Randomized and backward SDE representation for optimal control of non-Markovian SDEs

Marco Fuhrman
Politecnico di Milano, Dipartimento di Matematica
via Bonardi 9, 20133 Milano, Italy
marco.fuhrman@polimi.it

Huyên Pham
LPMA - Université Paris Diderot
Batiment Sophie Germain, Case 7012
13 rue Albert Einstein, 75205 Paris Cedex 13
and CREST-ENSAE
pham@math.univ-paris-diderot.fr

November 2013
Revised version: March 2014

Abstract

We study optimal stochastic control problem for non-Markovian stochastic differential equations (SDEs) where the drift, diffusion coefficients, and gain functionals are path-dependent, and importantly we do not make any ellipticity assumption on the SDE. We develop a controls randomization approach, and prove that the value function can be reformulated under a family of dominated measures on an enlarged filtered probability space. This value function is then characterized by a backward SDE with nonpositive jumps under a single probability measure, which can be viewed as a path-dependent version of the Hamilton-Jacobi-Bellman equation, and an extension to G-expectation.

Key words: Non-Markovian controlled SDEs, randomization of controls, dominated measures, backward stochastic differential equations.

MSC Classification (2010): 60H10, 93E20.

1 Introduction

We consider non-Markovian controlled stochastic differential equations (SDEs) of the form

$$dX_s \quad = \quad b_s(X, \alpha_s)\, ds + \sigma_s(X, \alpha_s)\, dW_s, \quad 0 \leq s \leq T, \tag{1.1}$$

where W is a n-dimensional Wiener process, α is a progressive control process, and the drift and diffusion coefficients b and σ may depend on the trajectory of the solution X valued in $\mathbb{R}^d$ in a non-anticipative way. Given initial conditions determined in our context by $t \in [0, T]$, and $x \in \mathbf{C}^d$, the set of continuous function from $[0, T]$ into $\mathbb{R}^d$, we denote by $X^{t,x,\alpha}$ the solution to (1.1) associated to the control α, and starting from $X_s = x(s)$ for $s \in [0, t]$. We are then interested in the value function for the optimal stochastic control problem:

$$v(t, x) \quad = \quad \sup_{\alpha} \mathbb{E}\Big[\int_t^T f_s(X^{t,x,\alpha}, \alpha_s) ds + g(X^{t,x,\alpha}) \Big], \tag{1.2}$$

where the running and terminal reward functionals f and g may also depend on the past trajectory of the solution X.

In the Markovian framework (see e.g. [18]), i.e. when b_s, σ_s, f_s depend on X only through its current value X_s, and g only on X_T, the value function also depends at time t only on the current state value $X_t = x(t)$, hence is a deterministic function on $[0, T] \times \mathbb{R}^d$. By the dynamic programming approach, the value function is then described by the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE), which is satisfied in general in the viscosity sense, and which characterizes the control problem once we have a uniqueness result for the HJB PDE. We refer to the monographs [12] or [24] for a detailed exposition of this theory of dynamic programming and viscosity solutions for stochastic optimal control.

The representation of stochastic control problem and HJB equation has been also developing by means of backward stochastic differential equations (BSDEs). In the Markovian case where the controller can affect only the drift coefficient, the HJB equation is a semi-linear PDE and is known to be related to a standard BSDE, see [21]. The controlled diffusion case, arising typically in finance in uncertain volatility models, leads to a fully nonlinear HJB PDE, and can be represented by second-order BSDE (2BSDE), as introduced in [3] and [26], whose basic idea is to require that the solution verifies the equation almost surely for every probability measure in a non dominated class of mutually singular measures. This theory is closely related to the notion of nonlinear and G-expectation, see [22], but requires a nondegeneracy condition on the diffusion coefficient together with some constraint between drift and diffusion. The general case without any ellipticity assumption on the controlled diffusion is addressed in [16], where it is proved that fully nonlinear HJB equations can be represented by a class of BSDE with nonpositive jumps. The basic idea, following [15] (see also [2] for optimal switching problem), is to randomize the control process α by replacing it by an uncontrolled pure jump process associated to a Poisson random measure, and then to constrain the jumps-component solution to the BSDE driven by Brownian motion and Poisson random measure, to remain nonpositive, by adding a nondecreasing process in a minimal way. A key feature of this class of BSDEs is its formulation under a single probability measure like for standard BSDE in contrast with 2BSDEs, thus avoiding technical issues in quasi-sure analysis. It is then proved in [16] that the minimal solution to the BSDE with nonpositive jumps satisfies the nonlinear HJB equation, so that it coincides with the value function of the Markovian stochastic control problem, once one has at disposal a uniqueness result for this HJB PDE (see [4] for a review on comparison results for viscosity solutions to nonlinear PDEs). Such Feynman-Kac type representation leads to new probabilistic approximation scheme for the numerical resolution of HJB equations in high dimension, as studied in [17].

The main goal of this paper is to extend the result of [16] to the non-Markovian framework. More precisely, we aim to prove that the value function in (1.2) may be represented in terms of a BSDE with nonpositive jumps, which can then be seen as a non-Markovian version of HJB equation. We use a controls randomization approach, by replacing the control process α by a pure-jump process associated to a Poisson random measure independent of the Wiener process, with fixed finite intensity measure. We then show that the value function in its weak formulation can be written equivalently as a randomized control problem under a family of dominated (and even equivalent) probability measures on an enlarged probability space, whose effect is to change the intensity measure of the Poisson random measure. By means of this randomized representation, we are finally able to relate the minimal solution to the BSDE with nonpositive jumps to the original value function (1.2). The arguments in this paper for proving this connection are quite different from the Markovian case studied in [16]. Indeed, this connection is shown in that paper through the HJB equation, which is satisfied both by the value function and the minimal solution, and thus requires a uniqueness result. Here, we prove this connection through the randomized control problem by purely probabilistic arguments. The main issue is to approximate continuous control processes by pure jump processes associated to random measures with compensator absolutely continuous with respect to a given finite intensity measure. In particular, we do not rely on the path-dependent HJB equation associated by dynamic programming principle to the value function in the non-Markovian context, thus circumventing delicate issues of dynamic programming (as originally studied in [7] for general non-Markovian stochastic control problems), viscosity solutions and comparison principles for fully nonlinear path-dependent PDEs, as recently studied in [23], [6] and [29], see also [10] for HJB equations in infinite dimension arising typically for stochastic systems with delays. This suggests in particular an original approach to derive the HJB equation for value function of stochastic control problem from the BSDE representation, hence without dynamic programming principle. Notice that compared to the paper [16], which treats a Markovian version of our problem with controlled jump-diffusion, we focus here only on the diffusion case, mainly for convenience. We believe that our randomization approach can be extended to the case of controlled Lévy-driven SDE, up to some technicalities inherent with jumps.

We mention that optimal control for path-dependent SDEs was also recently studied in [19] by adopting a quasi-sure formulation approach, which allows the author to prove a pathwise dynamic programming principle, and to derive a 2BSDE satisfied by the value function. However, the results are obtained essentially under a non degeneracy condition on the matrix diffusion coefficient and when control cannot affect independently drift and diffusion, see for details Assumption 2.1 and Remark 2.2 in [19]. Our results do not require any non degeneracy condition on σ, and include the case of control both on drift and diffusion coefficient arising for instance in portfolio optimization problems. Let us finally also point out the very general construction in [20], which proves in particular the dynamic programming principle for non-Markovian stochastic control, and leads, at least formally, to a representation of the value functions as a solution to 2BSDE. Such representation holds under general assumption (see their Assumption 2.1), which may not exclude the possibility of degenerate diffusion coefficient.

The rest of the paper is organized as follows. In Section 2, we detail the controlled path-dependent SDE, and introduce the corresponding value function in its weak formulation. Section 3 presents the main results of the paper. We formulate the value function by means of a randomized control problem over changes of dominated measures on an enlarged probability space. This randomized representation allows us to characterize the value function as the solution to a BSDE with nonpositive jumps. The proofs are reported in Section 4. Finally, we collect in Appendix some useful results about random measures and their compensators.

2 Control of path-dependent SDEs

We introduce in this section the path-dependent control setting and assumptions, and we define the value function for the associated optimal control problem.

2.1 Non-markovian controlled SDE

Let A, the control space, be a Lusin space (some authors call it a Borel space), i.e. a topological space homeomorphic to a Borel subset of a Polish space, endowed with a metric, denoted by ρ. As a subspace of a separable metric space, A is itself separable. We may assume without loss of generality that $\rho(a, a') < 1$, for any $a, a' \in A$, by replacing otherwise the initial metric by the equivalent one: $\rho/(1+\rho)$. We denote by $\mathcal{B}(A)$ the Borel σ-algebra of A. In the sequel, we also need to consider the set of all positive finite measures on $(A, \mathcal{B}(A))$ with full topological support, which will be denoted by $\mathcal{M}_f(A)$. This set is not empty: for instance it contains the measure $\sum_{n=1}^{\infty} 2^{-n}\delta_{a_n}$, where (a_n) is a dense sequence in A and δ_{a_n} denotes the Dirac measure at a_n. We note that A can be a finite or countable set, a Borel subset of $\mathbb{R}^q$, or more generally any Polish space.

In order to specify the measurability assumptions on the coefficients of the controlled equation, and of the control problem over a fixed horizon $T < \infty$, we introduce the path space $\mathbf{C}^d$ of continuous maps from $[0, T]$ to $\mathbb{R}^d$ and we equip $\mathbf{C}^d$ with the usual supremum norm $\|x\|_\infty = x_T^*$, where we set $x_t^* := \sup_{s\in[0,t]} |x(s)|$, for $t \in [0, T]$, and $x \in \mathbf{C}^d$. We define the filtration $(\mathcal{C}_t)_{t\in[0,T]}$, where $\mathcal{C}_t$ is the σ-algebra generated by the canonical coordinate maps $\mathbf{C}^d \to \mathbb{R}^d$, $x(\cdot) \mapsto x(s)$ up to time t:

$$\mathcal{C}_t \;:=\; \sigma\{x(\cdot) \mapsto x(s) \;:\; s \in [0, t]\}.$$

Let $Prog(\mathbf{C}^d)$ denote the progressive σ-algebra in $[0, T] \times \mathbf{C}^d$ with respect to $(\mathcal{C}_t)$.

The drift and diffusion coefficients

$$\begin{array}{ccc} [0,T]\times\mathbf{C}^d\times A & \longrightarrow & \mathbb{R}^d, \\ (t,x,a) & \longmapsto & b_t(x,a), \end{array} \qquad \begin{array}{ccc} [0,T]\times\mathbf{C}^d\times A & \longrightarrow & \mathbb{R}^{d\times n} \\ (t,x,a) & \longmapsto & \sigma_t(x,a) \end{array}$$

are $Prog(\mathbf{C}^d)\otimes\mathcal{B}(A)$-measurable. This measurability requirement is a standard one for ensuring that the stochastic differential equations (SDEs) with path-dependent coefficients is well-posed. We shall make the usual assumption:

(H1)

(i) For all $t \in [0, T]$, and $x \in \mathbf{C}^d$, the functions $b_t(x, a)$, and $\sigma_t(x, a)$ are continuous in A.

(ii) There exists a nonnegative constant K_1 such that

$$|b_t(x,a) - b_t(x',a)| + |\sigma_t(x,a) - \sigma_t(x',a)| \;\leq\; K_1(x-x')_t^*, \tag{2.1}$$

$$|b_t(0,a)| + |\sigma_t(0,a)| \;\leq\; K_1, \tag{2.2}$$

for all $(t, x, x', a) \in [0, T] \times \mathbf{C}^d \times \mathbf{C}^d \times A$.

We now formulate the controlled path-dependent SDE. Borrowing some terminology from [27], by an *admissible set-up* (or simply a set-up) we mean

$$\mathbb{A} \;=\; (\Omega, \mathcal{F}, \mathbb{G}, \mathbb{Q}, W),$$

where $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space equipped with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ satisfying the *usual conditions*, and $W = (W_t)_{0\leq t\leq T}$ is an n-dimensional standard $(\mathbb{Q}, \mathbb{G})$-Wiener process. Notice that $\mathbb{G}$ is not necessarily the natural filtration of W. We define the space of $\mathbb{A}$-admissible controls, denoted $\mathcal{A}(\mathbb{A})$, as the set of processes defined on $[0,T] \times \Omega$, valued in A, which are progressively measurable (for short, progressive) with respect to $\mathbb{G}$. For fixed $t \in [0,T]$, $x \in \mathbf{C}^d$, and given $\alpha \in \mathcal{A}(\mathbb{A})$, we consider the stochastic differential equation:

$$\left\{ \begin{array}{rcll} X_s &=& x(s), & s \in [0,t], \\ X_s &=& x(t) + \displaystyle\int_t^s b_u(X, \alpha_u)\, du + \int_t^s \sigma_u(X, \alpha_u)\, dW_u, & s \in [t,T]. \end{array} \right. \tag{2.3}$$

By standard results (see e.g. [27], Thm V. 11.2), under **(H1)**, there exists a unique $\mathbb{G}$-adapted strong solution $X = (X_s)_{0\leq s\leq T}$ to (2.3) with continuous trajectories and satisfying, for every $p \in [2, \infty)$,

$$\mathbb{E}^{\mathbb{Q}} \Big[\sup_{s\in[t,T]} |X_s|^p \Big] \quad \leq \quad C\Big(1 + (x_t^*)^p\Big), \tag{2.4}$$

(where $\mathbb{E}^{\mathbb{Q}}$ denotes of course the expectation under $\mathbb{Q}$) for some constant C depending only on p, T, and K_1 as defined in **(H1)**. We will denote the solution $X^{t,x,\mathbb{A},\alpha}$ to stress dependence on these parameters. Notice that $\{X_s^{t,x,\mathbb{A},\alpha},\ t \leq s \leq T\}$, depends on x only on the past trajectory $\{x(u), 0 \leq u \leq t\}$.

Remark 2.1 It is worth to mention that in contrast with [26], [19], no non-degeneracy assumption on the diffusion coefficient σ, nor specific condition between the drift and the diffusion coefficient is imposed. In particular, we may control independently both drift and diffusion, and it may well happen that some lines or columns of σ are equal to zero, and even $\sigma = 0$, in which case we have results for deterministic control problems.

Remark 2.2 One may consider a priori more general non-Markovian controlled equations of the form

$$dX_t \quad = \quad b_t(X, W, \alpha_t)\, dt + \sigma_t(X, W, \alpha_t)\, dW_t,$$

i.e. where the drift and diffusion coefficients are non-anticipative functionals of the trajectory of the Wiener process W as well. However, this generality is only apparent, since we can adopt the following standard procedure to reduce to the case presented above: we introduce a second state component $Y = W$ and consider the equivalent controlled system

$$\left\{ \begin{array}{rcl} dX_t &=& b_t(X, Y, \alpha_t)\, dt + \sigma_t(X, Y, \alpha_t)\, dW_t, \\ dY_t &=& dW_t \end{array} \right.$$

which is of the form considered above, but with an enlarged state (X, Y). It is easy to formulate assumptions on $b_t(X, W, \alpha_t)$, $\sigma_t(X, W, \alpha_t)$ that allow to verify the requirements in (2.1)-(2.2) on the resulting controlled system, so we omit the details. We only remark that the fact that the latter has degenerate noise does not prevent the possibility of applying our results, as noted above. □

2.2 The value function

We are given a running and terminal reward function:

$$\begin{array}{rcl} [0,T]\times \mathbf{C}^d \times A & \longrightarrow & \mathbb{R}, \\ (t,x,a) & \longmapsto & f_t(x,a), \end{array} \qquad \begin{array}{rcl} \mathbf{C}^d & \longrightarrow & \mathbb{R} \\ x & \longmapsto & g(x), \end{array}$$

which are respectively $Prog(\mathbf{C}^d)\otimes \mathcal{B}(A)$-measurable, and $\mathcal{C}_T$-measurable, and we assume:

(H2)

(i) For all $t\in[0,T]$, the function $f_t(x,a)$ is continuous in $(x,a)\in \mathbf{C}^d\times A$, and the function g is continuous on $\mathbf{C}^d$,

(ii) There exist nonnegative constants K_2 and m such that

$$|f_t(x,a)|+|g(x)| \quad \leq \quad K_2\big(1+\|x\|_\infty^m\big),$$

for all $(t,x,a)\in[0,T]\times\mathbf{C}^d\times A$.

We then define the gain functional:

$$J(t,x,\mathbb{A},\alpha) \quad = \quad \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T f_s(X^{t,x,\mathbb{A},\alpha},\alpha_s)\,ds + g(X^{t,x,\mathbb{A},\alpha})\Big],$$

for $(t,x)\in[0,T]\times\mathbf{C}^d$, $\mathbb{A}=(\Omega,\mathcal{F},\mathbb{G},\mathbb{Q},W)$ a set-up, $\alpha\in\mathcal{A}(\mathbb{A})$, and the value function in its weak formulation as the supremum over all admissible set-ups and controls:

$$v(t,x) \quad = \quad \sup_{\mathbb{A}}\ \sup_{\alpha\in\mathcal{A}(\mathbb{A})} J(s,x,\mathbb{A},\alpha), \quad (t,x)\in[0,T]\times\mathbf{C}^d. \tag{2.5}$$

Due to (2.4) and the polynomial growth condition on f,g in **(H2)**, it is easy to check that v is always finite, and satisfies actually:

$$|v(t,x)| \quad \leq \quad K(1+|x_t^*|^m), \quad (t,x)\in[0,T]\times\mathbf{C}^d, \tag{2.6}$$

for some positive constant K. Thus, v is a real-valued function defined on $[0,T]\times\mathbf{C}^d$. Moreover, since $\{X_s^{t,x,\mathbb{A},\alpha},\ t\leq s\leq T\}$, depends on x only on the past trajectory $\{x(u), 0\leq u\leq t\}$, this is also the case for $v(t,x)$. In other words, $v(t,.)$ is $\mathcal{C}_t$-measurable for all $t\in[0,T]$. We do not address here the question of joint measurability of v in its arguments, i.e. whether v is $Prog(\mathbf{C}^d)$-measurable, since it is not needed for our purpose. We simply mention that this issue is already not trivial in the Markovian case, and usually relies on a measurable selection theorem. Actually, it is proved in [9], that v is indeed measurable when A is a Polish space.

Remark 2.3 One could also consider the optimal control problem in the strong formulation, i.e. the search for an optimal control $\alpha\in\mathcal{A}(\mathbb{A})$ in a given set-up $\mathbb{A}$, and the corresponding value function (in general depending on $\mathbb{A}$) which is defined as in (2.5) but dropping the supremum with respect to $\mathbb{A}$: see e.g. [10] Section IV.2, [30] Sections 2.4.1-2.4.2, or [9] for detailed formulations. In the Markovian framework, when a verification theorem for the HJB equation holds under appropriate conditions, the value functions for the weak and strong formulations are known to be the same: see [10] Remark IV.3.2. In the general non Markovian context, we refer to [9] for a discussion about the equivalence between the weak and strong formulations[1] . □

[1] We would like to thank Xiaolu Tan for pointing us this reference.

Remark 2.4 Fix an initial condition $t = 0$, $x = 0$, a set-up $\mathbb{A}$, and given a control $\alpha \in \mathcal{A}(\mathbb{A})$, denote by $\mathbb{Q}^\alpha(\mathbb{A})$ the distribution of $X^{0,0,\mathbb{A},\alpha}$, which can be seen as a probability measure on the canonical space $\mathbf{C}^d$. Then, the family $\{\mathbb{Q}^\alpha(\mathbb{A}), \alpha \in \mathcal{A}(\mathbb{A})\}$ is not dominated in general when the diffusion coefficient depends on the control α. In particular, when $b = 0$, $\sigma_t(x, a) = a$, so that the SDE (2.3) degenerates to a stochastic integral, and for $f = 0$, we see that the value function (2.5) falls into the class of sublinear expectations studied in [5]. More precisely, by considering g as a random variable on the canonical space, we have:

$$v(0,0) \quad = \quad \sup_{\mathbb{A}} \sup_{\alpha \in \mathcal{A}(\mathbb{A})} \mathbb{E}^{\mathbb{Q}^\alpha(\mathbb{A})}[g],$$

so that the mapping $g \mapsto v(0,0)$ may be viewed as a generalization of G-expectation [22], where the volatility α_t of the canonical process is uncertain, valued in A. □

3 Randomized control problem and BSDE representation

In this section, we provide a randomized representation of the control problem (2.5) by randomization of the controls. This will allow us to characterize the value function as the solution to a backward stochastic differential equation (BSDE) with nonpositive jumps, formulated under a single probability measure. This should be understood as a non-Markovian analog of fully nonlinear Hamilton-Jacobi-Bellman equation, and an alternative to 2BSDE ([26], [19]) under more general conditions, see Remark 2.1. We shall assume that **(H1)** and **(H2)** hold throughout this section.

3.1 Randomized representation with dominated measures

To state the randomized control problem we initially fix a finite measure λ on $(A, \mathcal{B}(A))$ with full topological support, i.e. an element of $\mathcal{M}_f(A)$, according to our previous notation. By a *randomized admissible set-up* (or simply a randomized set-up) we mean

$$\mathbb{D}_\lambda \quad = \quad (\Omega, \mathcal{F}, \mathbb{P}, W, \mu_\lambda),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $W = (W_t)_{0 \le t \le T}$ is an n-dimensional standard Wiener process, $\mu_\lambda = \sum_{n \ge 1} \delta_{(S_n, \eta_n)}$ is a Poisson random measure in $[0, \infty) \times A$ with compensator $\lambda(da)dt$, independent of W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$ the completion of the natural filtration of the pair (W, μ_λ). Although it depends on $\mathbb{D}_\lambda$, we do not make it explicit in the notation. It is proved in [28], Lemma A1, that $\mathbb{F}$ is right-continuous and hence it satisfies the usual conditions.

Let $\mathcal{V}(\mathbb{D}_\lambda)$ be the class of random fields $\nu_t(\omega, a) : [0, \infty) \times \Omega \times A \to (0, \infty)$ which are $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(A)$-measurable, where $\mathcal{P}(\mathbb{F})$ denotes the predictable σ-algebra associated to $\mathbb{F}$, and essentially bounded with respect to the measure $dt \otimes d\mathbb{P} \otimes \lambda(da)$. For $\nu \in \mathcal{V}(\mathbb{D}_\lambda)$, the Doléans-Dade exponential process

$$L_t^\nu \quad := \quad \exp\Big(\int_0^t \int_A (1 - \nu_s(a))\, \lambda(da)\, ds \Big) \prod_{S_n \le t} \nu_{S_n}(\eta_n), \qquad t \ge 0, \tag{3.1}$$

is a positive $(\mathbb{P}, \mathbb{F})$-martingale on $[0, T]$, i.e. $\mathbb{E}[L_T^\nu] = 1$ (since ν is bounded and $\lambda(A) < \infty$), and defines a probability measure $\mathbb{P}^\nu$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, by setting $\mathbb{P}^\nu(d\omega) = L_T^\nu(\omega)\mathbb{P}(d\omega)$. We recall that, by Girsanov's theorem (see e.g. [14] Theorem 4.5), under $\mathbb{P}^\nu$ the random measure μ_λ admits compensator $\nu_t(a)\, \lambda(da)dt$ on $[0, T] \times A$, and W remains a Wiener process independent of μ_λ.

In order to define the gain functional, in addition to $\lambda \in \mathcal{M}_f(A)$ we also fix an element $a \in A$, a starting time $t \in [0,T]$ and an initial condition $x \in \mathbf{C}^d$. Then we define a pair of processes $(I,X) = (I_s, X_s)_{0 \le s \le T}$ as the unique strong solution to

$$\left\{ \begin{array}{lcll} I_s & = & a, & s \in [0,t], \\ I_s & = & a + \displaystyle\int_{(t,s]} \int_A (a' - I_{u-})\, \mu_\lambda(du, da'), & s \in [t,T], \\ X_s & = & x(s), & s \in [0,t], \\ X_s & = & x(t) + \displaystyle\int_t^s b_u(X, I_u)\, du + \int_t^s \sigma_u(X, I_u)\, dW_u, & s \in [t,T]. \end{array} \right. \tag{3.2}$$

We note that I is determined by the initial point a and the restriction of μ_λ to $(t,T] \times A$: more precisely, letting $N_t = \sum_{n \ge 1} 1_{S_n \le t}$ denote the number of jumps of μ_λ in the time interval $[0,t]$, we have the explicit formula

$$I_s \;=\; a\, 1_{[0, S_{N_s+1})}(s) + \sum_{n \ge N_s+1} \eta_n\, 1_{[S_n, S_{n+1})}(s), \qquad s \in [0,T].$$

The uncontrolled pure-jump process I valued in A, should be understood as a randomization of the control α in the primal problem defined in the previous paragraphs. Instead of (I,X), we may write $(I^{t,a,\lambda,\mathbb{D}_\lambda}, X^{t,x,a,\lambda,\mathbb{D}_\lambda})$ to stress dependence on these parameters.

We notice that the papers [11] or [8] also use a notion of "randomized" controls, which is however different from ours: the authors relaxed A-valued controls by a measure-valued process, but do not consider additional randomness and filtration as in our approach with the Poisson random measure.

We then introduce the randomized gain functional

$$J^*(t,x,a,\lambda,\mathbb{D}_\lambda,\nu) \;=\; \mathbb{E}^\nu\Big[\int_t^T f_s(X^{t,x,a,\lambda,\mathbb{D}_\lambda}, I_s^{t,a,\lambda,\mathbb{D}_\lambda})\, ds + g(X^{t,x,a,\lambda,\mathbb{D}_\lambda}) \Big],$$

(here $\mathbb{E}^\nu$ denotes the expectation under $\mathbb{P}^\nu$) for $(t,x,a) \in [0,T] \times \mathbf{C}^d \times A$, $\lambda \in \mathcal{M}_f(A)$, $\mathbb{D}_\lambda$ a randomized set-up, and $\nu \in \mathcal{V}(\mathbb{D}_\lambda)$. The intuitive interpretation is that in the randomized control problem we first replace the control α by a Poisson point process I, and we can then control the system by modifying the intensity of I. More precisely, we are able to change its compensator $\lambda(da)dt$ into $\nu_t(a)\,\lambda(da)dt$, by choosing the density ν in the class of bounded positive and predictable random fields.

We finally introduce two value functions for the randomized formulation. The first one is defined by

$$v_1^*(t,x,a,\lambda,\mathbb{D}_\lambda) \;=\; \sup_{\nu \in \mathcal{V}(\mathbb{D}_\lambda)} J^*(t,x,a,\lambda,\mathbb{D}_\lambda,\nu), \tag{3.3}$$

and corresponds to optimizing with respect to every choice of ν in the fixed set-up $\mathbb{D}_\lambda$ and for a fixed $\lambda \in \mathcal{M}_f(A)$, i.e. it corresponds to the strong formulation of the randomized optimal control problem. The second one is the value function in the weak formulation, where the set-up is part of the control:

$$v_2^*(t,x,a,\lambda) \;=\; \sup_{\mathbb{D}_\lambda} v_1^*(t,x,a,\lambda,\mathbb{D}_\lambda). \tag{3.4}$$

We shall see later that the strong and weak randomized formulations are equivalent, i.e. v_1^* does not depend on $\mathbb{D}_\lambda$, and so is equal to v_2^*. For the moment, we can easily check, as in the

original control problem, that v_1^* and v_2^* are always finite, satisfying actually the same growth condition (2.6), and $v_1^*(t,x,.)$ and $v_2^*(t,x,.)$ depend on $x \in \mathbf{C}^d$ only via the past trajectory $\{x(s), 0 \leq s \leq t\}$.

Our first main result is to connect the primal control problem to the weak randomized one.

Theorem 3.1 *We have*

$$v(t,x) \quad = \quad v_2^*(t,x,a,\lambda), \qquad (t,x) \in [0,T] \times \mathbf{C}^d,$$

for all $a \in A$, $\lambda \in \mathcal{M}_f(A)$. In particular, $v_2^*(t,x,a,\lambda)$ does not depend on (a,λ).

Remark 3.1 The randomized problem is a control problem over equivalent probability measures $\mathbb{P}^\nu$, whose effect is to change the intensity measure of the pure-jump component I. Theorem 3.1 then formally means that one can formulate the primal control problem originally written in a non dominated Wiener space framework into a dominated framework by enlarging the filtered probability space with an additional Poisson random measure. Moreover, the result is invariant with respect to the choice of the intensity measure for the Poisson random measure. □

3.2 BSDE characterization

Throughout this section, we fix $\lambda \in \mathcal{M}_f(A)$ and the initial conditions $t \in [0,T]$, $x \in \mathbf{C}^d$ and $a \in A$. We consider a randomized set-up $\mathbb{D}_\lambda = (\Omega, \mathcal{F}, \mathbb{P}, W, \mu_\lambda)$, denote by $\mathbb{F}$ the completion of the natural filtration of (W, μ_λ), and define a pair of processes (I, X) as the solution to the system (3.2), dropping their dependence on $t, x, a, \lambda, \mathbb{D}_\lambda$ in the notation.

Following [16], we say that a quadruple (Y, Z, U, K) is a solution to the BSDE with nonpositive jumps:

$$\left\{ \begin{array}{rcl} Y_s & = & g(X) + \displaystyle\int_s^T f_r(X, I_r)dr + K_T - K_s \\ & & \displaystyle - \int_s^T Z_r dW_r - \int_s^T \int_A U_r(a)\mu_\lambda(dr, da), \quad s \in [0,T], \\ U_s(a) & \leq & 0, \end{array} \right. \tag{3.5}$$

if $Y \in \mathcal{S}^2$, the space of càd-làg $\mathbb{F}$-adapted processes Y satisfying $\|Y\|^2 := \mathbb{E}[\sup_{s\in[0,T]} |Y_s|^2] < \infty$, identified up to indistinguishability, $Z \in L^2(W)$, the space of $\mathbb{F}$-predictable processes with values in $\mathbb{R}^n$ such that $\|Z\|^2_{L^2(W)} := \mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty$, identified up to $ds \otimes d\mathbb{P}$-a.e. equality, $U \in L^2(\tilde{\mu})$, the space of $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(A)$-measurable real-valued processes U such that $\|U\|^2_{L^2(\tilde{\mu})} := \mathbb{E}[\int_0^T \int_A |U_s(a)|^2 \lambda(da)\, ds] < \infty$, identified up to $ds \otimes d\mathbb{P} \otimes \lambda(da)$-a.e. equality, $K \in \mathcal{K}^2$, the subspace of $\mathcal{S}^2$ consisting of nondecreasing predictable processes such that $K_0 = 0$, $\mathbb{P}$-a.s., and the equation in (3.5) holds $\mathbb{P}$-a.s., while the nonpositive jump constraint holds on $[0,T] \times \Omega \times A$ a.e. with respect to the measure $ds \otimes \mathbb{P}(d\omega) \otimes \lambda(da)$.

Definition 3.1 *A minimal solution to the BSDE* (3.5) *is a quadruple $(Y, Z, U, K) \in \mathcal{S}^2 \times L^2(W) \times L^2(\tilde{\mu}) \times \mathcal{K}^2$ solution to* (3.5) *such that for any other solution (Y', Z', U', K') to the same BSDE, we have $\mathbb{P}$-a.s.:*

$$Y_s \quad \leq \quad Y'_s, \quad s \in [0,T].$$

As noticed in Remark 2.1 in [16], the minimal solution, when it exists, is unique as an element of $\mathcal{S}^2 \times L^2(W) \times L^2(\tilde{\mu}) \times \mathcal{K}^2$ By misuse of language, we say sometimes that Y (instead of the quadruple (Y, Z, U, K)) is the minimal solution to (3.5).

We state the main result of this paper.

Theorem 3.2 *For all $(t, x, a) \in [0, T] \times \mathbf{C}^d \times A$, $\lambda \in \mathcal{M}_f(A)$, and $\mathbb{D}_\lambda$ set-up, we have the following assertions:*

1. *There exists a unique minimal solution $Y = Y^{t,x,a,\lambda,\mathbb{D}_\lambda}$ to (3.5). Moreover, for $s \in [0, t]$, $Y_s^{t,x,a,\lambda,\mathbb{D}_\lambda}$ is deterministic and does not depend on $\mathbb{D}_\lambda$.*

2. *This minimal solution is related to the primal and randomized control problems by:*

$$v(t, x) \;=\; v_1^*(t, x, a, \lambda, \mathbb{D}_\lambda) \;=\; v_2^*(t, x, a, \lambda) \;=\; Y_t^{t,x,a,\lambda,\mathbb{D}_\lambda}. \tag{3.6}$$

In particular, v_1^* neither depends on $\mathbb{D}_\lambda$, i.e. the strong and weak randomized control problems coincide, and none of the functions in (3.6) depends on (a, λ).

Remark 3.2 The HJB equation for a stochastic control problem is usually derived from a dynamic programming principle on the value function. We note that in view of the above relation $v(t, x) = Y_t^{t,x,a,\lambda,\mathbb{D}_\lambda}$ (or simply $v(t, x) = Y_t^{t,x}$ since it does not depend on $a, \lambda, \mathbb{D}_\lambda$), which is proved by purely probabilistic arguments, this gives another method to derive the HJB equation from the minimal BSDE solution Y. In the Markovian case, it is shown in [16] that the solution Y^n to an appropriate penalized BSDE satisfies a semi linear PDE and, by passing to the limit, one obtains that $Y_t^{t,x}$ is solution to the nonlinear HJB equation. Such derivation does not resort to dynamic programming principle, which is known to be a delicate issue, and would be interesting to explore in the non-Markovian context. □

4 Proof of the main results

Throughout this section, we make the standing assumptions **(H1)** and **(H2)**.

4.1 Proof of Theorem 3.1

We start with the inequality $v(t, x) \geq v_2^*(t, x, a, \lambda)$, which is the easy part in the proof of Theorem 3.1 and it is to be expected since, intuitively, in the randomized control problem we control X through the intensity of the control process I, so we cannot have a better performance than choosing directly the control process α as we do in the primal problem.

4.1.1 Proof of the inequality $v \geq v_2^*$

Fix $t \in [0, T]$, $x \in \mathbf{C}^d$, $a \in A$, $\lambda \in \mathcal{M}_f(A)$ and an admissible randomized set-up $\mathbb{D}_\lambda = (\Omega, \mathcal{F}, \mathbb{P}, W, \mu_\lambda)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the completion of the natural filtration generated by (W, μ_λ). Choose a random field ν in the class $\mathcal{V}(\mathbb{D}_\lambda)$ and define the corresponding probability measure $\mathbb{P}^\nu$. Define (I, X), as the solution to (3.2) and consider the gain for the randomized control problem

$$J^*(t, x, a, \lambda, \mathbb{D}_\lambda, \nu) \;=\; \mathbb{E}^\nu\Big[\int_t^T f_s(X, I_s)\, ds + g(X)\Big].$$

Now we note that $\mathbb{A} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\nu, W)$ is an admissible set-up for the original control problem. Moreover, since the process I is progressive with respect to $\mathbb{F}$, it belongs to the class of $\mathbb{A}$-admissible controls, i.e. it is an element of $\mathcal{A}(\mathbb{A})$. The corresponding trajectory can be obtained solving equation (2.3) with $\alpha = I$, which coincides with the equations defining the process X in (3.2). It follows that, with this choice of $\mathbb{A}$ and α, the gain for the original and the randomized problems are the same, so we have

$$J^*(t, x, a, \lambda, \mathbb{D}_\lambda, \nu) \;=\; J(t, x, \mathbb{A}, I) \;\leq\; v(t, x).$$

The required conclusion follows by taking the supremum with respect to $\nu \in \mathcal{V}(\mathbb{D}_\lambda)$, and then with respect to $\mathbb{D}_\lambda$, in the left-hand side.

4.1.2 Proof of the inequality $v_2^* \geq v$

The required inequality will be a consequence of the following proposition:

Proposition 4.1 *Fix $t \in [0, T]$, $x \in \mathbf{C}^d$, $a \in A$, $\lambda \in \mathcal{M}_f(A)$. Then, for every admissible set-up $\mathbb{A}' = (\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$, for every admissible control $\alpha \in \mathcal{A}(\mathbb{A}')$ and for every number $\delta > 0$, there exist an admissible randomized set-up $\mathbb{D}_\lambda = (\Omega, \mathcal{F}, \mathbb{P}, W, \mu_\lambda)$ and an element $\nu \in \mathcal{V}(\mathbb{D}_\lambda)$ such that*

$$J^*(t, x, a, \lambda, \mathbb{D}_\lambda, \nu) \;\geq\; J(t, x, \mathbb{A}', \alpha) - \delta.$$

We first check that the required inequality $v_2^*(t, x, a, \lambda) \geq v(t, x)$ is an immediate consequence. Indeed, from the inequality of Proposition 4.1, it follows that

$$v_2^*(t, x, a, \lambda) \;\geq\; J(t, x, \mathbb{A}', \alpha) - \delta.$$

Taking the supremum with respect to $\alpha \in \mathcal{A}(\mathbb{A}')$, and then with respect to all admissible set-ups $\mathbb{A}'$, we conclude that $v_2^*(t, x, a, \lambda) \geq v(t, x) - \delta$ and finally the required inequality follows from the arbitrariness of δ.

The rest of this section is devoted to the proof of Proposition 4.1. Since the proof is rather technical, in order to simplify the notation we will limit the exposition to the case when $t = 0$. The general case can be proved in the same way, with slight and obvious changes. We fix elements $x \in \mathbf{C}^d$, $a \in A$, $\lambda \in \mathcal{M}_f(A)$. We also fix an admissible set-up $\mathbb{A}' = (\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$, an admissible control $\alpha \in \mathcal{A}(\mathbb{A}')$, and $\delta > 0$. The corresponding trajectory X is the unique solution to

$$X_t \;=\; x(0) + \int_0^t b_s(X, \alpha_s)\, ds + \int_0^t \sigma_s(X, \alpha_s)\, dW_s, \qquad t \in [0, T]. \tag{4.1}$$

Note that $x(\cdot)$ affects the trajectory X only through the value $x(0)$. The gain functional is

$$J(0, x, \mathbb{A}', \alpha) \;=\; \mathbb{E}^{\mathbb{Q}'}\Big[\int_0^T f_t(X, \alpha_t)\, dt + g(X)\Big].$$

Following [18], we introduce a metric in the set $\mathcal{A}(\mathbb{A}')$ of admissible controls. Recall that ρ denotes the metric in A, chosen such that $\rho < 1$. Next define, for any $\alpha^1, \alpha^2 \in \mathcal{A}(\mathbb{A}')$,

$$\tilde{\rho}(\alpha^1, \alpha^2) \;=\; \mathbb{E}^{\mathbb{Q}'}\Big[\int_0^T \rho(\alpha_t^1, \alpha_t^2)\, dt\Big].$$

Note that a sequence α^n converges in $\mathcal{A}(\mathbb{A}')$ to a limit α with respect to this metric if and only if $\alpha^n \to \alpha$ in $dt \otimes d\mathbb{Q}'$-measure, i.e. if and only if

$$\lim_{n\to\infty} (dt \otimes d\mathbb{Q}')(\{(t,\omega') \in [0,T] \times \Omega' \,:\, \rho(\alpha_t^n(\omega'), \alpha_t(\omega')) > \epsilon\}) \;=\; 0, \qquad \text{for any } \epsilon > 0.$$

In [18], the following continuity result of the gain functional with respect to the control is proved in the case of controlled diffusion processes. The extension to our non-Markovian situation is straightforward, so we only sketch its proof.

Lemma 4.1 *The map $\alpha \mapsto J(0,x,\mathbb{A}',\alpha)$ is continuous with respect to the metric $\tilde{\rho}$.*

Proof. In this proof we write $\mathbb{E}'$ instead of $\mathbb{E}^{\mathbb{Q}'}$ for short. Suppose $\alpha^n, \alpha \in \mathcal{A}(\mathbb{A}')$ and $\alpha^n \to \alpha$ in $dt \otimes d\mathbb{Q}'$-measure. Denote X^n, X the corresponding trajectories. Then, starting from the state equation (4.1), using usual arguments involving the Burkholder-Davis-Gundy inequalities and the Gronwall lemma, for every $p \in [2,\infty)$ we arrive at

$$\mathbb{E}'\Big[\sup_{t\in[0,T]} |X_t^n - X_t|^p\Big] \;\le\; C\Big\{\mathbb{E}'\Big[\int_0^T |b_t(X,\alpha_t^n) - b_t(X,\alpha_t)|^p + |\sigma_t(X,\alpha_t^n) - \sigma_t(X,\alpha_t)|^p\, dt\Big]\Big\},$$

for a suitable constant C, independent of n. Recalling the bound (2.4) on the solution X, by standard arguments we first conclude, by the dominated convergence theorem, under **(H1)**, that $\mathbb{E}'\Big[\sup_{t\in[0,T]} |X_t^n - X_t|^p\Big] \to 0$ as $n \to \infty$. Next we have

$$|J(0,x,\mathbb{A}',\alpha^n) - J(0,x,\mathbb{A}',\alpha)| \;\le\; \mathbb{E}'\Big[\int_0^T |f_t(X^n,\alpha_t^n) - f_t(X,\alpha_t)|\, dt\Big] + \mathbb{E}'|g(X^n) - g(X)|.$$

To finish the proof we show that the right-hand side tends to zero. Suppose on the contrary, that there exist $\eta > 0$ and a subsequence (denoted $(X^{n'}, \alpha^{n'})$) such that

$$\mathbb{E}'\Big[\int_0^T |f_t(X^{n'},\alpha_t^{n'}) - f_t(X,\alpha_t)|\, dt\Big] \;\ge\; \eta, \tag{4.2}$$

for every n'. Passing to a sub-subsequence, still denoted by the same symbol, we can assume that

$$\sup_{t\in[0,T]} |X_t^{n'} - X_t| \to 0, \quad d\mathbb{Q}' - a.s., \quad \rho(\alpha_t^n, \alpha_t) \to 0, \quad dt \otimes d\mathbb{Q}' - a.e.$$

as $n' \to \infty$, and by the assumed continuity properties of f it follows that $f_t(X^{n'},\alpha_t^{n'}) \to f_t(X,\alpha_t)$, $dt \otimes d\mathbb{Q}'$-a.e. Next we extract a further subsequence (n'_j) such that

$$\left(\mathbb{E}'\Big[\sup_{t\in[0,T]} |X_t^{n'_j} - X_t|^p\Big]\right)^{1/p} \;\le\; 2^{-j},$$

so that the random variable $\bar{X} := \sum_j \sup_{t\in[0,T]} |X_t^{n'_j} - X_t|$ satisfies $\mathbb{E}'|\bar{X}|^p < \infty$ as well as $|X_t^{n'_j}| \le |X_t| + |\bar{X}|$ for every t and j. Recalling the polynomial growth condition of f in **(H2)**, we obtain

$$\begin{aligned} |f_t(X^{n'_j},\alpha_t^{n'_j}) - f_t(X,\alpha_t)| \;&\le\; C(1 + \sup_{t\in[0,T]} |X_t^{n'_j}|^m + \sup_{t\in[0,T]} |X_t|^m) \\ &\le\; C(1 + |\bar{X}|^m + \sup_{t\in[0,T]} |X_t|^m) \end{aligned}$$

for a suitable constant C, and choosing $p = \max(m, 2)$, we conclude that the right-hand side is integrable, which gives a contradiction with (4.2) by the dominated convergence theorem. This shows that $\mathbb{E}'\big[\int_0^T |f_t(X^n, \alpha_t^n) - f_t(X, \alpha_t)|\, dt\big] \to 0$, and in a similar way one shows that $\mathbb{E}'|g(X^n) - g(X)| \to 0$. □

Remark 4.1 For further use we note that the metric $\tilde{\rho}$ can be defined on the set of all $\mathcal{B}([0,T]) \otimes \mathcal{F}'$-measurable A-valued processes. Now suppose that we have a sequence of filtrations $\mathbb{H}^m = (\mathcal{H}_t^m)$, satisfying $\mathcal{G}_t' \subset \mathcal{H}_t^m$ for every $t \geq 0$, such that W is a Wiener process with respect to each of them, and denote $\mathbb{A}^m = (\Omega', \mathcal{F}', \mathbb{H}^m, \mathbb{Q}', W)$ the corresponding set-ups. Then, given a sequence $\alpha^m \in \mathcal{A}(\mathbb{A}^m)$, and $\alpha \in \mathcal{A}(\mathbb{A}')$, the convergence $\tilde{\rho}(\alpha^m, \alpha) \to 0$ still implies $J(0, x, \mathbb{A}^m, \alpha^m) \to J(0, x, \mathbb{A}', \alpha)$. This is a slight extension of Lemma 4.1, that can be proved by the same arguments as before. □

The following lemma shows that we can replace any control process by another control which is a pure jump process, without changing the gain too much.

Lemma 4.2 *There exists an admissible control process $\bar{\alpha} \in \mathcal{A}(\mathbb{A}')$, such that*

$$J(0, x, \mathbb{A}', \bar{\alpha}) \;\geq\; J(0, x, \mathbb{A}', \alpha) - \delta, \tag{4.3}$$

and $\bar{\alpha}$ has the form $\bar{\alpha}_t = \sum_{n=0}^{N-1} \alpha_n 1_{[T_n, T_{n+1})}(t)$, where $0 = T_0 < T_1 < \ldots T_N = T$ is a deterministic partition of $[0, T]$, $\alpha_0, \ldots, \alpha_{N-1}$ are A-valued random variables that take only a finite number of values, and each α_n is $\mathcal{G}_{T_n}'$-measurable. In addition, we can choose $\bar{\alpha}$ satisfying $\alpha_0 = a$.

Proof. In [18] Lemma 3.2.6 it is proved that the set of admissible controls $\bar{\alpha}$ having the form specified in the lemma are dense in $\mathcal{A}(\mathbb{A}')$ with respect to the metric $\tilde{\rho}$. The Lemma is then a consequence of the continuity property of Lemma 4.1. The additional requirement that $\alpha_0 = a$ can be fulfilled by adding, if necessary, another point T' close to 0 to the subdivision and modifying $\bar{\alpha}$ setting $\bar{\alpha}_t = a$ for $t \in [0, T')$. This modification is as close as we wish to the original control with respect to the metric $\tilde{\rho}$, provided T' is chosen sufficiently small. □

Before proceeding further, we need to recall some useful facts, summarized in the following remark.

Remark 4.2 We start with an admissible set-up $\mathbb{A}' = (\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$, and we need to give new definitions or make suitable constructions using additional, *independent*, random variables or stochastic processes. If these random objects are not already defined on $(\Omega', \mathcal{F}', \mathbb{Q}')$ we can perform the following standard construction in order to reduce to this case. We consider another probability space $(\Omega'', \mathcal{F}'', \mathbb{Q}'')$ on which are defined these random objects, and we define $(\Omega, \mathcal{F}, \mathbb{Q})$ setting

$$\Omega \;=\; \Omega' \times \Omega'', \qquad \mathcal{F} \;=\; \mathcal{F}' \otimes \mathcal{F}'', \qquad \mathbb{Q} \;=\; \mathbb{Q}' \otimes \mathbb{Q}''.$$

We can also define a filtration $\mathbb{G} = (\mathcal{G}_t)$ in $(\Omega, \mathcal{F})$ setting $\mathcal{G}_t = \{B \times \Omega'' \,:\, B \in \mathcal{G}_t'\}$ for $t \geq 0$.

Any random variable Z (respectively, stochastic process (Y_t)) in $(\Omega', \mathcal{F}')$ admits a natural extension to a random variable (resp. stochastic process) on $(\Omega, \mathcal{F})$, still denoted by the same symbol, given by $Z(\omega) = Z(\omega')$ (resp. $Y_t(\omega) = Y_t(\omega')$), for $\omega = (\omega', \omega'') \in \Omega$. It easily verified that if Y is $\mathbb{G}'$-adapted (resp. optional, progressive, predictable) then its extension is $\mathbb{G}$-adapted (resp. optional, progressive, predictable). Moreover, the extension of W is a Wiener process with respect to $\mathbb{G}$ and $\mathbb{Q}$, so that we have constructed another admissible set-up $\mathbb{A} := (\Omega, \mathcal{F}, \mathbb{G}, \mathbb{Q}, W)$.

Note that the extension of an $\mathbb{A}'$-admissible control is $\mathbb{A}$-admissible, that the trajectory of an extended control process is the extension of the original trajectory, and that the corresponding gain functional has the same value.

Similar considerations hold for random objects originally defined on Ω''. For instance, if (V_n) denotes a random sequence on Ω'', its extension has the same law and is independent of $\mathcal{G}'_\infty$ (hence independent of W) under $\mathbb{Q}$.

We shall briefly describe this construction by saying that Ω is an *enlargement* of Ω', or that the set-up $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{Q}, W)$ is an *enlargement* of $(\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$.

In conclusion, starting with our admissible set-up $\mathbb{A}' = (\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$, we have proved the existence of an enlargement $\mathbb{A} = (\Omega, \mathcal{F}, \mathbb{G}, \mathbb{Q}, W)$ on which there exist random elements with arbitrary prescribed laws and independent of $\mathcal{G}'_\infty$ under $\mathbb{Q}$. □

In the proof of Proposition 4.1, we need a preliminary result, stated below as Lemma 4.3, where we basically prove that the marked point process $\bar{\alpha}$ in the statement of Lemma 4.2 can be slightly perturbed in such a way that its compensator becomes absolutely continuous with respect to the measure $\lambda(da)\,dt$. Then we shall see in Lemma 4.4 that the corresponding gain will also be changed slightly. The proof depends on a more general and technical result, reported in the Appendix as Lemma A.2.

Let us come back again to the original set-up $\mathbb{A}' = (\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$ in the statement of Proposition 4.1 and denote by $\bar{\alpha}$ the admissible control of Lemma 4.2. Let $\mathbb{A} = (\Omega, \mathcal{F}, \mathbb{G}, \mathbb{Q}, W)$ be an enlargement of the set-up $\mathbb{A}'$, as described in Remark 4.2, and denote the extension of the control $\bar{\alpha}$ still by the same symbol. It is convenient to extend further the definition of $\bar{\alpha}$ to $[0, \infty) \times \Omega$ in a trivial way setting

$$\alpha_n \quad = \quad \alpha_{N-1}, \;\; n \geq N; \qquad T_n = T + n - N, \;\; n > N,$$

and defining $\bar{\alpha}_t = \sum_{n=0}^{\infty} \alpha_n 1_{[T_n, T_{n+1})}(t)$. This way, $\bar{\alpha}$ is associated to the marked point process $(T_n, \alpha_n)_{n \geq 1}$ and $\alpha_0 = a$.

For every integer $m \geq 1$, on Ω we can find sequences $(U_n^m)_{n \geq 1}$, $(S_n^m)_{n \geq 1}$ of real random variables satisfying the following conditions:

1. every U_n^m is uniformly distributed on $(0, 1)$;
2. every S_n^m admits a density (denoted $f_n^m(t)$) with respect to the Lebesgue measure, and we have $0 < S_1^m < S_2^m < S_3^m < \ldots$ and $\sum_{n \geq 1} S_n^m < 1/m$ for every m;
3. for every m, the random variables U_n^m, S_k^m $(n, k \geq 1)$ are independent, and independent of $\mathcal{G}'_\infty$.

For every $m \geq 1$, let $\mathbf{B}(b, 1/m)$ denote the open ball of radius $1/m$, with respect to the metric ρ, centered at $b \in A$. Since $\lambda(da)$ has full support, we have $\lambda(\mathbf{B}(b, 1/m)) > 0$ and we can define a transition kernel $q^m(b, da)$ in A setting

$$q^m(b, da) \quad = \quad \frac{1}{\lambda(\mathbf{B}(b, 1/m))} \, 1_{\mathbf{B}(b, 1/m)}(a) \lambda(da).$$

We recall that we require A to be a Lusin space. It follows from Lemma A.1 that there exists a function $q^m : A \times [0, 1] \to A$, measurable with respect to $\mathcal{B}(A) \otimes \mathcal{B}([0, 1])$, such that for every $b \in A$ the measure $B \mapsto q^m(b, B)$ $(B \in \mathcal{B}(A))$ is the image of the Lebesgue measure on $[0, 1]$ under the mapping $u \mapsto q^m(b, u)$. Thus, if U is a random variable defined on some probability space and having uniform law on $[0, 1]$ then, for fixed $b \in A$, the random variable $q^m(b, U)$ has law $q^m(b, da)$ on A. The use of the same symbol q^m should not generate confusion.

Define

$$R_n^m \;=\; T_n + S_n^m, \quad \beta_n^m = q^m(\alpha_n, U_n^m), \qquad\qquad n, m \geq 1, \tag{4.4}$$

and set $R_0^m = 0$. Since we assume $S_n^m < S_{n+1}^m$ we see that $(R_n^m, \beta_n^m)_{n\geq 1}$ is a marked point process in A. Let $\kappa^m = \sum_{n\geq 1} \delta_{(R_n^m, \beta_n^m)}$ denote the corresponding random measure, and $(\mathcal{F}_t^{\kappa^m})$ the corresponding natural filtration. Finally set

$$\mathcal{H}_t^m \;=\; \mathcal{G}_t \vee \mathcal{F}_t^{\kappa^m}, \qquad t \geq 0. \tag{4.5}$$

Now Lemma A.2 in the Appendix provides us with the following explicit formula for the compensator $\tilde{\kappa}^m$ of κ^m with respect to the filtration $\mathbb{H}^m = (\mathcal{H}_t^m)$:

$$\tilde{\kappa}^m(dt, da) \;=\; \sum_{n\geq 1} 1_{(T_n \vee R_{n-1}^m, R_n^m]}(t)\, q^m(\alpha_n, da) \frac{f_n^m(t - T_n)}{1 - F_n^m(t - T_n)}\, dt,$$

where we denote by $F_n^m(s) = \int_{-\infty}^s f_n^m(t)dt$ the cumulative distribution function of S_n^m, with the convention that $\frac{f_n^m(s)}{1-F_n^m(s)} = 0$ if $F_n^m(s) = 1$.

We summarize the relevant properties of this construction in the following result.

Lemma 4.3 *With the previous notation, in the enlarged set-up $\mathbb{A}$ the following properties hold true:*

1. *$T_n < R_n^m$ and $\sum_{n\geq 1}(R_n^m - T_n) < 1/m$;*
2. *$\rho(\alpha_n, \beta_n^m) < 1/m$;*
3. *the $(\mathbb{Q}, \mathbb{H}^m)$-compensator of κ^m is absolutely continuous with respect to $\lambda(da)\, dt$, so that it can be written in the form*

$$\tilde{\kappa}^m(dt, da) \;=\; \phi_t^m(a)\, \lambda(da)\, dt$$

for a suitable nonnegative $\mathcal{P}(\mathbb{H}^m) \otimes \mathcal{B}(A)$-measurable function ϕ^m.

Proof. The first property follows from the fact that $S_n^m > 0$ and $\sum_{n\geq 1} S_n^m < 1/m$. Since, for every $b \in A$, $q^m(b, da)$ is supported in $\mathbf{B}(b, 1/m)$ we have $\rho(\alpha_n, \beta_n^m) < 1/m$. Finally, from the choice of the kernel $q^m(b, da)$ we obtain

$$\phi_t^m(a) \;=\; \sum_{n\geq 1} 1_{(T_n \vee R_{n-1}^m, R_n^m]}(t)\, \frac{1}{\lambda(\mathbf{B}(\alpha_n, 1/m))}\, 1_{\mathbf{B}(\alpha_n, 1/m)}(a) \frac{f_n^m(t - T_n)}{1 - F_n^m(t - T_n)}.$$

□

Now recall that we have fixed from the beginning $t = 0$, $x \in \mathbf{C}^d$, $a \in A$, $\lambda \in \mathcal{M}_f(A)$, a set-up $\mathbb{A}'$, $\alpha \in \mathcal{A}(A')$ and $\delta > 0$. Also recall the notation $J(0, x, \mathbb{A}', \alpha)$ for the gain functional.

Lemma 4.4 *There exists an admissible set-up $\mathbb{A}'' = (\Omega, \mathcal{F}, \mathbb{H}, \mathbb{Q}, W)$, which is an enlargement of $\mathbb{A}'$, and an admissible control $\hat{\alpha} \in \mathcal{A}(\mathbb{A}'')$ such that*

1. *$J(0, x, \mathbb{A}'', \hat{\alpha}) > J(0, x, \mathbb{A}', \alpha) - 2\delta$;*
2. *there exists an $\mathbb{H}$-marked point process $(R_n, \beta_n)_{n\geq 1}$ such that $\hat{\alpha}_t = \sum_{n\geq 0} \beta_n 1_{[R_n, R_{n+1})}(t)$ $(R_0 = 0, \beta_0 = a)$;*

3. *the* $(\mathbb{Q},\mathbb{H})$*-compensator of the corresponding random measure* $\kappa = \sum_{n\geq 1}\delta_{(R_n,\beta_n)}$ *is absolutely continuous with respect to* $\lambda(da)\,dt$*, so that it can be written in the form*

$$\tilde{\kappa}(dt,da) \;=\; \phi'_t(a)\,\lambda(da)\,dt$$

for a suitable nonnegative $\mathcal{P}(\mathbb{H})\otimes\mathcal{B}(A)$*-measurable function* ϕ'*.*

Proof. We first construct an appropriate enlargement $\mathbb{A} = (\Omega,\mathcal{F},\mathbb{G},\mathbb{Q},W)$ of $\mathbb{A}'$. Then we take the control $\bar{\alpha}$ of Lemma 4.2 and we extend it to $[0,\infty)\times\Omega$ as described above, so that it is associated to the $\mathbb{G}$-marked point process $(T_n,\alpha_n)_{n\geq 1}$. Finally, for every $m\geq 1$, we introduce $(R_n^m,\beta_n^m)_{n\geq 1}$ and the filtration $\mathbb{H}^m = (\mathcal{H}_t^m)$ defined by (4.4) and (4.5). Since the random variables S_n^m, U_n^m occurring in (4.4) are independent of $\mathcal{G}_\infty$ it follows that W is a Wiener process with respect to $\mathbb{H}^m$. Therefore $\mathbb{A}^m := (\Omega,\mathcal{F},\mathbb{H}^m,\mathbb{Q},W)$ is an admissible set-up. Next we define

$$\hat{\alpha}_t^m \;=\; \sum_{n\geq 0}\beta_n^m 1_{[R_n^m,R_{n+1}^m)}(t)$$

with the convention $R_0^m = 0, \beta_0^m = a$, and note that it is an admissible control, i.e. an element of $\mathcal{A}(\mathbb{A}^m)$. Now let us compare those controls with $\bar{\alpha}_t = \sum_{n=0}^\infty \alpha_n 1_{[T_n,T_{n+1})}(t)$. The first two conclusions of Lemma 4.3 show that $\hat{\alpha}_t^m$ converges to $\bar{\alpha}$ in $d\mathbb{Q}\otimes dt$-measure as $m\to\infty$, hence with respect to the metric $\tilde{\rho}$ introduced before. By Lemma 4.1 and Remark 4.1, this shows that $J(0,x,\mathbb{A}^m,\hat{\alpha}^m)\to J(0,x,\mathbb{A}',\bar{\alpha})$ as $m\to\infty$. So there exists M so large such that $J(0,x,\mathbb{A}^M,\hat{\alpha}^M) > J(0,x,\mathbb{A}',\bar{\alpha})-\delta$ and, by (4.3), $J(0,x,\mathbb{A}^M,\hat{\alpha}^M) > J(0,x,\mathbb{A}',\bar{\alpha})-2\delta$. We finally set

$$\mathbb{A}'' := \mathbb{A}^M,\quad \mathcal{H}_t := \mathcal{H}_t^M,\quad \beta_n := \beta_n^M,\quad R_n := R_n^M,\quad \hat{\alpha}_t := \hat{\alpha}_t^M,\quad \phi'_t(a) := \phi_t^M(a).$$

□

For the rest of the proof of Proposition 4.1, only Lemma 4.4 and Lemma 4.1 will be used. The idea is now to add to the control an additional *independent* Poisson measure with compensator $k^{-1}\lambda(da)\,dt$, with k large, i.e. with intensity so small that the gain is not much affected. The formal construction is as follows.

Let $\mathbb{A}'' = (\Omega,\mathcal{F},\mathbb{H},\mathbb{Q},W)$, $(R_n,\beta_n)_{n\geq 0}$, $\hat{\alpha}_t = \sum_{n\geq 0}\beta_n 1_{[R_n,R_{n+1})}(t)$, $\kappa = \sum_{n\geq 1}\delta_{(R_n,\beta_n)}$ with compensator $\tilde{\kappa}(dt,da) = \phi'_t(a)\,\lambda(da)\,dt$, denote the objects introduced in Lemma 4.4. By Remark 4.2 we can assume that, for every integer $k\geq 1$, on the probability space $(\Omega,\mathcal{F},\mathbb{Q})$ there exists a Poisson random measure on $(0,\infty)\times A$, denoted

$$\pi^k \;=\; \sum_{n\geq 1}\delta_{(T_n^k,\xi_n^k)},$$

admitting compensator $k^{-1}\lambda(da)\,dt$ with respect to its natural filtration $\mathbb{F}^{\pi^k} = (\mathcal{F}_t^{\pi^k})$, and independent of $\mathcal{H}_\infty$. Now we define another random measure setting

$$\mu^k = \kappa + \pi^k.$$

Note that the jumps times (R_n) are independent of the jump times (T_n^k), and the latter have absolutely continuous laws. It follows that, except possibly on a set of $\mathbb{Q}$ probability zero, their graphs are disjoint, i.e. κ and π^k have no common jumps. Therefore, the random measure μ^k admits a representation

$$\mu^k = \sum_{n\geq 1}\delta_{(S_n^k,\eta_n^k)}$$

where $(S_n^k, \eta_n^k)_{n\geq 1}$ is a marked point process, each S_n^k coincides with one of the times R_n or one of the times T_n^k, and each η_n^k coincides with one of the random variables ξ_n^k or one of the random variables β_n. Since κ and π^k are independent it follows from Proposition A.1 that μ^k has compensator $(\phi_t'(a) + k^{-1})\,\lambda(da)\,dt$ with respect to the filtration $\mathbb{H} \vee \mathbb{F}^{\pi^k} = (\mathcal{H}_t \vee \mathcal{F}_t^{\pi^k})$. Let us denote by $\mathbb{F}^k = (\mathcal{F}_t^k)$ the completion of the natural filtration generated by (W, μ^k). Clearly, W is a Wiener process with respect to $\mathbb{F}^k$.

Now we need to prove that the compensator of μ^k with respect to $\mathbb{F}^k$ remains absolutely continuous with respect to $\lambda(da)dt$, see Lemma 4.5 below. For its formulation, consider the measure space $([0,\infty) \times \Omega \times A, \mathcal{B}([0,\infty)) \otimes \mathcal{F} \otimes \mathcal{B}(A), dt \otimes \mathbb{Q}(d\omega) \otimes \lambda(da))$. Although this is not a probability space, one can define in a standard way the conditional expectation of any positive measurable function, given an arbitrary sub-σ-algebra. Let us denote by $\phi_t(\omega, a)$ the conditional expectation of the random field $\phi_t'(\omega, a)$ with respect to the σ-algebra $\mathcal{P}(\mathbb{F}^k) \otimes \mathcal{B}(A)$.

Lemma 4.5 *The compensator of μ^k with respect to $(\mathbb{Q}, \mathbb{F}^k)$ is $(\phi_t(a) + k^{-1})\,\lambda(da)\,dt$.*

Proof. Let H denote an arbitrary positive $\mathcal{P}(\mathbb{F}^k) \otimes \mathcal{B}(A)$-measurable function. Denote by $\mathbb{F}^{0,k} = (\mathcal{F}_t^{0,k})$ the uncompleted natural filtration generated by (W, μ^k). Then there exists a positive $\mathcal{P}(\mathbb{F}^{0,k}) \otimes \mathcal{B}(A)$-measurable function H' such that, for $\mathbb{Q}$-almost all ω, $H_t(\omega, a) = H_t'(\omega, a)$ for every t, a. Since clearly $\mathcal{F}_t^{0,k} \subset \mathcal{H}_t \vee \mathcal{F}_t^{\pi^k}$ for every $t \geq 0$, H' is also measurable with respect to $\mathcal{P}(\mathbb{H} \vee \mathbb{F}^{\pi^k}) \otimes \mathcal{B}(A)$ and we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty \int_A H_t(a)\,\mu^k(dt, da)\Big] &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty \int_A H_t'(a)\,\mu^k(dt, da)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty \int_A H_t'(a)\,(\phi_t'(a) + k^{-1})\,\lambda(da)\,dt\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty \int_A H_t(a)\,(\phi_t'(a) + k^{-1})\,\lambda(da)\,dt\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty \int_A H_t(a)\,(\phi_t(a) + k^{-1})\,\lambda(da)\,dt\Big],
\end{aligned}$$

which proves the required result. □

Let us define the pure jump process valued in A associated to the random measure μ^k by setting

$$I_t^k = \sum_{n\geq 0} \eta_n^k 1_{[S_n^k, S_{n+1}^k)}(t), \quad t \geq 0,$$

with the convention that $\eta_0^k = a$ ($a \in A$ was arbitrary and fixed above). Then $\mathbb{A}^k := (\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F}^k, W)$ is an admissible set-up and I^k is an admissible control, i.e. $I^k \in \mathcal{A}(\mathbb{A}^k)$. We can prove that, for large k, I^k is close to $\hat{\alpha}$ with respect to the metric $\tilde{\rho}$:

Lemma 4.6 *We have $\tilde{\rho}(I^k, \hat{\alpha}) \to 0$ as $k \to \infty$.*

Proof. We have to prove that $I^k \to \hat{\alpha}$ in $dt \otimes d\mathbb{Q}$-measure. Recall that the jump times of π^k are denoted T_n^k. Since T_1^k has exponential law with parameter $\lambda(A)/k$ the event $B_k = \{T_1^k > T\}$ has probability $e^{-\lambda(A)T/k}$, so that $\mathbb{Q}(B_k) \to 1$ as $k \to \infty$. Noting that on the set B_k, we have $\hat{\alpha}_t = I_t^k$ for all $t \in [0, T]$, the conclusion follows immediately. We remark that we have used the fact that $\alpha_0 = \eta_0^k = a$. □

Applying Lemma 4.1 and Remark 4.1 we conclude that $J(0,x,\mathbb{A}^k,I^k) \to J(0,x,\mathbb{A}'',\hat{\alpha})$ as $k \to \infty$. So there exists k large enough such that $J(0,x,\mathbb{A}^k,I^k) > J(0,x,\mathbb{A}'',\hat{\alpha}) - \delta$ and, by Lemma 4.4, $J(0,x,\mathbb{A}^k,I^k) > J(0,x,\mathbb{A}',\alpha) - 3\delta$. Introducing the notation

$$\mathbb{A}''' := \mathbb{A}^k, \quad \mathcal{F}_t := \mathcal{F}_t^k, \quad \eta_n := \eta_n^k, \quad S_n := S_n^k, \quad I_t := I_t^k, \quad t \geq 0,$$

we see that we have proved the following result, where we choose $\epsilon = k^{-1}$ (recall that we have fixed from the beginning $t = 0$, $x \in \mathbf{C}^d$, $a \in A$, $\lambda \in \mathcal{M}_f(A)$, set-up $\mathbb{A}'$, $\alpha \in \mathcal{A}(\mathbb{A}')$, and $\delta > 0$):

Lemma 4.7 *There exists an admissible set-up $\mathbb{A}''' = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}, W)$, which is an enlargement of $\mathbb{A}'$, and an admissible control $I \in \mathcal{A}(\mathbb{A}''')$ such that*

1. *$J(0,x,\mathbb{A}''',I) > J(0,x,\mathbb{A}',\alpha) - 3\delta$;*
2. *there exists an $\mathbb{F}$-marked point process $(S_n,\eta_n)_{n\geq 1}$ such that $I_t = \sum_{n\geq 0} \eta_n 1_{[S_n,S_{n+1})}(t)$ ($S_0 = 0, \eta_0 = a$);*
3. *$\mathbb{F}$ is the completion of the natural filtration of (W,μ), where $\mu = \sum_{n\geq 1} \delta_{(S_n,\eta_n)}$ is the corresponding random measure;*
4. *the $(\mathbb{Q},\mathbb{F})$-compensator of μ is absolutely continuous with respect to $\lambda(da)\,dt$ and it can be written in the form*

$$\tilde{\mu}(dt,da) \quad = \quad (\phi_t(a) + \epsilon)\,\lambda(da)\,dt$$

for some $\epsilon > 0$ and a nonnegative $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(A)$-measurable function ϕ.

We note for further use that the process I and the corresponding trajectory $(X_t)_{t\in[0,T]}$ are the solution to

$$\left\{ \begin{array}{lcll} I_t & = & a + \displaystyle\int_{(0,t]}\int_A (a' - I_{s-})\,\mu(ds,da'), & t \in [0,T], \\ X_t & = & x(0) + \displaystyle\int_0^t b_s(X,I_s)\,ds + \int_0^t \sigma_s(X,I_s)\,dW_s, & t \in [0,T]. \end{array} \right. \tag{4.6}$$

which coincides with (3.2) in the case $t = 0$ that we are addressing.

The final step in the proof of Proposition 4.1 consists in showing that the addition of the noise π^k above (a noise with intensity "of size $\epsilon = k^{-1}$") now makes it possible to make a Girsanov transformation and construct a randomized admissible set-up where μ is a Poisson random measure with compensator $\lambda(da)\,dt$ as required to fit the framework for the randomized control problem described in paragraph 3.1.

End of the proof of Proposition 4.1. Recall that we fix $x \in \mathbf{C}^d$, $a \in A$, $\lambda \in \mathcal{M}_f(A)$ and, without loss of generality, $t = 0$. We take arbitrary admissible set-up $\mathbb{A}' = (\Omega', \mathcal{F}', \mathbb{G}', \mathbb{Q}', W)$, admissible control $\alpha \in \mathcal{A}(\mathbb{A}')$ and $\delta > 0$. Next we consider again the set-up $\mathbb{A}''' = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}, W)$, the marked point process $(S_n,\eta_n)_{n\geq 1}$ (with $S_0 = 0, \eta_0 = a$), the corresponding admissible control $I_t = \sum_{n\geq 0} \eta_n 1_{[S_n,S_{n+1})}(t)$ and random measure $\mu = \sum_{n\geq 1} \delta_{(S_n,\eta_n)}$ and its compensator $\tilde{\mu}(dt,da)$ $= (\phi_t(a) + \epsilon)\,\lambda(da)\,dt$ in the statement of Lemma 4.7, and we recall that we have

$$J(0,x,\mathbb{A}''',I) > J(0,x,\mathbb{A}',\alpha) - 3\delta. \tag{4.7}$$

We want to show that there exist

1. a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that

$$\mathbb{D}_\lambda = (\Omega, \mathcal{F}, \mathbb{P}, W, \mu) \tag{4.8}$$

is an admissible randomized set-up;

2. an element $\bar{\nu} \in \mathcal{V}(\mathbb{D}_\lambda)$ such that

$$J^*(0, x, a, \lambda, \mathbb{D}_\lambda, \bar{\nu}) \ > \ J(0, x, \mathbb{A}''', I) - \delta. \tag{4.9}$$

By (4.7), and since δ is arbitrary, this is enough to finish the proof of Proposition 4.1.

To this end, let us define $\nu = \phi + \epsilon$, and note that ν is a positive $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(A)$-measurable random field. Since $\nu_t(a)\lambda(da)dt$ is the compensator of the non-explosive process μ, it follows easily that $\int_0^T \nu_t(a)\lambda(da)dt < \infty$ $\mathbb{Q}$-a.s., so that we can and will assume that ν takes only finite values on $[0, T] \times \Omega \times A$. Finally, since the inverse ν^{-1} is bounded and $\lambda(A) < \infty$, it follows from standard criterion (see e.g. [25]) that the Doléans-Dade exponential process

$$M_t \ := \ \exp\Big(\int_0^t \int_A (1 - \nu_s(a)^{-1})\, \nu_t(a)\lambda(da)\, ds \Big) \prod_{S_n \le t} \nu_{S_n}(\eta_n)^{-1}, \qquad t \ge 0, \tag{4.10}$$

is a strictly positive martingale on $[0, T]$ (with respect to $\mathbb{F}$ and $\mathbb{Q}$), and we can define an equivalent probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ setting $\mathbb{P}(d\omega) = M_T(\omega)\mathbb{Q}(d\omega)$. The expectation under $\mathbb{P}$ will be denoted simply $\mathbb{E}$ (while the expectation under $\mathbb{Q}$ is denoted $\mathbb{E}^{\mathbb{Q}}$). Now we make the following claims:

(i) μ (or more precisely its restriction to $(0, T] \times A$) has compensator $\lambda(da)\, dt$ (with respect to $\mathbb{F}$ and $\mathbb{P}$); in particular, it is a Poisson random measure.

This follows from a theorem of Girsanov type (see [14] Theorem (4.5)), which guarantees that under the new probability $\mathbb{P}$ the compensator of μ is given by $\nu_t(a)^{-1}\nu_t(a)\, \lambda(da)\, dt = \lambda(da)\, dt$.

(ii) W is a $(\mathbb{P}, \mathbb{F})$-Wiener process.

The proof is as follows. Since the probabilities $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the quadratic variation of X computed under $\mathbb{P}$ and $\mathbb{Q}$ is the same, and equals $\langle W \rangle_t = t$. So it is enough to show that W is a $(\mathbb{P}, \mathbb{F})$-local martingale, which is equivalent to the fact that MW is a $(\mathbb{Q}, \mathbb{F})$-local martingale. Finally, this follows from a general fact: since M is a $(\mathbb{Q}, \mathbb{F})$-martingale of finite variation, it is purely discontinuous and therefore orthogonal (under $\mathbb{Q}$) to W; thus, their product MW is a $(\mathbb{Q}, \mathbb{F})$-local martingale.

(iii) W and μ are independent under $\mathbb{P}$.

To prove this claim it is enough to show that, for any measurable $B \subset A$, the process

$$N_t^B \ := \ \int_0^t \int_B \mu(ds, da) \ = \ \sum_n 1_{\{S_n \le t\}} 1_{\{\eta_n \in B\}},$$

is independent from W under $\mathbb{P}$. From claims (i) and (ii) it follows that N^B is a Poisson process and W is a Wiener process, both with respect to $\mathbb{F}$ and $\mathbb{P}$. By a general result, see e.g. Theorem 11.43 in [13], to check the independence it is enough to note that their right bracket $[N^B, W]$ is null, which is obvious, since W is continuous and N^B has no continuous part.

From claims (i), (ii), (iii), and recalling that $\mathbb{F}$ is the completion of the natural filtration of (W,μ), we deduce that $\mathbb{D}_\lambda$ defined in (4.8) is indeed an admissible randomized set-up. Note that we have checked that the $\mathbb{P}$-compensator of μ is $\lambda(da)dt$, although we do not make it explicit in the notation.

Next we proceed to verify (4.9). Since in general we can not assert that the random field ν is bounded on $[0,T]\times\Omega\times A$, we can not conclude that it belongs to the class $\mathcal{V}(\mathbb{D}_\lambda)$. However, we can still define the process L^ν by formula (3.1), which defines a strictly positive local martingale hence supermartingale with respect to $\mathbb{P}$. It follows immediately from formulae (3.1) and (4.10) that L^ν_T is the inverse of M_T. It follows that $\mathbb{E}[L^\nu_T]=\mathbb{E}_{\mathbb{Q}}[M_T L^\nu_T]=1$, so that L^ν is indeed a $\mathbb{P}$-martingale on $[0,T]$ and we can define the corresponding probability $\mathbb{P}^\nu(d\omega):=L^\nu_T(\omega)\mathbb{P}(d\omega)$. Thus, the Girsanov transformation $\mathbb{P}\mapsto\mathbb{P}^\nu$ is the inverse to the transformation $\mathbb{Q}\mapsto\mathbb{P}$ made above, and changes back the probability $\mathbb{P}$ into $\mathbb{P}^\nu=\mathbb{Q}$ considered above. In addition, we recall that the control $I\in\mathcal{A}(\mathbb{A}''')$ constructed in Lemma 4.7 and the corresponding trajectory $(X_t)_{t\in[0,T]}$ are the solution to the system (4.6), which coincides with (3.2), since we are assuming $t=0$. It follows that

$$J(0,x,\mathbb{A}''',I) \;=\; \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T f_t(X,I_t)\,dt+g(X)\Big] \;=\; \mathbb{E}^{\nu}\Big[\int_0^T f_t(X,I_t)\,dt+g(X)\Big], \tag{4.11}$$

where $\mathbb{E}^\nu$ denotes the expectation under $\mathbb{P}^\nu$. If ν belongs to $\mathcal{V}(\mathbb{D}_\lambda)$, the right-hand side equals the gain $J^*(0,x,a,\lambda,\mathbb{D}_\lambda,\nu)$ for the randomized control problem and the desired inequality (4.9) obviously holds with $\bar\nu=\nu$. However, since in general we can not assert that $\nu\in\mathcal{V}(\mathbb{D}_\lambda)$, we revert to the following approximation procedure by truncation.

For any integer $k\geq 1$ define $\nu^k_t(a)=\nu_t(a)\wedge k$. Therefore $\nu^k\in\mathcal{V}(\mathbb{D}_\lambda)$, we can define the corresponding process L^{ν^k} by formula (3.1), the probability $\mathbb{P}^{\nu^k}(d\omega)=L^{\nu^k}_T(\omega)\,\mathbb{P}(d\omega)$, and compute the gain

$$J^*(0,x,a,\lambda,\mathbb{D}_\lambda,\nu^k) \;=\; \mathbb{E}^{\nu^k}\Big[\int_0^T f_t(X,I_t)\,dt+g(X)\Big],$$

where $\mathbb{E}^{\nu^k}$ denotes the expectation under $\mathbb{P}^{\nu^k}$. We claim that, for $k\to\infty$,

$$\mathbb{E}^{\nu^k}\Big[\int_0^T f_t(X,I_t)\,dt+g(X)\Big] \;\longrightarrow\; \mathbb{E}^{\nu}\Big[\int_0^T f_t(X,I_t)\,dt+g(X)\Big]. \tag{4.12}$$

If we can prove the claim, it follows from (4.11) that the inequality (4.9) is verified with $\bar\nu=\nu^k$ and k sufficiently large. So it remains to prove (4.12), that we re-write in the form

$$\mathbb{E}^{\nu^k}[\Phi] \;=\; \mathbb{E}[L^{\nu^k}_T\,\Phi] \;\longrightarrow\; \mathbb{E}\big[L^{\nu}_T\Phi\big] \;=\; \mathbb{E}^{\nu}\big[\Phi\big], \tag{4.13}$$

where we have set $\Phi:=\int_0^T f_t(X,I_t)\,dt+g(X)$. We note for further use that the assumption of polynomial growth in **(H2)** implies that there exists a constant C such that

$$|\Phi| \;\leq\; C(1+\sup_{t\in[0,T]}|X_t|^m). \tag{4.14}$$

For $N\geq 1$ we define $A_N=\{\sup_{t\in[0,T]}|X_t|>N\}$ and obtain

$$\big|\mathbb{E}^{\nu^k}[\Phi]-\mathbb{E}^{\nu}[\Phi]\big| \;\leq\; \mathbb{E}^{\nu^k}[1_{A_N}|\Phi|]+\mathbb{E}^{\nu}[1_{A_N}|\Phi|]+\mathbb{E}[|L^{\nu^k}_T-L^{\nu}_T|1_{A^c_N}|\Phi|].$$

By (4.14) we have

$$\mathbb{E}^{\nu^k}[1_{A_N}|\Phi|] \;\leq\; C\,\mathbb{E}^{\nu^k}\Big[1_{A_N}(1+\sup_{t\in[0,T]}|X_t|^m)\Big] \;\leq\; \frac{C}{N}\,\mathbb{E}^{\nu^k}\Big[\sup_{t\in[0,T]}|X_t|(1+\sup_{t\in[0,T]}|X_t|^m)\Big] \;\leq\; \frac{C'}{N}$$

for a suitable constant C', independent of k, where for the last inequality we have used the estimate (2.4) whose right-hand side is the same for all probabilities $\mathbb{P}^{\nu^k}$. A similar estimate holds for $\mathbb{E}^{\nu}[1_{A_N}|\Phi|]$ and we obtain

$$|\mathbb{E}^{\nu^k}[\Phi] - \mathbb{E}^{\nu}[\Phi]| \leq \frac{2C'}{N} + \mathbb{E}[|L_T^{\nu^k} - L_T^{\nu}|1_{A_N^c}|\Phi|]. \tag{4.15}$$

By the dominated convergence theorem we have

$$\int_0^T \int_A (1 - \nu_s^k(a))\, \lambda(da)\, ds \;\longrightarrow\; \int_0^T \int_A (1 - \nu_s(a))\, \lambda(da)\, ds,$$

a.s. (with respect to any of the equivalent probabilities $\mathbb{P}, \mathbb{P}^{\nu}, \mathbb{P}^{\nu^k}$) and

$$\prod_{S_n \leq T} \nu_{S_n}^k(\eta_n) \;\longrightarrow\; \prod_{S_n \leq T} \nu_{S_n}(\eta_n),$$

a.s., since the product has finitely many factors a.s. From formula (3.1) that defines L^{ν} and L^{ν^k} we obtain $L_T^{\nu^k} \to L_T^{\nu}$ a.s. Since $\mathbb{E}[L_T^{\nu^k}] = \mathbb{E}[L_T^{\nu}] = 1$, we even have $L_T^{\nu^k} \to L_T^{\nu}$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. By (4.14) the random variable $1_{A_N^c}|\Phi|$ is bounded a.s., so letting $k \to \infty$ in (4.15) we have

$$\limsup_{k\to\infty} |\mathbb{E}^{\nu^k}[\Phi] - \mathbb{E}^{\nu}[\Phi]| \leq \frac{2C'}{N}.$$

Letting N tend to infinity we conclude the proof of the claim (4.13), and the proof of Proposition 4.1 is also finished. □

4.2 Proof of Theorem 3.2

We fix initial conditions $(t, x, a) \in [0, T] \times \mathbf{C}^d \times A$, $\lambda \in \mathcal{M}_f(A)$, a randomized set-up $\mathbb{D}_\lambda = (\Omega, \mathcal{F}, \mathbb{P}, W, \mu)$, denote by $\mathbb{F}$ the completion of the natural filtration of (W, μ), and consider the pair of processes (I, X) as the solution to the system (3.2). Note that we write μ instead of μ_λ for simplicity, and because our results do not depend a posteriori on the choice of λ. Let us introduce the family of penalized BSDEs associated to (3.5), parametrized by integer $n \geq 1$:

$$\begin{aligned} Y_s^n &= g(X) + \int_s^T f_r(X, I_r)dr + n\int_s^T \int_A U_r^n(a)^+ \lambda(da)dr \\ &\quad - \int_s^T Z_r^n\, dW_r - \int_s^T \int_A U_r^n(a)\, \mu(dr, da), \quad 0 \leq s \leq T, \end{aligned} \tag{4.16}$$

where $u^+ = \max(u, 0)$. It follows from a result in [28], Lemma 2.4, that there exists a unique solution $(Y^n, Z^n, U^n) \in \mathcal{S}^2 \times L^2(W) \times L^2(\tilde{\mu})$ to (4.16), where we drop the dependence on $(t, x, a, \lambda, \mathbb{D}_\lambda)$. In that paper this result is proved by a classical argument in the theory of BSDE, namely a Picard iteration technique combined with a martingale representation theorem. This proof makes clear the following facts:

1. It is possible to find a $\mathbb{P}$-null set Ω_0 and to choose a version of the process Y^n (in the sense of indistinguishability) in such a way that $Y_s^n(\omega)$ is a constant for every $s \in [0, t]$ and $\omega \notin \Omega_0$. We will always consider such a version in the sequel, and we will say for short that the process Y_s^n is deterministic for $s \in [0, t]$. To verify this, denote by $\mathbb{F}^t = (\mathcal{F}_s^t)_{s \geq t}$ the completed filtration on $[t, \infty)$ generated by the restriction of the random measure μ to

$(t,\infty)\times A$ and by the increments of the Wiener process W on $[t,\infty)$. As noticed for the filtration $\mathbb{F}=\mathbb{F}^0$, $\mathbb{F}^t$ is right-continuous by the result in [28], Lemma A1. Then, recalling that $X_s=x(s)$ and $I_s=a$ for $s\in[0,t]$, one sees that (X,I) is $\mathbb{F}^t$-progressive and that the BSDE (4.16) can be solved on the time interval $[t,T]$ with respect to $\mathbb{F}^t$. In particular it follows that Y^n_t is $\mathcal{F}^t_t$-measurable, hence constant except possibly on a $\mathbb{P}$-null set Ω_0. Setting, for $s\in[0,t]$ and $\omega\notin\Omega_0$,

$$Z^n_s=0,\quad U^n_s(a)=0,\quad Y^n_s=Y^n_t+\int_s^t f_r(x,a)\,dr,$$

and defining (for instance) Y^n_s, Z^n_s, $U^n_s(a)$ to be zero on $\Omega_0\times[0,t]$, one obtains the solution to (4.16) on $[0,T]$, and Y^n has the required properties.

2. The law of the solution (Y^n,Z^n,U^n), hence in particular the (deterministic) values of Y^n_s, for $s\in[0,t]$, is determined by the coefficients b,σ,f,g as well as the fixed elements T,λ,a,t,x and n, but it does not depend on the particular choice of the probability space, the Wiener process and the Poisson process. Thus, Y^n_s ($s\in[0,t]$) has the same value if equation (4.16) is solved in another randomized admissible control system $\mathbb{D}'_\lambda=(\Omega',\mathcal{F}',\mathbb{P}',W',\mu')$, provided μ' has $\mathbb{P}'$-compensator $\lambda(da)dt$ with respect to the natural filtration generated by (W',μ').

We provide an explicit representation of the solution to the penalized BSDE in terms of a family of auxiliary randomized control problems. For every integer $n\geq 1$, let $\mathcal{V}^n(\mathbb{D}_\lambda)$ denote the subset of elements $\nu_t(\omega,a)$ in $\mathcal{V}(\mathbb{D}_\lambda)$ valued in $(0,n]$. The following result is a slight modification of [16], Proposition 4.1.

Lemma 4.8 *We have for all $n\geq 1$,*

$$Y^n_s \;=\; \operatorname*{ess\,sup}_{\nu\in\mathcal{V}^n(\mathbb{D}_\lambda)} \mathbb{E}^\nu\Big[\int_s^T f_r(X,I_r)\,dr+g(X)\big|\mathcal{F}_s\Big],\quad s\in[0,T],\qquad \mathbb{P}-a.s. \tag{4.17}$$

Proof. Fix $n\geq 1$, and consider (Y^n,Z^n,U^n) the solution to (4.16). For any $\nu\in\mathcal{V}^n(\mathbb{D}_\lambda)$, let $\mu^\nu(ds,da):=\mu(ds,da)-\nu_s(a)\lambda(da)ds$ denote the $\mathbb{P}^\nu$-compensated martingale measure of μ. It is shown in [16], Lemma 4.2 that for $Z^n\in L^2(W)$ and $U^n\in L^2(\tilde\mu)$, the processes

$$\int_0^\cdot Z^n dW,\qquad \int_0^\cdot\int_A U^n_s(a)\mu^\nu(ds,da),$$

are $\mathbb{P}^\nu$-martingales. Therefore, by taking the conditional expectation $\mathbb{E}^\nu$ given $\mathcal{F}_s$ in (4.16), we obtain:

$$\begin{aligned} Y^n_s \;&=\; \mathbb{E}^\nu\Big[\int_s^T f_r(X,I_r)\,dr+g(X)\big|\mathcal{F}_s\Big]\\ &\quad+\mathbb{E}^\nu\Big[\int_s^T\int_A [nU^n_r(a)^+-\nu_r(a)U^n_r(a)]\lambda(da)dr\big|\mathcal{F}_s\Big],\quad s\in[0,T]. \end{aligned}$$

From the elementary numerical inequality: $nu^+-\nu u\geq 0$ for all $u\in\mathbb{R}$, $\nu\in[0,n]$, we deduce that

$$Y^n_s \;\geq\; \operatorname*{ess\,sup}_{\nu\in\mathcal{V}^n(\mathbb{D}_\lambda)} \mathbb{E}^\nu\Big[\int_s^T f_r(X,I_r)\,dr+g(X)\big|\mathcal{F}_s\Big],\quad s\in[0,T].$$

For $\epsilon \in (0,1)$, define $\nu_s^\epsilon(a) = n1_{\{U_s^n(a)\geq 0\}} + \epsilon 1_{\{-1<U_s^n(a)<0\}} - \epsilon U_s^n(a)^{-1}1_{\{U_s^n(a)\leq -1\}}$. Then $\nu^\epsilon \in \mathcal{V}^n(\mathbb{D}_\lambda)$, and we have

$$nU_s^n(a)^+ - \nu_s^\epsilon(a)U_s^n(a) \quad \leq \quad \epsilon, \quad 0 \leq s \leq T,$$

so that

$$\begin{aligned} Y_s^n &\leq \mathbb{E}^{\nu^\epsilon}\Big[\int_s^T f_r(X, I_r)\, dr + g(X)\big|\mathcal{F}_s\Big] + \epsilon T\lambda(A) \\ &\leq \operatorname*{ess\,sup}_{\nu\in\mathcal{V}^n(\mathbb{D}_\lambda)} \mathbb{E}^{\nu}\Big[\int_s^T f_r(X, I_r)\, dr + g(X)\big|\mathcal{F}_s\Big] + \epsilon T\lambda(A), \end{aligned}$$

which is enough to conclude the proof. Note that we could not take $\nu_s(a) = n1_{\{U_t^n(a)\geq 0\}}$, since this process does not belong to $\mathcal{V}^n(\mathbb{D}_\lambda)$ because of the requirement of strict positivity. □

As a consequence of this explicit representation of the penalized BSDE, we obtain the following uniform estimate on the sequence $(Y^n)_n$:

Lemma 4.9 *The sequence $(Y^n)_n$ is monotonically increasing in n, and we have*

$$\sup_{s\in[0,T]} |Y_s^n| \quad \leq \quad C\Big(1 + \sup_{s\in[0,T]} |X_s|^m\Big), \qquad \mathbb{P}-a.s.$$

for some constant C depending only on T, m and on the constants K_1, K_2 as defined in **(H1)**, **(H2)**.

Proof. Monotonicity follows from the formula for Y^n presented in Lemma 4.8, since $\mathcal{V}^n(\mathbb{D}_\lambda) \subset \mathcal{V}^{n+1}(\mathbb{D}_\lambda)$. Then the inequality $Y_s^n \leq Y_s^{n+1}$ holds $\mathbb{P}$-a.s. for all $s \in [0,T]$ since these processes are càdlàg.

Below we denote by C a generic constant depending only on T, m, K_1, K_2, whose precise value may possibly change at each occurrence. Recalling the polynomial growth condition on f and g in **(H2)**, it follows from Lemma 4.8 that

$$|Y_s^n| \quad \leq \quad C \operatorname*{ess\,sup}_{\nu\in\mathcal{V}^n(\mathbb{D}_\lambda)} \mathbb{E}^{\nu}\Big[1 + \sup_{r\in[0,T]} |X_r|^m\big|\mathcal{F}_s\Big], \qquad s \in [0,T],\ \mathbb{P}-a.s.$$

Next we note that standard estimates on the stochastic equation (3.2) satisfied by X, based on the Lipschitz and linear growth conditions in **(H1)**, lead for every $s \in [0,T]$ to the inequality

$$\mathbb{E}^{\nu}\Big[\sup_{r\in[s,T]} |X_r|^m\big|\mathcal{F}_s\Big] \quad \leq \quad C\Big(1 + \sup_{r\in[0,s]} |X_r|^m\Big), \qquad \mathbb{P}-a.s.$$

which can be viewed as a conditional form of the estimate (2.4), and where the constant C can be chosen to be the same for every $\nu \in \mathcal{V}(\mathbb{D}_\lambda)$. It follows that $|Y_s^n| \leq C\big(1 + \sup_{r\in[0,s]} |X_r|^m\big)$, $\mathbb{P}$-a.s. and the required conclusion follows immediately. □

Remark 4.3 In [16], uniform estimates for (Y^n) in $\mathcal{S}^2$ are obtained in the general case where the generator f may also depend on Y^n, Z^n, but under the assumption that there exists a solution to (3.5). Here, in our specific control case (which leads to the explicit formula (4.17)), one derives directly from Lemma 4.9 and (2.4) a uniform estimate for (Y^n) in $\mathcal{S}^2$:

$$\mathbb{E}\Big[\sup_{s\in[0,T]} |Y_s^n|^2\Big] \quad \leq \quad C\big(1 + \|x\|_\infty^{2m}\big).$$

□

We are now in a position to complete the proof of Theorem 3.2.

End of the proof of Theorem 3.2.

• Lemma 4.9 corresponds to the statements of Lemma 3.1 and 3.2 in [16]. Following the same arguments used there, it can be proved that the sequence $(Y^n, Z^n, U^n, n\int_0^\cdot \int_A U^n(a)^+ \lambda(da)\, ds)$ converges, in the sense specified in [16], Theorem 3.1, to the required minimal solution (Y, Z, U, K) to the BSDE (3.5). Thus, for $s \in [0,T]$, the value $Y_s = \lim_{n\to\infty} \uparrow Y_s^n$ inherits from Y_s^n the property of being deterministic, and not dependent on the choice of the set-up $\mathbb{D}_\lambda$.

• Since Y_s^n, X_s, I_s are deterministic for $s \in [0,t]$, it follows from Lemma 4.8 that

$$Y_t^n \;=\; \sup_{\nu\in\mathcal{V}^n(\mathbb{D}_\lambda)} \mathbb{E}^\nu\Big[\int_t^T f_s(X, I_s)\, ds + g(X)\Big].$$

By sending n to infinity, this shows that

$$Y_t \;=\; \lim_{n\to\infty} \uparrow Y_t^n \;=\; \sup_{\nu\in\mathcal{V}(\mathbb{D}_\lambda)} \mathbb{E}^\nu\Big[\int_t^T f_s(X, I_s)\, ds + g(X)\Big] \;=\; v_1^*(t,x,a,\lambda,\mathbb{D}_\lambda). \quad (4.18)$$

Indeed, since $\mathcal{V}^n(\mathbb{D}_\lambda) \subset \mathcal{V}(\mathbb{D}_\lambda)$, we have $\lim_{n\to\infty} \uparrow Y_t^n \le v_1^*(t,x,a,\lambda,\mathbb{D}_\lambda)$. To prove the opposite inequality, take any $\nu \in \mathcal{V}(\mathbb{D}_\lambda)$ and define $\nu_t^n(a) = \nu_t(a) \wedge n$. By similar arguments to those used in the proof of equality (4.12), one can show that for $n \to \infty$

$$\mathbb{E}^{\nu^n}\Big[\int_t^T f_s(X, I_s)\, ds + g(X)\Big] \;\longrightarrow\; \mathbb{E}^\nu\Big[\int_s^T f_s(X, I_s)\, ds + g(X)\Big],$$

which implies that $\lim_{n\to\infty} Y_t^n \ge v_1^*(t,x,a,\lambda,\mathbb{D}_\lambda)$ since ν is arbitrary in $\mathcal{V}(\mathbb{D}_\lambda)$, and thus the equality (4.18). This shows in particular that v_1^* does not depend on the choice of $\mathbb{D}_\lambda$, and so is equal to v_2^*. Moreover, by combining with the result of Theorem 3.1, we obtain that Y_t represents also the value $v(t,x)$ of the original control problem. □

A Appendix: some facts on random measures and their compensators

We first recall the following fact, used in the previous sections.

Proposition A.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with two filtrations $(\mathcal{F}_t)$, $(\mathcal{K}_t)$, such that $\mathcal{F}_\infty$ and $\mathcal{K}_\infty$ are independent. Let $(T_n, \alpha_n)_{n\ge1}$ be a marked point process with respect to $(\mathcal{F}_t)$ and $(S_n, \beta_n)_{n\ge1}$ a marked point process with respect to $(\mathcal{K}_t)$. Denote $\mu = \sum_{n\ge1} \delta_{(T_n,\alpha_n)}$, $\kappa = \sum_{n\ge1} \delta_{(S_n,\beta_n)}$ the associated random measures, and $\tilde\mu(dt, da)$, $\tilde\kappa(dt, da)$ the respective $(\mathcal{F}_t)$- and $(\mathcal{K}_t)$-compensators. Finally assume that the processes have no common jumps.*

Then the random measure $\mu + \kappa$ admits $\tilde\mu(da\, dt) + \tilde\kappa(da\, dt)$ as a $(\mathcal{F}_t \vee \mathcal{K}_t)$-compensator.

The rest of this section is devoted to a technical result, stated below as Lemma A.2, that was used in the proof of Lemma 4.3.

We recall that in our paper we require A to be a Lusin space, and we denote by $\mathcal{B}(A)$ its Borel σ-algebra.

Lemma A.1 *Let $q(b, da)$ be a transition kernel on the Lusin space A. Then there exists a function $q : A \times [0,1] \to A$, measurable with respect to $\mathcal{B}(A) \otimes \mathcal{B}([0,1])$ and $\mathcal{B}(A)$, such that for*

every $b \in A$, the measure $B \mapsto q(b,B)$ ($B \in \mathcal{B}(A)$) is the image of the Lebesgue measure on $[0,1]$ under the mapping $u \mapsto q(b,u)$; equivalently,

$$\int_A k(a)\, q(b,da) \;=\; \int_0^1 k(q(b,u))\, du,$$

for every nonnegative measurable function k on A.

Proof. When A is a separable complete metric space (in particular, when A is the unit interval $[0,1]$) the result is known and follows from a construction of Skorohod: see e.g. [31], Theorem 3.1.1. The general case reduces to this one, since it is known that any Lusin space is either finite or countable (with the discrete topology) or isomorphic, as a measurable space, to the interval $[0,1]$: see e.g. [1], Corollary 7.16.1. □

From the lemma, it follows that if U is a random variable defined on some probability space and having uniform law on $[0,1]$ then, for fixed $b \in A$, the random variable $q(b,U)$ has law $q(b,da)$ on A. The use of the same letter q should not be a source of confusion.

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)$, let $(T_n, \alpha_n)_{n\geq 1}$ be a marked point process in A, with respect to $(\mathcal{F}_t)$, and let $\mu = \sum_{n\geq 1} \delta_{(T_n,\alpha_n)}$ the corresponding random measure. Assume that we can find sequences $(U_n)_{n\geq 1}$, $(S_n)_{n\geq 1}$ of real random variables defined on Ω and satisfying the following conditions:

1. every U_n is uniformly distributed on $[0,1]$;
2. every S_n admits a density (denoted $f_n(t)$) with respect to the Lebesgue measure, and we have $0 < S_1 < S_2 < S_3 < \ldots < \infty$;
3. the random variables U_n, S_k $(n,k \geq 1)$ are independent, and independent of $\mathcal{F}_\infty$.

Define

$$R_n = T_n + S_n, \quad \beta_n = q(\alpha_n, U_n), \qquad\qquad n \geq 1,$$

and set $R_0 = 0$. Since we assume $S_n < S_{n+1}$ we see that $(R_n, \beta_n)_{n\geq 1}$ is a marked point process in A. Let $\kappa = \sum_{n\geq 1} \delta_{(R_n,\beta_n)}$ denote the corresponding random measure, and $(\mathcal{F}_t^\kappa)$ the corresponding natural filtration. Finally set $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{F}_t^\kappa$. We wish to compute the compensator $\tilde{\kappa}$ of κ with respect to the filtration $\mathbb{H} = (\mathcal{H}_t)$. We use the notation $F_n(s) = \int_{-\infty}^s f_n(t)dt$ and the convention $\frac{f_n(s)}{1-F_n(s)} = 0$ if $F_n(s) = 1$.

Lemma A.2 *With the previous assumptions and notations, the compensator of the random measure κ with respect to $(\mathcal{H}_t)$ is given by the formula*

$$\tilde{\kappa}(dt,da) \;=\; \sum_{n\geq 1} 1_{(T_n \vee R_{n-1}, R_n]}(t)\, q(\alpha_n, da) \frac{f_n(t-T_n)}{1-F_n(t-T_n)}\, dt.$$

Proof. Let us first check that $\tilde{\kappa}(dt,da)$, defined by the formula above, is an $(\mathcal{H}_t)$-predictable random measure. We note that $T_n \vee R_{n-1}$ and R_n are $(\mathcal{H}_t)$-stopping times and that α_n and $\frac{f_n(t-T_n)}{1-F_n(t-T_n)}$ are $\mathcal{F}_{T_n}$-measurable and hence $\mathcal{F}_{T_n \vee R_{n-1}}$-measurable. It follows that for every $C \in \mathcal{B}(A)$ the process

$$1_{(T_n \vee R_{n-1}, R_n]}(t)\, q(\alpha_n, C) \frac{f_n(t-T_n)}{1-F_n(t-T_n)}$$

is $(\mathcal{H}_t)$-predictable and finally that $\tilde{\kappa}(dt,da)$ is an $(\mathcal{H}_t)$-predictable random measure.

To prove the Lemma we need now to verify that for every positive $\mathcal{P}(\mathbb{H})\otimes\mathcal{B}(A)$-measurable random field $H_t(\omega,a)$ we have

$$\mathbb{E}\Big[\int_0^\infty\int_A H_t(a)\,\kappa(dt\,da)\Big] \;=\; \mathbb{E}\Big[\int_0^\infty\int_A H_t(a)\,\tilde\kappa(dt\,da)\Big].$$

Since $\mathcal{H}_t=\mathcal{F}_t\vee\mathcal{F}_t^\kappa$, by a monotone class argument it is enough to consider H of the form

$$H_t(\omega,a)=H_t^1(\omega)H_t^2(\omega)k(a),$$

where H^1 is a positive $(\mathcal{F}_t)$-predictable random process, H^2 is a positive $(\mathcal{F}_t^\kappa)$-predictable random process and k is a positive $\mathcal{A}$-measurable function. Since $(\mathcal{F}_t^\kappa)$ is the natural filtration of κ, by a known result (see e.g. [14] Lemma (3.3)) H^2 has the following form:

$$\begin{aligned} H_t^2 \;&=\; b_1(t)1_{(0,R_1]}(t)+b_2(\beta_1,R_1,t)1_{(R_1,R_2]}(t)+b_3(\beta_1,\beta_2,R_1,R_2,t)1_{(R_2,R_3]}(t)+\\ &\quad\ldots+b_n(\beta_1,\ldots,\beta_{n-1},R_1,\ldots,R_{n-1},t)1_{(R_{n-1},R_n]}(t)+\ldots,\end{aligned}$$

where each b_n is a positive measurable deterministic function of $2n-1$ real variables. Since

$$\mathbb{E}\Big[\int_0^\infty\int_A H_t(a)\,\kappa(dt\,da)\Big] \;=\; \mathbb{E}\Big[\sum_{n\ge1}H_{R_n}(\beta_n)\Big]$$

to prove the thesis it is enough to check that for every $n\ge1$ we have the equality

$$\mathbb{E}\big[H_{R_n}(\beta_n)\big] \;=\; \mathbb{E}\Big[\int_0^\infty\int_A H_t(a)\,q(\alpha_n,da)\frac{f_n(t-T_n)}{1-F_n(t-T_n)}\,1_{T_n\vee R_{n-1}<t\le R_n}\,dt\Big],$$

which can also be written

$$\begin{aligned}&\mathbb{E}\big[H^1_{R_n}b_n(\beta_1,\ldots,\beta_{n-1},R_1,\ldots,R_{n-1},R_n)k(\beta_n)\big]=\\ &\mathbb{E}\Big[\int_0^\infty\int_A H_t^1b_n(\beta_1,\ldots,\beta_{n-1},R_1,\ldots,R_{n-1},t)k(a)q(\alpha_n,da)\frac{f_n(t-T_n)}{1-F_n(t-T_n)}\,1_{T_n\vee R_{n-1}<t\le R_n}\,dt\Big].\end{aligned}$$

We use the notation

$$K_n(t) \;=\; H_t^1\,b_n(\beta_1,\ldots,\beta_{n-1},R_1,\ldots,R_{n-1},t)$$

to reduce the last equality to

$$\mathbb{E}\,[K_n(R_n)k(\beta_n)]=\mathbb{E}\int_0^\infty\int_A K_n(t)\,k(a)\,q(\alpha_n,da)\frac{f_n(t-T_n)}{1-F_n(t-T_n)}\,1_{T_n\vee R_{n-1}<t\le R_n}\,dt. \tag{A.1}$$

By the definition of R_n and β_n, we have $\mathbb{E}[K_n(R_n)k(\beta_n)]=\mathbb{E}[K_n(T_n+S_n)k(q(\alpha_n,U_n))]$. As noted above, since U_n has uniform law on $(0,1)$, the random variable $q(b,U_n)$ has law $q(b,da)$ on A, for any fixed $b\in A$. Recalling that S_n has density f_n and noting that U_n, S_n and $(\mathcal{F}_\infty,\beta_1,\ldots,\beta_{n-1},R_1,\ldots,R_{n-1})$ are all independent we obtain

$$\mathbb{E}\,[K_n(R_n)k(\beta_n)] \;=\; \mathbb{E}\Big[\int_0^\infty\int_A K_n(T_n+s)\,k(a)\,q(\alpha_n,da)\,f_n(s)\,ds\Big]. \tag{A.2}$$

Using again the independence of S_n and $(\mathcal{F}_\infty,\beta_1,\ldots,\beta_{n-1},R_1,\ldots,R_{n-1})$ we also have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^\infty\int_A K_n(T_n+s)\,k(a)\,q(\alpha_n,da)\,\frac{f_n(s)}{1-F_n(s)}\,1_{S_n\ge s}\,ds\Big]\\ =\;&\mathbb{E}\Big[\int_0^\infty\int_A K_n(T_n+s)\,k(a)\,q(\alpha_n,da)\,\frac{f_n(s)}{1-F_n(s)}\,\mathbb{P}(S_n\ge s)\,ds\Big]\end{aligned}$$

and since $\mathbb{P}(S_n \geq s) = \int_s^\infty f_n(r)\,dr = 1 - F_n(s)$, this coincides with the right-hand side of (A.2). By a change of variable we arrive at the equality

$$\begin{aligned} \mathbb{E}\left[K_n(R_n)k(\beta_n)\right] &= \mathbb{E}\Big[\int_{T_n}^\infty \int_A K_n(t)\,k(a)\,q(\alpha_n, da)\,\frac{f_n(t-T_n)}{1-F_n(t-T_n)}\,1_{S_n \geq t-T_n}\,dt\Big] \\ &= \mathbb{E}\Big[\int_0^\infty \int_A K_n(t)\,k(a)\,q(\alpha_n, da)\,\frac{f_n(t-T_n)}{1-F_n(t-T_n)}\,1_{T_n < t \leq R_n}\,dt\Big]. \end{aligned} \tag{A.3}$$

We finally claim that

$$f_n(t-T_n)\,1_{T_n<t\leq R_n} \;=\; f_n(t-T_n)1_{T_n\vee R_{n-1}<t\leq R_n}, \qquad \mathbb{P}\otimes dt - a.s.$$

If we can prove the claim, we conclude that (A.3) coincides with (A.1) and the proof will be finished. To prove the claim, we show that the following integral is zero:

$$\mathbb{E}\Big[\int_0^\infty f_n(t-T_n)\,|1_{T_n<t\leq R_n} - 1_{T_n\vee R_{n-1}<t\leq R_n}|\,dt\Big] \;=\; \mathbb{E}\Big[\int_0^\infty f_n(t-T_n)\,1_{T_n<t\leq T_n\vee R_{n-1}}\,dt\Big]. \tag{A.4}$$

Since $T_n \vee R_{n-1} = T_n \vee (T_{n-1} + S_{n-1}) \leq T_n \vee (T_n + S_{n-1}) = T_n + S_{n-1}$, the right-hand side of (A.4) is smaller or equal to

$$\mathbb{E}\Big[\int_0^\infty f_n(t-T_n)\,1_{T_n<t\leq T_n+S_{n-1}}\,dt\Big] \;=\; \mathbb{E}\Big[\int_0^\infty f_n(s)\,1_{s\leq S_{n-1}}\,ds\Big].$$

Since S_n and S_{n-1} are independent, and $S_{n-1} < S_n$ a.s., we finally have

$$\mathbb{E}\Big[\int_0^\infty f_n(s)\,1_{s\leq S_{n-1}}\,ds\Big] \;=\; \mathbb{E}\big[1_{S_n\leq S_{n-1}}\big] \;=\; 0.$$

□

References

[1] Bertsekas, D.P. and S.E. Shreve (1978): Stochastic optimal control. The discrete time case. *Mathematics in Science and Engineering*, 139. Academic Press.

[2] Bouchard, B. (2009): "A stochastic target formulation for optimal switching problems in finite horizon", *Stochastics*, **81**, 171-197.

[3] Cheridito, P., Soner, M., Touzi, N and N. Victoir (2007): "Second-order backward stochastic differential equations and fully nonlinear PDEs", *Communication in Pure and Applied Mathematics*, **60**, 1081-1110.

[4] Crandall M., Ishii H. and P.L. Lions (1992) : "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.*, **27**, 1-67.

[5] Denis, L., Hu, M. and S. Peng (2011): "Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths", *Potential Anal.*, **34**, 139-161.

[6] Ekren, I., Keller, K, Touzi, N. and and J. Zhang (2011): "On Viscosity Solutions of Path Dependent PDEs", to appear in *Annals of Probability.*

[7] El Karoui, N. (1981): Les aspects probabilistes du contrôle stochastique, *Ecole d'été de probabilités de St-Flour*, vol. 876, Lect. Notes in Maths, 73-238, Springer.

[8] El Karoui N., Huu Nguyen D. and M. JeanblancPicqué (1988): "Existence of an optimal Markovian filter for the control under partial observations", *SIAM J. Cont. Optim.*, **26**, 1025-1061.

[9] El Karoui, N. and X. Tan (2013): "Capacities, Measurable Selection and Dynamic Programming Part II: Application in Stochastic Control Problems", Preprint arXiv:1310.3364

[10] Fabbri, G., Gozzi, F. and A. Święch: Stochastic Optimal Control in Infinite Dimensions: Dynamic Programming and HJB Equations, book in preparation, draft.

[11] Fleming W (1978): "Generalized solutions in optimal stochastic control", in Differential games and control theory, Lect. Notes in Pure and Applied Math, **30**, Dekker.

[12] Fleming, W. and M. Soner (2006): Controlled Markov processes and viscosity solutions, Springer, Series SMAP, vol. 25.

[13] He, S.W., Wang, J.G. and J.A. Yan (1992): Semimartingale theory and stochastic calculus. CRC Press, Boca Raton, FL.

[14] Jacod, J. (1975): "Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales", *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **31**, 235-253.

[15] Kharroubi I., J. Ma, H. Pham, and J. Zhang (2010): "Backward SDEs with constrained jumps and Quasi-variational inequalities", *Annals of Probability*, **38**, 794-840.

[16] Kharroubi, I. and H. Pham (2012): "Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE", to appear in *Annals of Probability.*

[17] Kharroubi I., Langrené N. and H. Pham (2013): "Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps", Preprint arXiv: 1311.4505

[18] Krylov, N. V. (1980): Controlled diffusion processes. *Stochastic Modelling and Applied Probability*, **14**. Springer-Verlag, Berlin.

[19] Nutz M. (2012): "A quasi-sure approach to the control of non-markovian stochastic differential equations", to appear in *Electronic Journal of Probability.*

[20] Nutz M. and R. Van Handel (2013): "Constructing sub linear expectations on path space", *Stochastic Processes and their Applications*, **123**, 3100-3121.

[21] Pardoux E. and S. Peng (1992): "Backward stochastic differential equation and quasilinear parabolic partial differential equations", in *Stochastic partial differential equations and their applications*, B. Rozovskii and R. Sowers (eds), Lect. Notes in Cont. Inf. Sci., **176**, 200-217.

[22] Peng S. (2006): "G-expectation, G-Brownian motion and related stochastic calculus of Itô type", Proceedings of 2005, Abel symposium, Springer.

[23] Peng S. (2011): "Note on viscosity solutions of path-dependent PDE and G-martingales, preprint arXiv: 1106.1144v1

[24] Pham H. (2009): Continuous-time stochastic control and optimization with financial applications, Springer, Series SMAP, vol, 61.

[25] Protter P. and K. Shimbo (2008): "No arbitrage and general semimartingale", in *Festschrift for Thomas Kurtz.*

[26] Soner M., Touzi N., and J. Zhang (2011): "The wellposedness of second order backward SDEs", *Probability Theory and Related Fields*, **153**, 149-190.

[27] Rogers, L.C.G. and D. Williams (2000): Diffusions, Markov processes, and martingales. Vol. 2. Itô calculus. *Cambridge University Press*, Cambridge.

[28] Tang, S. and X. Li (1994): "Necessary conditions for optimal control of stochastic systems with random jumps", *SIAM J. Control Optim.*, **32**, no. 5, 1447-1475.

[29] Tang S. and F. Zhang (2013): "Path-dependent optimal stochastic control and viscosity solution of associated Bellman equations", preprint arxiv: 1210.2078

[30] Yong, J. and Zhou, X.Y. (1999). Stochastic controls. Hamiltonian systems and HJB equations. Applications of Mathematics 43. Springer, New York.

[31] Zabczyk, J. (1996): Chance and decision. Stochastic control in discrete time. *Quaderni Scuola Normale Superiore*, Pisa.