

Seven-dimensional forest fires

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Abstract

We show that in high dimensional Bernoulli bond percolation, removing from a thin infinite cluster a much thinner infinite cluster leaves an infinite component. This observation has implications for the van den Berg-Brouwer forest fire process, also known as self-destructive percolation, for dimension high enough.

1 Introduction

Think about the open vertices of supercritical site percolation as if they were trees, and about the infinite cluster as a forest. Suddenly a fire breaks out and the entire forest is cleared. New trees then start growing randomly. When can one expect a new infinite cluster to appear? The surprising conjecture in [vdBB04] is that in the two-dimensional case, even if the original forest were extremely thin, still a considerable amount of trees must be added to create a new infinite cluster. Heuristically, the conjecture claims that the infinite cluster might occupy a very low proportion of vertices but they sit in a way that separates the remaining finite clusters by gaps that cannot be easily bridged. This conjecture is still open. See [vdBB04, vdBBV08, vdBdL09] for connections to other models of forest fires and more.

Let us define the model formally, in three steps. The model was originally introduced as a site percolation model, but we will define it for bonds, as some of the auxiliary results we need have only been proved for bond percolation. We are given a graph G , a probability $p \in [0, 1]$ (“the original density”) and a probability $\varepsilon \in [0, 1]$ (“the recovered density”). Let \mathbb{P}_p be the Bernoulli bond percolation measure on G with parameter p .

1. Assign independent uniformly distributed values from $[0, 1]$ to the edges of G . Let $\omega_p \in \{0, 1\}^{E(G)}$ denote the set of edges with value at most p . The configuration ω_p is distributed as \mathbb{P}_p , and a *cluster* refers

to a maximal connected component of edges. It will be of importance below that as p ranges over $[0, 1]$, we obtain a simultaneous coupling of Bernoulli configurations on G such that $\omega_{p_1} \subset \omega_{p_2}$ when $p_1 \leq p_2$.

- Let $\tilde{\mathbb{P}}_p$ be the law of the configuration $\tilde{\omega}_p$ constructed as follows: for any edge e ,

$$\tilde{\omega}_p(e) = \begin{cases} \omega_p(e) & \text{if } e \text{ is in a finite cluster of } \omega_p, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\tilde{\mathbb{P}}_{p,\varepsilon}$ be the law of $\tilde{\omega}_{p,\varepsilon}$ where $\tilde{\omega}_{p,\varepsilon}$ is defined as follows: for any edge e , $\tilde{\omega}_{p,\varepsilon}(e) = \max\{\tilde{\omega}_p(e), \omega'_\varepsilon(e)\}$, where ω'_ε is a percolation configuration with edge-weight ε , which is independent of ω_p .

We can now define our property of interest.

Definition Let $p_c(G)$ denote the critical threshold for bond percolation on a graph G . We say that G *recovers from fires* if for every $\varepsilon > 0$, there exists $p > p_c(G)$ such that $\tilde{\mathbb{P}}_{p,\varepsilon}$ has an infinite connected component, with probability 1. We say that G *site-recovers from fires* if the analogous definitions for site percolation hold.

In [vdBB04] the authors showed that a binary tree site-recovers from fires and conjectured that \mathbb{Z}^2 lattice does *not* site-recover from fires. The binary tree is an example of a non-amenable graph, that is, a graph in which the boundary of a (finite) set of vertices is comparable in size to the set itself. Recovery from fires, both in edge and site sense, was proven in [AST13] for a large class of non-amenable transitive graphs. Our result concerns hyper-cubic lattices.

Theorem 1. *For d sufficiently large, \mathbb{Z}^d recovers from fires.*

Here and below, \mathbb{Z}^d refers to the \mathbb{Z}^d nearest neighbour lattice. The main property of \mathbb{Z}^d that we will use is that $\mathbb{P}_{p_c}(0 \leftrightarrow \partial B(0, r)) \leq Cr^{-2}$ (see below for a discussion on this condition, and also for the notations). This was proved in [KN11] based on results of Hara, van der Hofstad & Slade [HvdHS03, Har08]. These establish the necessary estimate for d sufficiently large (19 seems to be enough, though this can be improved) and also for *stretched-out* lattices in $d > 6$. The number 6 is actually meaningful and is the limit of the technique involved, lace expansion. Our proof easily extends to the stretched-out 7-dimensional lattice (hence the title of the article), but for simplicity we will prove the theorem only for nearest-neighbour percolation in d sufficiently high. In fact, our proof provides further information in the supercritical percolation regime. Recall the common notation $\mathcal{C}_\infty(\omega_p)$ for the infinite cluster of edges present in ω_p .

Theorem 2. *For every $\varepsilon > 0$ and d sufficiently large, there exists $p > p_c$ such that $\omega_{p_c+\varepsilon} \setminus \mathcal{C}_\infty(\omega_p)$ contains an infinite cluster almost surely.*

Theorem 1 is clearly a corollary of Theorem 2. Another consequence is that for every $\varepsilon > 0$, the critical probability for percolation on the random graph obtained from \mathbb{Z}^d by removing a sufficiently ‘thin’ supercritical percolation cluster, that is $\mathcal{C}_\infty(\omega_{p_c+\delta})$ for small enough $\delta = \delta(\varepsilon) > 0$, is almost surely at most $p_c + \varepsilon$. Theorem 2 and the last statement cannot possibly hold for site percolation on \mathbb{Z}^2 , since an infinite cluster cuts space up into finite pieces.

Proof sketch We will show that for every $\varepsilon > 0$, there exists some $p > p_c$ such that when removing the infinite cluster of p -percolation from $(p_c + \varepsilon)$ -percolation, the remainder still percolates. The proof proceeds by a renormalization procedure.

1. We first choose $\ell \in \mathbb{N}$ sufficiently large such that for any $L \geq \ell$, connectivity properties of boxes of size $L^2 \times \ell^{d-2}$ in $(p_c + \varepsilon)$ -percolation behave like $(1 - \eta)$ -percolation on a coarse grain lattice for some small η . This is a standard application of Grimmett-Marstrand [GM90] and renormalization theory.
2. We then use the fact that the one-arm exponent in high dimensions is 2 to note that for any L , only a small number M of vertices in a box of size $L^2 \times \ell^{d-2}$ can connect to distance L in *critical percolation*.
3. Picking L sufficiently large, one can argue that these M points do not alter the connectivity properties of boxes of size $L^2 \times \ell^{d-2}$ for $(p_c + \varepsilon)$ -percolation. In particular, the coarse grain percolation still behaves like $(1 - \eta)$ -percolation even after removing that small number of vertices.
4. We now pick p sufficiently close to p_c that the behaviour (for ω_p) at scale L is not altered by moving from p_c to p . Since there are less sites in $\mathcal{C}_\infty(\omega_p)$ than sites connected to distance L in ω_p , this p gives the result.

Examining this a little shows that what the proof really needs is that the one-arm exponent is bigger than 1, i.e. that

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B(0, r)) \leq r^{-1-c} \quad c > 0.$$

The number of points removed in the second renormalization step will in this case no longer be bounded independently of L , but would still be too small to block the cluster of the boxes at scale ℓ . This is interesting as it

is conjectured to hold also below 6 dimensions. While nothing is proved, simulations hint that it might hold for \mathbb{Z}^5 [AS94, §2.7]. On the other hand, let us note that in \mathbb{Z}^3 this probability is larger than cr^{-1} (this is well-known but we are not aware of a precise reference – compare to [vdBK85, (3.15)] and [Kes82, Theorem 5.1]). Hence, the approach used here has no hope of working in \mathbb{Z}^3 (though, of course, this does not preclude the possibility that \mathbb{Z}^3 does recover from fires). We remark that a similar renormalization technique was recently used in [GHK], also under the assumption that the one-arm exponent is bigger than 1.

Notations Identify \mathbb{Z}^2 with the subgraph of \mathbb{Z}^d of points with the $d-2$ last coordinates equal to 0. Let $S_\ell = \{x \in \mathbb{Z}^d : |x_i| \leq \ell \ \forall i \geq 3\}$ be the two-dimensional slab of height $2\ell + 1$. Recall also the following standard notations: For a subgraph G of \mathbb{Z}^d , we say that x is connected to y in G if they are in the same connected component of G . We denote this by $x \xleftrightarrow{G} y$ or simply $x \longleftrightarrow y$ when the context is clear. We will use the notation $x \longleftrightarrow A$ to denote the fact that $x \longleftrightarrow y$ for some y in $A \subset \mathbb{Z}^d$.

Let $\|\cdot\|_\infty$ be the infinity norm on \mathbb{R}^d defined by

$$\|x\|_\infty = \max\{|x_i| : i = 1, \dots, d\}.$$

We consider the hypercubic lattice \mathbb{Z}^d for some large but fixed d . For $\ell, L > 0$, define the ball $B_x(L) = \{y \in \mathbb{Z}^d : \|y - x\|_\infty \leq L\}$ and let $\partial B_x(L)$ be its inner vertex boundary.

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2 Proof

From now on, d is fixed and large enough. For $x \in \mathbb{Z}^2$, let $\mathcal{A}(x, \ell, L, M)$ be the event that there are less than M sites y in the $(6L + 1) \times (6L + 1) \times (2\ell + 1)^{d-2}$ box $S_\ell \cap B_x(3L)$ that are connected to a site at distance L from themselves. Note that we do not assume that this connection is contained in the slab S_ℓ , the connection may be anywhere in $B_y(L)$.

Lemma 3. *Let $\eta > 0$ and $\ell > 0$. There exists $M > 0$ such that for any integer L , there exists $p > p_c$ such that*

$$\mathbb{P}_p(\mathcal{A}(x, \ell, L, M)) \geq 1 - \eta.$$

Proof. By [KN11], there exists $C > 0$ such that (for large enough d)

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial B_0(n)) \leq \frac{C}{n^2}. \quad (1)$$

Choose M in such a way that $\frac{49(2\ell+1)^{d-2}C}{M} < \eta$. For any integer L , Markov's inequality implies

$$\begin{aligned} \mathbb{P}_{p_c} \left[|\{y \in S_\ell \cap B_x(3L) : y \leftrightarrow \partial B_y(L)\}| \geq M \right] \\ \leq \frac{1}{M} \sum_{y \in S_\ell \cap B_x(3L)} \mathbb{P}_{p_c}(y \leftrightarrow \partial B_y(L)). \end{aligned}$$

By (1) and the choice of M , the right-hand side is thus strictly smaller than η . By choosing p close enough to p_c , we obtain that

$$\mathbb{P}_p \left[|\{y \in S_\ell \cap B_x(3L) : y \leftrightarrow \partial B_y(L)\}| \geq M \right] \leq \eta. \quad \square$$

For a set $S \subset \mathbb{Z}^d$, let ω^S be the configuration obtained from ω by closing each edge adjacent to some site in S . Let $\mathcal{B}(x, \ell, L, M)$ be the event that for any set S of M sites contained in $B_x(3L)$, ω^S contains

- a cluster crossing from $\partial B_x(L)$ to $\partial B_x(3L)$ contained in the slab S_ℓ ,
- a unique cluster in the box $S_\ell \cap B_x(3L)$ of radius larger than L .

Lemma 4. *Let $\eta > 0$ and $\varepsilon > 0$. There exists $\ell > 0$ such that for any $M > 0$, there is $L > 0$ so that*

$$\mathbb{P}_{p_c+\varepsilon}(\mathcal{B}(x, \ell, L, M)) \geq 1 - \eta.$$

Proof. For a given ℓ and L denote by $E = E(x, \ell, L)$ the event that:

1. There is a crossing from $\partial B_x(L)$ to $\partial B_x(3L)$ in S_ℓ .
2. There is exactly one cluster in $S_\ell \cap B_x(3L)$ of radius larger than L .

Shortly, the event E is just \mathcal{B} without the set S , or, if you want, \mathcal{B} is the event that E occurred in ω^S for all S with $|S| \leq M$.

We claim that for ℓ sufficiently large, $\mathbb{P}_{p_c+\varepsilon}(\neg E) \leq \exp(-cL)$ for some $c = c(\varepsilon, \ell) > 0$ independent of L . Finding such an ℓ is a standard exercise in renormalization theory, but let us give a few details nonetheless. Call a box of side-length $2\ell+1$ *good* if it contains crossings between opposite faces in all directions, and if all clusters of diameter at least $\frac{1}{4}\ell$ connect inside the box. By choosing ℓ large, we can require that a box is good with arbitrarily high probability (see e.g. the appendix of [BBHK08]). Considering such boxes centered around the sites in $\ell\mathbb{Z}^2$. The events that these boxes are good are 2-dependent (in the sense of [LSS97], i.e. disjoint boxes are good

independently), and hence by [LSS97], if the probability that a box is good is sufficiently large, then the good boxes stochastically dominate two-dimensional site percolation at density, say, $\frac{9}{10}$. Now, a cluster of good boxes contains a cluster in the underlying percolation, since the crossings of adjacent boxes must intersect. This means that if either of the conditions in the definition of E fail, then there is an ℓ_∞ -cluster of bad boxes containing at least L/ℓ boxes. (Here an ℓ_∞ -cluster refers to a maximal set of connected sites with respect to ℓ_∞ -distance, as opposed to ℓ_1 -distance used elsewhere.) But the probability for that, from Peierls' argument, is at most $(8/10)^{L/\ell} \cdot (6L/\ell)^2$. This shows the claim.

Fix $M > 0$. Let F_M be the set of configurations in $B_x(3L)$ for which there exists $S \subset B_x(3L)$ with $|S| = M$ and $\omega^S \notin E$. We have

$$\begin{aligned} \mathbb{P}_{p_c+\varepsilon}(F_M) &\leq \sum_{S \subset B_x(3L): |S|=M} \mathbb{P}_{p_c+\varepsilon}(\omega^S \notin E) \\ &\leq \sum_{S \subset B_x(3L): |S|=M} (1 - p_c - \varepsilon)^{-2dM} \mathbb{P}_{p_c+\varepsilon}(\neg E) \\ &\leq (1 - p_c - \varepsilon)^{-2dM} (6L + 1)^{dM} \mathbb{P}_{p_c+\varepsilon}(\neg E) \\ &\leq (1 - p_c - \varepsilon)^{-2dM} (6L + 1)^{dM} \exp(-cL). \end{aligned}$$

For L large enough, this quantity is smaller than η . The lemma follows from the fact that if $\omega \notin \mathcal{B}(x, \ell, L, M)$, then there exists $S \subset B_x(3L)$ with $|S| = M$ and $\omega^S \notin E$, i.e. $\omega \in F_M$. \square

In order to prove Theorem 1 and 2, we will use Lemma 4 to construct an infinite cluster at density $p_c + \varepsilon$, and Lemma 3 to make sure that the infinite cluster present at the lower density p does not interfere too much with this construction.

Proof of Theorems 1 and 2. Recall the notations ω_p , $\tilde{\omega}_p$ and ω'_ε from page 1. We need to show that for any $\varepsilon > 0$, there exists $p > p_c$ such that $\tilde{\omega}_{p,\varepsilon}$ has an infinite component. Note that $(\omega_{p_c} \cup \omega'_\varepsilon) \setminus \mathcal{C}_\infty(\omega_p)$ is stochastically dominated by $\tilde{\omega}_{p,\varepsilon}$. Thus, it suffices to show that for every $\varepsilon > 0$, there is $p > p_c$ such that $\omega_{p_c+\varepsilon} \setminus \mathcal{C}_\infty(\omega_p)$ contains an infinite component. That is, Theorem 1 follows from Theorem 2, and it suffices to prove the latter.

Let therefore $\varepsilon > 0$. Fix $\eta > 0$ such that $1 - 2\eta$ exceeds the critical parameter for any 8-dependent percolation on vertices of \mathbb{Z}^2 . Define successively ℓ, M, L and p as follows. Fix $\ell = \ell(\varepsilon, \eta) > 0$ as defined in Lemma 4. Pick $M = M(\eta, \ell) > 0$ as defined in Lemma 3. This defines $L = L(\eta, \varepsilon, \ell, M) > 0$ by Lemma 4, and then $p = p(\eta, \ell, M, L) > p_c$ by Lemma 3.

Let \mathbf{P} denote the joint law of $(\omega_p, \omega_{p_c+\varepsilon})$ under the increasing coupling described above. A site $x \in L\mathbb{Z}^2$ is said to be *good* if $\omega_p \in \mathcal{A}(x, \ell, L, M)$ and $\omega_{p_c+\varepsilon} \in \mathcal{B}(x, \ell, L, M)$. By definition,

$$\mathbf{P}[\mathcal{A}(x, \ell, L, M) \cap \mathcal{B}(x, \ell, L, M)] \geq 1 - 2\eta.$$

Since these events depend on edges in $B_x(4L)$ only, the site percolation (on $L\mathbb{Z}^2$) thus obtained is 8-dependent. As a consequence, there exists an infinite cluster of good sites on the coarse grained lattice $L\mathbb{Z}^2$.

On the event that there exists an infinite cluster of good sites on the coarse grained lattice, there exists an infinite path in $\omega_{p_c+\varepsilon} \setminus \mathcal{C}_\infty(\omega_p)$. Indeed, by induction, consider a path of adjacent good sites x_1, \dots, x_n . Consider C_i to be a cluster in

$$[\omega_{p_c+\varepsilon} \setminus \mathcal{C}_\infty(\omega_p)] \cap [B_{x_i}(3L) \setminus B_{x_i}(L)]$$

of radius larger than L . By the definition of \mathcal{A} there are at most M sites in the box $S_l \cap B_{x_i}(3L)$ connected to distance L in ω_p . Hence the same box also contains no more than M sites in $\mathcal{C}_\infty(\omega_p)$ since any site connected to infinity must be connected to distance L . Using the definition of \mathcal{B} with S being exactly $\mathcal{C}_\infty(\omega_p) \cap S_l \cap B_{x_i}(3L)$ we see that $\omega_{p_c+\varepsilon} \setminus \mathcal{C}_\infty(\omega_p)$ contains a crossing cluster for the box $S_l \cap B_{x_i}(3L)$ with all the properties listed before Lemma 4. In particular, the uniqueness property ensures two such crossing clusters in two neighbouring boxes must intersect. The result follows readily. \square

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