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TRACER DIFFUSION AT LOW TEMPERATURE IN KINETICALLY CONSTRAINED MODELS

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Abstract We describe the motion of a tracer in an environment given by a kinetically constrained spin model (KCSM) at equilibrium. We check convergence of its trajectory properly rescaled to a Brownian motion and positivity of the diffusion coefficient D as soon as the spectral gap of the environment is positive (which coincides with the ergodicity region under general conditions). Then we study the asymptotic behaviour of D when the density $1-q$ of the environment goes to 1 in two classes of KCSM. For non-cooperative models, the diffusion coefficient D scales like a power of q, with an exponent that we compute explicitly. In the case of the Fredrickson-Andersen one-spin facilitated model, this proves a prediction made in Jung, Garrahan and Chandler (2004). For the East model, instead we prove that the diffusion coefficient is comparable to the spectral gap, which goes to zero faster than any power of q. This result contradicts the prediction of physicists (Jung, Garrahan and Chandler (2004),Jung, Garrahan and Chandler (2005)), based on numerical simulations, that suggested $D \sim \text{gap}^{\xi}$ with $\xi < 1$.

1. Introduction. Kinetically constrained models (KCSM) have been introduced in the physics literature to model glassy dynamics. They are Markov processes on $\{0,1\}^{\mathbb{Z}^d}$ (or more generally on the set of configurations on a graph), where zeros mark empty sites, and ones mark sites occupied by a particle. The dynamics is of Glauber type: with rate one, each site refreshes its occupation variable: to a zero with probability q, and to a one with probability $1-q$, on the condition that a specific constraint be satisfied by the configuration around the to-be-updated site. This constraint takes the form "a certain set of zeros should be present in a fixed neighbourhood", but does not involve the configuration *at* the to-be-updated site, so that the product Bernoulli measure on $\mathbb{Z}^d$ with parameter $1-q$ is reversible for the dynamics.

A tracer particle evolves in an environment given by a KCSM. The environment is not influenced by the tracer, which performs a simple random walk constrained to jumping only between two empty sites. Properly rescaled, the tracer trajectory is expected to converge to a Brownian motion with a diffusion coefficient depending on the environment. Standard results and strategy (Kipnis and Varadhan (1986),De Masi et al. (1989), Spohn (1990)) allow us to show that in the ergodic regime for the environment there is indeed convergence to a Brownian motion, and to give a variational formula for the diffusion coefficient (see Proposition 3.1 and Lemma 3.2). A general argument then implies that, as soon as the environment has a positive spectral gap, the diffusion coefficient is also positive, so that the convergence result is non-degenerate (Proposition 3.4). Note that the ergodicity regime of KCSM has been identified in Cancrini et al. (2008), and has been shown to coincide with the region of positivity of the spectral gap in great generality, including all the models we consider. Thus we prove in fact positivity of the diffusion coefficient in the ergodic regime of the dynamical environment. The variational formula also yields an immediate upper bound on the

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diffusion coefficient. A similar study was carried in Bertini and Toninelli (2004) with environments given by some non-cooperative constrained models with Kawasaki dynamics.

The main focus of this paper is to compute the asymptotics of the diffusion coefficient when $q \to 0$. This study is inspired by the papers Jung, Garrahan and Chandler (2004) and Jung, Garrahan and Chandler (2005), which in turn have the following physical motivation. In homogeneous liquid systems, physicists argue that the relaxation time τ (measured as the viscosity of the liquid), the temperature T and the diffusion coefficient D of a particle moving inside the system satisfy the following relation, called the Stokes-Einstein relation

$$D \propto T\tau^{-1}, \tag{1}$$

This relation is well obeyed in liquids at high enough temperature. Instead, in supercooled liquids it is experimentally observed (see for instance Edmond et al. (2012), Cicerone and Ediger (1996), Chang and Sillescu (1997), Swallen et al. (2003)) that $D\tau/T$ increases by 2-3 orders of magnitude when decreasing T towards the glass transition temperature. In particular both D and τ^{-1} decrease faster than any power law when the temperature is lowered and for many supercooled liquids a good fit of data is

$$D \propto \tau^{-\xi} \text{ with } \xi < 1. \tag{2}$$

In other words, the self-diffusion of particles becomes much faster than structural relaxation and the Stokes Einstein relation is violated. This decoupling between translational diffusion and global relaxation is interpreted as a landmark of dynamical heterogeneities in glassy systems, namely the existence of spatially correlated regions of relatively high or low mobility that persist for a finite lifetime in the liquid, and that grow in size as one approaches the glass transition. More precisely, the decoupling should be due to the fact that diffusion is dominated by the fastest regions whereas structural relaxation is dominated by the slowest regions.

In order to investigate the possible violation of the Stokes Einstein relation in KCM, which are used as simplified models of glassy dynamics, in [JGC04] and [JGC05] the authors run simulations of a tracer in two systems with constrained dynamics in one dimension: the FA-1f model (in which the constraint requests that at least one neighbour be empty), and the East model (in which the constraint is satisfied if the neighbour in the East direction is empty). They predict in both cases a breakdown of the Stokes-Einstein relation. More precisely, they predict that in the FA-1f model in one dimension

$$D \sim q^2 \sim \text{gap}^{2/3} \tag{3}$$

and in the East model

$$D \approx \text{gap}^{\xi} \quad \text{with} \quad \xi \approx 0.73. \tag{4}$$

Our results confirm (3) but invalidate (4). Indeed we prove that for the East model $D \approx$ gap up to polynomial corrections (Theorem 4.1). For this model simulations are much harder to run than for FA model due to the very fast divergence of the relaxation time when $q \to 0$ (faster than any power of $1/q$; see (63)), thus accounting for the wrong numerical prediction.

More generally we show that, in any dimension, if the model is defined by the constraint "there should be at least k zeros in a ball of radius k around the to-be-updated site" the diffusion coefficient is of order q^{k+1} ($k = 1$ corresponds to the FA-1f model, so the result confirms the conjecture in Jung, Garrahan and Chandler (2004); see Theorem 4.1). The proof of this result relies on the

introduction of an auxiliary dynamics whose diffusion coefficient gives a lower bound for D. This dynamics is similar to that in Spohn (1990), though it is less immediate to derive because it does not appear by just suppressing terms in the variational formula. The very construction of this auxiliary dynamics is in fact quite informative about the effective dynamics of the tracer, and can be generalized to other non-cooperative models (see Definition 2.1). Back to the FA-1f model, in dimension 2, our result and the estimate of the spectral gap in Cancrini et al. (2008) (Theorem 6.4) show that $D \propto$ gap. When $d \geq 3$, whereas we do know the asymptotic behaviour of D, for want of precision in the estimates on the spectral gap of the FA-1f model, we cannot decide whether $D \propto \text{gap}^{\xi}$ for some exponent ξ, but our results do imply that ξ cannot be strictly smaller than one.

We also study the diffusion coefficient when the environment is given by the East model, which does not belong to the non-cooperative class. As mentioned above we prove in this case $D \approx gap$ up to polynomial corrections (Theorem 5.2), contradicting (4). The strategy used in that context is very different from the one we designed for the "k-zeros" model, because the dynamics of the East model is cooperative, so that restricting the dynamics only to a neighbourhood of the tracer is not relevant. The proof relies instead on precise estimates of the energy barriers that have to be overcome in order for the tracer to cross the typical distance between two zeros at equilibrium, $1/q$. These estimates have been established mostly in Cancrini et al. (2008) and Chleboun, Faggionato and Martinelli (2012). As an extension of results in these two papers, we provide in particular a better estimate on the spectral gap in infinite volume (Lemma 5.5).

The paper is organized as follows. In section 2, we define the processes of the environment, the tracer dynamics and the environment seen from the tracer. In section 3, we prove convergence of the tracer trajectory to a Brownian motion with positive diffusion coefficient in the ergodic regime. Section 4 is devoted to retrieve the right asymptotics for the diffusion coefficient when the density goes to 1 in non-cooperative models. Finally, in section 5, we show that asymptotically the diffusion coefficient in the East model is of the same order as the spectral gap, up to polynomial corrections.

2. Models and notations. Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. For $\omega \in \Omega$, $x \in \mathbb{Z}^d$ we define ω^x the configuration such that

$$\omega^x_y = \begin{cases} \omega_y & \text{if } y \neq x \\ 1-\omega_x & \text{if } y = x. \end{cases} \tag{5}$$

A KCSM is defined by its equilibrium density $p = 1-q$ and constraints $(c_x(\omega))_{x\in\mathbb{Z},\omega\in\Omega}$, taking values 0 and 1. We require that the constraints be translation invariant, that c_x depend on a fixed finite neighborhood of x and not on ω_x (i.e. $c_x(\omega) = 1$ iff $c_x(\omega^x) = 1$). We also want the constraints to be monotone (if $\forall x \in \mathbb{Z}^d,\ \omega_x \leq \omega'_x$, then $\forall x \in \mathbb{Z}^d,\ c_x(\omega) \geq c_x(\omega')$). We will denote by $\mathcal{L}_E$ the generator of the environment process: for f a local function on $\{0,1\}^{\mathbb{Z}}$

$$\mathcal{L}_E f(\omega) = \sum_{y\in\mathbb{Z}} c_y(\omega)((1-q)(1-\omega_y) + q\omega_y)\left[f(\omega^y) - f(\omega)\right]. \tag{6}$$

In words, a zero (resp. each one) at site x in configuration η turns into a one (resp. a zero) at rate $(1-q)$ (resp. q), provided the constraint is satisfied at x, i.e. $c_x(\eta) = 1$. This process satisfies the detailed balance property w.r.t. μ the product Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ of parameter $1-q$, so it is reversible.

A transition $\omega \to \omega^x$ is *legal* if $c_x(\omega) = 1$. Note that $\omega \to \omega^x$ is legal iff $\omega^x \to \omega$ is. A KCSM is *non-cooperative* if a finite empty set is enough to empty the whole configuration through legal transitions. More precisely

Definition 2.1 *A KCSM is* non-cooperative *if the following holds:*

There exists a finite set $A \subset \mathbb{Z}^d$ such that for every $\omega \in \Omega$, if $\omega_{|A} \equiv 0$, for every $x \in \mathbb{Z}^d$ such that $\omega_x = 1$, there is a finite sequence $\omega^{(0)}, ..., \omega^{(n)}$ such that $\omega^{(0)} = \omega$, $\left(\omega^{(n)}\right)_x = 0$, and for all $i = 1, ..., n$, $\omega^{(i)} = \left(\omega^{(i-1)}\right)^{x_i}$ where $x_i \in \mathbb{Z}^d$ such that $c_{x_i}\left(\omega^{(i-1)}\right) = 1$.

The ergodic regime for KCSM was identified in Cancrini et al. (2008). In general, there is a critical parameter $q_c \in [0,1]$ such that the process is ergodic for $q > q_c$ and non-ergodic for $q < q_c$. $p_c = q_c$ is characterized as the critical density of an appropriate bootstrap percolation model; basically, it is the density above which blocked clusters (i.e. clusters of occupied sites that cannot be emptied through legal transitions) appear with positive probability. A non-cooperative model is ergodic at every density $p = 1 - q \in (0,1)$ ($q_c = 0$).

We consider an environment given by a KCSM, and we inject a tracer at the origin. The tracer jumps at rate one to one of its nearest neighbours, provided that both the site where it sits and the site where it wants to jump are empty (for the environment). More formally, let $(\omega(t), X_t)$ be the joint evolution of the KCSM and the tracer. It is a Markov process on $\{0,1\}^{\mathbb{Z}^d} \times \mathbb{Z}^d$ given by the generator

$$\begin{aligned} \mathcal{L}_0 f(\omega, x) &= \sum_{y \in \mathbb{Z}^d} c_y(\omega)((1-q)(1-\omega_y) + q\omega_y)\left[f(\omega^y, x) - f(\omega, x)\right] \\ &\quad + \sum_{i=1}^{d} \sum_{\alpha=\pm 1} (1-\omega_x)(1-\omega_{x+\alpha e_i})\left[f(\omega, x+\alpha e_i) - f(\omega, x)\right]. \end{aligned} \tag{7}$$

We consider the process $\eta(t)$ of the environment seen from the tracer, whose generator is given by

$$\begin{aligned} \mathcal{L} f(\eta) &= \sum_{y \in \mathbb{Z}^d} c_y(\eta)((1-q)(1-\eta_y) + q\eta_y)\left[f(\eta^y) - f(\eta)\right] \\ &\quad + \sum_{i=1}^{d} \sum_{\alpha=\pm 1} (1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i + \cdot}) - f(\eta)\right], \end{aligned} \tag{8}$$

where $\eta_{y+\cdot}$ denotes the configuration such that $(\eta_{y+\cdot})_x = \eta_{y+x}$. This is again a reversible process w.r.t. μ the product Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ of parameter $1-q$ (it satisfies detailed balance).

3. Convergence to a non-degenerate Brownian motion. We follow the strategy of Kipnis and Varadhan (1986), De Masi et al. (1989), Spohn (1990) to establish

Proposition 3.1 *If the environment process is ergodic ($q > q_c$), we have*

$$\lim_{\epsilon \to 0} \epsilon X_{\epsilon^{-2} t} = \sqrt{2D} B_t, \tag{9}$$

where B_t is the standard Brownian motion, the convergence holds in the sense of weak convergence of path measures on $D([0,\infty), \mathbb{R}^d)$ and the diffusion matrix D is given by

$$u.Du = q^2 \|u\|_2^2 - \int_0^\infty \mu\left(j_u e^{\mathcal{L}t} j_u\right) dt, \tag{10}$$

where for any $u = (u_1, ..., u_d) \in \mathbb{Z}^d$

$$j_u(\eta) = (1-\eta_0) \sum_{i=1}^{d} \sum_{\alpha=\pm 1} (1-\eta_{\alpha e_i}) \alpha u_i. \tag{11}$$

Proof

Considering the martingale

$$M_t^u = u.X_t - \int_0^t j_u(\eta(s))ds \tag{12}$$

and following the steps of De Masi et al. (1989), Spohn (1990), using reversibility, we get

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[(u.X_t)^2\right] = \sum_{i=1}^d \sum_{\alpha=\pm 1} u_i^2 \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\right) - 2\int_0^\infty \mu\left(j_u e^{t\mathcal{L}} j_u\right) dt \tag{13}$$

In particular, $\int_0^\infty \mu\left(j_u e^{t\mathcal{L}} j_u\right) dt < \infty$, so that, since the process of generator $\mathcal{L}$ is ergodic, Theorem 1.8 of Kipnis and Varadhan (1986) applies to $\int_0^t j_u(\eta_s)ds$, yielding

$$\epsilon u.X_{\epsilon^{-2}t} = \epsilon\left(M^u_{\epsilon^{-2}t} + N_{\epsilon^{-2}t}\right) + Q^\epsilon(t), \tag{14}$$

where $M_t + N_t$ is a martingale in $L^2(\mathbb{P})$ with stationary increments and $Q^\epsilon(t)$ is an error term that vanishes when ϵ goes to 0. This implies the convergence of $\epsilon X_{\epsilon^{-2}t}$ to $\sqrt{2D}B_t$ with D given by (10). ✓

For the previous result to be meaningful, we need to prove $D > 0$. We show that this is true as soon as the KCSM has a positive spectral gap. In Cancrini et al. (2008), it has been proved for a large class of KCSM that the spectral gap is positive in the whole ergodic regime, so that this requirement is not a big restriction. In particular, the spectral gap is positive at every density $p = 1 - q \in (0, 1)$ for the East model and non-cooperative models. A first step in the direction of proving $D > 0$ is to give a variational formula for D, which is the adaptation to our context of Prop. 2 in Spohn (1990).

Lemma 3.2

$$\begin{aligned} u.Du &= \frac{1}{2}\inf_f \Bigg\{ \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)((1-q)(1-\eta_y) + q\eta_y)\left[f(\eta^y) - f(\eta)\right]^2\right) \\ &\quad + \sum_{i=1}^d \sum_{\alpha=\pm 1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i +\cdot}) - f(\eta)\right]^2\right) \Bigg\}, \end{aligned} \tag{15}$$

where the infimum is taken over local functions f on Ω.

Proof

We notice, as in Spohn (1990), that

$$\int_0^\infty \mu\left(j_u e^{t\mathcal{L}} j_u\right) dt = -\inf\left\{-2\mu(j_u f) - \mu(f\mathcal{L}f)\right\}, \tag{16}$$

where the infimum is taken over local functions on Ω. Then, using detailed balance, notice that we can write

$$-4\mu(j_u f) = 2\sum_{i=1}^d \sum_{\alpha=\pm 1} \alpha u_i \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i +\cdot}) - f(\eta)\right]\right). \tag{17}$$

Moreover,

$$
\begin{aligned}
-2\mu(f\mathcal{L}f) &= \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)(p(1-\eta_y)+(1-p)\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right) \\
&\quad + \sum_{i=1}^{d}\sum_{\alpha=\pm1} \mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot})-f(\eta)\right]^2\right) \qquad (18)
\end{aligned}
$$

Inserting (17) and (18) into (13) and rearranging the terms, we get (15). ✓

Now we can prove $D>0$ when the spectral gap of the environment is positive. Recall its definition

Definition 3.3

$$
\mathrm{gap}(\mathcal{L}_E) = \inf \frac{-\mu(f\mathcal{L}_E f)}{Var_\mu(f)}, \qquad (19)
$$

where the infimum is taken over all functions in $L^2(\mu)$ with $Var_\mu(f)\neq 0$.

Proposition 3.4

$$
q^2\|u\|_2^2 \geq u.Du \geq \frac{\mathrm{gap}(\mathcal{L}_E)}{4d+\mathrm{gap}(\mathcal{L}_E)} q^2\|u\|_2^2 \qquad (20)
$$

Proof

The upper bound follows directly from (10), since the second term is non-negative.

For the lower bound, consider the expression of D given in (15). The first sum in the infimum is $-2\mu(f\mathcal{L}_E f)$, so that by definition of the spectral gap

(21)

$$
u.2Du \geq \inf\left\{2\mathrm{gap}(\mathcal{L}_E)Var_\mu(f) + \sum_{i=1}^{d}\sum_{\alpha=\pm1}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i+\cdot})-f(\eta)\right]^2\right)\right\}.
$$

To bound the double sum, we use the inequality $(a+b)^2 \geq \gamma a^2 - \frac{\gamma}{1-\gamma}b^2$ for $\gamma<1$. This yields

$$
\begin{aligned}
\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha u_i + f(\eta_{\alpha e_i})-f(\eta)\right]^2\right) &\geq \gamma q^2 u_i^2 - \frac{\gamma}{1-\gamma}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot})-f(\eta)\right]^2\right) \\
&\geq \gamma q^2 u_i^2 - 4\frac{\gamma}{1-\gamma}Var_\mu(f)
\end{aligned}
$$

So that, injecting this in (21), we get

$$
u.Du \geq \inf\left\{\left(\mathrm{gap}(\mathcal{L}_E) - 4d\frac{\gamma}{1-\gamma}\right)Var_\mu(f) + \gamma q^2\|u\|_2^2\right\}. \qquad (22)
$$

Choosing $\gamma = \frac{\mathrm{gap}(\mathcal{L}_E)}{4d+\mathrm{gap}(\mathcal{L}_E)} < 1$, we get the desired lower bound. ✓

Note that at high density ($q\to 0$), the spectral gap of the East model is of order higher than any polynom in q, so that the term q^2 is negligible. In fact, for the East model, the lower bound here is quite accurate (Theorem 5.2). For non-cooperative models however, we are able to do much better. In particular, for FA-1f in one dimension, this gives $D\geq Cq^5$, which is pretty poor, given that D is in fact of order q^2, as predicted in Jung, Garrahan and Chandler (2004). Except in the FA-1f model, the upper bound also needs refinement. Designing more precise bounds on D when $q\to 0$ is the object of the next sections.

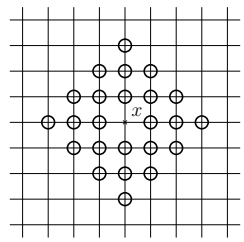

FIGURE 1. $\mathcal{N}_3(x)$, *the* 3*-neighbourhood of* x *in* $\mathbb{Z}^2$.

4. Correct order of D for small q in non-cooperative models. For the sake of simplicity, we give the following result only in the case of a particular class of non-cooperative models. However, we expect our method to work more generally for non-cooperative models, and give the right order in q at high density.

Let us define a class of non-cooperative KCSM, which we will call "k-zeros" for a positive integer k. Let $\|.\|_1$ denote the 1-norm on $\mathbb{Z}^d$, i.e. the norm induced by the graph distance. Let

$$\mathcal{N}_k(x) = \Big\{ y \in \mathbb{Z}^d \mid 0 < \|y - x\|_1 \leq k \Big\} \tag{23}$$

be the k-neighbourhood of x (see Figure 1).

The model "k-zeros" in $\mathbb{Z}^d$ is defined by the following constraints (recall (6))

$$c_x(\omega) = \begin{cases} 1 & \text{if} \quad \sum_{y \in \mathcal{N}_k(x)} (1 - \omega_y) \geq k \\ 0 & \text{else} \end{cases} , \tag{24}$$

i.e. the constraint is satisfied if there are at least k zeros within distance k. It is non-cooperative since it is enough to empty $0, e_1, 2e_1, ..., (k-1)e_1$ to empty the whole lattice through legal transitions. For $k = 1$, the "1-zero" model is better known as the one-flip Fredrickson-Andersen (or FA-1f) model.

Theorem 4.1 *For the tracer diffusion in the model "k-zeros", there exist constants* $0 < c \leq C < \infty$ *depending only on* d *such that for all* $u \in \mathbb{Z}^d$

$$cq^{k+1}\|u\|_2^2 \leq u.Du \leq Cq^{k+1}\|u\|_2^2. \tag{25}$$

Remark 4.2 *We believe that the techniques developed below can be adapted to show the equivalent of Theorem 4.1 for any non-cooperative model,* k *being the minimal number of zeros needed to empty the whole lattice (see Definition 2.1), and* 1 *being replaced by* m *the minimal number of extra zeros needed to move a minimal cluster around. We propose a heuristic for the order* q^{k+m}*. Consider for a moment a simple symmetric random walk on the interval* $\{-1/(2q), ..., 1/(2q)\}$ *of length* $1/q$*. For large times* T*, the time spent in* 0 *by the random walk is approximately* Tq*. Since* $1/q$ *is the typical distance between two zeros under the product Bernoulli measure* μ*, the fraction of time during which there is a zero at* 0 *before time* T *is approximately* Tq*. When that happens, a tracer sitting in* 0 *has a probability of order* q *to jump, which gives a diffusion coefficient for the tracer in the FA-1f model of order* $Tq \times q/T = q^2$*. How does this adapt to another non-cooperative environment, where* $k \geq 1$*,* $m \geq 1$ *(for instance the "k-zeros" model,* $k > 1$*, in which case* $m = 1$*)? A single zero cannot move on its own in such a model, but a group of* k *zeros can, and since the number of extra zeros it needs to move is* m*, the diffusion coefficient of such a group is of order* q^m*.*

So we have to consider the fraction of time spent in 0 *by a group of* k *zeros performing a random walk on* $\{-1/(2q^k), ..., 1/(2q^k)\}$ *before time* T *(*$1/q^k$ *being the typical distance between two such groups under* μ*), that is* Tq^k*. During the time the group of* k *zeros is in contact with the tracer (i.e. at site* 0*), the tracer diffuses with it, which means with rate* q^m*. In the end, the diffusion coefficient of the tracer should therefore be of order* $Tq^k \times q^m/T$*.*

4.1. *Lower bound in Theorem 4.1.* The key to the proof of the lower bound we give below is that we are able to come down to studying a *local* dynamics (see Lemma 4.3 and the description of the dynamics in the proof of Lemma 4.4). The possibility of doing this simplification is strongly related to the fact that we are working with non-cooperative models.

For the sake of simplicity, this proof is written for $k = 3$, but it generalizes without difficulty to any $k \geq 1$. It is widely inspired by the fourth section in Spohn (1990).

The first step is to give a lower bound on D in terms of the diffusion coefficient $\overline{D}$ of another dynamics (Lemma 4.3), for which we can prove positivity (Lemma 4.4). In the auxiliary dynamics, the only allowed transitions are jumps of the tracer between empty sites and swaps of its left and right neighbourhood, which can be reconstructed using only transitions that are allowed in the initial dynamics (see Figures 2 and 3). We need some notations to be more specific.

Let $\mu^{(3)}$ be the product Bernoulli measure *on* $\mathbb{Z}$ conditioned to having at least three consecutive zeros, one of which at the origin, i.e. let $A \subset \Omega$ be defined as

$$A = \left\{ \eta \in \Omega \mid \eta_0 = 0 \text{ and } (1-\eta_1)(1-\eta_2) + (1-\eta_{-1})(1-\eta_1) + (1-\eta_{-2})(1-\eta_{-1}) \geq 1 \right\} \tag{26}$$

and

$$\mu^{(3)} = \mu\left(. \mid A\right). \tag{27}$$

Also, if $\eta \in \Omega$, denote by $\eta^{\leftrightarrow}$ the configuration obtained by exchanging the occupation numbers in sites -1 and $+1$, and -2 and $+2$

$$\eta_y^{\leftrightarrow} = \begin{cases} \eta_1 & \text{if } y = -1 \\ \eta_{-1} & \text{if } y = 1 \\ \eta_2 & \text{if } y = -2 \\ \eta_{-2} & \text{if } y = 2 \\ \eta_y & \text{else.} \end{cases} \tag{28}$$

We also generalize the notation η^x by defining $\eta^{x_1,...,x_n}$ as the configuration η flipped at sites $x_1, ..., x_n$ (the x_i being distinct).

We can now state

Lemma 4.3 *If* $\overline{D}$ *is defined by*

$$\begin{aligned} (\overline{29}) = \; & \frac{1}{2} \inf_f \Big\{ \mu^{(3)} \left(\left(1 - (1-\eta_1)(1-\eta_{-1})\right) \left[f\left(\eta^{\leftrightarrow}\right) - f(\eta)\right]^2 \right) \\ & + \mu^{(3)} \left((1-\eta_1) \left[1 + f(\eta_{1+.}) - f(\eta)\right]^2 \right) + \mu^{(3)} \left((1-\eta_{-1}) \left[-1 + f(\eta_{-1+.}) - f(\eta)\right]^2 \right) \Big\}, \end{aligned}$$

where the infimum is taken over local functions on Ω*, then we have*

$$e_1.De_1 \geq \frac{1+2p}{4} q^4 \overline{D}. \tag{30}$$

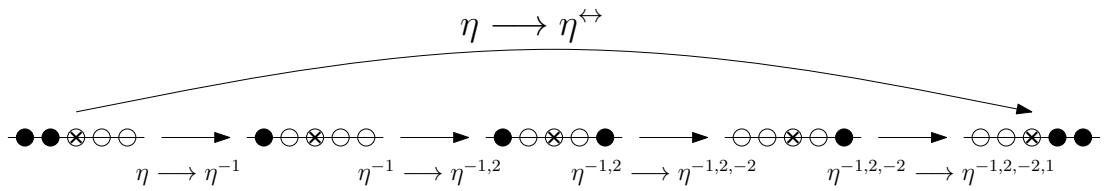

FIGURE 2. *The four legal flips used to reconstruct the swap $\eta \longrightarrow \eta^{\leftrightarrow}$ when $\eta_{-1} = \eta_{-2} = (1-\eta_1) = (1-\eta_2) = 1$. The cross recalls that the tracer is sitting at the origin.*

Proof of Lemma 4.3

For briefness, we define

$$r_x(\eta) = (1-q)(1-\eta_x) + q\eta_x. \tag{31}$$

Then we have, given the definition of $\mu^{(3)}$ (27), for every local function f

$$\begin{aligned}\mu^{(3)}\Big((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big) &= \mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big)\\ &\quad+\mu^{(3)}\Big((1-\eta_{-1})(1-\eta_{-2})\eta_1\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big)\end{aligned} \tag{32}$$

Our aim is to reconstruct the swap changing η into $\eta^{\leftrightarrow}$, using only legal (for the "3-zeros" model dynamics) flips. The first term of the r.h.s. in (32) can be rewritten as

$$\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big)+\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}(1-\eta_{-2})\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big). \tag{33}$$

Let us focus on the first term. See in Figure 2 a representation of the successive flips used to reconstruct the swap. Writing that, when $\eta_{-1} = \eta_{-2} = (1-\eta_1) = (1-\eta_2) = 1$

$$\begin{aligned}f\left(\eta^{\leftrightarrow}\right)-f(\eta) &= f\left(\eta^{-1,2,-2,1}\right)-f\left(\eta^{-1,2,-2}\right)+f\left(\eta^{-1,2,-2}\right)-f\left(\eta^{-1,2}\right)\\ &\quad+f\left(\eta^{-1,2}\right)-f\left(\eta^{-1}\right)+f\left(\eta^{-1}\right)-f(\eta)\end{aligned} \tag{34}$$

and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big) \leq\ & 4\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1,2,-2,1}\right)-f\left(\eta^{-1,2,-2}\right)\right]^2\Big)\\ &+4\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1,2,-2}\right)-f\left(\eta^{-1,2}\right)\right]^2\Big)\\ &+4\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1,2}\right)-f\left(\eta^{-1}\right)\right]^2\Big)\\ &+4\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1}\right)-f(\eta)\right]^2\Big)\end{aligned} \tag{35}$$

Note that all the flips involved are legal for the dynamics "3-zeros": there are always at least three zeros in the 3-neighbourhood of the site that is flipped. Then, we make a change of variables in the first three terms above to get

$$\mu^{(3)}\Big((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\Big)$$

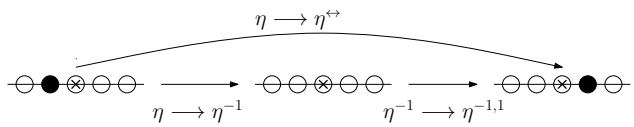

FIGURE 3. *The two legal flips used to reconstruct the swap $\eta \longrightarrow \eta^{\leftrightarrow}$ when $\eta_{-1} = \eta_{-2} = (1-\eta_1) = \eta_2 = 1$.*

$$
\begin{aligned}
&\leq 4\frac{1-q}{q}\mu^{(3)}\left((1-\eta_1)\eta_2(1-\eta_{-1})(1-\eta_{-2})\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)\\
&\quad + 4\mu^{(3)}\left((1-\eta_1)\eta_2(1-\eta_{-1})\eta_{-2}\left[f\left(\eta^{-2}\right)-f\left(\eta\right)\right]^2\right)\\
&\quad + 4\frac{1-q}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_2)(1-\eta_{-1})\eta_{-2}\left[f\left(\eta^2\right)-f\left(\eta\right)\right]^2\right)\\
&\quad + 4\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}\eta_{-2}\left[f\left(\eta^{-1}\right)-f\left(\eta\right)\right]^2\right) \qquad (36)
\end{aligned}
$$

In the same way (following the strategy represented in Figure 3), we get

$$
\begin{aligned}
&\mu^{(3)}\left((1-\eta_1)(1-\eta_2)\eta_{-1}(1-\eta_{-2})\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\right)\\
&\leq \frac{2(1-q)}{q}\mu^{(3)}\left((1-\eta_{-1})(1-\eta_2)(1-\eta_{-2})(1-\eta_1)\left[f(\eta^1)-f(\eta)\right]^2\right)\\
&\quad + 2\mu^{(3)}\left((1-\eta_1)(1-\eta_2)(1-\eta_{-2})\eta_{-1}\left[f(\eta^{-1})-f(\eta)\right]^2\right) \qquad (37)
\end{aligned}
$$

Combining (33), (36) and (37), and doing the same for the second term in (32), we get:

$$
\begin{aligned}
&\mu^{(3)}\left((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\right)\\
&\leq \frac{4}{q}\mu^{(3)}\left((1-\eta_{-1})(1-\eta_{-2})r_1(\eta)\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)\\
&\quad + \frac{4}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_{-1})r_{-2}(\eta)\left[f\left(\eta^{-2}\right)-f\left(\eta\right)\right]^2\right)\\
&\quad + \frac{4}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_{-1})r_2(\eta)\left[f\left(\eta^2\right)-f\left(\eta\right)\right]^2\right)\\
&\quad + \frac{4}{q}\mu^{(3)}\left((1-\eta_1)(1-\eta_2)r_{-1}(\eta)\left[f\left(\eta^{-1}\right)-f\left(\eta\right)\right]^2\right) \qquad (38)
\end{aligned}
$$

Now notice that we have:

$$
\begin{aligned}
&\mu^{(3)}\left((1-\eta_{-1})(1-\eta_{-2})r_1(\eta)\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)\\
&\qquad = \frac{1}{\mu(A)}\mu\left((1-\eta_0)(1-\eta_{-1})(1-\eta_{-2})r_1(\eta)\left[f\left(\eta^1\right)-f\left(\eta\right)\right]^2\right)
\end{aligned}
$$

and similarly for the other terms in (38), so that we have proved the following inequality – recalling that $\mu(A) = q^3(1+2p)$:

$$
\begin{aligned}
&\textstyle\sum_{y\in\mathbb{Z}^d}\mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right)\\
&\qquad \geq q^4\frac{(1+2(1-q))}{4}\mu^{(3)}\left((1-(1-\eta_1)(1-\eta_{-1}))\left[f\left(\eta^{\leftrightarrow}\right)-f(\eta)\right]^2\right). \qquad (39)
\end{aligned}
$$

We are almost done: it remains to notice that

$$\mu^{(3)}\left((1-\eta_1)\left[1+f(\eta_{1+\cdot})-f(\eta)\right]^2\right) \quad \leq \quad \frac{1}{\mu(A)}\mu\left((1-\eta_0)(1-\eta_1)\left[1+f(\eta_{1+\cdot})-f(\eta)\right]^2\right)$$

and similarly with 1 replaced by -1, so that a fortiori:

$$\mu^{(3)}\left((1-\eta_1)\left[1+f(\eta_{1+\cdot})-f(\eta)\right]^2\right)+\mu^{(3)}\left((1-\eta_{-1})\left[-1+f(\eta_{-1+\cdot})-f(\eta)\right]^2\right)$$

$$\leq \frac{4}{q^4(1+2(1-q))}\sum_{i=1}^{d}\sum_{\alpha=\pm 1}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[\alpha\delta_{1i}+f(\eta_{\alpha e_i+\cdot})-f(\eta)\right]^2\right). \tag{40}$$

Combining (39) and (40), and recalling (15), we get the lemma. ✓

Of course there is nothing special about the direction e_1, and the lemma is valid in all directions. Notice that it does not depend on the dimension. Now we need to see how to obtain a universal lower bound for $\overline{D}$.

Lemma 4.4

$$\overline{D} \geq 4/9 \tag{41}$$

Proof of Lemma 4.4

Following the same lines as in the proof of Proposition 3.1 and Lemma 3.2, we see that $\overline{D}$ is the diffusion coefficient of the dynamics reversible w.r.t. $\mu^{(3)}$ described below

- with rate 1, if $\eta_1 = 0$, the tracer jumps to the right, i.e. we go from η to $\eta_{1+\cdot}$.
- with rate 1, if $\eta_{-1} = 0$, the tracer jumps to the left, i.e. we go from η to $\eta_{-1+\cdot}$.
- with rate 1, if either $\eta_1 = 1$ or $\eta_{-1} = 1$, $\{-2,-1\}$ and $\{2,1\}$ are swapped, i.e. we go from η to $\eta^{\leftrightarrow}$.

As in Spohn (1990), starting from a configuration η chosen after $\mu^{(3)}$, we can index by $\mathbb{Z}$ all the configurations that can be reached by this dynamics in the following way. $\eta^{(0)} = \eta$ is the initial configuration, that is almost surely in A. Then we define inductively $\eta^{(n)}$, $n \in \mathbb{Z}$. If $\eta_1^{(n)} = 0$, $\eta^{(n+1)} = \eta_{1+\cdot}^{(n)}$. If $\eta_1^{(n)} = 1$, $\eta^{(n+1)} = \left(\eta^{(n)}\right)^{\leftrightarrow}$. Similarly, if $\eta_{-1}^{(n)} = 0$, $\eta^{(n-1)} = \eta_{-1+\cdot}^{(n)}$. If $\eta_{-1}^{(n)} = 1$, $\eta^{(n-1)} = \left(\eta^{(n)}\right)^{\leftrightarrow}$. Note that this definition is consistent ($\eta^{(n+1-1)} = \eta^{(n)}$).

Using this labelling with integers of all attainable configurations, the dynamics described above can be equivalently defined in the following way: if the system is in the configuration $\eta^{(n)}$, it goes to $\eta^{(n+1)}$ with rate one, and to $\eta^{(n-1)}$ also with rate one. So we can rewrite the process starting from η as $\eta(t) = \eta^{N_t}$ where $(N_t)_{t\geq 0}$ is a simple random walk on $\mathbb{Z}$.

Now to conclude, we just need to notice that if X_t is the position of the tracer at time t in this dynamics, we have

$$X_t \geq \left\lfloor \frac{2}{3}N_t \right\rfloor,$$

since two out of three times N moves to the right, X also jumps by one.

$$2\overline{D} = \lim_{t\to+\infty}\frac{1}{t}\mathbb{E}\left[X_t^2\right] \geq \frac{4}{9}\lim_{t\to+\infty}\frac{1}{t}\mathbb{E}\left[N_t^2\right] = 8/9.$$

✓

To deduce Theorem 4.1 from Lemma 4.3 and Lemma 4.4, let $u \in \mathbb{R}^d$ be such that $\|u\|_2 = 1$ and notice that we can use comparisons with the auxiliary dynamics above in all directions to get

$$
\begin{aligned}
2u.Du &\geq \sum_{i=1}^{d} \inf_{f_i} \Bigg\{ \frac{1}{d} \sum_{x \in \mathbb{Z}^d} \mu \left(c_x(\eta) r_x(\eta) \left[f_i(\eta^x) - f_i(\eta) \right]^2 \right) && (42)\\
&\quad + \sum_{\alpha = \pm 1} \mu \left((1-\eta_0)(1-\eta_{\alpha e_i}) \left[\alpha u_i + f_i(\eta_{\alpha e_i + \cdot}) - f_i(\eta) \right]^2 \right) \Bigg\} && (43)\\
&\geq \sum_{i=1}^{d} u_i^2 \inf_{f_i} \Bigg\{ \frac{1}{d} \sum_{x \in \mathbb{Z} \cdot e_i} \mu \left(c_x^i(\eta) r_x(\eta) \left[f_i(\eta^x) - f_i(\eta) \right]^2 \right) && (44)\\
&\quad + \frac{1}{d} \sum_{\alpha = \pm 1} \mu \left((1-\eta_0)(1-\eta_{\alpha e_i}) \left[\alpha + f_i(\eta_{\alpha e_i + \cdot}) - f_i(\eta) \right]^2 \right) \Bigg\} && (45)\\
&\geq \frac{2}{d} D_1, && (46)
\end{aligned}
$$

where $c_x^i(\eta)$ is one iff the constraint is satisfied using only zeros in the direction i, D_1 is the diffusion coefficient in one dimension and we used $\sum_{i=1}^{d} u_i^2 = 1$. Theorem 4.1 follows from this inequality and the two previous Lemmas.

Remark 4.5 *This strategy can be applied to other non-cooperative models. However, the auxiliary dynamics (the one involving swaps around the origin and jumps of the tracer) will be model dependant and may not be strictly one-dimensional. It may be encoded by a random walks on graphs slightly more complex than $\mathbb{Z}$, but still with a uniformly positive diffusion coefficient. We believe that this technique allows to retrieve the correct exponent at low temperature for non-cooperative models.*

4.2. *Upper bound in Theorem 4.1.* In view of (15), to find an upper bound on D, we need to find an appropriate test function. As a warming, suppose that $d = 1$. Then, looking for a function that cancels the second line in (15), we find that a natural function to consider is

$$
f(\eta) = \min \{ x \in \mathbb{N} \mid \eta_x = 1 \} . \tag{47}
$$

Then it is not too difficult to check that if we plug this function in the first line of (15), we get an expression of order q^{k+1}: the factor q^k comes from the constraint, and the extra q comes from the extra empty site we need in order to evolve.

In higher dimension, we are going to find a good test function to evaluate $e_1.De_1$. Define $C(\eta)$ the connected cluster of zeros containing the origin in the configuration η ($C(\eta) = \varnothing$ if $\eta_0 = 1$). See Figure 4 for an example.

Now we can define our test function.

$$
f(\eta) = \min \left\{ x \in \mathbb{N} \mid C(\eta) \subset (-\infty, x-1] \times \mathbb{Z}^{d-1} \right\} . \tag{48}
$$

For instance, if $\eta_0 = 1$, $f(\eta) = 0$. In Figure 4, $f(\eta) = 4$. Note that this function coincides with that in (47) when $d = 1$. This function cancels the second line in (15) when $u = e_1$. Indeed, when

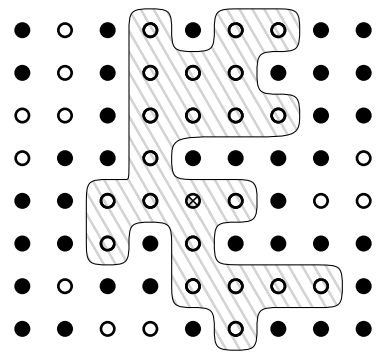

FIGURE 4. *An example of $C(\eta)$. Zeros are represented by empty circles, ones by filled disks and the origin is marked by a cross. The cluster of zeros containing the origin is circled by a line and tiled in gray. In this case, $f(\eta) = 4$.*

$(1-\eta_0)(1-\eta_{\alpha e_i} \neq 0$, 0 and αe_i belong to the same cluster of zeros. So what we need to do is show that

$$\sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right) \leq Cq^{k+1} \tag{49}$$

for some finite C. Let us split the l.h.s. in two terms and treat them separately: we need to show that

$$S_0 = \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)(1-\eta_y)\left[f(\eta^y)-f(\eta)\right]^2\right) \leq Cq^{k+1} \tag{50}$$

$$S_1 = \sum_{y\in\mathbb{Z}^d} \mu\left(c_y(\eta)\eta_y\left[f(\eta^y)-f(\eta)\right]^2\right) \leq Cq^{k} \tag{51}$$

Thanks to detailed balance, $(1-q)S_0 = qS_1$, so we only need to show (50).

Let us now study S_0. The mechanism involved here is the removal of part of the cluster of zeros around the origin. In particular, when $(1-\eta_y)\left[f(\eta^y)-f(\eta)\right]^2 \neq 0$, we certainly have $\left[f(\eta^y)-f(\eta)\right]^2 \leq |C(\eta)|^2$, where $|C(\eta)|$ is the cardinal of $C(\eta)$. So that

$$S_0 \leq \mu\left(|C(\eta)|^2 \sum_{y\in C(\eta)} c_y(\eta)(1-\eta_y)\right) \tag{52}$$

$$\leq \sum_{n\geq 0} \mu\left(|C(\eta)|^2\, \mathbf{1}_{0\leftrightarrow\partial B_n,\ 0\not\leftrightarrow\partial B_{n+1}} \sum_{y\in C(\eta)} c_y(\eta)(1-\eta_y)\right), \tag{53}$$

where ∂B_n denotes the set of points at distance n from 0, and $\{0\leftrightarrow\partial B_n\}$ is the event that there is a site at distance n from 0 in $C(\eta)$. Since on the event $\{0\leftrightarrow\partial B_n,\ 0\not\leftrightarrow\partial B_{n+1}\}$, $C(\eta)\subset B_1(0,n)$, we have

$$S_0 \leq \sum_{n\geq 0}(2n+1)^{2d} \sum_{y\in B_1(0,n)} \mu\left(c_y(\eta)(1-\eta_y)\mathbf{1}_{0\leftrightarrow\partial B_n,\ 0\not\leftrightarrow\partial B_{n+1}}\right). \tag{54}$$

On the one hand, for any y, we have for some constant C depending only on d

$$\mu\left(c_y(\eta)(1-\eta_y)\right) \leq Cq^{k+1}, \tag{55}$$

since the constraint requires at least k zeros to be satisfied, and c_y is independent from η_y. On the other hand, if $0 \leftrightarrow \partial B_n$, there is a self-avoiding walk of length n starting at 0 which is empty. So a rough bound on the number of self-avoiding walks of length n yields

$$\mu\left(0 \leftrightarrow \partial B_n,\ 0 \nleftrightarrow \partial B_{n+1}\right) \leq (2d)^n q^n. \tag{56}$$

Putting together (55) and (56), we get

$$S_0 \ \leq \ \sum_{n\geq 0}(2n+1)^{3d}\left[Cq^{k+1} \wedge (2dq)^n\right] \leq C'q^{k+1} \tag{57}$$

for q small enough. So we have proved (50).

A general argument allows to retrieve the upper bound in Theorem 4.1 for any $u \in \mathbb{R}^d$ from the result for $e_1, ..., e_d$. Write $u = \sum_{i=1}^d u_i e_i$ and compute

$$u.Du \ = \ \sum_{i=1}^d u_i^2 e_i.De_i + \sum_{i\neq j} u_i u_j e_i.De_j \tag{58}$$

Notice that D is symmetric and positive (by Proposition 3.4), so that the application $(u,v) \mapsto u.Dv$ is a scalar product. We can therefore apply Cauchy-Schwarz inequality to the terms $e_i.De_j$ and get

$$u.Du \ \leq \ Cq^{k+1}\left(\sum_{i=1}^d |u_i|\right)^2 \leq C'q^{k+1}, \tag{59}$$

where C' depends only on d by equivalence of the norms in finite dimension.

5. In the East model, $D \approx$ gap. In this section, the environment process is given by the East model, a one-dimensional KCSM for which the constraint is that the East neighbour of the to-be-updated site be vacant. The corresponding generator is

$$\mathcal{L}_E f(\omega) = \sum_{y\in\mathbb{Z}}(1-\omega_{y+1})((1-q)(1-\omega_y)+q\omega_y)\left[f(\omega^y)-f(\omega)\right]. \tag{60}$$

Before getting into the results concerning the tracer, let us recall briefly the definition and basic property of the so-called distinguished zero, a very useful tool for the study of the East model, which was introduced in Aldous and Diaconis (2002).

Definition 5.1 *Consider $\omega \in \Omega$ a configuration with $\omega_x = 0$ for some $x \in \mathbb{Z}$. Define $\xi(0) = x$. Call $T_1 = \inf\{t \geq 0 \mid$ the clock in x rings and $\omega_{x+1}(t) = 0\}$, the time of the first legal ring at x. Let $\xi(s) = x$ for $s < T_1$, $\xi(T_1) = x+1$ and start again to define recursively $(\xi(s))_{s\geq 0}$.*

Notice that for any $s \geq 0$, $\omega_{\xi(s)}(s) = 0$, and that $\xi : \mathbb{R}^+ \to \mathbb{Z}$ is almost surely càdlàg and increasing by jumps of 1.

This distinguished zero has an important property: as it moves forward, it leaves equilibrium on its left (see Lemma 4 in Aldous and Diaconis (2002) or Lemma 3.5 of Cancrini et al. (2010)). In particular, if ω is such that $\omega_x = 0$ and A an event depending only on the configuration restricted to $[x_-, x_+]$, with $x_+ < x$, letting $V = \{x_-, ..., x-1\}$, then we have the following estimate

$$\mathbb{P}_\omega\left(\omega(t) \in A\right) \leq \mu_V(\omega_{|V})^{-1}\mathbb{P}_{\mu_V\cdot\omega}\left(\omega(t) \in A\right) = \mu(\omega_{|V})^{-1}\mu(A), \tag{61}$$

where μ_V is the Bernoulli$(1-q)$ product measure on $\{0,1\}^V$, $\mu_V \cdot \omega$ denotes the law of a random configuration equal to ω on $\mathbb{Z}\backslash V$ and chosen with law μ_V on V. In the above estimate, the factor $\mu_V(\omega_{|V})$ comes from a change of measure to start from μ in V, and the last equality comes from the property of the distinguished zero mentioned above.

For briefness, in this section, we will denote the spectral gap of the East process by gap (see 19).

Theorem 5.2 *When the environment is given by the East model, there exist constants $C, c > 0$ and α such that*

$$cq^2\text{gap} \leq D \leq Cq^{-\alpha}\text{gap}. \tag{62}$$

Remark 5.3 *In Aldous and Diaconis (2002) and Cancrini et al. (2008), it was established that*

$$\lim_{q\to 0} \frac{\log(1/\text{gap})}{(\log(1/q))^2} = (2\log 2)^{-1}. \tag{63}$$

In particular, this means that the powers of q appearing in (62) are merely corrections to the correct asymptotic for D, which is governed by the spectral gap of the East model. (62) is therefore incompatible with the prediction in Jung, Garrahan and Chandler (2004) that $D \approx \text{gap}^{\xi}$ for some $\xi < 1$.

Proof of Theorem 5.2

The first inequality was already contained in Proposition 3.4.

For the proof of the second inequality, fix $t > 0$ and $\tau \ll t$ to be chosen later, such that t/τ is an integer and $\tau \lesssim \text{gap}^{-1}$ (more precisely, $\tau = q^{\beta}\text{gap}^{-1}$).

Then we can write

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\left(\sum_{k=1}^{t/\tau} X_{k\tau} - X_{(k-1)\tau}\right)^2\right] \\
&= \sum_{k=1}^{t/\tau} \mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)^2\right] + \sum_{k\neq k'} \mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)\left(X_{k'\tau} - X_{(k'-1)\tau}\right)\right] \\
&= \frac{t}{\tau}\mathbb{E}\left[X_\tau^2\right] + \sum_{k\neq k'} \mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)\left(X_{k'\tau} - X_{(k'-1)\tau}\right)\right] \qquad (64)
\end{aligned}$$

We need to show that (64) is smaller that $tq^{-\alpha}\text{gap}$ for some α when τ is well chosen. We are going to bound the first term using the fact that energy barriers make it very costly to cross a distance greater than $1/q$ in time $\tau \lesssim \text{gap}^{-1}$. To bound the second term, we use the symmetry of the model and the fact that the process seen from the tracer has a positive spectral gap.

Proposition 5.4 *There exists $\beta, C < \infty$ such that, if $\tau = q^{\beta}\text{gap}^{-1}$,*

$$\mathbb{E}\left[X_\tau^2\right] \leq Cq^{-C}. \tag{65}$$

First we need two lemmas, that rely on precise estimates on the spectral gap of the East model on lengths of order at most $1/q$, and related energy barriers, that have been established in Chleboun, Faggionato and Martinelli (2012). We start by showing a precise comparison between the relaxation

time in infinite volume and the relaxation time in volume $1/q$. Recall that it was shown in Cancrini et al. (2008) that for any $\delta > 0$

$$\text{gap}^{-1} \leq C_\delta \left(\frac{1}{q}\right)^{\log_2(1/q)/(2-\delta)}. \tag{66}$$

Lemma 5.5 *Let $n = \lceil \log_2(1/q) \rceil$ and $T_{rel}(L)$ be the relaxation time of the East model on length L with empty boundary condition. Then there exist finite constants C, C' such that*

$$\text{gap}^{-1} \leq C q^{-C} T_{rel}(1/q) \leq C' q^{-C'} \frac{n!}{q^n 2^{\binom{n}{2}}}. \tag{67}$$

Proof

The second inequality follows immediately from Theorem 2 in Chleboun, Faggionato and Martinelli (2012). To prove the first one, we refine the bisection technique used in Cancrini et al. (2008) to prove (66). Let $\delta(q) = 10/\log(1/q)$, $l_k = 2^k$, $\delta_k = \left\lfloor l_k^{1-\delta/2} \right\rfloor$, $s_k = \left\lfloor l_k^{\delta/6} \right\rfloor$. These are the same definitions as in Cancrini et al. (2008), except that instead of a fixed $\delta > 0$, we take δ to 0 with q. With these definitions, we have for every $k \geq k_\delta := 6/\delta$ the following estimate[1] (see (6.3) in Cancrini et al. (2008))

$$\text{gap}^{-1} \leq T_{\text{rel}}(l_k + l_k^{1-\delta/6}) \prod_{j=k}^{\infty} \left(\frac{1}{1 - p^{\delta_j/2}}\right) \prod_{j=k}^{\infty} \left(1 + s_j^{-1}\right). \tag{68}$$

As in Cancrini et al. (2008), let

$$j_* = \min\left\{ j \mid p^{\delta_j/2} \leq e^{-1} \right\} \approx \log_2(1/q)/(1 - \delta/2). \tag{69}$$

As long as $j_* \geq k_\delta$, which is true thanks to our choice of δ, we can replace k by j_* in (68). Now we have (see the computations in Cancrini et al. (2008), top of page 484 for the first estimate)

$$\prod_{j=j_*}^{\infty} \left(\frac{1}{1 - p^{\delta_j/2}}\right) \leq C \tag{70}$$

$$\prod_{j=j_*}^{\infty} \left(1 + s_j^{-1}\right) \leq q^{-C}, \tag{71}$$

for C some constant not depending on q. Noticing that $l_{j_*} + l_{j_*}^{1-\delta/6} \leq d/q$ for some constant d, we get

$$\text{gap}^{-1} \leq C q^{-C} T_{\text{rel}}(d/q). \tag{72}$$

Now it is enough to recall Theorem 4 in Chleboun, Faggionato and Martinelli (2012), that states that there is no time scale separation on scale $1/q$

$$T_{\text{rel}}(d/q) \sim T_{\text{rel}}(1/q) \tag{73}$$

✓

Now we can use Lemma 5.5 to prove the following estimate, which basically means that in times smaller than gap^{-1}, it will be extremely difficult for the system to erase a row of $1/q$ ones.

[1] This condition is not necessary, but sufficient; it comes from the fact that Lemma 4.2 in Cancrini et al. (2008) has to be satisfied in order to apply the bisection technique.

Lemma 5.6 *Recall that $\tau = q^{\beta}\mathrm{gap}^{-1}$. Let $l = 1/q$ and $\mathbb{P}_{\mathbf{1}0}(\cdot)$ denote (abusively) the law of the East process starting from a configuration equal to one on $\{1, ..., l\}$, with a zero in $l+1$. Let T_0 be the first time there is a zero at 1. Independently of the choice of the initial configuration outside $\{1, ..., l, l+1\}$, we have, if β is large enough (independently of q)*

$$\mathbb{P}_{\mathbf{1}0}(T_0 \leq \tau) \leq Cq. \tag{74}$$

Proof of Lemma 5.6

In Chleboun, Faggionato and Martinelli (2012)[2], the authors define a certain set ∂A_* of configurations in $\{0,1\}^l$ that has two interesting properties (it is defined in paragraph 5.2.1 of Chleboun, Faggionato and Martinelli (2012), the properties below are stated in Remark 5.8 and Corollary 5.10)

- Starting from a configuration equal to one on $\{1, ..., l\}$, with a zero in 0, in order to put a one in 0 before time τ, the dynamics restricted to $\{1, ..., l\}$ has to go through the set ∂A_* at some time $s \leq \tau$.
- For some $\alpha' < \infty$, if $n = \lceil \log_2 l \rceil$

$$\mu(\partial A_*) \leq \frac{q^n 2^{\binom{n}{2}}}{n!} q^{-\alpha'}. \tag{75}$$

Put another way, ∂A_* is a bottleneck separating the events $\{\eta_0 = \eta_{l+1} = 0,\ \eta_1 = ... = \eta_l = 1\}$ and $\{\eta_0 = 1\}$ in the East dynamics.

Call τ_0 the first time there is a one in 0. Denote (abusively) by $0\mathbf{1}0$ any configuration equal to zero in 0 and $l+1$, and to one on $\{1, ..., l\}$, by T an exponential variable of parameter 2 independent of T_0, and by τ_0 the first time at which there is a one in position 0. Notice that, once there is a zero in 1, if the clock attached to site 0 rings before that attached to 1, and if the associated Bernoulli variable is a one, then the configuration at site 0 takes value one. So that

$$\frac{1-q}{2}\mathbb{P}_{\mathbf{1}0}(T_0 + T \leq \tau + 1/2) \leq \mathbb{P}_{0\mathbf{1}0}(\tau_0 \leq \tau + 1/2), \tag{76}$$

where $\mathbf{1}0$ and $0\mathbf{1}0$ are equal except maybe in 0. The constant 1/2 appears to allow the following estimate

$$\mathbb{P}_{\mathbf{1}0}(T_0 + T \leq \tau + 1/2) \geq \mathbb{P}_{\mathbf{1}0}(T_0 \leq \tau)\,\mathbb{P}(T \leq 1/2) = (1 - e^{-1})\mathbb{P}_{\mathbf{1}0}(T_0 \leq \tau). \tag{77}$$

(76) and (77) yield

$$\mathbb{P}_{\mathbf{1}0}(T_0 \leq \tau) \leq \frac{2}{(1-q)(1-e^{-1})}\mathbb{P}_{0\mathbf{1}0}(\tau_0 \leq \tau + 1/2). \tag{78}$$

Now we use the first property of ∂A_* to get

$$\mathbb{P}_{0\mathbf{1}0}(\tau_0 \leq \tau + 1/2) \quad \leq \quad \mathbb{P}_{\mathbf{1}0}\Big(\exists\ s \leq \tau + 1/2 \quad \text{s.t.}\ (\omega(s))_{[1,l]} \in \partial A_*\Big) \tag{79}$$

[2]Note that the orientation convention is reversed in that paper: contrary to here, the constraint that has to be satisfied to update x is that $x-1$ should be empty.

To evaluate the r.h.s., we condition on $N_{\tau+1/2}$ the number of rings occurring in $[1,l]$ before time $\tau+1/2$ in the graphical construction with a union bound to get

$$
\begin{aligned}
(80)\quad \mathbb{P}_{\mathbf{1}0}\left(\exists\, s\le \tau+1/2 \quad \text{s.t. } (\omega(s))_{[1,l]}\in\partial A_*\right) &\le \mathbb{E}\left[N_{\tau+1/2}\right]\sup_{s\le\tau+1/2}\mathbb{P}_{\mathbf{1}0}\left((\omega(s))_{[1,l]}\in\partial A_*\right)\\
(81)\qquad &\le (\tau+1/2)\,l\sum_{\sigma\in\partial A_*}\sup_{s\le\tau+1/2}\mathbb{P}_{\mathbf{1}0}\left((\omega(s))_{[1,l]}=\sigma\right)\\
(82)\qquad &\le (\tau+1/2)\,l\sum_{\sigma\in\partial A_*}p^{-l}\mu(\sigma)\\
&\le (\tau+1/2)\,l(1-q)^{-l}\mu(\partial A_*)\\
(83)\qquad &\le (\tau+1/2)\,(1-q)^{-l}\frac{q^n2^{\binom{n}{2}}}{n!}q^{-(\alpha'+1)},
\end{aligned}
$$

where we used (61) with the distinguished zero starting at $l+1$ to get the third inequality, and the second property of ∂A_* (75) to get the last one. Now collect (78), (79) and (83) to get

$$
(84)\qquad \mathbb{P}_{\mathbf{1}0}\left(T_0\le\tau\right)\le\frac{2}{(1-q)(1-e^{-1})}\,(\tau+1/2)\,(1-q)^{-l}\frac{q^n2^{\binom{n}{2}}}{n!}q^{-(\alpha'+1)}.
$$

For q small enough and $\tau=q^{\beta}\text{gap}^{-1}$, $\tau+1/2\le q^{\beta-1}\text{gap}^{-1}$, so that Lemma 5.5 yields

$$
(85)\qquad \mathbb{P}_{\mathbf{1}0}\left(T_0\le\tau\right)\le Cq^{-\alpha''}q^{\beta-1},
$$

for some C,α'' independent of q.

✓

Proof of Proposition 5.4

First of all, let us reformulate what we want to show.

$$
\begin{aligned}
\mathbb{E}\left[X_\tau^2\right] &= \sum_{x=1}^{\infty}(2x-1)\mathbb{P}\left(|X_\tau|\ge x\right)\\
&= 2\sum_{x=1}^{\infty}(2x-1)\mathbb{P}\left(X_\tau\ge x\right)\\
(86)\qquad &\le 4\sum_{m=1}^{\infty}q^{-m}\mathbb{P}\left(X_\tau\ge q^{-m}\right)
\end{aligned}
$$

In light of Lemma 5.6, we can now notice that in order to have $X_\tau\ge q^{-m}$ for $m\ge 2$, the system will have to overcome a large number of energy barriers (i.e. rows of ones of length larger than $1/q$), so that the probability of this event will become very small.

Fix $m>2$, and let us study $\mathbb{P}\left(X_\tau\ge q^{-m}\right)$. Throughout the proof, to simplify the notations, if $C(q)$ is a quantity going to infinity when $q\to 0$, we will not make the distinction between $C(q)$ and $\lfloor C(q)\rfloor$. We divide $\{0,...,q^{-m}\}$ into $q^{-m+2}(3m)^{-1}$ groups of $3m$ blocks of length q^{-2}. Given a configuration, we say that a block of q^{-2} sites is well-behaved if we can find a row of consecutive ones of length at least $1/q$ that ends with a zero inside it. We can estimate the probability of a block having this property by

$$
(87)\qquad \mu(\text{a given block is not well-behaved})\le\left(1-q(1-q)^{1/q}\right)^{1/q}\le c<1
$$

for some constant c.

Let A be the event that in all of these $q^{-m+2}(3m)^{-1}$ groups of blocks, there is one of the $3m$ blocks that is not well-behaved. With this definition, on A^c, there is a group of $3m$ well-behaved blocks. Let us estimate the probability of A under μ using (87)

$$\begin{aligned}
\mu(A) &\leq \left(1-\mu(\text{a given block is well-behaved})^{3m}\right)^{q^{-m+2}(3m)^{-1}} \\
&\leq (1-(1-c)^{3m})^{q^{-m+2}(3m)^{-1}} \qquad (88)\\
&\leq e^{-Cq^2q^{-\gamma m}}, \qquad (89)
\end{aligned}$$

with $\gamma > 0$, $C < \infty$.

So we can write

$$\mathbb{P}\left(X_\tau \geq q^{-m}\right) \leq \mu(A) + \mu\left(\mathbf{1}_{A^c}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right). \qquad (90)$$

Denote by B_1 the first block of length q^{-2}, $B_2 = B_1 + q^{-2}$,..., $B_{3m} = B_1 + (3m-1)\,q^{-2}$. We have the following estimate

$$\mu\left(\mathbf{1}_{A^c}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right) \leq q^{-m+2}(3m)^{-1}\mu\left(\prod_{i=1}^{3m}\mathbf{1}_{B_i \text{ well-behaved}}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right). \qquad (91)$$

Let η be a configuration in which all the B_i are well-behaved. Let x_i be the starting point of the first row of $1/q$ ones ended by a zero in B_i, and T_i the first time this site is empty. We denote by $(\xi_i(s))_{s\leq\tau}$ the trajectory of the distinguished zero started from the position of the zero at the end of the row of ones starting at x_i, up to time τ.

$$\begin{aligned}
\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right) &\leq \mathbb{P}_\eta\left(\forall i = 1,...,3m \quad T_i \leq \tau\right) \\
&\leq \mathbb{P}_\eta\left(T_{3m} \leq \tau\right)\mathbb{P}_\eta\left(\forall i = 1,...,3m-1 \quad T_i \leq \tau \mid T_{3m} \leq \tau\right) \\
&\leq \mathbb{P}_\eta\left(T_{3m} \leq \tau\right)\mathbb{E}_\eta\left[\mathbb{P}_\eta\left(\forall i = 1,...3m-1\ T_i \leq \tau \mid \left(\xi_{3m-1}(s)\right)_{s\leq\tau}\right) \mid T_{3m} \leq \tau\right], \qquad (92)
\end{aligned}$$

since the dynamics on the left of $x_{3m-1} + 1/q$ knowing $(\xi_{3m-1}(s))_{s\leq\tau}$ does not depend on what happens on the right of $(\xi_{3m-1}(s))_{s\leq\tau}$.

Let us show iteratively that, uniformly in the trajectory $(\xi_k(s))_{s\leq\tau}$,

$$\mathbb{P}_\eta\left(\forall i = 1,...k\ T_i \leq \tau \mid \left(\xi_k(s)\right)_{s\leq\tau}\right) \leq (Cq)^k. \qquad (93)$$

For $k = 1$, *mutatis mutandis*, the proof of Lemma 5.6 applies. Let $k > 1$. $\mathbb{P}_\eta\left(\forall i = 1,...k\ T_i \leq \tau \mid \left(\xi_k(s)\right)_{s\leq\tau}\right)$ is also

$$\mathbb{P}_\eta\left(T_k \leq \tau \mid \left(\xi_k(s)\right)_{s\leq\tau}\right)\mathbb{P}_\eta\left(\forall i = 1,...k-1\ T_i \leq \tau \mid \left(\xi_k(s)\right)_{s\leq\tau}, T_k \leq \tau\right), \qquad (94)$$

which can be rewritten

(95)

$$\mathbb{P}_\eta\left(T_k \leq \tau \mid \left(\xi_k(s)\right)_{s\leq\tau}\right)\mathbb{E}_\eta\left[\mathbb{P}_\eta\left(\forall i = 1,...k-1\ T_i \leq \tau \mid \left(\xi_{k-1}(s)\right)_{s\leq\tau}\right) \mid \left(\xi_k(s)\right)_{s\leq\tau}, T_k \leq \tau\right],$$

and the induction hypothesis applies.

Putting together (92), (93) and (91), we get for some constant C

$$\mu\left(\mathbf{1}_{A^c}(\eta)\mathbb{P}_\eta\left(X_\tau \geq q^{-m}\right)\right) \leq (Cq)^{2m}. \tag{96}$$

Recalling (90), (89) and (86), we get Proposition 5.4. ✓

What now remains is to show there is enough decorrelation to bound the second sum in (64). This is not difficult, once we make the following remark.

Lemma 5.7 *Denote by* gap_T *the spectral gap of the process seen from the tracer (recall* (8)*)*

$$\mathrm{gap}_T = \inf \frac{-\mu(f\mathcal{L}f)}{Var_\mu(f)}, \tag{97}$$

where the infimum is taken over non-constant functions $f \in L^2(\mu)$*. Then we have*

$$\mathrm{gap}_T \geq \mathrm{gap}. \tag{98}$$

Proof

This follows directly from (18) and the definition of gap and gap_T (recall (19)). ✓

Now we are armed to study the terms $\mathbb{E}\left[\left(X_{k\tau} - X_{(k-1)\tau}\right)\left(X_{k'\tau} - X_{(k'-1)\tau}\right)\right]$. First of all, by stationarity, this quantity depends only on τ and $|k-k'|$. So we only need to study $\mathbb{E}\left[(X_\tau)\left(X_{k\tau} - X_{(k-1)\tau}\right)\right]$ for $k \geq 2$. In fact, using Cauchy-Schwarz inequality and Proposition 5.4, we only need to study this term for $k \geq 3$, which allows some decorrelation to take place between times τ and $(k-1)\tau$. Let us denote by $\left(P_s^T\right)_{s\geq 0}$ the semigroup associated to $\mathcal{L}$. $\mathbb{E}_{(\omega,x)}[\cdot]$ will denote the law of the process with generator $\mathcal{L}_0$ starting from the configuration ω with the tracer in position x ($\mathbb{E}[\cdot]$ is still the law of the process starting from μ and the tracer at the origin). Using successively the Markov property at time τ, we can write

$$\mathbb{E}\left[(X_\tau)\left(X_{k\tau} - X_{(k-1)\tau}\right)\right] = \mathbb{E}\left[X_\tau \mathbb{E}_{(\omega(\tau),X_\tau)}\left[X'_{(k-1)\tau} - X'_{(k-2)\tau}\right]\right], \tag{99}$$

where $(X'_s)_{s\geq 0}$ denotes the trajectory of the tracer under the law $\mathbb{E}_{(\omega(\tau),X_\tau)}[\cdot]$. Now we use successively Cauchy-Schwarz inequality and stationarity of the process seen from the tracer to get

$$\mathbb{E}\left[(X_\tau)\left(X_{k\tau} - X_{(k-1)\tau}\right)\right]^2 \leq \mathbb{E}\left[X_\tau^2\right]\mathbb{E}\left[\mathbb{E}_{(\omega(\tau),X_\tau)}\left[X_{(k-1)\tau} - X_{(k-2)\tau}\right]^2\right] \tag{100}$$

$$\leq \mathbb{E}\left[X_\tau^2\right]\mathbb{E}\left[\mathbb{E}_{\left((\omega(\tau))_{X_\tau+\cdot},0\right)}\left[X_{(k-1)\tau} - X_{(k-2)\tau}\right]^2\right] \tag{101}$$

$$\leq \mathbb{E}\left[X_\tau^2\right]\mu\left(\mathbb{E}_{(\omega,0)}\left[X_{(k-1)\tau} - X_{(k-2)\tau}\right]^2\right), \tag{102}$$

Let us focus on $\mathbb{E}_{(\omega,0)}\left[X_{(k-1)\tau} - X_{(k-2)\tau}\right]$. Using the Markov property at time $(k-2)\tau$, we get

$$\begin{aligned}\mathbb{E}_{(\omega,0)}\left[X_{(k-1)\tau} - X_{(k-2)\tau}\right] &= \mathbb{E}_{(\omega,0)}\left[\mathbb{E}_{\left(\omega((k-2)\tau),X_{(k-2)\tau}\right)}\left[X'_\tau - X'_0\right]\right] \\ &= \mathbb{E}_{(\omega,0)}\left[\mathbb{E}_{\left((\omega((k-2)\tau))_{X_{(k-2)\tau}+\cdot},0\right)}\left[X'_\tau\right]\right] \\ &= P^T_{(k-2)\tau}g(\omega),\end{aligned} \tag{103}$$

where $g(\omega) = \mathbb{E}_{(\omega,0)}[X_\tau]$, and the X'_s in the first and second line denote respectively the trajectory of the tracer under the laws $\mathbb{E}_{\left(\omega((k-2)\tau),X_{(k-2)\tau}\right)}[\cdot]$ and $\mathbb{E}_{\left((\omega((k-2)\tau))_{X_{(k-2)\tau}+\cdot},0\right)}[\cdot]$. Therefore, using the

spectral gap inequality, and the fact that g is a mean-zero function in $L^2(\mu)$ thanks to stationarity and Proposition 5.4, we get

$$\begin{aligned}\mathbb{E}\left[(X_\tau)\left(X_{k\tau}-X_{(k-1)\tau}\right)\right]^2 &\leq \mathbb{E}\left[X_\tau^2\right]\mu\left(\left(P^T_{(k-2)\tau}g\right)^2\right)\\ &\leq \mathbb{E}\left[X_\tau^2\right]^2 e^{-2(k-2)\tau\mathrm{gap}_T}\\ &\leq \mathbb{E}\left[X_\tau^2\right]^2 e^{-2(k-2)q^\beta} \qquad (104)\end{aligned}$$

Since $\sum_{k\geq 1} e^{-kq^\beta} \lesssim q^{-\beta}$, the second term in (64) is

$$\sum_{k\neq k'} \mathbb{E}\left[\left(X_{k\tau}-X_{(k-1)\tau}\right)\left(X_{k'\tau}-X_{(k'-1)\tau}\right)\right] \leq C\left\lfloor t/\tau\right\rfloor \mathbb{E}\left[X_\tau^2\right] q^{-\beta}. \qquad (105)$$

Putting this into (64) together with Proposition 5.4, we get Theorem 5.2. ✓

APPENDIX A: AN ALTERNATIVE PROOF IN THE FA-1F MODEL

When the environment is given by the one-spin Fredrickson-Andersen model (FA-1f), in which $c_x(\eta) = 1 - \prod_{i=1}^d \eta_{e_i}\eta_{-e_i}$ (the constraint requires at least one nearest neighbour to be empty), the diffusion coefficient at low density is of order q^2. This means that in this particular case, the correct order is already given by the first term in (10), which allows to design another strategy to find the lower bound in Theorem 4.1 when $k = 1$. Since the diffusion coefficient is of order lower than q^2 in the k-zeros model with $k > 1$, this technique does not apply. For simplicity, we write the proof in dimension $d = 1$.

We follow the strategy devised to prove Lemma 6.25 in Komorowski, Landim and Olla (2012), i.e. we prove that

$$\sup\left\{2\mu(jf) - \mathcal{D}(f)\right\} \leq cq^2, \qquad (106)$$

where $c < 1$ does not depend on q and $\mathcal{D}(f) = -\mu\left(f\mathcal{L}f\right)$. Seeing (10) and (16), this is sufficient to prove Theorem 4.1 when $k = 1$, $d = 1$. To obtain that result, we define

$$\mathcal{D}_{jump}(f) = \frac{1}{2}\mu\left((1-\eta_0)(1-\eta_{\alpha e_i})\left[f(\eta_{\alpha e_i+\cdot}) - f(\eta)\right]^2\right) \qquad (107)$$

$$\mathcal{D}_{FA}(f) = \frac{1}{2}\sum_{y\in\mathbb{Z}}\mu\left(c_y(\eta)((1-q)(1-\eta_y)+q\eta_y)\left[f(\eta^y) - f(\eta)\right]^2\right), \qquad (108)$$

so that $\mathcal{D}(f) = \mathcal{D}_{jump}(f) + \mathcal{D}_{FA}(f)$, and we show separately that for all f

$$2\mu(jf) - \mathcal{D}_{jump}(f) \leq q^2 \qquad (109)$$

$$2\mu(jf) - \mathcal{D}_{FA}(f) \leq Cq^2, \qquad (110)$$

where $C \geq 1$ is a constant that does not depend on q. To get the result from (109) and (110), we write that for any $\lambda > 0$, for any local function f

$$\lambda^{-1}\left(2\mu(jf) - \mathcal{D}_{jump}(f) - \mathcal{D}_{FA}(f)\right) = 2\mu\left(j\lambda^{-1}f\right) - \lambda\mathcal{D}_{jump}\left(\lambda^{-1}f\right) - \lambda\mathcal{D}_{FA}\left(\lambda^{-1}f\right)$$

So that

$$\lambda^{-1}\sup\left\{2\mu(jf) - \mathcal{D}_{jump}(f) - \mathcal{D}_{FA}(f)\right\} \leq \sup\left\{2\mu\left(jg\right) - \lambda\mathcal{D}_{jump}\left(g\right) - \lambda\mathcal{D}_{FA}\left(g\right)\right\}$$

Take for instance $\lambda = C/(C+1)$. We have $\lambda \geq 1-\lambda$, so that

$$\begin{aligned}\lambda^{-1}\sup\left\{2\mu(jf) - \mathcal{D}_{jump}(f) - \mathcal{D}_{FA}(f)\right\} &\leq \sup\left\{2\mu\left(jg\right) - \lambda\mathcal{D}_{jump}\left(g\right) - (1-\lambda)\mathcal{D}_{FA}\left(g\right)\right\} \\ &\leq \left[\lambda + (1-\lambda)C\right]q^2 = q^2,\end{aligned}$$

using (109) and (110), so that (106) is proven.

1. Proof of (109).
For any local function f, we can rewrite $\mu(jf)$ in terms of the "jumps" $\eta \to \eta_{1+\cdot}$ and $\eta \to \eta_{-1+}$

$$2\mu(jf) = -\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta_{1+\cdot}) - f(\eta)\right]\right) + \mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta_{-1+\cdot}) - f(\eta)\right]\right)$$

Now using the inequality $ab \leq (a^2+b^2)/2$, the Dirichlet form $\mathcal{D}_{jump}(f)$ appears in the r.h.s.

$$\begin{aligned}2\mu(jf) &\leq q^2 + \frac{1}{2}\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta_{1+\cdot}) - f(\eta)\right]^2\right) + \frac{1}{2}\mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta_{-1+\cdot}) - f(\eta)\right]^2\right) \\ &\leq q^2 + \mathcal{D}_{jump}(f)\end{aligned}$$

2. Proof of (110).
We need only to prove it for small q. First we make a few computations to express $\mu(jf)$ in terms of allowed flips ($\eta \to \eta^1$ or $\eta \to \eta^{-1}$). Then we use the same optimization technique performed in the proof of Lemma 6.13 in Komorowski, Landim and Olla (2012) to get the desired bound. We have the following equalities

$$\begin{aligned}\mu((1-\eta_0)(1-\eta_1)f(\eta)) &= \frac{q}{1-2q}\mu\left((1-\eta_0)\left[f(\eta^1) - f(\eta)\right]\right) + \frac{q}{1-q}\mu\left((1-\eta_0)\eta_1 f(\eta)\right), \quad (111)\\ \mu((1-\eta_0)(1-\eta_{-1})f(\eta)) &= \frac{q}{1-2q}\mu\left((1-\eta_0)\left[f(\eta^{-1}) - f(\eta)\right]\right) + \frac{q}{1-q}\mu\left((1-\eta_0)\eta_{-1} f(\eta)\right), \quad (112)\\ \mu\left((1-\eta_0)\eta_1 f(\eta)\right) &= (1-q)\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1) - f(\eta)\right]\right) + (1-q)\mu\left((1-\eta_0)f(\eta)\right), \quad (113)\\ \mu\left((1-\eta_0)\eta_{-1} f(\eta)\right) &= (1-q)\mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta^{-1}) - f(\eta)\right]\right) + (1-q)\mu\left((1-\eta_0)f(\eta)\right) \quad (114)\end{aligned}$$

So that, computing differences, we get

$$\begin{aligned}\tilde{\mu}(jf) &= \frac{q}{p-q}\left[\mu\left((1-\eta_0)\left[f(\eta^1) - f(\eta)\right]\right) - \mu\left((1-\eta_0)\left[f(\eta^{-1}) - f(\eta)\right]\right)\right] \quad (115)\\ &\quad + q\left[\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1) - f(\eta)\right]\right) - \mu\left((1-\eta_0)(1-\eta^{-1})\left[f(\eta^{-1}) - f(\eta)\right]\right)\right]\end{aligned}$$

Assume $q < 1/2$. Using the inequality $ab \leq (a^2+b^2)/2$, we get for any $\alpha, \beta > 0$

$$\begin{aligned}\frac{\mu(jf)}{q} &\leq \frac{1}{1-2q}\left\{\alpha q + \frac{1}{2\alpha}\left[\mu\left((1-\eta_0)\left[f(\eta^1) - f(\eta)\right]^2\right) + \mu\left((1-\eta_0)\left[f(\eta^{-1}) - f(\eta)\right]^2\right)\right]\right\} + \beta \\ &\quad + \frac{1}{2\beta}\left[\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1) - f(\eta)\right]^2\right) + \mu\left((1-\eta_0)(1-\eta_{-1})\left[f(\eta^{-1}) - f(\eta)\right]^2\right)\right]\end{aligned}$$

We insert the missing rates to recover terms appearing in $\mathcal{D}_{FA}(f)$. For instance, since we assumed $q < 1/2$

$$\mu\left((1-\eta_0)\left[f(\eta^1) - f(\eta)\right]^2\right) \leq \frac{1}{q}\mu\left((1-\eta_0)((1-q)(1-\eta_1) + q\eta_1)\left[f(\eta^1) - f(\eta)\right]^2\right) \quad (116)$$

and

$$\mu\left((1-\eta_0)(1-\eta_1)\left[f(\eta^1) - f(\eta)\right]^2\right) \leq \frac{1}{p}\mu\left((1-\eta_0)((1-q)(1-\eta_1) + q\eta_1)\left[f(\eta^1) - f(\eta)\right]^2\right) \quad (117)$$

So that we get

$$\mu(jf) \leq \frac{q}{1-2q}\left\{\alpha q + \frac{1}{\alpha q}\mathcal{D}_{FA}(f)\right\} + q\left\{\beta + \frac{1}{\beta(1-q)}\mathcal{D}_{FA}(f)\right\} \tag{118}$$

Optimizing in α, β, this yields

$$\mu(jf) \leq \frac{2q}{1-2q}\sqrt{\mathcal{D}_{FA}(f)} + 2\sqrt{\mathcal{D}_{FA}(f)/(1-q)} \tag{119}$$

This is enough to prove (110) for small q (see section 6.3 of Komorowski, Landim and Olla (2012)).

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