.3 we have that

(33) G_i is invertible for $i = 1, \ldots, N$.

Moreover, the identities (31) and (32) specialize in the present case to

(34) $-Q_i \;=\; G_i^{-1} + \Lambda + \Phi(\widehat{G}_i)$ and

(35) $R \;=\; \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i + R\mathbf{e}_i^* G_i^{-1}\mathbf{e}_i R,$

respectively. From the latter identity we deduce a bound

(36) $N\left[\!\left[G - \widehat{G}_i \right]\!\right] \leq N\left[\!\left[G - \widehat{G}_i \right]\!\right]_1 \leq \left[\!\left[R - \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i \right]\!\right]_1 \leq \left[\!\left[G_i^{-1} \right]\!\right] \left[\!\left[R\mathbf{e}_i^* \right]\!\right]_2 \left[\!\left[\mathbf{e}_i R \right]\!\right]_2$

via the matrix Hölder inequality and the following lemma.

Lemma 3.5. *For* $A \in \mathrm{Mat}_{3N}$ *one has* $\sum_{i=1}^N \left[\!\left[\mathbf{e}_i A\mathbf{e}_i^* \right]\!\right]_1 \leq \left[\!\left[A \right]\!\right]_1$.

Proof. It is well-known that $\left[\!\left[A \right]\!\right]_1 = \sup \sum_{i=1}^N |v_i A w_i^*|$ where the supremum is extended over orthonormal bases $\{v_i\}_{i=1}^N$ and $\{w_i\}_{i=1}^N$ for $\mathrm{Mat}_{1\times N}$. A suitable choice of $\{v_i\}$ and $\{w_i\}$ gives the desired inequality.   □

3.6. **Motivation for the definition of Φ.** Suppose for the moment that U and V are random and satisfy (1), (2), (3) and (4). We claim that the random matrix X has the following properties:

(37) $\displaystyle \sup_{p\in[2,\infty)} p^{-\alpha_0} \bigvee_{i,j=1}^N \left\| \left[\!\left[\mathbf{e}_i X\mathbf{e}_j \right]\!\right] \right\|_p < \alpha_2$

                         for a constant α_2 depending only on α_0 and α_1.

(38)   The family $\{\mathbf{e}_i X\mathbf{e}_j\}_{1\leq i\leq j\leq N}$ is independent.

(39) $\mathbf{E}X = 0.$

(40) $\mathbf{E}\mathbf{e}_i X\mathbf{e}_j^* A\mathbf{e}_k^* X\mathbf{e}_i = \delta_{jk}\Phi(A)$

                         for $i, j, k = 1, \ldots, N$ s.t. $i \notin \{j,k\}$ and $A \in \mathrm{Mat}_3$.

The first three claims are clear. We just prove the last. We have in any case

$$\mathbf{e}_i X\mathbf{e}_j^* = \left(\frac{U-V}{\sqrt{2}}(i,j) \right) (e_{12} + e_{21}) + \left(\frac{-U-V}{\sqrt{2}}(i,j) \right) (e_{13} + e_{31})$$

by direct appeal to the definitions. Now by assumptions (2), (3) and (4), for any fixed distinct indices $i, j = 1, \ldots, N$, the two $\mathbb{C}$-valued random variables $\frac{U-V}{\sqrt{2}}(i,j)$

and $\frac{-U-V}{\sqrt{2}}(i,j)$ form an orthonormal system. Formula (40) then follows by the definition of Φ. The claims are proved. It follows for $i = 1, \ldots, N$ that

(41) $\qquad\qquad\qquad \sigma(\hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^*)$ and $\sigma(\mathbf{e}_i X)$ are independent,

(42) $\qquad\qquad\qquad R_i$ and $\widehat{G}_i$ are $\sigma(\hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^*)$-measurable and

(43) $\qquad\qquad\qquad \mathbf{E}(Q_i | \hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^*) = 0$ a.s..

The system of relations (37)—(43) exhibits X as a Wigner-matrix-like array of 3-by-3 blocks and opens the way toward an analysis of R by methods analogous to those used to study the resolvent of a Wigner matrix, especially those of [9]. In particular, the motivation for the definition Φ is now clear: to achieve (40) and thus also (43) we are forced to define Φ as we have.

We will consider the generalized resolvent formalism again in §7, after a long digression to consider the Schwinger-Dyson equation from several angles.

4. Stability of a general form of the Schwinger-Dyson equation

For background see e.g. [1], [2], [3, Chap. 5], [13] or [16].

4.1. Basic definitions.

4.1.1. *The Schwinger-Dyson equation.* Let $\mathcal{S}$ be a finite-dimensional unital Banach algebra. A triple

$$(\Lambda, M, \Phi) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$$

is said to satisfy the *Schwinger-Dyson equation* if

(44) $$1_{\mathcal{S}} + (\Lambda + \Phi(M))M = 0,$$

in which case M is necessarily invertible. (In a finite-dimensional unital algebra existence of a left inverse implies existence of a two-sided inverse.) We emphasize that in our (somewhat eccentric) usage, a solution of the Schwinger-Dyson equation is not a function; rather, it is just a point in the space $\mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$.

4.1.2. *Nondegeneracy.* Now let $(\Lambda, M, \Phi) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$ be any solution of the Schwinger-Dyson equation. If the linear map

(45) $$\left(x \mapsto M^{-1}x - \Phi(x)M\right) \in B(\mathcal{S})$$

is invertible we say that (Λ, M, Φ) is *nondegenerate* in which case we let

$$\kappa = \kappa_{\Lambda, M, \Phi}$$

denote the inverse of the linear map (45) and we also say with slight abuse of terminology that the quadruple

$$(\Lambda, M, \Phi, \kappa) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S}) \times B(\mathcal{S})$$

is a *nondegenerate* solution of the Schwinger-Dyson equation. If we need to empha-size the role of $\mathcal{S}$ we say that $(\Lambda, M, \Phi, \kappa)$ is a solution *defined over* $\mathcal{S}$ but we omit the epithet when (usually) context permits.

4.1.3. *The stability radius.* Recall our notation $x_\bullet = 1 \vee x$. Given a nondegenerate solution of the Schwinger-Dyson equation $(\Lambda, M, \Phi, \kappa)$ as above, we call the quantity $\frac{1}{8[\![\kappa]\!]_\bullet [\![\Phi]\!]_\bullet}$ the *stability radius* of $(\Lambda, M, \Phi, \kappa)$. The meaning of the stability radius will be explained by Proposition 4.4 below.

The next proposition describes the class of nondegenerate solutions of the Schwinger-Dyson equation connected with the (local) semicircle law.

Proposition 4.2. *Fix $z \in \mathfrak{h}$ and let $m = \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4-t^2}\, dt}{t-z}$. (i) One has*

$$(46) \qquad \Im m > 0, \quad z = -m - m^{-1} \quad \text{and} \quad |m| \leq 1 \wedge \frac{1}{\Im z}.$$

(ii) The quadruple

$$(47) \qquad (z, m, 1, (m^{-1} - m)^{-1})$$

is a nondegenerate solution of the Schwinger-Dyson equation defined over $\mathbb{C}$. (iii) The stability radius of the solution (47) satisfies the lower bound

$$(48) \qquad \frac{1}{8|(m^{-1} - m)^{-1}|_\bullet |1|_\bullet} \geq \frac{\sqrt{1 \wedge |z-2| \wedge |z+2|}}{c}$$

where c is an absolute constant.

One could, say, take $c_{4.2} = 8$. But we prefer the inexplicit notation $c_{4.2}$ for being more informative.

Proof. (i) Well-known. (ii) Taking $\mathcal{S} = \mathbb{C} = B(\mathcal{S})$ in the general definition, it is clear that $(z, m, 1)$ is a solution of the Schwinger-Dyson equation. Since the linear map (45) in the case of $(z, m, 1)$ becomes multiplication by $m^{-1} - m$ and the latter is not zero, we have $\kappa_{z,m,1} = (m^{-1} - m)^{-1}$ as claimed. (iii) The estimate follows straightforwardly from the algebraic identity $(m^{-1} - m)^2 = z^2 - 4$. $\qquad\qquad\square$

For our study of anticommutators the following more exotic examples of nondegenerate solutions of the Schwinger-Dyson equation will be needed.

Proposition 4.3. *For each $z \in \mathfrak{h}$ the triple (Λ, M, Φ) defined in (16), (22) and (23), respectively, is a nondegenerate solution of the Schwinger-Dyson equation defined over Mat_3. (Recall that the matrices Λ and M depend on z but the notation does not show it.) Furthermore, we have bounds*

$$(49) \qquad [\![\Lambda]\!] \leq 1 + |z|, \quad [\![\Phi]\!] \leq 8, \quad [\![M]\!] \leq 2 \quad \text{and} \quad [\![M + \Lambda^0]\!] \leq 2\left(1 \wedge \frac{1}{\Im z}\right)$$

where Λ^0 is as defined on line (28). Let $\kappa = \kappa_{\Lambda, M, \Phi} \in B(\mathrm{Mat}_3)$. (As do Λ and M, the linear map κ depends on z but the notation does not show it.) Finally, the nondegenerate solution $(\Lambda, M, \Phi, \kappa)$ of the Schwinger-Dyson equation has stability radius satisfying the lower bound

$$(50) \qquad \frac{1}{8[\![\kappa]\!]_\bullet [\![\Phi]\!]_\bullet} \geq \frac{\sqrt{h}}{c}$$

where h is as defined on line (8) above and $c \geq 1$ is an absolute constant.

The elementary but long and computationally intensive proof is postponed to §5. The constant $c_{4.3}$ has a crucial role to play in the proof of Theorem 1.2.

The main result of this section is the following.

Proposition 4.4. *Let $\mathcal{S}$ be a finite-dimensional unital Banach algebra. Let*

$$(\Lambda_0, M_0, \Phi_0, \kappa_0)$$

be a nondegenerate solution of the Schwinger-Dyson equation defined over $\mathcal{S}$. Fix $G_0 \in \mathcal{S}$ and let

$$E_0 = 1_{\mathcal{S}} + (\Lambda_0 + \Phi_0(G_0))G_0 \in \mathcal{S}.$$

We then have

$$(51) \qquad [\![G_0 - M_0]\!] \leq \frac{1}{8[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet} \Rightarrow [\![G_0 - M_0]\!] \leq 20[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet [\![M_0]\!]_\bullet^2 [\![E_0]\!].$$

The proof takes up the rest of this section. Statement (51) provides the promised interpretation of the stability radius. The proof is by a routine deployment of the Banach fixed point theorem, with care taken over making the constants explicit. Our estimates are relatively crude; doubtless our approach could be refined.

4.5. Abbreviated terminology for the proof of Proposition 4.4. Until the end of the proof the linear map $\Phi_0 \in B(\mathcal{S})$ is fixed. Accordingly, we drop reference to Φ_0 in the terminology, saying, e.g., that the triple $(\Lambda_1, M_1, \kappa_1)$ is a nondegenerate solution of the Schwinger-Dyson equation if the quadruple $(\Lambda_1, M_1, \Phi_0, \kappa_1)$ is.

4.6. The deformation equation associated to a nondegenerate solution of the Schwinger-Dyson equation. As in the statement of Proposition 4.4, let

$$(\Lambda_0, M_0, \kappa_0) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$$

be a nondegenerate solution of the Schwinger-Dyson equation. We say that a pair $(\Theta, H) \in \mathcal{S} \times \mathcal{S}$ satisfies the *deformation equation* associated with the triple $(\Lambda_0, M_0, \kappa_0)$ if

$$(52) \qquad\qquad\qquad H = \kappa_0 \left(\Theta M_0 + \Theta H + \Phi_0(H)H \right).$$

Lemma 4.7. *As in the statement of Proposition 4.4, let $(\Lambda_0, M_0, \kappa_0)$ be a nondegenerate solution of the Schwinger-Dyson equation. Fix $(\Lambda_1, M_1) \in \mathcal{S} \times \mathcal{S}$ and write $(\Theta, H) = (\Lambda_1 - \Lambda_0, M_1 - M_0)$. Then the pair (Λ_1, M_1) is a solution of the Schwinger-Dyson equation if and only if the pair (Θ, H) is a solution of the deformation equation (52) associated with the triple $(\Lambda_0, M_0, \kappa_0)$.*

Proof. We first prove the implication ($\Rightarrow$). We have

$$
\begin{aligned}
0 &= 1 + (\Lambda_1 + \Phi_0(M_1))M_1 = 1 + (\Lambda_0 + \Theta + \Phi_0(M_0 + H))(M_0 + H) \\
&= 1 + (\Lambda_0 + \Phi_0(M_0))M_0 + (\Theta + \Phi_0(H))H \\
&\quad + (\Theta + \Phi_0(H))M_0 + (\Lambda_0 + \Phi_0(M_0))H \\
&= \Theta H + \Theta M_0 + \Phi_0(H)H + \Phi_0(H)M_0 - M_0^{-1}H
\end{aligned}
$$

and hence

$$M_0^{-1}H - \Phi_0(H)M_0 = \Theta M_0 + \Theta H + \Phi_0(H)H.$$

Thus the deformation equation (52) holds. The steps of the preceding argument are reversible. Thus the converse ($\Leftarrow$) also holds. $\qquad\square$

Lemma 4.8. *Again, as in the statement of Proposition 4.4, let $(\Lambda_0, M_0, \kappa_0)$ be a nondegenerate solution of the Schwinger-Dyson equation. Fix constants ϵ and δ such that*

$$0 \leq \epsilon \leq \frac{1}{4[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet} \quad and \quad 0 \leq \delta \leq \frac{\epsilon}{4[\![\kappa_0]\!]_\bullet [\![M_0]\!]_\bullet}.$$

Fix $\Lambda \in \mathrm{Ball}_{\mathcal{S}}(\Lambda_0, \delta)$. *(For the latter notation see* §2.5.) *Then there exists unique* $M \in \mathrm{Ball}_{\mathcal{S}}(M_0, \epsilon)$ *such that the pair* (Λ, M) *is a solution of the Schwinger-Dyson equation.*

Proof. Let

$$\Theta = \Lambda - \Lambda_0 \in \mathrm{Ball}_{\mathcal{S}}(0, \delta)$$

and consider the quadratic mapping

$$Q := (x \mapsto \kappa_0 (\Theta M_0 + \Theta x + \Phi_0(x)x)) : \mathcal{S} \to \mathcal{S}.$$

By Lemma 4.7, an element $M \in \mathcal{S}$ has the property that the pair (Λ, M) is a solution of the Schwinger-Dyson equation if and only if the difference $M - M_0$ is a fixed point of Q. Thus our task is transformed to that of proving the existence of a unique fixed point of Q in $\mathrm{Ball}_{\mathcal{S}}(0, \epsilon)$. For achieving the latter goal the Banach fixed point theorem is the natural tool.

We turn now to the analysis of Q restricted to $\mathrm{Ball}_{\mathcal{S}}(0, \epsilon)$. For $x \in \mathrm{Ball}_{\mathcal{S}}(0, \epsilon)$ we have

$$
\begin{aligned}
\llbracket Q(x) \rrbracket &= \llbracket \kappa_0 (\Theta M_0 + \Theta x + \Phi_0(x)x) \rrbracket \\
&\leq \llbracket \kappa_0 \rrbracket \llbracket M_0 \rrbracket \delta + \llbracket \kappa_0 \rrbracket \delta \epsilon + \llbracket \kappa_0 \rrbracket \llbracket \Phi_0 \rrbracket \epsilon^2 \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \epsilon.
\end{aligned}
$$

Thus we have

(53) $$Q(\mathrm{Ball}_{\mathcal{S}}(0, \epsilon)) \subset \mathrm{Ball}_{\mathcal{S}}(0, \epsilon).$$

For $x_1, x_2 \in \mathrm{Ball}_{\mathcal{S}}(0, \epsilon)$ we have

$$
\begin{aligned}
&\llbracket Q(x_1) - Q(x_2) \rrbracket \\
={}& \llbracket \kappa_0 (\Theta M_0 + \Theta x_1 + \Phi_0(x_1)x_1) - \kappa_0 (\Theta M_0 + \Theta x_2 + \Phi_0(x_2)x_2) \rrbracket \\
\leq{}& \llbracket \kappa_0 \rrbracket \llbracket \Theta(x_1 - x_2) + \Phi_0(x_1 - x_2)x_1 + \Phi_0(x_2)(x_1 - x_2) \rrbracket \\
\leq{}& (\llbracket \kappa_0 \rrbracket \delta + \llbracket \kappa_0 \rrbracket \llbracket \Phi_0 \rrbracket \epsilon + \llbracket \kappa_0 \rrbracket \llbracket \Phi_0 \rrbracket \epsilon) \llbracket x_1 - x_2 \rrbracket \\
\leq{}& \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) \llbracket x_1 - x_2 \rrbracket = \frac{3}{4} \llbracket x_1 - x_2 \rrbracket.
\end{aligned}
$$

Thus we have

(54) $$x_1, x_2 \in \mathrm{Ball}_{\mathcal{S}}(0, \epsilon) \Rightarrow \llbracket Q(x_1) - Q(x_2) \rrbracket \leq \frac{3}{4} \llbracket x_1 - x_2 \rrbracket.$$

By (53) and (54) the map Q induces a contraction mapping of the complete metric space $\mathrm{Ball}_{\mathcal{S}}(0, \epsilon)$ to itself. By the Banach fixed point theorem Q indeed has a unique fixed point in $\mathrm{Ball}_{\mathcal{S}}(0, \epsilon)$. $\qquad\square$

4.9. **Proof of Proposition 4.4.** We may assume that

(55) $$\llbracket E_0 \rrbracket \leq \frac{1}{64 \llbracket \kappa_0 \rrbracket_{\bullet}^2 \llbracket M_0 \rrbracket_{\bullet}^2 \llbracket \Phi_0 \rrbracket_{\bullet}^2},$$

since otherwise (51) already holds and there is nothing to prove. Now by the hypothesis of (51) we have $\llbracket G_0 \rrbracket \leq 2\llbracket M_0 \rrbracket_{\bullet}$ and furthermore by (55) we have $\llbracket E_0 \rrbracket \leq \frac{1}{2}$. Thus $\Lambda_0 + \Phi_0(G_0)$ is invertible and its inverse satisfies the bound

(56) $$\llbracket (\Lambda_0 + \Phi_0(G_0))^{-1} \rrbracket \leq 2\llbracket G_0 \rrbracket \leq 4\llbracket M_0 \rrbracket_{\bullet}.$$

Let

$$M = -(\Lambda_0 + \Phi_0(G_0))^{-1} \text{ and } \Lambda = \Lambda_0 + \Phi_0(G_0 - M).$$

The pair (Λ, M) is a solution of the Schwinger-Dyson equation because

$$
\begin{aligned}
1 + (\Lambda + \Phi_0(M))M &= 1 + (\Lambda_0 + \Phi_0(G_0 - M) + \Phi_0(M))M \\
&= 1 + (\Lambda_0 + \Phi_0(G_0))M = 1 - 1 = 0.
\end{aligned}
$$

By (56) and the definitions we have

$$
\begin{aligned}
(57) \qquad [\![G_0 - M]\!] &= [\![G_0 + (\Lambda_0 + \Phi_0(G_0))^{-1}]\!] \\
&= [\![(\Lambda + \Phi_0(G_0))^{-1}E_0]\!] \leq 4[\![M_0]\!]_\bullet [\![E_0]\!].
\end{aligned}
$$

By hypothesis of (51) along with (55) and (57) we have

$$
\begin{aligned}
[\![M - M_0]\!] &\leq [\![G_0 - M]\!] + [\![G_0 - M_0]\!] \\
&\leq 4[\![M_0]\!]_\bullet [\![E_0]\!] + \frac{1}{8[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet} \leq \frac{1}{4[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet}.
\end{aligned}
$$

By (55) and (57) we also have

$$
(58) \qquad [\![\Lambda - \Lambda_0]\!] = [\![\Phi_0(G_0 - M)]\!] \leq 4[\![\Phi_0]\!]_\bullet [\![M_0]\!]_\bullet [\![E_0]\!] \leq \frac{1}{16[\![\kappa_0]\!]_\bullet^2 [\![M_0]\!]_\bullet [\![\Phi_0]\!]_\bullet}.
$$

Applying Lemma 4.8 in the case

$$
(\delta, \epsilon) = \left(\frac{1}{16[\![\kappa_0]\!]_\bullet^2 [\![M_0]\!]_\bullet [\![\Phi_0]\!]_\bullet}, \frac{1}{4[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet} \right),
$$

we conclude that M is the unique element of $\mathrm{Ball}_\mathcal{S}\left(M_0, \frac{1}{4[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet}\right)$ such that (Λ, M) is a solution of the Schwinger-Dyson equation. By applying Lemma 4.8 again in the case

$$
(\delta, \epsilon) = ([\![\Lambda - \Lambda_0]\!], 4[\![\kappa_0]\!]_\bullet [\![M_0]\!]_\bullet [\![\Lambda - \Lambda_0]\!])
$$

we find that in fact

$$
[\![M - M_0]\!] \leq 4[\![\kappa_0]\!]_\bullet [\![M_0]\!]_\bullet [\![\Lambda - \Lambda_0]\!].
$$

Thus by (57) and (58) we have

$$
\begin{aligned}
[\![G_0 - M_0]\!] &\leq [\![G_0 - M]\!] + [\![M - M_0]\!] \\
&\leq 4[\![M_0]\!]_\bullet [\![E_0]\!] + (4[\![\kappa_0]\!]_\bullet [\![M_0]\!]_\bullet)(4[\![M_0]\!]_\bullet [\![\Phi_0]\!]_\bullet)[\![E_0]\!],
\end{aligned}
$$

which suffices to prove (51). $\qquad \square$

5. Proof of Proposition 4.3

The plan of proof is as follows. We first state a result about equation (6). (See Proposition 5.2 below.) We then use this result to derive Proposition 4.3. Finally we prove Proposition 5.2. The only tool we use here is high school algebra.

5.1. **Key estimates involving equation** (6). Let

$$
(59) \qquad \omega = \sqrt{\sqrt{5} - 2} \cong 0.4858682712,
$$

which is the unique positive root of the polynomial

$$
(60) \qquad m^4 + 4m^2 - 1 = (m - \omega)(m + \omega)(m - \mathrm{i}/\omega)(m + \mathrm{i}/\omega).
$$

Repeating (7) for the reader's convenience, let

$$
(61) \qquad\qquad \zeta = \sqrt{\frac{11 + 5\sqrt{5}}{2}} \cong 3.330190676,
$$

which is the unique positive root of the polynomial

$$
(62) \qquad\qquad z^4 - 11z^2 - 1 = (z - \zeta)(z + \zeta)(z - \mathrm{i}/\zeta)(z + \mathrm{i}/\zeta).
$$

It can be shown that the system of equations

$$
(63) \qquad
\begin{aligned}
zm^3 - m^2 - zm - 1 &= 0 \\
\tfrac{\partial}{\partial m}\left(zm^3 - m^2 - zm - 1\right) &= 0
\end{aligned}
$$

has exactly four complex solutions, namely

$$
(64) \qquad\qquad (z, m) = (-\zeta, \omega), (\zeta, -\omega), (-\mathrm{i}/\zeta, \mathrm{i}/\omega), (\mathrm{i}/\zeta, -\mathrm{i}/\omega).
$$

(We omit the proof of this fact since we do not actually use it in the sequel.) These four points in $\mathbb{C}^2$ are where the Implicit Function Theorem fails to yield locally a solution $m = m(z)$ of (6) depending analytically on z. Thus the numbers ζ and ω are not pulled out of thin air; rather, they naturally call attention to themselves in connection with the geometry of the plane algebraic curve (6).

Our main technical result in this section, by means of which we will prove Proposition 4.3, is the following.

Proposition 5.2. *If $z \in \mathfrak{h}$ and $m = m_{\{\mathbf{uv}\}}(z)$, then*

$$
(65) \qquad\qquad |m| \ \leq\ 1 \wedge \frac{1}{\Im z},
$$

$$
(66) \qquad\qquad |\Re m| \ \leq\ \omega < \frac{1}{2} \ \text{ and }
$$

$$
(67) \qquad\qquad |m^2 - \omega^2| \ \geq\ \frac{\sqrt{h}}{c},
$$

where $c \geq 1$ is an absolute constant and h is the quantity on line (8).

The proof of the proposition takes up the rest of this section after we have made the application to the proof of Proposition 4.3.

5.3. **Remark.** The recent paper [18] sheds light on the more delicate properties of the laws of self-adjoint polynomials in free semicircular variables, including lack of atoms and algebraicity of Stieltjes transforms. It is an open problem to refine this theory to yield a general analogue of Proposition 4.3. Such an analogue would make it possible to prove a local limit law for self-adjoint polynomials in Wigner matrices. We consciously overkill the proofs of Propositions 4.3 and 5.2 here in the hope that some among the details could provide clues for the theory (partly algebraic geometry and partly operator theory) we would like to have.

5.4. **Proof of Proposition 4.3 with Proposition 5.2 granted.** We break the proof down into several steps.

5.4.1. *Proof that* (Λ, M, Φ) *solves the Schwinger-Dyson equation.* Since $m = m_{\{uv\}}(z)$ the pair (z, m) satisfies (6). Note that equation (6) can be rewritten as

$$(68) \qquad z = \frac{m^2 + 1}{m^3 - m} = \frac{1}{m-1} + \frac{1}{m+1} - \frac{1}{m}.$$

Recall that

$$\Lambda = \begin{bmatrix} z & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} m & 0 & 0 \\ 0 & -\frac{1}{m-1} & 0 \\ 0 & 0 & -\frac{1}{m+1} \end{bmatrix}.$$

Using (24) to calculate the action of Φ and also exploiting (68), we have

$$\Phi(M) = \begin{bmatrix} -\frac{1}{m-1} - \frac{1}{m+1} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad \text{and}$$

$$\Lambda + \Phi(M) = \begin{bmatrix} -\frac{1}{m} & 0 & 0 \\ 0 & m-1 & 0 \\ 0 & 0 & m+1 \end{bmatrix} = -M^{-1}.$$

Thus (Λ, M, Φ) is indeed a solution of the Schwinger-Dyson equation.

5.4.2. *Proof of the bounds* (49). The first bound is clear. The second bound is proved as follows:

$$[\![\Phi(A)]\!] \le ([\![e_{12} + e_{21}]\!]^2 + [\![e_{13} + e_{31}]\!]^2)[\![A]\!] \le 8[\![A]\!].$$

The third and fourth bounds are equivalent to the statements

$$|m| \vee \left|\frac{1}{m-1}\right| \vee \left|\frac{1}{m-1}\right| \le 2 \quad \text{and} \quad |m| \vee \left|\frac{m}{m-1}\right| \vee \left|\frac{m}{m+1}\right| \le 2\left(1 \wedge \frac{1}{\Im z}\right),$$

respectively. Both bounds follow easily from (65) and (66).

5.4.3. *Proof of nondegeneracy.* Abusing notation since we haven't yet proved invertibility, let κ^{-1} denote the linear map (45). Then, making use of formula (24) again, we have

$$\kappa^{-1}\left(\begin{bmatrix} x_1 & x_4 & x_6 \\ x_5 & x_2 & x_8 \\ x_7 & x_9 & x_3 \end{bmatrix}\right) = \begin{bmatrix} 1/m & 0 & 0 \\ 0 & -(m-1) & 0 \\ 0 & 0 & -(m+1) \end{bmatrix}\begin{bmatrix} x_1 & x_4 & x_6 \\ x_5 & x_2 & x_8 \\ x_7 & x_9 & x_3 \end{bmatrix}$$

$$- \begin{bmatrix} x_2 + x_3 & x_5 & x_7 \\ x_4 & x_1 & 0 \\ x_6 & 0 & x_1 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & -\frac{1}{m-1} & 0 \\ 0 & 0 & -\frac{1}{m+1} \end{bmatrix}.$$

With respect to the basis for Mat_3 dual to the peculiar numbering of matrix entries in (24), the matrix for κ^{-1} is block diagonal with diagonal blocks

$$(69) \qquad \begin{bmatrix} 1/m & -m & -m \\ \frac{1}{m-1} & -(m-1) & 0 \\ \frac{1}{m+1} & 0 & -(m+1) \end{bmatrix}, \quad \begin{bmatrix} 1/m & \frac{1}{m-1} \\ -m & -(m-1) \end{bmatrix},$$

$$\begin{bmatrix} 1/m & \frac{1}{m+1} \\ -m & -(m+1) \end{bmatrix}, \quad \begin{bmatrix} -(m-1) & 0 \\ 0 & -(m+1) \end{bmatrix},$$

respectively. The determinants of these blocks are

$$(70) \qquad -\frac{m^4 + 4m^2 - 1}{m(m-1)(m+1)}, \quad \frac{2m-1}{m(m-1)}, \quad -\frac{2m+1}{m(m+1)}, \quad (m-1)(m+1),$$

respectively. By (65) and (66) and the lower bound $\Im m > 0$, the determinants on the list (70) are finite and nonzero. It follows that (Λ, M, Φ) is nondegenerate and hence κ is well-defined.

5.4.4. *Proof of the bound* (50). The inverses of the diagonal blocks on the list (69) are

$$\frac{\begin{bmatrix} -(m^2-1)^2 m & m^2(m^2-1)(m+1) & m^2(m^2-1)(m-1) \\ -(m+1)^2 m & (2m+1)(m-1) & m^2(m+1) \\ -(m-1)^2 m & m^2(m-1) & -(2m-1)(m+1) \end{bmatrix}}{(m^4 + 4m^2 - 1)},$$

$$\frac{\begin{bmatrix} -(m-1)^2 m & -m \\ m^2(m-1) & m-1 \end{bmatrix}}{2m-1}, \quad \frac{\begin{bmatrix} (m+1)^2 m & m \\ -m^2(m+1) & -(m+1) \end{bmatrix}}{2m+1},$$

$$\begin{bmatrix} -\frac{1}{m-1} & 0 \\ 0 & -\frac{1}{m+1} \end{bmatrix},$$

respectively. By (65) and (66) along with the factorization (60), the entries of the matrices above are bounded in absolute value by $c_1/|m^2-\omega^2|$. It follows by Proposition 5.5 appearing immediately after this proof that $[\![\kappa]\!] \le c_2/|m^2-\omega^2|$. Finally, the bound (50) follows via (67). The proof of Proposition 4.3 is now complete modulo Propositions 5.2 and 5.5. $\qquad\square$

Proposition 5.5. *Let $\psi \in B(\mathrm{Mat}_n)$ be any linear map. Let $\{e_{ij}\}_{i,j=1}^n$ be the standard basis of Mat_n consisting of elementary matrices. Write*

$$\psi(e_{i_2 j_2}) = \sum_{i_1, j_1} \psi(i_1, j_1, i_2, j_2) e_{i_1 j_1}$$

for scalars $\psi(i_1, j_1, i_2, j_2)$. Then

$$[\![\psi]\!] \le \sqrt{n} \sum_{i_1, j_1, i_2, j_2} |\psi(i_1, j_1, i_2, j_2)|.$$

We omit the routine proof.

5.6. **Algebraic identities.** We prepare for the proof of Proposition 5.2 by deriving a certain algebraic identity. Let a and t be independent (commuting) algebraic variables. The following polynomial congruences hold modulo $a^4 + 4a^2 - 1$ and were derived with the help of a computer algebra system:

$$(71) \qquad t^4 - 11t^2 - 1 \Big|_{t=\frac{3a^3+13a}{2}} \equiv 0,$$

$$(72) \qquad \left(\frac{3a^3 + 13a}{2}\right)\left(\frac{a^3 + a}{2}\right) \equiv 1,$$

$$(73) \qquad (t^3 - t) \pm \left(\frac{a^3 + a}{2}\right)(t^2 + 1) \equiv \left(t \pm \frac{a^3 + 5a}{2}\right)(t \mp a)^2.$$

In particular, the numerical identities

$$\zeta = \frac{3\omega^3 + 13\omega}{2} \quad \text{and} \quad \frac{1}{\zeta} = \frac{\omega^3 + \omega}{2}$$

follow from (71) and (72), respectively. Let

$$\rho = \frac{\omega^3 + 5\omega}{2} \cong 1.272019648.$$

Now let $z \in \mathfrak{h}$ and $m = m_{\{\mathbf{uv}\}}(z)$ as in Proposition 5.2, then substitute $(t, a) = (m, \omega)$ in (73) and finally take the product over the two choices of signs. We thus obtain the identity

$$(1 - z^2/\zeta^2)(m^3 - m)^2 = (m^2 - \rho^2)(m^2 - \omega^2)^2.$$

Taking absolute values on both sides and then square roots we obtain the relation

(74) $$\frac{1}{|m^2 - \omega^2|} = \frac{|m^2 - \rho^2|^{1/2}}{|m^2 - 1|} \frac{1}{|m|} \frac{\zeta}{|z^2 - \zeta^2|^{1/2}}$$

after some rearrangement.

5.7. **The quadrant-lifting diagram.** We continue preparation for the proof of Proposition 5.2. We introduce a visual aid to explain the geometry of equation (6). Let $m = u + iv$ with u and v real. Then for $m^3 - m \neq 0$ we have formulas

(75) $$\Re \frac{m^2 + 1}{m^3 - m} = \frac{u((u^2 + v^2)^2 - 4v^2 - 1)}{|m^3 - m|^2},$$

(76) $$\Im \frac{m^2 + 1}{m^3 - m} = -\frac{v((u^2 + v^2)^2 + 4u^2 - 1)}{|m^3 - m|^2}.$$

It follows that

(77) $$\left\{ m \in \mathbb{C} \setminus \{-1, 0, 1\} \middle| \Re \frac{m^2 + 1}{m^3 - m} = 0 \right\} \cup \{-1, 1\}$$

$$= \left\{ \pm \left(\sqrt{\sqrt{1 + 4t^2} - t^2} + it \right) \middle| |t| \leq \frac{1}{\omega} \right\} \cup i\mathbb{R},$$

(78) $$\left\{ m \in \mathbb{C} \setminus \{-1, 0, 1\} \middle| \Im \frac{m^2 + 1}{m^3 - m} = 0 \right\} \cup \{0\}$$

$$= \left\{ \pm \left(t + i\sqrt{\sqrt{1 - 4t^2} - t^2} \right) \middle| |t| \leq \omega \right\} \cup \mathbb{R}.$$

By plotting the sets (77) and (78) in the complex plane and also keeping track of the signs of $\Re \frac{m^2+1}{m^3-m}$ and $\Im \frac{m^2+1}{m^3-m}$ we obtain Figure 2 in which each of the twelve regions is labeled by the quadrant of the complex plane to which it is sent by the map $m \mapsto \frac{m^2+1}{m^3-m}$. Accordingly, we call this figure the *quadrant lifting diagram* associated with equation (6).

5.8. **Proof of Proposition 5.2.** Since $z, m \in \mathfrak{h}$ and (z, m) satisfies (6), a glance at the quadrant lifting diagram reveals that m belongs to the set in the complex plane bounded below by the interval $[-\omega, \omega]$ and above by the contour

$$t \mapsto t + i\sqrt{\sqrt{1 - 4t^2} - t^2} \quad \text{for } |t| \leq \omega.$$

This observation overkills the proof that $|m| \leq 1$ and $|\Re m| \leq \omega$. We already have $|m| \leq 1/\Im z$ because m is the value at z of a Stieltjes transform. Thus the bounds
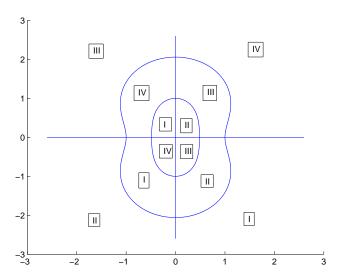
FIGURE 2. The quadrant lifting diagram

(65) and (66) hold. It remains only to prove the bound (67). Bounding the right side of (74) by means of (65) and (66) we find that

$$\frac{1}{|m^2 - \omega^2|} \leq \frac{c_1}{|m|\sqrt{|z^2 - \zeta^2|}}.$$

Using (65), (66) and (68) we deduce a bound

$$\frac{1}{|m|} \leq |z| + \frac{2}{1 - \omega} \leq c_2(1 + |z|).$$

We have finally a bound

$$\frac{1 + |z|}{\sqrt{|z^2 - \zeta^2|}} \leq \frac{c_3}{\sqrt{1 \wedge |z - \zeta| \wedge |z + \zeta|}}.$$

Combining displayed lines above we obtain the desired bound (67). The proof of Proposition 5.2 is complete. (Thus also the proof of Proposition 4.3 is complete.) □

6. A GENERAL MATRIX-VALUED SELF-CONSISTENT EQUATION

We prove a technical result similar in intent to [9, Lemma 4.3] if superficially different in form. (See Proposition 6.2 below.) The result is an elaboration and refinement of Proposition 4.4. Our methods here are deterministic and algebraic.

6.1. **Setup for the technical result.** Fix a finite-dimensional unital Banach algebra $\mathcal{S}$. Fix a nondegenerate solution

$$(\Lambda_0, M_0, \Phi_0, \kappa_0)$$

of the Schwinger-Dyson equation defined over $\mathcal{S}$ for which (recall) $\frac{1}{8[\![\kappa_0]\!]_\bullet[\![\Phi_0]\!]_\bullet}$ is by definition the stability radius. Fix a family

$$\{G_i, \widehat{G}_i\}_{i=1}^N$$

of elements of $\mathcal{S}$ where all the G_i are invertible. (It is not necessary to assume that the $\widehat{G}_i$ are invertible.) Consider the statistic

$$\mathfrak{E} \;=\; \bigvee_{i=1}^{N} \frac{\left[\!\left[G_i^{-1} + \Lambda_0 + \Phi_0(\widehat{G}_i) \right]\!\right]}{\left[\!\left[\widehat{G}_i \right]\!\right]_{\bullet}^{1/2}} \vee \bigvee_{i=1}^{N} \sqrt{\frac{\left[\!\left[\widehat{G}_i - \frac{1}{N}\sum_{i=1}^{N} G_i \right]\!\right]}{\left[\!\left[G_i \right]\!\right]_{\bullet}\left[\!\left[G_i^{-1} \right]\!\right]}},$$

which is a gauge of error in this situation. The idea to emphasize the statistic $\mathfrak{E}$ clearly derives from [9, Lemma 4.3] and the related constellation of identities and estimates. The following is the main result of this section.

Proposition 6.2. *Notation and assumptions are as above. We have*

$$(79) \qquad\qquad \bigvee_{i=1}^{N} [\![G_i - M_0]\!] \le \frac{1}{8[\![\kappa_0]\!]_{\bullet}[\![\Phi_0]\!]_{\bullet}}$$

$$\Rightarrow \quad \bigvee_{i=1}^{N} [\![G_i - M_0]\!] \le 2^{14}(1 + [\![M_0]\!])^{7}([\![\Phi_0]\!]_{\bullet} \vee [\![\Lambda_0]\!]_{\bullet})^{4}[\![\kappa_0]\!]_{\bullet}\mathfrak{E}.$$

Proof. Let

$$\mathfrak{G} = \bigvee_{i=1}^{N} [\![G_i]\!]_{\bullet}, \quad G = \frac{1}{N}\sum_{i=1}^{N} G_i \;\text{ and }\; E = 1 + (\Lambda_0 + \Phi_0(G))G.$$

Also to abbreviate notation let

$$\mathfrak{M} \;=\; 1 + [\![M_0]\!] \;\text{ and }\; \mathfrak{F} = [\![\Phi_0]\!]_{\bullet} \vee [\![\Lambda_0]\!]_{\bullet}.$$

We temporarily assume that

$$(80) \qquad\qquad [\![E]\!] \vee \bigvee_{i=1}^{N} [\![G_i - G]\!] \le 2^{8}\mathfrak{G}^{5}\mathfrak{F}^{3}\mathfrak{E}.$$

Presently we will explain how to lift this assumption. By the hypothesis of (79) we have $[\![G - M_0]\!] \le \frac{1}{8[\![\kappa_0]\!]_{\bullet}[\![\Phi_0]\!]_{\bullet}}$ and hence $\mathfrak{G} \le \mathfrak{M}$. Thus by (51) and (80) we have

$$\bigvee_{i=1}^{N} [\![G_i - M_0]\!] \;\le\; [\![G - M_0]\!] + \bigvee_{i=1}^{N} [\![G_i - G]\!] \;\le\; 20[\![\kappa_0]\!]_{\bullet}[\![\Phi_0]\!]_{\bullet}[\![M_0]\!]_{\bullet}^{2}[\![E]\!] + 2^{8}\mathfrak{G}^{5}\mathfrak{F}^{3}\mathfrak{E}$$

$$\le\; (2^{5}[\![\kappa_0]\!]_{\bullet}\mathfrak{F}\mathfrak{M}^{2} + 1)2^{8}\mathfrak{M}^{5}\mathfrak{F}^{3}\mathfrak{E} \le 2^{14}\mathfrak{M}^{7}\mathfrak{F}^{4}[\![\kappa_0]\!]_{\bullet}\mathfrak{E},$$

i.e., the conclusion of (79) holds.

It remains now only to prove (80). We will not need the hypothesis of (79) for that purpose. We may assume that

$$(81) \qquad\qquad \mathfrak{E}^{2} \le \mathfrak{E} \le \frac{1}{2^{6}\mathfrak{G}^{3}\mathfrak{F}^{2}} \le 1$$

because the left side of (80) is trivially bounded by $2^{2}\mathfrak{G}^{2}\mathfrak{F}$.

We first bound $\left[\!\left[G - \widehat{G}_i\right]\!\right]$. We calculate as follows.

$$
\begin{aligned}
\left[\!\left[G_i^{-1}\right]\!\right] &\leq \left[\!\left[G_i^{-1} + \Lambda_0 + \Phi_0(\widehat{G}_i)\right]\!\right] + \mathfrak{F} + \mathfrak{F}\left[\!\left[\widehat{G}_i\right]\!\right] \\
&\leq \mathfrak{E}\left[\!\left[\widehat{G}_i\right]\!\right]_\bullet^{1/2} + 2\mathfrak{F}\left[\!\left[\widehat{G}_i\right]\!\right]_\bullet \leq 4\mathfrak{F}\left[\!\left[\widehat{G}_i\right]\!\right]_\bullet \\
&\leq 4\mathfrak{F}[\![G]\!]_\bullet + 4\mathfrak{F}\left[\!\left[G - \widehat{G}_i\right]\!\right] \leq 4\mathfrak{G}\mathfrak{F} + 4\mathfrak{F}\mathfrak{E}^2[\![G_i]\!]_\bullet\left[\!\left[G_i^{-1}\right]\!\right] \\
&\leq 4\mathfrak{G}\mathfrak{F} + 4\mathfrak{G}\mathfrak{F}\mathfrak{E}\left[\!\left[G_i^{-1}\right]\!\right].
\end{aligned}
$$

Since $4\mathfrak{G}\mathfrak{F}\mathfrak{E} \leq \frac{1}{2}$ by (81) and hence $\left[\!\left[G_i^{-1}\right]\!\right] \leq 8\mathfrak{G}\mathfrak{F}$ we have

$$
\text{(82)} \qquad\qquad \left[\!\left[G - \widehat{G}_i\right]\!\right] \leq \mathfrak{E}^2[\![G_i]\!]_\bullet\left[\!\left[G_i^{-1}\right]\!\right] \leq 8\mathfrak{G}^2\mathfrak{F}\mathfrak{E}.
$$

We next bound $[\![E]\!]$. We calculate as follows.

$$
\begin{aligned}
\left[\!\left[G_i^{-1} + \Lambda_0 + \Phi_0(G)\right]\!\right] &\leq \mathfrak{E}\left[\!\left[\widehat{G}_i\right]\!\right]_\bullet^{1/2} + \mathfrak{F}\left[\!\left[G - \widehat{G}_i\right]\!\right] \\
&\leq \mathfrak{E}[\![G]\!]_\bullet^{1/2} + \mathfrak{E}\left[\!\left[G - \widehat{G}_i\right]\!\right]^{1/2} + \mathfrak{F}\left[\!\left[G - \widehat{G}_i\right]\!\right] \\
&\leq \mathfrak{E}[\![G]\!]_\bullet^{1/2} + \mathfrak{E}^2 + \left[\!\left[G - \widehat{G}_i\right]\!\right] + \mathfrak{F}\left[\!\left[G - \widehat{G}_i\right]\!\right] \\
&\leq 2\mathfrak{G}^{1/2}\mathfrak{E} + 2\mathfrak{F}\left[\!\left[G - \widehat{G}_i\right]\!\right] \\
&\leq 2\mathfrak{G}^{1/2}\mathfrak{E} + 16\mathfrak{G}^2\mathfrak{F}^2\mathfrak{E} \leq 2^5\mathfrak{G}^2\mathfrak{F}^2\mathfrak{E}.
\end{aligned}
$$

We used the arithmetic-geometric mean inequality at the third step above and (82) at the penultimate step. We conclude that

$$
\text{(83)} \qquad\qquad [\![E]\!] \leq \bigvee_{i=1}^{N} [\![1 + (\Lambda_0 + \Phi_0(G))G_i]\!] \leq 2^5\mathfrak{G}^3\mathfrak{F}^2\mathfrak{E}.
$$

Finally we bound $[\![G - G_i]\!]$. By (81), the left side of (83) is bounded by $\frac{1}{2}$. Thus $\Lambda_0 + \Phi_0(G)$ is invertible and we have $\left[\!\left[(\Lambda_0 + \Phi_0(G))^{-1}\right]\!\right] \leq 2[\![G]\!]$. In turn we have by (83) that

$$
\left[\!\left[(\Lambda_0 + \Phi_0(G))^{-1} + G\right]\!\right] \vee \left[\!\left[(\Lambda_0 + \Phi_0(G))^{-1} + G_i\right]\!\right] \leq 2^6\mathfrak{G}^4\mathfrak{F}^2\mathfrak{E}
$$

and hence

$$
\text{(84)} \qquad\qquad \bigvee_{i=1}^{N} [\![G - G_i]\!] \leq 2^7\mathfrak{G}^4\mathfrak{F}^2\mathfrak{E}.
$$

The bound (80) follows now from (83) and (84). The proof of Proposition 6.2 is complete. $\qquad\square$

7. Analysis of the generalized resolvent

We return to the deterministic setting of §3 exclusive of §3.6. We continue the analysis of objects related to the generalized resolvent R.

Here is our main result in §7.

Theorem 7.1. *Notation and assumptions are as set forth in §3. Consider the compact rectangle*

$$(85) \qquad \mathcal{R} = \left\{ z \in \mathfrak{h} \,\middle|\, |\Re z| \leq \tau \ \text{and} \ \frac{1}{N} \leq \Im z \leq \tau \right\}$$

where $\tau \geq 1$ is an absolute constant. We write $\mathfrak{K}(z)$ to show z-dependence, it being understood that U and V are held fixed as z varies. Let

$$K = \theta \sup_{z \in \mathcal{R}} \mathfrak{K}(z) < \infty$$

where $\theta \geq 1$ is another absolute constant. Let h be as defined on line (8). Consider also the compact (possibly empty) set

$$\mathcal{X} = \left\{ z \in \mathcal{R} \,\middle|\, \frac{K^2}{N} \leq h^2 \Im z \right\}.$$

Then we have

$$(86) \qquad [\![U]\!] \vee [\![V]\!] \leq 4 \ \text{and} \ z \in \mathcal{X} \ \Rightarrow \ \bigvee_{i=1}^{N} [\![G_i - M]\!] \leq \frac{K}{\sqrt{Nh\Im z}}$$

provided that τ is sufficiently large and θ is sufficiently large depending on τ.

The proof will be completed in §7.5 after some preparation.

7.2. **An *a priori* bound.** We have in general a bound

$$(87) \qquad \bigvee_{i=1}^{N} [\![G_i - M]\!] \leq \frac{2^7([\![U]\!] \vee [\![V]\!] \vee 1)^2}{\Im z}$$

obtained by combining (19), (30) and (49). We emphasize that the hypothesis of (86) is not used here. Use of the bound (87) in this paper turns out to be precisely the technical innovation that permits us to avoid the cumbersome two-variable resolvent apparatus of [1].

The next result meshes the self-adjoint linearization trick with the self-improving sort of estimate exploited in [9].

Proposition 7.3. *For $i = 1, \ldots, N$ we have*

$$(88) \qquad \left[\!\!\left[G - \widehat{G}_i \right]\!\!\right] \ \leq \ 16[\![W]\!]^2 \frac{(\Im z)_\bullet}{N\Im z} [\![G_i]\!]_\bullet \left[\!\!\left[G_i^{-1} \right]\!\!\right] \ \text{and}$$

$$(89) \qquad \left[\!\!\left[G_i^{-1} + \Lambda + \Phi(\widehat{G}_i) \right]\!\!\right] \ \leq \ 4\mathfrak{K}[\![W]\!] \sqrt{\frac{(\Im z)_\bullet}{N\Im z}} \left[\!\!\left[\widehat{G}_i \right]\!\!\right]_\bullet^{1/2}.$$

We emphasize that the hypothesis of (86) is not used here.

Proof. By (19), (29) and the matrix Hölder inequality, we have

$$\operatorname{tr} \frac{\Im G_i}{\Im z} = \operatorname{tr} \mathbf{e}_i \frac{\Im R}{\Im z} \mathbf{e}_i^* = [\![\mathbf{e}_i W r]\!]_2^2 \geq \frac{[\![\mathbf{e}_i W r W^*]\!]_2^2}{[\![W^*]\!]^2} = \frac{[\![\mathbf{e}_i (R + \Lambda^0 \otimes \mathbf{I}_N)]\!]_2^2}{[\![W]\!]^2}$$

and similarly

$$\operatorname{tr} \frac{\Im G_i}{\Im z} \geq \frac{[\![(R + \Lambda^0 \otimes \mathbf{I}_N)\mathbf{e}_i^*]\!]_2^2}{[\![W]\!]^2}.$$

It follows that

$$\sqrt{2} + [\![W]\!]\sqrt{\operatorname{tr}\frac{\Im G_i}{\Im z}} \geq [\![\mathbf{e}_i R]\!]_2 \vee [\![R\mathbf{e}_i^*]\!]_2.$$

It follows in turn by using the convexity bound $2(x^2 + y^2) \geq (x+y)^2$ that

$$(90) \quad 16[\![W]\!]^2 \frac{(\Im z)_\bullet}{\Im z}[\![G_i]\!]_\bullet \geq 4 + 2[\![W]\!]^2 \operatorname{tr}\frac{\Im G_i}{\Im z} \quad \geq \quad [\![\mathbf{e}_i R]\!]_2^2 \vee [\![R\mathbf{e}_i^*]\!]_2^2, \text{ hence}$$

$$16[\![W]\!]^2 \frac{(\Im z)_\bullet}{\Im z}[\![G]\!]_\bullet \geq 4 + 2[\![W]\!]^2 \operatorname{tr}\frac{\Im G}{\Im z} \quad \geq \quad \frac{[\![R]\!]_2^2}{N}, \text{ similarly}$$

$$16[\![W]\!]^2 \frac{(\Im z)_\bullet}{\Im z}[\![\widehat{G}_i]\!]_\bullet \geq 4 + 2[\![W]\!]^2 \operatorname{tr}\frac{\Im \widehat{G}_i}{\Im z} \quad \geq \quad \frac{[\![R_i]\!]_2^2}{N} \text{ and hence}$$

$$(91) \quad 4[\![W]\!]\sqrt{\frac{(\Im z)_\bullet}{N\Im z}}[\![\widehat{G}_i]\!]_\bullet^{1/2} \quad \geq \quad \frac{1}{\sqrt{N}}\left(1 \vee \frac{[\![R_i]\!]_2}{\sqrt{N}}\right).$$

Statements (36) and (90) prove (88). Statements (34) and (91) along with the definition of $\mathfrak{K}$ prove (89). $\qquad\square$

The following key result combines Propositions 4.3, 6.2 and 7.3.

Proposition 7.4. *We have*

$$(92) \quad \bigvee_{i=1}^{N}[\![G_i - M]\!] \leq \frac{\sqrt{h}}{c_{4.3}} \Rightarrow \bigvee_{i=1}^{N}[\![G_i - M]\!] \leq \frac{C(1+|z|)^5[\![W]\!]}{\sqrt{Nh\Im z}}\mathfrak{K}$$

where C is an absolute constant.

We emphasize that the hypothesis of (86) is not used here.

Proof. Proposition 6.2 specialized to the present setup is the assertion that

$$\bigvee_{i=1}^{N}[\![G_i - M]\!] \leq \frac{1}{8[\![\kappa]\!]_\bullet[\![\Phi]\!]_\bullet}$$

$$\Rightarrow \quad \bigvee_{i=1}^{N}[\![G_i - M]\!] \leq 2^{14}(1 + [\![M]\!])^7([\![\Phi]\!]_\bullet \vee [\![\Lambda]\!]_\bullet)^4[\![\kappa]\!]_\bullet \mathfrak{E}$$

where the quantity $\mathfrak{E}$ satisfies

$$\mathfrak{E} \leq 4\mathfrak{K}[\![W]\!]\sqrt{\frac{(\Im z)_\bullet}{N\Im z}}$$

by Proposition 7.3 and the definition of $\mathfrak{K}$. We obtain (92) after simplifying by means of Proposition 4.3. $\qquad\square$

7.5. **Proof of Theorem 7.1.** Whereas above we abstained from using the hypothesis of (86), we now enforce it throughout the remainder of the argument.

7.5.1. *Setup for application of Proposition 1.5.* In the triple (U, V, z) we hold U and V fixed subject to the condition $[\![U]\!] \vee [\![V]\!] \leq 4$. We allow z to vary but we constrain it to the space $\mathcal{X} \subset \mathfrak{h}$. On the space $\mathcal{X}$ we consider the three continuous functions

$$f_1 = \bigvee_{i=1}^{N}[\![G_i - M]\!], \quad f_2 = \frac{\sqrt{h}}{c_{4.3}} \text{ and } f_3 = \frac{K}{c_{4.3}\sqrt{Nh\Im z}}.$$

The rest of the proof is a matter of checking hypotheses in Proposition 1.5. The process of checking naturally dictates choices for the absolute constants τ and θ.

7.5.2. $\mathcal{X}$ *is connected if not empty.* Let $\rho = \frac{K^2}{N}$. If $\rho > \tau$ then for $z \in \mathcal{X}$ we have $1 \leq \tau < \rho \leq h^2 \Im z = \Im z$, in contradiction to the definition of $\mathcal{R}$, and hence $\mathcal{X}$ is empty. Assume that $\rho \leq \tau$ hereafter. Then the set $\mathcal{X}$ contains the top side of $\mathcal{R}$, i.e., the horizontal line segment $\{x + i\tau \mid -\tau \leq x \leq \tau\}$. Furthermore, since the function $h^2 \Im z$ is monotone increasing on vertical line segments in $\mathcal{R}$, each point of $\mathcal{X}$ is connected to the top side of $\mathcal{R}$ by a vertical line segment contained in $\mathcal{X}$. Thus $\mathcal{X}$ is indeed connected if nonempty.

7.5.3. *Checking hypothesis* (12). Consider the statement

$$(93) \qquad \left.\bigvee_{i=1}^{N} [\![G_i - M]\!] \right|_{z=i\tau} \leq \left.\frac{2^{11}}{\Im z}\right|_{z=i\tau} = \frac{2^{11}}{\tau} < \frac{1}{c_{4.3}} = \left.\frac{\sqrt{h}}{c_{4.3}}\right|_{z=i\tau}.$$

The first inequality holds by (87). The third inequality can be made to hold by choosing τ large enough. So now we fix $\tau \geq 1$ large enough to make the statement (93) hold. Then hypothesis (12) of Proposition 1.5 is verified.

7.5.4. *Checking hypothesis* (13). We next choose θ so that

$$\theta \geq 2c_{4.3}C_{7.4}(1 + 2\tau)^5 2^5 \geq 2c_{4.3}C_{7.4}(1 + 2\tau)^5 [\![W]\!],$$

where the second inequality holds by (19). Then by Proposition 7.4 we have

$$\bigvee_{i=1}^{N} [\![G_i - M]\!] \leq \frac{\sqrt{h}}{c_{4.3}} \Rightarrow \bigvee_{i=1}^{N} [\![G_i - M]\!] \leq \frac{K}{2c_{4.3}\sqrt{Nh\Im z}}.$$

With θ thus fixed, hypothesis (13) of Proposition 1.5 is verified.

7.5.5. *Checking hypothesis* (14). Finally we have

$$\frac{K}{2c_{4.3}\sqrt{Nh\Im z}} \leq \frac{\sqrt{h}}{2c_{4.3}} < \frac{\sqrt{h}}{c_{4.3}}$$

by definition of $\mathcal{X}$. Thus hypothesis (14) of Proposition 1.5 is verified. The conclusion (15) of Proposition 1.5 is then the same as the conclusion (86) of Theorem 7.1. The proof of Theorem 7.1 is complete. $\qquad\square$

The following technical result is needed in §8 for construction of the random variable $\mathbf{K}$ figuring in Theorem 1.2.

Proposition 7.6. *For $i = 1, \ldots, N$ and distinct $z_1, z_2 \in \mathfrak{h}$ we have*

$$(94) \qquad [\![U]\!] \vee [\![V]\!] \leq 4 \text{ and } (\Im z_1) \wedge (\Im z_2) \geq \frac{1}{N} \Rightarrow \frac{|\mathfrak{K}_i(z_1) - \mathfrak{K}_i(z_2)|}{|z_1 - z_2|} \leq cN^{7/2}$$

where c is an absolute constant.

Proof. Temporarily we write $R(z)$, $r(z)$, $R_i(z)$ and $Q_i(z)$ to show z-dependence with U and V held fixed. Consider the functions

$$f, g : \{z \in \mathfrak{h} \mid \Im z \geq 1/N\} \to [0, \infty)$$

given by the formulas

$$f(z) = \sqrt{N} \vee [\![R_i(z)]\!]_2 \text{ and } g(z) = N[\![Q_i(z)]\!].$$

Our task is to estimate the Lipschitz constant of $1 \vee \frac{g}{f}$. It is enough to estimate the Lipschitz constant of $\frac{g}{f}$. Let

$$C = 1 + [\![\Phi]\!] + [\![X]\!] + [\![X]\!]^2 + 3[\![W]\!]^2.$$

By (19), under the assumption $[\![U]\!] \vee [\![V]\!] \leq 4$, the quantity C is bounded by an absolute constant. Thus it will suffice to to bound the Lipschitz constant of $\frac{g}{f}$ by a polynomial in C times $N^{7/2}$.

By the first of the identities on line (29) and the standard resolvent bound $[\![r(z)]\!] \leq 1/\Im z$ we have for $\Im z \geq 1/N$ that

$$[\![R(z)]\!] \leq 2 + [\![W]\!]^2 [\![r(z)]\!] \leq 3[\![W]\!]^2 N \leq CN, \text{ similarly } [\![R_i(z)]\!] \leq CN$$

and hence

$$[\![Q_i(z)]\!] \leq ([\![X]\!]^2 + [\![\Phi]\!])[\![R_i(z)]\!] + [\![X]\!] \leq C^2 N.$$

Thus g is bounded by $C^2 N^2$. Obviously f is lower-bounded by $\sqrt{N}$.

By the second of the identities on line (29), for distinct $z_1, z_2 \in \mathfrak{h}$ such that $\Im z_1 \wedge \Im z_2 \geq 1/N$, we have

$$\frac{[\![R(z_1) - R(z_2)]\!]}{|z_1 - z_2|} \leq N^2 [\![W]\!]^2 \leq CN^2 \text{ and similarly } \frac{[\![R_i(z_1) - R_i(z_2)]\!]}{|z_1 - z_2|} \leq CN^2.$$

It follows that

$$\frac{[\![R_i(z_1) - R_i(z_2)]\!]_2}{|z_1 - z_2|} \leq CN^{5/2}.$$

It follows in turn that the Lipschitz constant of f is bounded by $CN^{5/2}$. It also follows that

$$\frac{[\![Q_i(z_1) - Q_i(z_2)]\!]}{|z_1 - z_2|} \leq ([\![X]\!]^2 + [\![\Phi]\!])CN^2 \leq C^2 N^2.$$

Thus the Lipschitz constant of g is bounded by $C^2 N^3$.

Using the identity

$$\frac{g(z_1)}{f(z_1)} - \frac{g(z_2)}{f(z_2)} = \frac{g(z_1) - g(z_2)}{f(z_1)} + g(z_2)\frac{f(z_2) - f(z_1)}{f(z_1)f(z_2)}$$

we deduce that

$$\frac{\left|\frac{g(z_1)}{f(z_1)} - \frac{g(z_2)}{f(z_2)}\right|}{|z_1 - z_2|} \leq \frac{C^2 N^3}{\sqrt{N}} + \frac{(C^2 N^2)(CN^{5/2})}{N} \leq 2C^3 N^{7/2},$$

which finishes the proof. $\square$

8. Proof of Theorem 1.2

8.1. **Construction of K.** We fix absolute constants $\tau \geq 64$ and $\theta \geq 1$ once and for all so that the conclusion (86) of Theorem 7.1 holds. We work simultaneously in the settings of Theorem 1.2 and Theorem 7.1. In particular, U and V are now random. Let $\mathcal{R}$ be the rectangle (85). By Proposition 7.6 we know that conditioned on $[\![U]\!] \vee [\![V]\!] \leq 4$ the quantity $\mathfrak{K}_i(z)$ depends Lipschitz-continuously on $z \in \mathcal{R}$ with Lipschitz constant bounded by $cN^{7/2}$. Recall also that $\mathfrak{K}_i$ is by definition bounded below by 1. Thus for suitable absolute constants β_1 and β_3 and a suitable net $\mathcal{R}_0 \subset \mathcal{R}$ of at most $\beta_3 N^{\beta_1 - 1}$ points we have

$$(95) \qquad 2\bigvee_{z_0 \in \mathcal{R}_0} \bigvee_{i=1}^{N} \mathfrak{K}_i(z_0) = 2\bigvee_{z_0 \in \mathcal{R}_0} \mathfrak{K}(z_0) \geq \sup_{z \in \mathcal{R}} \mathfrak{K}(z)$$

conditioned on $[\![U]\!] \vee [\![V]\!] \leq 4$. We define $\mathbf{K}$ to equal the left side of (95) multiplied by θ. It follows immediately from (26) and (86) that the random variable $\mathbf{K} \geq 1$ thus defined has the desired property (9). It remains only to prove that $\mathbf{K}$ has

property (10). The latter task is a matter of revisiting the topic of [9, Appendix B]. We will handle the details a bit differently than in the cited reference, basing our proof instead on a classical result from [25].

8.2. **Remark.** In the proof of the local semicircle law [9, Thm. 3.1] the Lipschitz continuity of the various functions in play is frequently invoked while marching toward the real axis. It might have seemed we were trying to avoid such considerations here by using Proposition 1.5. Certainly we have avoided their use in a dynamical way. But ultimately our reworking of the method of [9] has merely displaced the use of Lipschitz continuity to the phase of the argument presented here in §8 in which we construct $\mathbf{K}$.

We begin the proof that $\mathbf{K}$ has property (10) by recalling the relationship between moment bounds of the form (1) and exponentially light tails.

Proposition 8.3. *Fix constants $\alpha, c > 0$ and $C \geq 1$. Let Z be a nonnegative random variable.*

   (i) *If $\sup\limits_{p \in [2,\infty)} p^{-\alpha} \|Z\|_p \leq c$, then $\Pr(Z > t^\alpha) \leq \exp\left(\alpha\left(2 - \frac{t}{c^{1/\alpha} e}\right)\right)$ for $t > 0$.*

   (ii) *If $\Pr(Z > t^\alpha) \leq C e^{-t/c^{1/\alpha}}$ for $t > 0$, then $\sup\limits_{p \in [1,\infty)} p^{-\alpha} \|Z\|_p \leq cC\,(\alpha + 1)^\alpha$.*

Proof. (i) In the Markov bound $\Pr(Z > t^\alpha) \leq \frac{\|Z\|_p^p}{t^{\alpha p}} \leq \left(\frac{c^{1/\alpha} p}{t}\right)^{\alpha p}$ we substitute $p = \frac{t}{c^{1/\alpha} e}$ if $\frac{t}{c^{1/\alpha} e} \geq 2$ and simplify. (ii) For the Γ-function $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$ one has a functional equation $s\Gamma(s) = \Gamma(s+1)$, a bound $\Gamma(s) \leq 1$ for $1 \leq s \leq 2$ and (hence) an elementary inequality $\Gamma(1+s) \leq (1+s)^s$ for $s \geq 0$. For $p \geq 1$ we then have

$$\mathbf{E} Z^p = \alpha p \int_0^\infty \Pr(Z > t^\alpha)\, t^{\alpha p - 1}\, dt \leq \alpha p C \int_0^\infty e^{-t/c^{1/\alpha}} t^{\alpha p - 1}\, dt \leq c^p C (p\alpha + 1)^{p\alpha}$$

and thus $p^{-\alpha} \|Z\|_p \leq cC^{1/p} (\alpha + 1/p)^\alpha$ for $p \geq 1$. $\qquad\square$

We next recall a classical result. Let $\Theta(s) = \frac{2^{s/2}}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right)$ for $s \geq 0$.

Theorem 8.4 (Whittle [25]). *Let $Y_1, \ldots, Y_n$ be independent real random variables in L^2 and of mean zero. Fix $p \in [2, \infty)$. Let $v \in \mathbb{R}^n$ be a real vector. Let $B \in \mathrm{Mat}_n$ be a matrix with real entries. If $\bigvee_{i=1}^n \|Y_i\|_p < \infty$, then*

$$(96) \qquad \left\| \sum_{i=1}^n v(i) Y_i \right\|_p \leq 2\Theta(p)^{\frac{1}{p}} \left(\sum_{i=1}^n v(i)^2 \|Y_i\|_p^2 \right)^{1/2}.$$

Furthermore, if $\bigvee_{i=1}^n \|Y_i\|_{2p} < \infty$, then

$$(97) \qquad \left\| \sum_{i,j=1}^N B(i,j)(Y_i Y_j - \mathbf{E}[Y_i Y_j]) \right\|_p$$

$$\leq 2^3 \Theta(p)^{\frac{1}{p}} \Theta(2p)^{\frac{1}{2p}} \left(\sum_{i,j=1}^N B(i,j)^2 \|Y_i\|_{2p}^2 \|Y_j\|_{2p}^2 \right)^{1/2}.$$

We hasten to point out that one has an elementary bound

(98)
$$\sup_{s \geq 2} \frac{\Theta(s)^{\frac{1}{s}}}{\sqrt{s}} \leq 1.$$

Thus the estimates (96) and (97) can be simplified nicely.

From Proposition 8.3 and Theorem 8.4 we then get the following tail-bound.

Proposition 8.5. *Fix constants $\gamma_0 > 0$ and $\gamma_1 \geq 1$. Fix a positive integer k. Let $Y_0, \ldots, Y_{2N} \in \mathrm{Mat}_k$ have L^2 random entries of mean zero. Assume that the family*

$$\{\sigma(Y_0)\} \cup \{\sigma(Y_i, Y_{i+N})\}_{i=1}^N$$

of σ-fields is independent. Assume that

$$\sup_{p \geq 2} p^{-\gamma_0} \bigvee_{i=0}^{2N} \|[\![Y_i]\!]\|_p \leq \sqrt{\frac{\gamma_1}{N}}.$$

Let

$$Y = \begin{bmatrix} Y_1 & \ldots & Y_N \end{bmatrix} \in \mathrm{Mat}_{k \times kN} \quad and \quad \widehat{Y} = \begin{bmatrix} Y_{N+1} & \ldots & Y_{2N} \end{bmatrix} \in \mathrm{Mat}_{k \times kN}.$$

Let $B \in \mathrm{Mat}_{kN}$ be any constant matrix. Then for every $t > 0$ we have

$$\Pr\left(\frac{\left[\!\!\left[Y B \widehat{Y}^* - Y_0 - \mathbf{E}(Y B \widehat{Y}^*) \right]\!\!\right]}{\frac{\gamma_1}{\sqrt{N}} \left(1 \vee \frac{[\![B]\!]_2}{\sqrt{N}} \right)} > t^{2\gamma_0 + 1} \right) \leq \gamma_2 e^{-\gamma_3 t}$$

for constants $\gamma_2 \geq 1$ and $\gamma_3 > 0$ depending only on γ_0 and k.

Proof. By Proposition 8.3 it is enough to prove that

(99)
$$\sup_{p \geq 2} p^{-(1+2\gamma_0)} \left\| \left[\!\!\left[Y B \widehat{Y}^* - Y_0 - \mathbf{E}(Y B \widehat{Y}^*) \right]\!\!\right] \right\|_p \leq \frac{\gamma_4 \gamma_1}{\sqrt{N}} \left(1 \vee \frac{[\![B]\!]_2}{\sqrt{N}} \right)$$

where $\gamma_4 \geq 1$ is a constant depending only on γ_0 and k. Without loss of generality we may assume that $Y_0 = 0$, B has real entries and that the random matrices Y_i have real entries. We may then in turn assume that $k = 1$. By (96) we may assume that every diagonal entry of B vanishes. We may also obviously assume that $N \geq 2$. Now let $I \subset \{1, \ldots, N\}$ be any subset of cardinality $\lfloor \frac{N}{2} \rfloor$ and let I^c denote the complement of I. Let

$$B_I(i,j) = B(i,j) \mathbf{1}_{i \in I} \mathbf{1}_{j \in I^c},$$

thus defining a matrix $B_I \in \mathrm{Mat}_N$ supported on the set

$$I \times I^c \subset \{1, \ldots, N\}^2.$$

Let

$$\widetilde{Y}_I(i) = \begin{cases} Y_i & \text{if } i \in I, \\ Y_{i+N} & \text{if } i \in I^c. \end{cases}$$

Note that the entries of $\widetilde{Y}_I$ are independent. Note also that

$$Y B_I \widehat{Y}^* = \widetilde{Y}_I B_I \widetilde{Y}_I^*.$$

Thus we have

(100)
$$\sup_{p \geq 2} p^{-(1+2\gamma_0)} \left\| \left[\!\!\left[Y B_I \widehat{Y}^* - \mathbf{E}(Y B_I \widehat{Y}^*) \right]\!\!\right] \right\|_p \leq \frac{\gamma_1 \gamma_4}{4} \frac{[\![B_I]\!]_2}{N} \leq \frac{\gamma_1 \gamma_4}{4} \frac{[\![B]\!]_2}{N}$$

by Theorem 8.4 and the upper bound (98). Now the average of B_I over I equals qB for some constant $q \geq \frac{1}{4}$. Thus, averaging over I on the left side of (100) and using Jensen's inequality, we obtain (99). $\qquad\square$

8.6. **End of the proof.** Proposition 8.5 and the summary of properties of the random matrix X in §3.6 together provide us with constants $\beta_2 \geq 1$ and $\beta_4 > 0$ depending only on α_0 and α_1 such that for $i = 1, \ldots, N$ and any $z_0 \in \mathfrak{h}$ we have a conditional tail bound

$$\Pr\left(2\theta\mathfrak{K}_i(z_0) > t^{\frac{1}{2\alpha_0+1}} \middle| \hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^* \right) \geq \beta_4 \exp(-\beta_2 t) \text{ a.s..}$$

uniform in z_0. The latter combined with the evident union bound over $\beta_3 N^{\beta_1}$ events yields (10) with $\beta_0 = \beta_3 \beta_4$. The proof of Theorem 1.2 is complete. $\qquad\square$

9. Appendix: The deterministic local semicircle law

We state and prove the semicircular analogue of Theorem 7.1. The proof will apply Propositions 1.5, 3.3, 4.2 and 6.2 above, none of which have anything specifically to do with anticommutators.

9.1. Setup for the result.

9.1.1. *Basic data.* Fix a hermitian matrix $X \in \mathrm{Mat}_N$ and a point $z \in \mathfrak{h}$ arbitrarily.

9.1.2. *Specialized matrix notation.* Let e_i denote the ith row of $\mathbf{I}_N$ and let $\hat{e}_i$ denote the result of deleting the ith row of $\mathbf{I}_N$.

9.1.3. *Functions of z.* For $i = 1, \ldots, N$ let

$$R = (X - z\mathbf{I}_N)^{-1} \in \mathrm{Mat}_N, \quad G_i = e_i R e_i^* = R(i,i) \in \mathfrak{h},$$

$$G = \frac{1}{N}\mathrm{tr}\, R = \frac{1}{N}\sum_{i=1}^N G_i,$$

$$R_i = (\hat{e}_i X \hat{e}_i^* - z\mathbf{I}_{N-1})^{-1} \in \mathrm{Mat}_{N-1}, \quad \widehat{G}_i = \frac{1}{N}\mathrm{tr}\, R_i \in \mathfrak{h},$$

$$Q_i = e_i X \hat{e}_i^* R_i \hat{e}_i X e_i^* - X(i,i) - \widehat{G}_i \in \mathbb{C},$$

$$\mathfrak{K}_i = 1 \vee \frac{|Q_i|}{\frac{1}{\sqrt{N}}\left(1 \vee \frac{[\![R_i]\!]_2}{\sqrt{N}}\right)} \in [1,\infty), \quad \mathfrak{K} = \bigvee_{i=1}^N \mathfrak{K}_i.$$

All these objects depend on (X, z) but the notation does not show it. We will write $\mathfrak{K}(z)$ to show z-dependence, it being understood that X is held fixed as z varies. Also let

$$m = \frac{1}{2\pi}\int_{-2}^{2} \frac{\sqrt{4 - t^2}\, dt}{t - z} \in \mathfrak{h} \text{ and } h = 1 \wedge |z + 2| \wedge |z - 2| > 0.$$

Both m and h depend on z but the notation does not show it.

Here then is the *deterministic local semicircle law.*

Theorem 9.2. *Notation and assumptions are as above. Let $\tau \geq 1$ and $\theta \geq 1$ be absolute constants. Consider the rectangle*

$$\mathcal{R} = \left\{z \in \mathfrak{h} \middle| |\Re z| \leq \tau \text{ and } \frac{1}{N} \leq \Im z \leq \tau\right\}$$

and let

$$K = \theta \sup_{z \in \mathcal{R}} \mathfrak{K}(z) < \infty.$$

Consider also the closed (possibly empty) set

$$\mathcal{X} = \left\{ z \in \mathcal{R} \,\middle|\, \frac{K^2}{N} \le h^2 \Im z \right\}.$$

Then we have

(101) $$z \in \mathcal{X} \Rightarrow \bigvee_{i=1}^{N} |G_i - m| \le \frac{K}{\sqrt{N h \Im z}}$$

provided that τ is large enough and θ is large enough depending on τ.

We follow the outline of the proof of Theorem 7.1, finishing up in §9.6 below. The hypothesis of (101) is not used until §9.6.

9.3. **An *a priori* bound.** Clearly we have

(102) $$\bigvee_{i=1}^{N} |G_i - m| \le \frac{2}{\Im z}.$$

Proposition 9.4. *We have*

(103) $$|G - \widehat{G}_i| \quad \le \quad \frac{(\Im z)_\bullet}{N \Im z} |G_i|_\bullet |G_i|^{-1},$$

(104) $$|G_i^{-1} + z + \widehat{G}_i| \quad \le \quad \mathfrak{K} \sqrt{\frac{(\Im z)_\bullet}{N \Im z}} |\widehat{G}_i|_\bullet^{1/2}.$$

Proof. Since

$$\frac{\Im R}{\Im z} = R R^* = R^* R,$$

we have

(105) $$\frac{\Im G_i}{\Im z} \quad = \quad [\![e_i R]\!]_2^2 = [\![R e_i]\!]_2^2 \text{ and similarly}$$

(106) $$\frac{\Im \widehat{G}_i}{\Im z} \quad = \quad \frac{[\![R_i]\!]_2^2}{N}.$$

By Proposition 3.3 we have

(107) $$Q_i \quad = \quad -G_i^{-1} - z - \widehat{G}_i \text{ and}$$

(108) $$R \quad = \quad \hat{e}_i^* R_i \hat{e}_i + R e_i^* G_i^{-1} e_i R.$$

By (105), (108) and Cauchy-Schwarz we have

$$|G - \widehat{G}_i| = \frac{1}{N} |\mathrm{tr}\, R - \mathrm{tr}\, R_i| \le \frac{[\![R e_i^*]\!]_2 [\![e_i R]\!]_2}{N |G_i|} \le \frac{1}{N \Im z},$$

which is enough to prove (103). By (106), (107) and the definition of $\mathfrak{K}$ we have

$$|G_i^{-1} + z + \widehat{G}_i| \le \frac{\mathfrak{K}}{\sqrt{N}} \left(1 \vee \frac{[\![R_i]\!]_2}{\sqrt{N}} \right) \le \frac{\mathfrak{K}}{\sqrt{N}} \left(1 \vee \sqrt{\frac{\Im \widehat{G}_i}{\Im z}} \right),$$

which is enough to prove (104). $\square$

Proposition 9.5. *We have*

$$(109) \qquad \bigvee_{i=1}^{N} |G_i - m| \leq \frac{\sqrt{h}}{c_{4.2}} \Rightarrow \bigvee_{i=1}^{N} |G_i - m| \leq \frac{C(1 + |z|)^5 \mathfrak{K}}{\sqrt{Nh\Im z}}$$

where C is an absolute constant.

Proof. Let $\kappa = (m^{-1} - m)^{-1}$. By Proposition 4.2 the quadruple $(z, m, 1, \kappa)$ is a nondegenerate solution of the Schwinger-Dyson equation defined over $\mathbb{C}$ and furthermore $|m| < 1$. By Proposition 9.4 we have $\mathfrak{E} \leq \sqrt{\frac{(\Im z)_\bullet}{N\Im z}} \mathfrak{K}$. Thus we have

$$\bigvee_{i=1}^{N} |G_i - m| \leq \frac{1}{8|\kappa|_\bullet} \Rightarrow \bigvee_{i=1}^{N} |G_i - m| \leq \frac{2^{21}|z|_\bullet^4 \sqrt{(\Im z)_\bullet} \mathfrak{K}}{\sqrt{N\Im z}} |\kappa|_\bullet$$

by substituting into Proposition 6.2. We then obtain (109) via (48). $\qquad\square$

9.6. **Proof of Theorem 9.2.** The hypothesis of (101) will be enforced now until the end of the proof.

9.6.1. *Setup for application of Proposition 1.5.* We hold X fixed now. We allow z to vary but constrain z to the space $\mathcal{X} \subset \mathfrak{h}$. On the space $\mathcal{X}$ we consider the three continuous functions

$$f_1 = \bigvee_{i=1}^{N} |G_i - m|, \ \ f_2 = \frac{\sqrt{h}}{c_{4.2}} \ \text{ and } \ f_3 = \frac{K}{2c_{4.2}\sqrt{Nh\Im z}}.$$

It remains only to check hypotheses in Proposition 1.5. The process of checking will dictate the choices of τ and θ.

9.6.2. *$\mathcal{X}$ is connected if nonempty.* (Here one simply repeats §7.5.2 verbatim.)

9.6.3. *Checking hypothesis* (12)*.* Consider the statement

$$(110) \qquad \bigvee_{i=1}^{N} |G_i - m| \bigg|_{z=i\tau} \leq \frac{2}{\Im z}\bigg|_{z=i\tau} = \frac{2}{\tau} < \frac{1}{c_{4.2}} = \frac{\sqrt{h}}{c_{4.2}}\bigg|_{z=i\tau}.$$

The first inequality holds by (102). The third inequality holds for τ large enough. Now fix $\tau \geq 1$ to make (110) hold. Then hypothesis (12) of Proposition 1.5 holds.

9.6.4. *Checking hypothesis* (13)*.* Choose θ so that

$$\theta \geq 2c_{4.2}C_{9.5}(1 + 2\tau)^5.$$

Then we have

$$\bigvee_{i=1}^{N} |G_i - m| \leq \frac{\sqrt{h}}{c_{4.2}} \Rightarrow \bigvee_{i=1}^{N} |G_i - m| \leq \frac{K}{2c_{4.2}\sqrt{Nh\Im z}}$$

by Proposition 9.5. Thus hypothesis (13) of Proposition 1.5 holds.

9.6.5. *Checking hypothesis* (14)*.* We have

$$z \in \mathcal{X} \Rightarrow \frac{K}{2c_{4.2}\sqrt{Nh\Im z}} \leq \frac{\sqrt{h}}{2c_{4.2}} < \frac{\sqrt{h}}{c_{4.2}}$$

by definition of $\mathcal{X}$. Thus hypothesis (14) of Proposition 1.5 holds. The conclusion (15) of Proposition 1.5 and conclusion (101) of Theorem 9.2 are then the same. The proof of Theorem 9.2 is complete. $\qquad\square$

9.7. **Remark.** By studying the generalized resolvent

$$
\left[
\begin{array}{cc}
-z\mathbf{I}_p & X \\
X^* & -\mathbf{I}_q
\end{array}
\right]^{-1}
\quad (X \in \mathrm{Mat}_{p \times q})
$$

one can obtain a similar deterministic local Marcenko-Pastur law.

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