EXPLICIT SOLUTION OF AN INVERSE FIRST-PASSAGE TIME PROBLEM FOR LÉVY PROCESSES AND COUNTERPARTY CREDIT RISK

M.H.A. DAVIS AND M.R. PISTORIUS

Abstract. For a given Markov process X and survival function $\overline{H}$ on $\mathbb{R}^+$, the *inverse first-passage time problem* (IFPT) is to find a barrier function $b : \mathbb{R}^+ \to [-\infty, +\infty]$ such that the survival function of the first-passage time $\tau_b = \inf\{t \geq 0 : X(t) \leq b(t)\}$ is given by $\overline{H}$. In this paper we consider a version of the IFPT problem where the barrier is *fixed at zero* and the problem is to find an initial distribution μ and a time-change I such that for the time-changed process $X \circ I$ the IFPT problem is solved by a constant barrier at the level zero. For any Lévy process X satisfying an exponential moment condition, we derive the solution of this problem in terms of λ-*invariant distributions* of the process X killed at the epoch of first entrance into the negative half-axis. We provide an explicit characterization of such distributions, which is a result of independent interest. For a given multi-variate survival function $\overline{H}$ of generalized frailty type we construct subsequently an explicit solution to the corresponding IFPT with the barrier level fixed at zero. We apply these results to the valuation of financial contracts that are subject to counterparty credit risk.

1. Introduction

Financial models incorporating the idea that a firm defaults on its debt when the value of the debt exceeds the value of the firm were originally introduced by Merton [32]. Black & Cox [6] extended the Merton model by modelling the time of default as the first time that the value of the firm less the value of its debt becomes negative. Because 'firm value' cannot be directly measured, later contributors such as Longstaff & Schwartz [31] and Hull & White [19] have moved to stylized models in which default occurs when some process $Y(t)$ – interpreted as 'distance to default' – crosses a given, generally time-varying, barrier $b(t)$. The risk-neutral distribution of the default time can be inferred from the firm's credit default swap spreads, and Hull & White [19] provide a numerical algorithm to determine $b(t)$ such that the first hitting time distribution H is equal to this market-implied default time distribution when $Y(t)$ is Brownian motion.

As we will show, these calculations are greatly simplified if, instead of starting at a fixed point $Y(0) = x > 0$ and calibrating the barrier $b(t)$ we fix the barrier at $b(t) \equiv 0$ and start Y at a random point $Y(0) = Y_0$, where Y_0 has a distribution function F on $\mathbb{R}^+$, to be chosen. If we combine this with a deterministic time change then it turns out that essentially any continuous distribution H can be realized in this way, often with closed-form expressions for F.

In precise terms, the *inverse first-passage time* (IFPT) problem may be described as follows. Let (Y, P^μ) be a real-valued Markov process with càdlàg[1] paths that has initial distribution μ on $\mathbb{R}^+\backslash\{0\}$ (*i.e.*, $P^\mu(Y_0 \in \mathrm{d}x) = \mu(\mathrm{d}x)$). Given a CDF H on $\mathbb{R}^+$, the IFPT for the process (Y, P^μ) is to find a barrier function $b : \mathbb{R}^+ \to [-\infty, +\infty]$ such that the first-passage time τ_b^Y of the process Y below the barrier b has CDF H:

$$P^\mu(\tau_b^Y \leq t) = H(t), \qquad t \in \mathbb{R}^+, \tag{1.1}$$
$$\text{with } \tau_b^Y = \inf\{t \in \mathbb{R}^+ : Y_t \in (-\infty, b(t))\}.$$

Recently there has been a renewed interest in the IFPT problem, in good part motivated by the above questions of credit risk modeling. Chen *et al.* [12] prove existence and uniqueness of the IFPT of an arbitrary continuous CDF

Date: this version: March 25, 2014.

2010 *Mathematics Subject Classification.* 60J75, 91G40, 91G80.

Key words and phrases. Inverse first-passage problem, Lévy process, quasi-invariant distribution, credit risk, counterparty risk, multi-variate first-passage times.

Acknowledgement. We thank two anonymous referees for their careful reading of the paper and their very constructive comments.

[1]càdlàg = right-continuous with left-limits

on $\mathbb{R}^+$ for a diffusion with smooth bounded coefficients and strictly positive volatility function. In [1, 18, 19, 37, 38] a number of methods have been developed to compute this boundary, which is in general non-linear. Zucca & Sacerdote [38] analyse a Monte Carlo approximation method and a method based on the discretization of the Volterra integral equation satisfied by the boundary, which was derived in Peskir [35], while related integral equations are studied in Jaimungal *et al.* [22]. Avellaneda & Zhu [1] derive a free boundary problem for the density of a diffusion killed upon first hitting the boundary, where the free boundary is the solution to the IFPT, and Cheng *et al.* [13] established the existence and uniqueness of a solution to this free-boundary problem. A related "smoothed" version of the IFPT problem is considered in Ettinger *et al.* [16]: for any prescribed life-time it is shown that there exists a unique continuously differentiable boundary for which a standard Brownian motion killed at a rate that is a given function of this boundary has the prescribed life-time.

In this paper we consider a related inverse problem where the barrier is *fixed* to be equal to zero, and the problem is to identify in a given family a stochastic process whose first-passage time below the level zero has the given probability distribution. For a given Markov process X, the class of stochastic processes that we consider consists of the collection $(P^\mu, X \circ I)$ that is obtained by time-changing X by a continuous increasing function I and by varying the initial distribution μ of X over the set of all probability measures on the positive half-line. Here $I : \mathbb{R}^+ \to [0, \infty]$ is a function that is continuous and increasing on its domain, *i.e.* at all t for which $I(t)$ is finite, and the time-changed process $X \circ I = \{(X \circ I)(t), t \in \mathbb{R}^+\}$ is defined by $(X \circ I)(t) = X(I(t))$ if $I(t)$ is finite, and by $\limsup_{t\to\infty} X(t)$ otherwise.

Definition 1.1. For a continuous CDF H on $\mathbb{R}^+$, the *randomized and time-changed inverse first-passage problem* (RIFPT) is to find a probability measure μ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and an increasing continuous function $I : \mathbb{R}^+ \to [0, \infty]$ such that for the time-changed process $Y = X \circ I$ the first-passage time into the negative half-line $(-\infty, 0)$ has CDF H:

$$P^\mu(\tau_0^Y \le t) = H(t), \qquad t \in \mathbb{R}^+, \tag{1.2}$$
$$\text{with } \tau_0^Y = \inf\{t \in \mathbb{R}^+ : Y_t \in (-\infty, 0)\}.$$

The fact that the boundary is constant and known is helpful for practical implementation of the model, *e.g.* in subsequent counterparty risk valuation computations and for the matching of model and market prices.

In this paper we concentrate on the case where X is a Lévy process satisfying an exponential moment condition. The class of Lévy processes has been extensively deployed in financial modeling; see the monograph Cont & Tankov [14]. For the general theory of Lévy processes we refer to the monographs Applebaum [3], Bertoin [5], Kyprianou [27] and Sato [36].

The key step is to determine, for some $\lambda \in \mathbb{R}_+$, a λ-invariant distribution for the process X killed at the first hitting time of 0, which is a result of independent interest; see Definition 2.4 below. If μ is λ-invariant then under P^μ the first-passage time τ_0^X is exponentially distributed with parameter λ, so (μ, I) with $I(t) = t$ solves the RIFPT problem when H is $\mathrm{Exp}(\lambda)$. The solution for other continuous distribution functions H is then obtained by an obvious deterministic time change.

The paper is structured as follows. In Section 2 we formulate the problem and state the main results for the RIFPT problem, Theorems 2.2 and 2.6. The proof of Theorem 2.2 is also given, together with an illustrative example where the Lévy process is drifting Brownian motion. In Section 3 a multi-dimensional version of the RIFPT theorem is stated for a specific class of multivariate default-time distributions; its proof follows quite easily given the results of Section 2. The proof of Theorem 2.6, which is presented in Section 5, involves the relationship between first-passage times and the so-called Wiener-Hopf factors; these matters are discussed in Section 4. In Section 6 the results of Theorem 2.6 are illustrated explicitly for the special case of *mixed-exponential Lévy processes.* The concluding Section 7 demonstrates the application of our results to a problem of counterparty risk valuation.

2. IFPT Problem formulation and main results

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered probability space with completed filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\ge 0}$, and X be an $\mathbf{F}$-Lévy process, i.e., an $\mathbf{F}$-adapted stochastic process with càdlàg paths that has stationary independent increments, with $X_0 = 0$ and the property that for each $s \le t < u$ the increment $X_u - X_t$ is independent of $\mathcal{F}_s$. Let $\{P_x, x \in \mathbb{R}\}$

be the family of probability measures corresponding to shifts of the Lévy process X by x and, more generally, denote by P^μ the family of measures with initial distribution (the distribution of X_0) equal to μ; thus $P_x = P^{\delta_x}$ where δ_x is the Dirac measure at x and $P = P_0$. To avoid degeneracies we exclude throughout the case that X has monotone paths. As standing notation we denote $X_*(t) = \inf_{s\le t} X(s)$ and $X^*(t) = \sup_{s\le t} X(s)$. Below we describe a solution to the RIFPT problem under the following conditions:

Assumption 2.1. The Gaussian coefficient σ^2 and Lévy measure ν of X satisfy at least one of the following conditions:

$$\text{(i) } \sigma^2 > 0, \qquad \text{(ii) } \nu(-1,1) = +\infty, \qquad \text{(iii) } \nu \text{ has no atoms and } S^\nu \cap (-\infty, 0) \neq \emptyset,$$

where S^ν denotes the support of ν.

When only Assumptions 2.1(iii) holds, the process X is of the form $X_t = \mathtt{d}t + \sum_{s\in(0,t]} \Delta X_s$, where $\Delta X_s = X_s - X_{s^-}$ denotes the jump-size of X at time s, for some constant $\mathtt{d}$, which is called the infinitesimal drift of X.

The first observation is that for any initial distribution there exists a unique time-change that solves the RIFPT problem. For a given probability measure μ on the positive real line, define the function $I_\mu : \mathbb{R}^+ \to [0,\infty]$ by

$$I_\mu(t) = \overline{F}_\mu^{-1}(\overline{H}(t)), \qquad t \in \mathbb{R}^+, \tag{2.1}$$

$$\text{with } \overline{F}_\mu^{-1}(x) = \inf\{t \in \mathbb{R}^+ : \overline{F}_\mu(t) < x\}, \tag{2.2}$$

where $\overline{H} = 1 - H$ and $\overline{F}_\mu$ denote the survival functions corresponding to the CDF H and to the CDF of the first-passage time τ_0^X of X into the negative half-line $(-\infty, 0)$ under the probability measure P^μ. Here and throughout this paper, we use the convention $\inf \emptyset = +\infty$.

Theorem 2.2. *Let H be a continuous CDF on $\mathbb{R}^+$, and let μ be a probability measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ with $\mu(\{0\}) = 0$. Assume Assumption 2.1 holds and that μ has no atoms if only Assumption 2.1(iii) is satisfied. Then the function I_μ defined in* (2.1) *is the unique time-change such that (μ, I_μ) is a solution of the RIFPT problem.*

For the proof, we need some properties of the distribution of the running infimum.

Lemma 2.3. (i) *If X satisfies Assumption 2.1(i) or (ii), the CDF of $X_*(t)$ is continuous, for any $t > 0$.*
(ii) *Alternatively, if only Assumption 2.1(iii) holds, then for any $t > 0$ the CDF of $X_*(t)$ is continuous on the set $\mathbb{R}_- \setminus \min\{\mathtt{d}t, 0\}$, with $\mathbb{R}_- = (-\infty, 0]$.*

The proof of Lemma 2.3(i) can be found in Sato [36, Lemma 49.3] and Pecherskii & Rogozin [34, Lemma 1], while Lemma 2.3(ii) follows by conditioning on the first jump of the process X.

Proof of Theorem 2.2. Denote by $\mathtt{c}$ the value 0 or $\max\{-\mathtt{d}, 0\}$ according to whether or not X satisfies at least one of the Assumptions 2.1(i) and (ii). The key observation in the proof is that for any $x > 0$ the map $t \mapsto P_x(\tau_0^X > t)$ is (a) strictly decreasing and (b) continuous at any t satisfying $\mathtt{c}t \neq x$. To verify claim (b) it suffices to show that $P_x(\tau_0^X = t)$ is zero for any non-negative t that is such that $\mathtt{c}t \neq x$. The latter follows as consequence of the bound $P_x(\tau_0^X = t) \le P_0(X_*(t) = -x)$ that holds for any strictly positive x and t, and the fact (from Lemma 2.3) that the CDF of $X_*(t)$ is continuous on $(-\infty, 0]\backslash\{-\mathtt{c}t\}$. To see that claim (a) is true, we observe that, by the Markov property, we have for strictly positive x, t and s

$$\begin{aligned} P_x\left(\tau_0^X > t\right) - P_x\left(\tau_0^X > t+s\right) &= P_x\left(\tau_0^X > t, \tau_0^X \le t+s\right) \\ &\ge E\left(\mathbf{1}_{\{X_*(t)>-x\}} P(X_*(s) < -x - z)|_{z=X_t}\right). \end{aligned} \tag{2.3}$$

Since for any strictly positive epoch s the random variable X_s has an infinitely divisible distribution and the support of an infinitely divisible distribution not corresponding to the sum of a subordinator and a deterministic drift is unbounded from below (*e.g.*, [36, Corollary 24.4]), it follows that under Assumptions 2.1 we have

$$P(X_*(s) < -x) \ge P(X_s < -x) > 0, \qquad s > 0, x \ge 0. \tag{2.4}$$

By combining (2.3) and (2.4) we have for any strictly positive x, t and s,

$$P_x\left(\tau_0^X > t\right) > P_x\left(\tau_0^X > t+s\right),$$

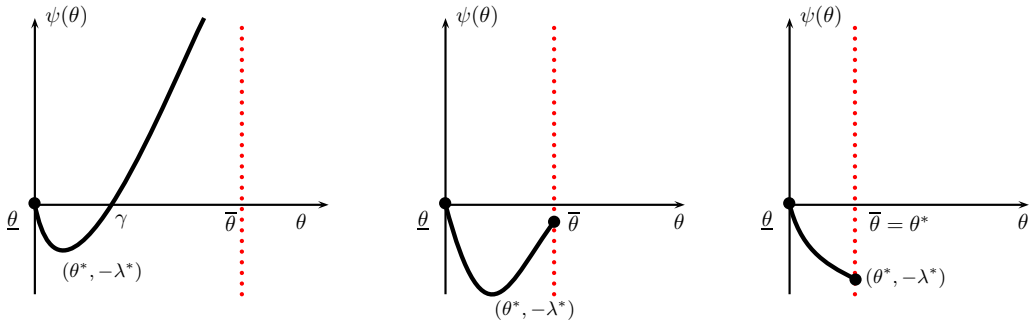

FIGURE 1. Three graphs of Laplace exponents ψ of Lévy processes satisfying Assumption 2.5, with $-\lambda^* = \min_{\theta\in[\underline{\theta},\overline{\theta}]} \psi(\theta) = \psi(\theta^*)$, where $[\underline{\theta},\overline{\theta}]$ denotes the closure of the domain of ψ. In the left-hand figure, γ denotes the largest root of the Cramér-Lundberg equation $\psi(\theta) = 0$ and $\theta^* < \overline{\theta}$ satisfies the equation $\psi'(\theta) = 0$. In the right-hand figure θ^* and $\overline{\theta}$ coincide.

and hence (b) holds true.

The above key observation in conjunction with Lebesgue's Dominated Convergence Theorem and the assumption that μ has no atoms if X does not satisfy Assumption 2.1(i) and (ii) imply that the map $t \mapsto \overline{F}_\mu(t)$ is continuous and strictly decreasing. Denote by Y^μ the time-changed process $X \circ I_\mu$. Since I_μ is monotone increasing and continuous, we have

$$P^\mu\left(\tau_0^{Y^\mu} \geq t\right) = P^\mu\left(\tau_0^X \geq I_\mu(t)\right) = \overline{F}_\mu\left(\overline{F}_\mu^{-1}(\overline{H}(t))\right) = \overline{H}(t) \tag{2.5}$$

for $t \in \mathbb{R}^+$, where we used in the final equality that $\overline{F}_\mu$ is continuous. □

We next turn to the specification of the second degree of freedom, the initial distribution μ. By an appropriate choice of the randomisation μ the form of the function $\overline{F}_\mu$ in the specification of the time-change I_μ in (2.1) can be considerably simplified. In particular, the function $\overline{F}_\mu$ is equal to an exponential if μ is taken to be equal to any quasi-invariant distribution of the process X killed at the epoch of first-passage below the level 0, the definition of which, we recall, is as follows:

Definition 2.4. For given $\lambda \in \mathbb{R}^+$, the probability measure μ on the measurable space $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ is a λ-*invariant distribution* for the process X killed at the epoch of first entrance into the negative half-axis $(-\infty, 0)$ if

$$P^\mu\left(X_t \in A, t < \tau_0^X\right) = \mu(A)\mathrm{e}^{-\lambda t} \qquad \text{for all } A \in \mathcal{B}(\mathbb{R}^+). \tag{2.6}$$

The probability measure μ is a *quasi-invariant distribution of* $\{X_t, t < \tau_0^X\}$ if μ is a $\lambda-$invariant distribution of $\{X_t, t < \tau_0^X\}$ for some $\lambda \in \mathbb{R}^+$.

To guarantee existence of quasi-invariant distributions we restrict ourselves in the subsequent analysis to Lévy processes X satisfying an exponential integrability condition.

Assumption 2.5. The distribution of X_1 satisfies the following exponential moment condition:

$$E[\mathrm{e}^{\epsilon X_1}] < 1 \qquad \text{for some } \epsilon \in (0, \infty),$$

where $E[\cdot]$ denotes the expectation under the probability measure $P(= P_0)$.

Under Assumption 2.5, there exists a continuum of quasi-invariant distributions of the process X killed upon the first moment of entrance into the negative half-axis, which are given in terms of the Laplace exponent and the positive Wiener-Hopf factor of X.

The Laplace exponent $\psi : \mathbb{R} \to (-\infty, \infty]$ of X, given by $\psi(\theta) = \log E[\mathrm{e}^{\theta X_1}]$ for real θ, is finite valued and convex when restricted to the interior $(\underline{\theta}, \overline{\theta})$ of its maximal domain, where $\overline{\theta} = \sup\{\theta \in \mathbb{R} : E[\exp\{\theta X_1\}] < \infty\}$ and $\underline{\theta} = \inf\{\theta \in \mathbb{R} : E[\exp\{\theta X_1\}] < \infty\}$ (see Figure 1 for plots of Laplace exponents of Lévy processes satisfying Assumption 2.5.) Since ψ is a convex lower-semicontinuous function that under Assumption 2.5 takes a strictly

negative value at some $\epsilon > 0$, it follows that the infimum of ψ is strictly negative and is attained at some $\theta^* \in [\underline{\theta}, \overline{\theta}]$, *i.e.*,

$$-\lambda^* := \inf_{\theta \in [\underline{\theta}, \overline{\theta}]} \psi(\theta) = \psi(\theta^*) < 0. \tag{2.7}$$

On the interval $(\underline{\theta}, \theta^*]$ the function ψ is continuous and strictly monotone decreasing with inverse denoted by

$$\bar{\phi} : [-\lambda^*, \psi(\underline{\theta})) \to (\underline{\theta}, \theta^*]\,. \tag{2.8}$$

In particular, we note $\psi'(0+) \in [-\infty, 0)$ so that the mean $E[X_1]$ of X_1 is strictly negative.

The positive Wiener-Hopf factor is the function $\Psi^+ : (0, \infty) \times \mathbb{D}^+ \to \mathbb{C}$ with $\mathbb{D}^+ := \{u \in \mathbb{C} : \Im(u) \geq 0\}$ given by

$$\Psi^+(q, \theta) = E[\exp(\mathtt{i}\theta X^*_{e(q)})], \qquad q > 0,\ \theta \in \mathbb{D}^+, \tag{2.9}$$

with $e(q)$ denoting an $\mathrm{Exp}(q)$ random time that is independent of X. In Lemma 4.2 we show that the function Ψ^+ can be uniquely extended to the set $\{(q, \theta) : \Re(q) \geq -\lambda^*, \Im(\theta) \geq -\theta^*\} \backslash \{(-\lambda^*, -\theta^*)\}$ (by analytical continuation and continuous extension); this extension is also denoted by Ψ^+.

Consider for any $\lambda \in (0, \lambda^*]$ the function $\widehat{\mu}_\lambda : \mathbb{R}^+ \to \mathbb{C}$ given by

$$\widehat{\mu}_\lambda(\theta) = \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda) + \theta} \cdot \Psi^+(-\lambda, \mathtt{i}\theta), \tag{2.10}$$

where $\bar{\phi}$ denotes the inverse of the Laplace exponent as described above. The function $\widehat{\mu}_\lambda$ is the Laplace transform of some probability measure μ_λ—an explicit expression for μ_λ is given in Lemma 5.1. The members of the family $\{\mu_\lambda, \lambda \in (0, \lambda^*]\}$ are quasi-invariant distributions of $\{X_t, t < \tau_0^X\}$:

Theorem 2.6. *Assume that X is a Lévy process satisfying $E[\exp(-\epsilon X_1)] < 1$ for some $\epsilon \in (0, \infty)$. Then, for any $\lambda \in (0, \lambda^*]$, $\widehat{\mu}_\lambda$ is the Laplace transform of some probability measure μ_λ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, which is the unique λ-invariant distribution of $\{X_t, t < \tau_0^X\}$, the process X that is killed upon the epoch of first-passage into the negative half-line $(-\infty, 0)$.*

In the case that X is a mixed-exponential Lévy process the measures μ_λ, $\lambda \in (0, \lambda^*]$, can be shown to be equal to certain mixed-exponential distributions—see Sections 6.

Under any of the initial distributions μ_λ given in Theorem 2.6 the distribution of the first-passage time τ_0^X is exponential and thus the corresponding survival function $\overline{F}_{\mu_\lambda}$ and time change I_{μ_λ} defined in (2.1) take explicit forms:

$$\begin{aligned} \overline{F}_{\mu_\lambda}(t) &= \exp(-\lambda t), \qquad t \in \mathbb{R}^+, \lambda \in (0, \lambda^*], \\ I_{\mu_\lambda}(t) &= -\frac{1}{\lambda} \log \overline{H}(t). \end{aligned}$$

When the survival function $\overline{H}$ is continuous, $I_{\mu_\lambda}(t)$ is equal to a multiple of the cumulative hazard rate integrated over the interval $[0, t]$.

The combination of Theorems 2.2 and 2.6 immediately yields the following result:

Corollary 2.7. *For any given continuous survival function $\overline{H}$ and $\lambda \in (0, \lambda^*]$, the RIFPT problem is solved by the pair $(\mu_\lambda, I_{\mu_\lambda})$,* i.e.,

$$P^{\mu_\lambda}\left(\tau_0^{Y^{\mu_\lambda}} > t\right) = \overline{H}(t), \qquad t \in \mathbb{R}^+.$$

2.1. **Example.** As a simple example, let us consider the case where X_t is Brownian motion with drift, with initial distribution μ, or equivalently $X_t = X_0 + W_t + \eta t$ where W_t is a standard Brownian motion, $\eta \in \mathbb{R}$ and $X_0 \sim \mu$ is a random variable independent of $\{W_t, t \in \mathbb{R}^+\}$. In this case

$$\psi(\theta) = \log E[\mathrm{e}^{\theta X_1}] = \eta\theta + \frac{1}{2}\theta^2$$

and $\underline{\theta} = -\infty, \overline{\theta} = +\infty$, so the coefficients in (2.7) are $\theta^* = -\eta$, $\lambda^* = \frac{1}{2}\eta^2$ and the inverse of ψ to the left of θ^* is

$$\overline{\phi}(y) = -\eta - \sqrt{\eta^2 + 2y}.$$

The positive Wiener-Hopf factor is

$$\Psi^+(q,\theta) = \frac{-\mathtt{i}(\eta - \sqrt{\eta^2+2q})}{\theta - \mathtt{i}(\eta - \sqrt{\eta^2+2q})}.$$

The Laplace transform of the λ-invariant distribution is therefore given by

$$\begin{aligned} \widehat{\mu}_\lambda(\theta) &= \left(\frac{-\eta - \sqrt{\eta^2-2\lambda}}{\theta - (\eta + \sqrt{\eta^2-2\lambda})}\right)\left(\frac{-\eta + \sqrt{\eta^2-2\lambda}}{\theta - (\eta - \sqrt{\eta^2-2\lambda})}\right) \\ &= \frac{2\lambda}{\theta_+ - \theta_-}\left(\frac{1}{\theta-\theta_+} - \frac{1}{\theta-\theta_-}\right), \end{aligned} \tag{2.11}$$

where $\theta_\pm = \eta \pm \sqrt{\eta^2-2\lambda}$. The condition $\eta \in [-\sqrt{2\lambda}, 0)$ is necessary and sufficient for the expression at (2.11) to be the Laplace transform of a probability measure on $\mathbb{R}^+$, and we note that this is the same as the condition $\lambda \in (0, \lambda^*]$ of Theorem 2.6. Under this condition μ_λ is a mixture of exponentials (or a gamma distribution if $\eta = -\sqrt{2\lambda}$). This special case was presented in our earlier paper [15].

3. Multi-dimensional RIFPT

Given a joint survival function $\overline{H} : (\mathbb{R}^+)^d \to [0,1]$ and a d-dimensional Lévy process, a d-dimensional version of the RIFPT problem is phrased as the problem to find a probability measure on $\mathbb{R}^d$ and a collection of increasing continuous functions $I^1, \ldots, I^d$ such that the following identity holds:

$$P^\mu\left(\tau^{Y^1} > t_1, \ldots, \tau^{Y^d} > t_d\right) = \overline{H}(t_1, \ldots, t_d), \qquad \text{for all } t_1, \ldots, t_d \in \mathbb{R}^+, \tag{3.1}$$

$$Y^i := X \circ I^i \text{ for } i = 1, \ldots, d. \tag{3.2}$$

In order to present a solution we will impose some structure on the joint survival function $\overline{H}$, assuming that it is from the class of multivariate *generalised frailty survival functions* that is defined as follows:

Definition. A joint survival function $\overline{H} : \mathbb{R}^d_+ \to [0,1]$ is called a (d-dimensional) *generalised frailty distribution* if there exists a random vector $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_m)$ for some $m \in \mathbb{N}$ such that we have

$$\overline{H}(t_1, \ldots, t_d) = E\left[\prod_{i=1}^d \overline{H}_i(t_i|\Upsilon)\right], \qquad t_1, \ldots, t_d \in \mathbb{R}^+,$$

where $\overline{H}_i(\cdot|u) : \mathbb{R}^+ \to [0,1]$, $i = 1, \ldots, d$, $u \in \mathbb{U}^m$ denotes a collection of survival functions, where $\mathbb{U}^m$ denotes the image of the random vector Υ.

When we denote by $(T_1, \ldots, T_d)$ a random vector with joint survival function $\overline{H}$, the condition in the definition can be phrased as the requirement that there exists a finite-dimensional random vector Υ such that, conditional on Υ, the random variables $T_1, \ldots, T_d$ are mutually independent. In the context of credit risk modeling, for example, one may interpret the vector Υ as the common factors driving the solvency of a collection of d companies (such as economic environment, as opposed to idiosyncratic factors).

We remark that the terminology "generalised frailty" is extracted from the theory of survival analysis (*e.g.*, Kalbfleisch & Prentice [26]) in which *frailty* refers to a common factor driving the survival probabilities of the individual entities. One of the commonly studied models is that of *multiplicative frailty* where the frailty appears as a multiplicative factor in the individual hazard functions, in which case the conditional individual survival functions $\overline{H}_i(\cdot|u)$ take the form $\overline{H}_i(\cdot)^u$ for $u \in \mathbb{R}^+$.

Assume henceforth that $\overline{H}$ is a d-dimensional generalised frailty survival function, and denote the corresponding collection of conditional survival functions by $\{\overline{H}_i(\cdot|u), i = 1, \ldots, d, u \in \mathbb{U}^m\}$ for some $m \in \mathbb{N}$. A solution to the multi-dimensional IFPT of the survival function $\overline{H}$ can be constructed by application of the construction that was used in Corollary 2.7 to the conditional survival functions $\overline{H}_i(\cdot|u)$. To formulate this result, let $\{X^{i|u}, i \in \{1, \ldots, d\}, u \in \mathbb{U}^m\}$ be a collection of independent Lévy processes, each satisfying Assumption 2.5, and denote by $\{\mu_i(\cdot|u), i \in \{1, \ldots, d\}, u \in \mathbb{U}^m\}$ the probability distributions that have Laplace transforms $\widehat{\mu}_i(\cdot|u)$ given by

$$\widehat{\mu}_i(\theta|u) = \frac{\bar{\phi}_{i|u}(-\lambda_{i|u})}{\bar{\phi}_{i|u}(-\lambda_{i|u}) + \theta} \cdot \Psi^+_{i|u}(-\lambda_{i|u}, \mathtt{i}\theta), \qquad \text{for some } \lambda_{i|u} \in (0, \lambda^*_{i|u}],$$

where $\bar{\phi}_{i|u}$, $\Psi^+_{i|u}$, $\lambda^*_{i|u}$ are the corresponding left-inverse of the Laplace exponent, positive Wiener-Hopf factor and minimum of the Laplace exponent of $X^{i|u}$, respectively. Finally, let $\{I_i(\cdot|u), i \in \{1,\dots,m\}, u \in \mathbb{U}^m\}$ denote the collection of time-changes given by

$$I_i(t|u) = -\frac{1}{\lambda_{i|u}} \log \overline{H}_i(t|u), \qquad t \in \mathbb{R}^+.$$

The solution of the multi-dimensional IFPT is given as follows:

Theorem 3.1. *It holds*

$$P\left(\tau_0^{Y^1} > t_1, \dots, \tau_0^{Y^d} > t_d\right) = \overline{H}(t_1,\dots,t_d), \qquad t_1,\dots,t_d \in \mathbb{R}^+, \qquad \textit{with}$$
$$Y^i(t) = Y_0^{i|\Upsilon} + X^{i|\Upsilon}\left(I_i(t|\Upsilon)\right), \qquad i = 1,\dots,d,$$

where, conditional on $\Upsilon = u \in \mathbb{U}^m$, the random variable $Y_0^{i|u}$ follows the probability distribution $\mu_i(\cdot|u)$ and is independent of the vector $(X^{1|u},\dots,X^{d|u})$ of Lévy processes.

Proof. By the tower-property of conditional expectations and the fact that, conditional on the random variable Υ, the set $\{Y^{i|\Upsilon}, i = 1,\dots,d\}$ forms a collection of independent random variables, we have for any vector $(t_1,\dots,t_d) \in (\mathbb{R}^+)^d$

$$P\left(\tau_0^{Y^1} > t_1, \dots, \tau_0^{Y^d} > t_d\right) = E\left[\prod_{i=1}^d P\left(\tau_0^{Y^i} > t_i \middle| \Upsilon\right)\right]$$
$$= E\left[\prod_{i=1}^d P^{\mu_i(\cdot|\Upsilon)}\left(\tau_0^{X^{i|\Upsilon}} > I_i(t_i|\Upsilon)\right)\right] = E\left[\prod_{i=1}^d \overline{H}_i(t_i|\Upsilon)\right] = \overline{H}(t_1,\dots t_d),$$

where in the second line we used Corollary 2.7. □

4. Wiener-Hopf factorization and first-passage times

4.1. **Preliminaries.** In this subsection we set the notation and recall some basic results concerning the Wiener-Hopf factorization of X. We refer to Sato [36, Ch. 9] for a self-contained account of classical Wiener-Hopf factorization theory of Lévy processes and further references; see also Kuznetsov [25] for a recent derivation using analytical arguments.

Denote by Ψ the characteristic exponent of X, *i.e.*, the unique map $\Psi : \mathbb{R} \to \mathbb{C}$ that satisfies $E[\exp\{\mathtt{i}\theta X_t\}] = \exp\{t\Psi(\theta)\}$ for any $t \in \mathbb{R}^+$. According to the Lévy-Khintchine formula, the characteristic exponent is given by

$$\Psi(\theta) = \mathtt{i}\eta\theta - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} [\mathrm{e}^{\mathtt{i}\theta z} - 1 - \mathtt{i}\theta z \mathbf{1}_{\{|z|<1\}}]\nu(\mathrm{d}z), \qquad \theta \in \mathbb{R}, \tag{4.1}$$

where $\eta \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$ is the variance of the continuous martingale part of X, and ν denotes the Lévy measure of X. Under Assumption 2.5 the random variable X_1 has negative mean and the Lévy measure ν of X satisfies the condition (*e.g.*, Sato [36, Thm. 25.3])

$$\int_{(1,\infty)} \mathrm{e}^{\epsilon x}\nu(\mathrm{d}x) < \infty. \tag{4.2}$$

Furthermore, under this condition Ψ can be analytically extended to the strip

$$\mathcal{S} = \{\theta \in \mathbb{C} : \Im(\theta) \in \Theta^o \cup \{0\}\},$$

where $\Im(\theta)$ denotes the imaginary part of θ and where Θ^o is the interior of the set $\Theta = \{\theta \in \mathbb{R} : \psi(\theta) < \infty\}$ which is a non-empty interval given Assumption 2.5. This analytical extension of Ψ will also be denoted by Ψ. The characteristic exponent Ψ is related to the Laplace exponent ψ of X by $\psi(\theta) = \Psi(-\mathtt{i}\theta)$ for $\theta \in \Theta$.

The probability distributions of the running supremum $X^*(t)$ and infimum $X_*(t)$ of X up to time t are related to the characteristic exponent Ψ by the Wiener-Hopf factorization of X, which expresses Ψ as the product of the Wiener-Hopf factors Ψ^+ and Ψ^- as follows:

$$\frac{q}{q - \Psi(\theta)} = \Psi^+(q,\theta)\Psi^-(q,\theta), \qquad \theta \in \mathbb{R},\ q > 0, \tag{4.3}$$

with $\Psi^+(q,\theta)$ given in (2.9) and the function $\Psi^- : (0,\infty)\times\mathbb{D}^- \to \mathbb{C}$ with $\mathbb{D}^- = \{u\in\mathbb{C} : \Im(u)\le 0\}$, given by $\Psi^-(q,\theta) = E[\exp\{\mathtt{i}\theta X_*(e(q))\}]$ for $\theta\in\mathbb{D}^-$, where, as before, $e(q)$ denotes an independent exponential random variable with mean q^{-1} that is independent of X (*e.g.*, Sato [36, Thms. 45.2, 45.7, Rem. 45.9]).

Since, as noted before, X has negative mean under Assumption 2.5, $\lim_{t\to\infty} X_t^*$ is almost surely finite, and Lebesgue's Dominated Convergence Theorem implies that $E[\exp(\mathtt{i}\theta X^*_{e(q)})] \to E[\exp(\mathtt{i}\theta X^*_\infty)]$ so that $\Psi^+(0,\theta) := \lim_{q\searrow 0}\Psi^+(q,\theta)$ is well-defined and equal to $E[\exp(\mathtt{i}\theta X^*_\infty)]$. It follows thus from the Wiener-Hopf factorization (4.3) that the limit $\Psi^-(q,\theta)/q$ for $q\searrow 0$ exists and is equal to

$$\Psi^-(0,\theta)/0 := \lim_{q\downarrow 0} q^{-1}\Psi^-(q,\theta) = -\Psi(\theta)^{-1}\cdot\Psi^+(0,\theta)^{-1}, \qquad \theta\in\mathbb{R}. \tag{4.4}$$

The function $\Psi^+(q,\cdot)$ with $q\in\mathbb{R}^+$ admits an analytical extension to the domain $\mathcal{S}^+ := \{\theta\in\mathbb{C} : \Im(\theta) > -\overline{\theta}\}$, while the function $\Psi^-(q,\cdot)/q$ with $q\in\mathbb{R}^+$, may be extended analytically to $\mathcal{S}^- := \{\theta\in\mathbb{C} : \Im(\theta)\in(-\infty,-\underline{\theta})\}$. Denoting these analytical extensions also by $\Psi^+(q,\,\cdot\,)$ and $\Psi^-(q,\,\cdot\,)/q$ the Wiener-Hopf factorization (4.3) continues to hold for all θ in the strip $\mathcal{S}$.

4.2. **Wiener-Hopf factorization under the Esscher-transform.** In order to establish that $\Psi^+(q,s)$ admits an analytical extension in q as stated in the introduction we first provide a 'change-of-variable' formula relating Ψ^+ to its counterparts under Esscher-transforms of P. We recall that the *Esscher transform* $P_x^{(\theta)}$ of the probability measure P_x for $x\in\mathbb{R}^+$ and $\theta\in\Theta := \{\theta\in\mathbb{R} : \psi(\theta)<\infty\}$ is the probability measure that is absolutely continuous with respect to P_x with Radon-Nikodym derivative on $\mathcal{F}_t$ given by

$$\left.\frac{\mathrm{d}P_x^{(\theta)}}{\mathrm{d}P_x}\right|_{\mathcal{F}_t} = \exp(\theta(X_t - x) - t\psi(\theta)), \qquad \theta\in\Theta, x\in\mathbb{R}^+.$$

Under the measure $P_x^{(\theta)}$ the process $X - X_0$ is still a Lévy process with a Laplace exponent $\psi^{(\theta)}$ that is given in terms of ψ by

$$\psi^{(\theta)}(s) = \psi(s+\theta) - \psi(\theta), \qquad s+\theta\in\Theta, \tag{4.5}$$

and a positive Wiener-Hopf factor denoted by Ψ_θ^+.

Lemma 4.1. *For any $q\in\mathbb{R}^+$ and $\theta\in\Theta$ with $\psi(\theta)<q$ we have*

$$\Psi^+(q,s) = \frac{\Psi_\theta^+(q-\psi(\theta), s+\mathtt{i}\theta)}{\Psi_\theta^+(q-\psi(\theta),\mathtt{i}\theta)}, \qquad \Psi^-(q,s) = \frac{\Psi_\theta^-(q-\psi(\theta), s+\mathtt{i}\theta)}{\Psi_\theta^-(q-\psi(\theta),\mathtt{i}\theta)}, \tag{4.6}$$

for $s\in\mathcal{S}^+$ and $s\in\mathcal{S}^-$, respectively. In particular, we have for any $q\in\mathbb{R}^+$ and $\lambda\in(0,\lambda^*]$

$$\Psi^\pm(q,s) = \frac{\Psi_r^\pm(q+\lambda, s+\mathtt{i}r)}{\Psi_r^\pm(q+\lambda,\mathtt{i}r)}, \qquad r = \bar{\phi}(-\lambda), \tag{4.7}$$

with $s\in\mathcal{S}^+$ and $s\in\mathcal{S}^-$, respectively.

Proof. By changing measure from P to $P^{(\theta)}$ we find with $\zeta = q-\psi(\theta)$

$$\begin{aligned}
\Psi^+(q,s) &= \int_0^\infty q\mathrm{e}^{-qt}E[\mathrm{e}^{\mathtt{i}sX_t^*}]\mathrm{d}t = \frac{q}{\zeta}\int_0^\infty \zeta\mathrm{e}^{-\zeta t}E^{(\theta)}[\mathrm{e}^{-\theta X_t}\mathrm{e}^{\mathtt{i}sX_t^*}]\mathrm{d}t\\
&= \frac{q}{\zeta}E^{(\theta)}[\mathrm{e}^{-\theta(X_{e(\zeta)}-X^*_{e(\zeta)})}\mathrm{e}^{\mathtt{i}(s+\mathtt{i}\theta)X^*_{e(\zeta)}}]\\
&= \frac{q}{\zeta}E^{(\theta)}[\mathrm{e}^{-\theta(X_{e(\zeta)}-X^*_{e(\zeta)})}]E^{(\theta)}[\mathrm{e}^{\mathtt{i}(s+\mathtt{i}\theta)X^*_{e(\zeta)}}]\\
&= \frac{q}{\zeta}\Psi_\theta^-(\zeta,\mathtt{i}\theta)\Psi_\theta^+(\zeta,s+\mathtt{i}\theta) = \Psi_\theta^+(\zeta,\mathtt{i}\theta)^{-1}\Psi_\theta^+(\zeta,s+\mathtt{i}\theta),
\end{aligned}$$

where we used Wiener-Hopf factorization (4.3) and the form (4.5) of ψ_θ in the third and fourth lines. The identity concerning Ψ^- is derived in an analogous manner. Finally, the equality (4.7) follows by taking $\theta = r$ in (4.6). □

Lemma 4.2. *The functions $\Psi^+(u,v)$ and $\Psi^-(u,w)$ can be uniquely extended by analytical continuation and continuous extension to the respective domains*

$$\mathbb{V}_+ := \{(u,v) \in \mathbb{C}^2 : \Re(u) \geq -\lambda^*, \Im(v) \geq -\theta^*\} \backslash \{(-\lambda^*, -\mathtt{i}\theta^*)\},$$
$$\mathbb{V}_- := \{(u,w) \in \mathbb{C}^2 : \Re(u) \geq -\lambda^*, \Im(w) \leq 0\}.$$

In particular, denoting these extensions again by Ψ^+ and Ψ^- we have continuity in λ of $\Psi^+(-\lambda, \mathtt{i}u)$ on $(0, \lambda^*]$ for each $u \in \mathbb{R}^+$ and it holds

$$\Psi^+(-\lambda, \mathtt{i}u) = \frac{\Psi_r^+(0, \mathtt{i}(u+r))}{\Psi_r^+(0, \mathtt{i}r)}, \qquad r = \bar{\phi}(-\lambda), \qquad \lambda \in (0, \lambda^*), \tag{4.8}$$

$$\frac{\lambda}{\lambda + \Psi(u)} = \Psi^+(-\lambda, u)\Psi^-(-\lambda, u), \qquad \lambda \in (0, \lambda^*]. \tag{4.9}$$

Proof. The Wiener-Hopf factor $\Psi^+(q,s)$ is well-known to be holomorphic and non-zero on the domain $D := \{(q,s) \in \mathbb{C}^2 : \Re(q) > 0, \Im(s) > 0\}$ and continuous on the closure $\overline{D}$. The identity in (4.6) implies that at any $(q,s) \in \overline{D}$ the power series in (q,s) of $\Psi^+(q,s)$ and $L(q,s) := \Psi_{\theta^*}^+(q - \psi(\theta^*), s + \mathtt{i}\theta^*)/\Psi_{\theta^*}^+(q - \psi(\theta^*), \mathtt{i}\theta^*)$ are equal. Since L is holomorphic on the interior $(\mathbb{V}^+)^o$ of $\mathbb{V}^+$ and continuous on $\mathbb{V}^+$, it follows that the function $\Psi^+(q,s)$ can be uniquely extended by analytical continuation and continuous extension to the set $\mathbb{V}_+$. In particular, it follows that the function $\lambda \mapsto \Psi^+(-\lambda, \mathtt{i}\theta)$ is continuous on $(0, \lambda^*]$, and we have consistency with (4.8) by construction of the extension. The proof of the extension of Ψ^- to $\mathbb{V}^-$ is similar and omitted. By multiplying the functions in (4.7) with $q = -\lambda$ and using the form of $\Psi_r^-(0,\theta)/0$ [see (4.4)] it follows that the product in the rhs of (4.9) is equal to $\{-\Psi_r(u + \mathtt{i}r)\}^{-1}\{-\Psi_r(\mathtt{i}r)\} = \lambda/[\Psi(u) + \lambda]$ [in view of (4.5)]. □

For later reference we give next expressions in terms of Bromwich-type integrals for the joint Laplace transform of τ_0^X and the overshoot $X_{\tau_0^X}$ and the Laplace transform of $X(e(q))$ on the set $\{X_*(e(q)) \geq 0\}$, both under a given initial distribution μ, and use this to derive an integral equation for the Laplace transform of a λ-invariant distribution. To derive these expressions, we first express the Laplace transform of the function $K_{\theta,q} : \mathbb{R}^+ \to \mathbb{R}$ given by

$$K_{\theta,q}(x) = E_x[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{\tau_0^X > e(q)\}}], \qquad x \in \mathbb{R}^+,$$

for given positive q and θ, in terms of the Wiener-Hopf factors Ψ^+ and Ψ^-.

Lemma 4.3. (i) *For $\theta, q > 0$ and $x \in \mathbb{R}^+$ we have*

$$K_{\theta,q}(x) = \Psi^+(q, \mathtt{i}\theta) \cdot E_0[\mathrm{e}^{-\theta X_*(e(q))}\mathbf{1}_{\{X_*(e(q)) \geq -x\}}]. \tag{4.10}$$

(ii) *The Laplace transform $\widehat{K}_{\theta,q}$ of $K_{\theta,q}$ is given by*

$$\widehat{K}_{\theta,q}(u) = \frac{\Psi^+(q, \mathtt{i}\theta)\Psi^-(q, -\mathtt{i}u)}{\theta + u}, \qquad u \in \mathbb{R}^+. \tag{4.11}$$

Proof. (i) The independence of the random variables $(X - X_*)(e(q))$ and $X_*(e(q))$ (from the Wiener-Hopf factorization (4.3)) and the fact that the events $\{\tau_0^X > e(q)\}$ and $\{X_*(e(q)) \geq 0\}$ are equal P_x-a.s. for any nonnegative x (*i.e.*, the probability $P_x(\Delta)$ of the difference Δ of these two sets is 0) imply that we have

$$\begin{aligned} K_{\theta,q}(x) &= E_x[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{\tau_0^X > e(q)\}}] = \mathrm{e}^{-\theta x}E_0[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{X_*(e(q)) \geq -x\}}] \\ &= \mathrm{e}^{-\theta x}E_0[\mathrm{e}^{-\theta\{(X-X_*)(e(q)) + X_*(e(q))\}}\mathbf{1}_{\{X_*(e(q)) \geq -x\}}] \\ &= \mathrm{e}^{-\theta x}E_0[\mathrm{e}^{-\theta(X-X_*)(e(q))}]E_0[\mathrm{e}^{-\theta X_*(e(q))}\mathbf{1}_{\{X_*(e(q)) \geq -x\}}] \end{aligned}$$

for any nonnegative real x, which yields (4.10) in view of the fact that the Laplace transform of $(X - X_*)(e(q))$ is given by $\Psi^+(q, \mathtt{i}\theta)$.

(ii) In view of (4.10) the Laplace transform $\widehat{K}_{\theta,q}$ is equal to

$$\begin{aligned}\widehat{K}_{\theta,q}(u) &= \Psi^+(q,\mathtt{i}\theta)E_0\left[\int_0^\infty \mathrm{e}^{-(u+\theta)x}\mathrm{e}^{-\theta X_*(e(q))}\mathbf{1}_{\{X_*(e(q))\geq -x\}}\mathrm{d}x\right]\\ &= \Psi^+(q,\mathtt{i}\theta)E_0\left[\mathrm{e}^{-\theta X_*(e(q))}\int_{-X_*(e(q))}^\infty \mathrm{e}^{-(u+\theta)x}\mathrm{d}x\right]\\ &= \Psi^+(q,\mathtt{i}\theta)\frac{1}{\theta+u}E_0[\mathrm{e}^{uX_*(e(q))}], \qquad u\in\mathbb{R}^+,\end{aligned}$$

which yields (4.11) in view of the definition of the Wiener-Hopf factor Ψ^-. □

Proposition 4.4. *Let μ be a probability measure on $\mathbb{R}^+\backslash\{0\}$ without atoms and denote by $\widehat{\mu}$ its Laplace transform. Assume that there are $c>0$, $C>0$ and $a\in\Theta^o$ satisfying $\widehat{\mu}(-a)<\infty$ and*

$$|\widehat{\mu}(-u)(1+|u|^c)|<C \qquad \text{for all } u \text{ with } \Re(u)=a. \tag{4.12}$$

(i) *For any $q,\theta\in\mathbb{R}^+$, $q\neq 0$, we have the identities*

$$E^\mu[\mathrm{e}^{-\theta X(e(q))}\mathbf{1}_{\{X_*(e(q))\geq 0\}}] = \Psi^+(q,\mathtt{i}\theta)\cdot\frac{1}{2\pi\mathtt{i}}\int_{a-\mathtt{i}\infty}^{a+\mathtt{i}\infty}\widehat{\mu}(-u)\Psi^-(q,-\mathtt{i}u)\frac{\mathrm{d}u}{u+\theta} \tag{4.13}$$

$$E^\mu[\mathrm{e}^{-q\tau_0^X+\theta(X_{\tau_0^X}-X_0)}] = \frac{1}{2\pi\mathtt{i}}\int_{a-\mathtt{i}\infty}^{a+\mathtt{i}\infty}\widehat{\mu}(-u+\theta)\left(1-\frac{\Psi^-(q,-\mathtt{i}u)}{\Psi^-(q,-\mathtt{i}\theta)}\right)\frac{\mathrm{d}u}{u-\theta}. \tag{4.14}$$

(ii) *Let $\lambda\in(0,\lambda^*]$. The measure μ is a λ-invariant distribution of the process $\{X_t, t<\tau_0^X\}$ if and only if $\widehat{\mu}$ satisfies the collection of equations*

$$\widehat{\mu}(\theta)\cdot\frac{q}{q+\lambda} = \Psi^+(q,\mathtt{i}\theta)\cdot\frac{1}{2\pi\mathtt{i}}\int_{a-\mathtt{i}\infty}^{a+\mathtt{i}\infty}\widehat{\mu}(-u)\Psi^-(q,-\mathtt{i}u)\frac{\mathrm{d}u}{u+\theta}, \qquad q>0. \tag{4.15}$$

Remark 4.5. The identity in (4.13) is also valid if instead of (4.12) we require $|q^{-1}\Psi^-(q,-\mathtt{i}u)|(1+|u|^c)|<C$ uniformly over all $q>0$ and u with $\Re(u)=a$. We note that the boundedness of $|\Psi^-(q,-\mathtt{i}u)|(1+|u|)$ over the set of $q>0$ and u with $\Re(u)=a$ is equivalent to the condition that the Lévy process X creeps downwards. This observation follows from the fact that X creeps downwards precisely if the descending ladder height process has non-zero infinitesimal drift.

Proof. It follows from (4.12) that the function $x\mapsto \mathrm{e}^{\theta x}K_{\theta,q}(x)$ is non-decreasing on $\mathbb{R}^+$ (and has thus at most countably many points of discontinuity). The Laplace Inversion Theorem yields that, at any point of continuity x, $K_{\theta,q}(x)$ is equal to the integral of the rhs of the identity in (4.11) over the Bromwich contour $\Re(u)=a$, *i.e.*,

$$K_{\theta,q}(x) = \Psi^+(q,\mathtt{i}\theta)\cdot\frac{1}{2\pi\mathtt{i}}\int_{a-\mathtt{i}\infty}^{a+\mathtt{i}\infty}\mathrm{e}^{ux}\Psi^-(q,-\mathtt{i}u)\frac{\mathrm{d}u}{u+\theta}. \tag{4.16}$$

The identity in (4.13) follows by integrating (4.16) against $\mu(\mathrm{d}x)$ and interchanging the order of integration. This interchange follows by an application of Fubini's theorem which is justified in view of the estimate

$$\int_{(0,\infty)}\int_{0-\mathtt{i}\infty}^{0+\mathtt{i}\infty}\left|\mathrm{e}^{ux}\frac{\Psi^-(q,-\mathtt{i}u)}{u+\theta}\right|\mathrm{d}u\mu(\mathrm{d}x) \leq \int_{(0,\infty)}\mu(\mathrm{d}x)\cdot\int_{\mathbb{R}}C\frac{\theta+|u|}{(u^2+\theta^2)(1+|u|^c)}\mathrm{d}u<\infty. \tag{4.17}$$

To derive this estimate, we used the bound in (4.12), that μ is a probability measure and the observations (a) $1/(u+d)=(\bar{u}+d)/(|\Im(u)|^2+|\Re(u)+d|^2)$ for any $d\in\mathbb{R}$ and $u\in\mathbb{C}$, with $\bar{u}$ denoting the complex conjugate of u, and (b) $|\exp\{ux\}|=\exp\{\Re(u)x\}$ for any $x\in\mathbb{R}$ and $u\in\mathbb{C}$. Hence, the proof of the identity in (4.13) is complete.

The identity in (4.14) can be proved by an analogous line of reasoning (the details of which are omitted) by deploying the *Pecherskii-Rogozin identity*

$$\int_0^\infty \mathrm{e}^{-ux}E_x[\mathrm{e}^{-q\tau_0^X+\theta X_{\tau_0^X}}]\mathrm{d}x = \frac{1}{u-\theta}\left(1-\frac{\Psi^-(q,-\mathtt{i}u)}{\Psi^-(q,-\mathtt{i}\theta)}\right), \qquad u\in\mathbb{R}^+;$$

for a proof see *e.g.*, Sato [36, Thm. 49.2] or Alili & Kyprianou [2, Section 3.1] for a probabilistic proof.

(ii) The assertion follows from Definition 2.4 by noting that (a) the lhs and rhs of (4.15) are equal to the double Laplace transforms in (t,x) of the measures $m_t^{(1)}$ and $m_t^{(2)}$ on $(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$ given by $m_t^{(1)}(\mathrm{d}x)=\exp(-\lambda t)\mu_\lambda(\mathrm{d}x)$ and $m_t^{(2)}(\mathrm{d}x)=P^{\mu_\lambda}(X_t\in\mathrm{d}x, t<\tau_0^X)$ respectively (by (4.13)) and (b) for any Borel set A, $m_t^{(1)}(A)$ and $m_t^{(2)}(A)$ are continuous and càdlàg at any $t>0$, respectively. □

5. Proof of Theorem 2.6

We show first that $\widehat{\mu}_\lambda$ is the Laplace transform of a probability measure μ_λ and identify this measure in terms of the invariant distribution of the reflected process $Z := X - X_* \wedge 0$ (with $x \wedge 0 = \min\{x, 0\}$ for $x \in \mathbb{R}$) under a certain Esscher transform.

We recall that, since Z_t and X_t^* have the same distribution under P_0 for each fixed $t \geq 0$ (by the time-reversal property of Lévy processes, *e.g.*, Bertoin [5, Prop. VI.3]) and, under Assumption 2.5, X_t^* converges to an almost surely finite limit X_∞^* as $t \to \infty$, the limit $P(Z_\infty \in \mathrm{d}x) := \lim_{t\to\infty} P_0(Z_t \in \mathrm{d}x)$ is well-defined and has characteristic function $\Psi^+(0,\theta) = E[\exp(\mathtt{i}\theta X_\infty^*)]$. It is straightforward to verify that the measure $P(Z_\infty \in \mathrm{d}x)$ is the unique invariant probability distribution of the reflected process Z.

For any $\lambda \in (0, \lambda^*)$ we specify the measure μ_λ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ by

$$\begin{aligned} &\mu_\lambda(\mathrm{d}x) = c_r \cdot r\exp(-rx)P^{(r)}(Z_\infty \leq x)\mathrm{d}x, \qquad x \in \mathbb{R}^+, \text{ with} \\ &c_r = 1/E^{(r)}[\exp(-rZ_\infty)], \qquad r = \bar{\phi}(-\lambda), \end{aligned} \tag{5.1}$$

where $\bar{\phi}$ denotes the inverse of the Laplace exponent as described above, and where we used that the mean $E^{(r)}[X_1]$ of X is strictly negative under $P^{(r)}$. Here the normalising constant c_r is such that any of the measures μ_λ has unit mass. We also define a measure μ_{λ^*} as the limit in distribution of μ_λ for $\lambda \nearrow \lambda^*$ (the existence of this limit is verified in Lemma 5.1(ii)). We next verify that the function $\widehat{\mu}_\lambda$ defined in (2.10) is equal to the Laplace transforms of the measures μ_λ.

Lemma 5.1. (i) *For any $\lambda \in (0, \lambda^*)$, the Laplace transform of μ_λ is given by*

$$\int_0^\infty \mathrm{e}^{-\theta x}\mu_\lambda(\mathrm{d}x) = \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda)+\theta} \cdot \frac{\Psi_r^+(0, \mathtt{i}(\theta + r))}{\Psi_r^+(0, \mathtt{i}r)}, \qquad r = \bar{\phi}(-\lambda), \text{ and} \tag{5.2}$$

$$\Psi^+(-\lambda, \mathtt{i}\theta) = \frac{\Psi_r^+(0, \mathtt{i}(\theta + r))}{\Psi_r^+(0, \mathtt{i}r)}, \tag{5.3}$$

where $\bar{\phi}$ denotes the inverse of the Laplace exponent as described above.

(ii) *$\widehat{\mu}_{\lambda^*} := \lim_{\lambda\nearrow\lambda^*} \widehat{\mu}_\lambda$ is the Laplace transform of a probability measure and*

$$\widehat{\mu}_{\lambda^*} = \frac{\bar{\phi}(-\lambda^*)}{\bar{\phi}(-\lambda^*)+\theta} \cdot \Psi^+(-\lambda^*, \mathtt{i}\theta). \tag{5.4}$$

Proof. (i) It is straightforward to verify from (5.1) that μ_λ is equal to a convolution:

$$\mu_\lambda(\mathrm{d}x) = c_r \int_{[0,x]} r\exp(-r(x-y))E^{(r)}\left[\exp(-rZ_\infty)I_{\{Z_\infty \in \mathrm{d}y\}}\right]\mathrm{d}x, \quad x \in \mathbb{R}^+, \quad r = \bar{\phi}(-\lambda), \tag{5.5}$$

so that we obtain the expression (2.10) by taking Laplace transform in x in (5.5). Eqn. (5.3) directly follows from Lemma 4.2.

(ii) From (5.3) we see that the function $\lambda \to \Psi^+(-\lambda, \mathtt{i}\theta)$ is continuous. As $\bar{\phi}(-\lambda)$ is also continuous, we have thus from (2.10) and (4.8) that $\widehat{\mu}_\lambda(\theta)$ converges to the expression on the rhs of (5.4) as $\lambda \nearrow \lambda^*$, for any $\theta \in \mathbb{R}^+$. Since $\widehat{\mu}_{\lambda^*}(0) \to 1$ when $\theta \searrow 0$, the Continuity Theorem (*e.g.*, Feller [17, Thm. XIII.1.2]) implies that $\widehat{\mu}_{\lambda^*}$ is the Laplace transform of a probability measure, μ_{λ^*} say, and μ_λ converges weakly to μ_{λ^*}. □

We next establish for any $\lambda \in (0, \lambda^*]$ the λ-invariance of the measure μ_λ for the killed process $\{X_t, t < \tau_0^X\}$ by showing that, under the initial distribution μ_λ, the running infimum $X_*(t)$ and the distance of X_t from $X_*(t)$ are asymptotically independent as t tends to infinity, conditional on $X_*(t)$ being positive, and that the corresponding asymptotic distribution of $X_*(t)$ is exponential with parameter $r = \phi(-\lambda)$.

Proposition 5.2. *Let $\lambda \in (0, \lambda^*]$, $r = \bar{\phi}(-\lambda)$ and $\theta, \eta \in \mathbb{R}^+$.*

(i) *As $t \to \infty$, $P^{\mu_\lambda}(X_*(t) \geq y | X_*(t) \geq 0) \to \mathrm{e}^{-ry}$ for $y \in \mathbb{R}^+$, and we have*

$$E^{\mu_\lambda}[\mathrm{e}^{-\theta Z_t - \eta X_*(t)} | X_*(t) \geq 0] \quad \longrightarrow \quad \Psi^+(-\lambda, \theta) \cdot \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda)+\eta}, \tag{5.6}$$

$$\mathrm{e}^{\lambda t}P^{\mu_\lambda}(X_*(t) \geq 0) \quad \longrightarrow \quad 1. \tag{5.7}$$

(ii) *The probability measure μ_λ is a λ-invariant distribution for $\{X_t, t < \tau_0^X\}$.*

Proof. (i) We consider first the case $\lambda \in (0, \lambda^*)$. We find by inserting the definition (5.1) of μ_λ, changing measure from P to the Esscher transform $P^{(r)}$ and interchanging the order of integration (justified by Fubini's Theorem)

$$\begin{aligned}
&E^{\mu_\lambda}[\mathrm{e}^{-\theta Z_t - \eta X_*(t)}\mathbf{1}_{\{X_*(t)\geq 0\}}] \\
&= \int_{\mathbb{R}^+}\int_{[0,x]} r\mathrm{e}^{-rx} c_r P^{(r)}(Z_\infty \in \mathrm{d}y) E_0[\mathrm{e}^{-\theta Z_t - \eta(X_*(t)+x)}\mathbf{1}_{\{X_*(t)\geq -x\}}]\mathrm{d}x \\
&= c_r \int_{\mathbb{R}^+}\int_{[0,x]} r\mathrm{e}^{-(r+\eta)x} P^{(r)}(Z_\infty \in \mathrm{d}y)\mathrm{e}^{-\lambda t} E_0^{(r)}[\mathrm{e}^{-(\theta+r)Z_t - (\eta+r)X_*(t)}\mathbf{1}_{\{X_*(t)\geq -x\}}]\mathrm{d}x \\
&= \mathrm{e}^{-\lambda t}\cdot c_r \int_{\mathbb{R}^+}\int_y^\infty r\mathrm{e}^{-(r+\eta)x} E_0^{(r)}[\mathrm{e}^{-(\theta+r)Z_t - (\eta+r)X_*(t)}\mathbf{1}_{\{X_*(t)\geq -x\}}]\mathrm{d}x P^{(r)}(Z_\infty \in \mathrm{d}y) \\
&= \mathrm{e}^{-\lambda t}\cdot \frac{r}{r+\eta}\cdot \int_{\mathbb{R}^+} c_r E_0^{(r)}[\mathrm{e}^{-(\theta+r)Z_t - (\eta+r)\{(y+X_*(t))\vee 0\}}]P^{(r)}(Z_\infty \in \mathrm{d}y), \qquad (5.8)
\end{aligned}$$

for $\theta \in \mathbb{R}^+$, with $x \vee 0 = \max\{x, o\}$ for $x \in \mathbb{R}^+$ and, as before, $c_r = 1/E^{(r)}[\exp(-rZ_\infty)]$ and $r = \bar{\phi}(-\lambda)$. Since the integrand in (5.8) tends to $c_r E_0^{(r)}[\mathrm{e}^{-(r+\theta)Z_\infty}]$ when t tends to infinity (which is equal to $\Psi^+(-\lambda, \mathtt{i}\theta)$ by (4.8)), we deduce by an application of Lebesgue's Dominated Convergence Theorem that the integral also tends to this constant. Taking $\theta = 0$ in (5.8) yields (5.7), and subsequently dividing (5.8) by $P^{\mu_\lambda}(X_*(t) \geq 0)$ yields (5.6). Finally, we note that the first assertion in (i) is a direct consequence of the Continuity Theorem (*e.g.*, Feller [17, Thm. XIII.1.2]) and (5.6) [with $\theta = 0$].

The case $\lambda = \lambda^*$ can be treated by following the line of reasoning in the previous paragraph, replacing throughout the measure $\mathbf{1}_{\mathbb{R}^+}(y)c_r\mathrm{e}^{-ry}P^r(Z_\infty \in \mathrm{d}y)$ by the one with Laplace transform $\Psi^+(-\lambda^*, \mathtt{i}\theta)$.

(ii) The Continuity Theorem and (5.6) [with $\eta = \theta$] implies that we have $E^{\mu_\lambda}[f(X_t)|X_*(t) \geq 0] \to \int_{\mathbb{R}^+} f(x)\mu_\lambda(\mathrm{d}x)$ as $t \to \infty$ for any continuous bounded function f on $\mathbb{R}^+$. The Skorokhod Embedding Theorem implies that this convergence remains valid for any function f that is bounded and continuous on $\mathbb{R}^+\backslash C$ with C a countable set, which satisfies $\mu_\lambda(C) = 0$ by absolutely continuity of μ_λ. Thus, by the Markov property and (5.7) we have for $t, \theta \in \mathbb{R}^+$

$$\begin{aligned}
E^{\mu_\lambda}[\mathrm{e}^{-\theta X_t}\mathbf{1}_{\{X_*(t)\geq 0\}}] &= \lim_{s\to\infty} \mathrm{e}^{\lambda s}\int_{\mathbb{R}^+} E_x[\mathrm{e}^{-\theta X_t}\mathbf{1}_{\{X_*(t)\geq 0\}}]P^{\mu_\lambda}(X_s \in \mathrm{d}x, X_*(s) \geq 0) \\
&= \lim_{s\to\infty} \mathrm{e}^{\lambda s} E^{\mu_\lambda}[\mathrm{e}^{-\theta X_{t+s}}\mathbf{1}_{\{X_*(t+s)\geq 0\}}] \\
&= \mathrm{e}^{-\lambda t}\lim_{s\to\infty} E^{\mu_\lambda}[\mathrm{e}^{-\theta X_{t+s}}|X_*(t+s) \geq 0] = \mathrm{e}^{-\lambda t}\cdot \widehat{\mu}_\lambda(\theta). \qquad (5.9)
\end{aligned}$$

Inverting Laplace transforms on the lhs and rhs of (5.9) shows that the measure μ_λ satisfies (2.6) in Definition 2.4, and the proof is complete. □

With the above results in hand, we now move to the question of uniqueness of the quasi-invariant distributions.

Proposition 5.3. *For any λ in the interval $(0, \lambda^*]$, there exists a unique probability measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ that satisfies the relation*

$$\mu(A) = \frac{q+\lambda}{q}P^\mu[X_{e(q)} \in A, e(q) < \tau_0^X] \qquad A \in \mathcal{B}(\mathbb{R}^+), \qquad q > 0. \qquad (5.10)$$

The proof rests on a contraction argument.

Proof of Proposition 5.3. Again we consider first the case $\lambda \in (0, \lambda^*)$. By changing measure from P to the Esscher transform $P^{(\theta^*)}$ the rhs of (5.10) can be expressed as

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} (q+\lambda)\mathrm{e}^{-qt}\mathrm{e}^{-\lambda^* t}E_x^{(\theta^*)}[\mathrm{e}^{-\theta^*(X_t - x)}\mathbf{1}_{\{t<\tau_0^X\}}]\mathrm{d}t\mu(\mathrm{d}x).$$

Denote by $\mathcal{M}$ the collection of measures m on the measure space $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ that satisfy the conditions

(5.11) the measure $\tilde{m}$ given by $\tilde{m}(\mathrm{d}x) := \mathrm{e}^{-\theta^* x}m(\mathrm{d}x)$ satisfies $\tilde{m}(\mathbb{R}^+) = 1$, and

$$\int_{\mathbb{R}^+} P_x(e(q) < \tau_0^X)\tilde{m}(\mathrm{d}x) = \frac{q}{q+\lambda}. \qquad (5.12)$$

The set $\mathcal{M}$ is non-empty as it contains the measure $m_\lambda := \mathrm{e}^{\theta^* x}\mu_\lambda(\mathrm{d}x)$ (which is the case since μ_λ is a λ-invariant distribution of $\{X_t, t < \tau_0^X\}$ by Proposition 5.2). Furthermore, $\mathcal{M}$ is a closed subset of the space $\mathcal{P}^*$ of measures m

on $\mathbb{R}^+$ satisfying the integrability condition $\int_{\mathbb{R}^+} \mathrm{e}^{-\theta^* x} m(\mathrm{d}x) < \infty$, which is a Banach space under the norm given by $\|\pi - \pi'\| := \sup_{\Upsilon} |\pi(f) - \pi'(f)|$ with $\Upsilon := \{f \in L^0(\mathbb{R}^+) : |f(x)| \leq \mathrm{e}^{-\theta^* x} \ \forall x \in \mathbb{R}^+\}$ which is contained in the set $L^0(\mathbb{R}^+)$ of real-valued Borel-functions with domain $\mathbb{R}^+$.

Next we let $\mathcal{H}$ be the operator $\mathcal{H} : \mathcal{M} \to \mathcal{P}^*$ given by

$$(\mathcal{H}m)(A) = \frac{q+\lambda}{q^*} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} q_* \mathrm{e}^{-q_* t} P_x^{\theta_*}(X_t \in A, t < \tau_0^X) \mathrm{d}t \, m(\mathrm{d}x), \qquad A \in \mathcal{B}(\mathbb{R}^+), m \in \mathcal{M}, \tag{5.13}$$

where $q^* = q + \lambda^*$. We note that any λ-invariant distribution μ of $\{X_t, t < \tau_0^X\}$ gives rise to a fixed point of $\mathcal{H}$ in $\mathcal{M}$: denoting by m_* the Borel measure on $\mathbb{R}^+$ given by $m_*(\mathrm{d}x) = \mathrm{e}^{\theta^* x} \mu(\mathrm{d}x)$, it is straightforward to verify by a change-of-measure argument that the equality in (5.10) can be equivalently rephrased as $m_* = \mathcal{H} m_*$. We show next that the operator $\mathcal{H}$ is a contraction on $\mathcal{M}$.

First, we verify that $\mathcal{H}$ maps $\mathcal{M}$ to itself. We show that, for any $m \in \mathcal{M}$, the measure m' on $\mathbb{R}^+$ given by $m'(\mathrm{d}x) = \mathrm{e}^{-\theta^* x}(\mathcal{H}m)(\mathrm{d}x)$ (a) has unit mass and (b) satisfies the condition in (5.12). To see that (a) holds we observe that, by changing the measure back from $P^{(\theta^*)}$ to P, we get

$$m'(A) = \frac{q+\lambda}{q} P^{\tilde{m}}(X_{e(q)} \in A, e(q) < \tau_0^X) = P^{\tilde{m}}(X_{e(q)} \in A | e(q) < \tau_0^X),$$

with the measure $\tilde{m}$ defined in (5.11), where the second equality follows from (5.12). Furthermore, an application of the Markov property shows

$$\begin{aligned} P^{m'}(\tau_0^X > e(q)) &= E^{\tilde{m}}[P_{X_{e(q)}}(\tau_0^X > e(q)) | \tau_0^X > e(q)] \\ &= P^{\tilde{m}}(\tau_0^X > e(q) + e'(q) | \tau_0^X > e(q)) = P^{\tilde{m}}(\tau_0^X > e'(q)) = \frac{q}{q+\lambda}, \end{aligned}$$

where $e'(q)$ and $e(q)$ denote independent $\mathrm{Exp}(q)$-random times that are independent of X, and where the second line holds as $\tau_0^X \sim \mathrm{Exp}(\lambda)$ under $P^{\tilde{m}}$ (since $\tilde{m}$ satisfies (5.12)). Hence, property (b) also holds true.

Secondly, we note that the definition of $\mathcal{H}$ directly yields the estimate

$$\|\mathcal{H}m_1 - \mathcal{H}m_2\| \leq \frac{q+\lambda}{q^*} \|m_1 - m_2\| < \|m_1 - m_2\|, \qquad m_1, m_2 \in \mathcal{M},$$

where in the second inequality we used that $q + \lambda$ is strictly smaller than q^*.

Thus, an application of Banach's Contraction Theorem shows that there exists a unique measure π^* in $\mathcal{M}$ that satisfies the relation $\pi^* = \mathcal{H}\pi^*$, which implies the asserted uniqueness for $\lambda \in (0, \lambda^*)$.

We next consider the boundary case $\lambda = \lambda^*$. The proof in this case follows by a modification of above argument. Since the function $\upsilon \to \Psi(-\mathtt{i}\upsilon)$ is analytic in a neighbourhood of θ^* in the complex plane and $-\lambda^* = \Psi(-\mathtt{i}\theta^*)$, and non-constant analytic functions map open sets to open sets, it follows that for any sufficiently small $\epsilon > 0$ and any λ_ϵ satisfying $\lambda_\epsilon - \lambda^* \in (0, \epsilon]$ there exists an υ in a neighbourhood of θ^* in the complex plane such that $\Psi(-\mathbf{i}\upsilon) = -\lambda_\epsilon$. Fix such an ϵ and a corresponding λ_ϵ and $\upsilon = \upsilon_\epsilon$. By repeating above argument, replacing the Esscher-transform $P^{(\theta^*)}$ by the complex-valued change of measure $P^{(\upsilon_\epsilon)}$, we find that the corresponding map $\mathcal{H}^\epsilon$ (defined by the rhs of (5.13) with (λ^*, θ^*) replaced by $(\lambda_\epsilon, \upsilon_\epsilon)$) is still a contraction but now on the set $\mathcal{M}_\epsilon$ of complex valued measures $m = m_1 + \mathtt{i}m_2$ satisfying the condition $\tilde{m}(\mathbb{R}^+) = 1$ and (5.12) with the Borel-measure $\tilde{m}$ on $\mathbb{R}^+$ now given by $\tilde{m}(\mathrm{d}x) = \mathrm{e}^{-\upsilon_\epsilon x} m(\mathrm{d}x)$.

Specifically, $\mathcal{H}^\epsilon$ is a contraction in the Banach space $\mathcal{P}^\epsilon$ of complex valued measures m satisfying the condition $|\int_{\mathbb{R}^+} \mathrm{e}^{-\upsilon_\epsilon x} m(\mathrm{d}x)| < \infty$, with respect to the norm $\|\pi - \pi'\|_\epsilon := \sup_{\Upsilon_\epsilon} |\pi(f) - \pi'(f)|$ where the supremum is taken over the subset $\Upsilon_\epsilon := \{f \in L^0(\mathbb{C}) : |f(x)| \leq |\mathrm{e}^{-\upsilon_\epsilon x}| \ \forall x \in \mathbb{R}^+\}$ of the set $L^0(\mathbb{C})$ of complex-valued Borel-functions with domain $\mathbb{R}^+$. Thus, also in the case $\lambda = \lambda^*$, Banach's Contraction Theorem yields the existence of a unique probability measure satisfying (5.10), and the proof is complete. □

Proof of Theorem 2.6. Let λ in $(0, \lambda^*]$ be arbitrary. In Lemma 5.1 it is shown that μ_λ is the Laplace transform of the probability measure μ_λ Furthermore, it follows by combining Propositions 5.2 and 5.3 that the probability measure μ_λ is the unique λ-invariant distribution for the process $\{X_t, t < \tau_0^X\}$. □

6. Mixed-exponential Lévy processes

We next identify explicitly the quasi-invariant distributions for the class of mixed-exponential Lévy processes that are killed upon first entrance into the negative half-axis. We recall that a *mixed-exponential Lévy process* $X = \{X_t, t \in \mathbb{R}^+\}$ is a jump-diffusion given by

$$X_t = X_0 + \eta t + \sigma W_t + \sum_{j=1}^{N_t} U_j, \qquad t \in \mathbb{R}^+, \tag{6.1}$$

where W is a Wiener process, $\eta \in \mathbb{R}$ and $\sigma > 0$ denote the drift and the volatility, and N is a Poisson process with rate ℓ that is independent of W. The series $(U_j)_{j\in\mathbb{N}}$ consists of IID random variables that are independent of W and N and follow a *double-mixed-exponential distribution*, which is a probability distribution on $\mathbb{R}$ with PDF given by

$$f(x) = pf_+(x) + (1-p)f_-(x), \qquad \text{with} \qquad f_\pm(x) = \sum_{k=1}^{m_\pm} a_k^\pm \alpha_k^\pm \mathrm{e}^{-\alpha_k^\pm |x|} \mathbf{1}_{\mathbb{R}^+}(\pm x), \qquad x \in \mathbb{R},$$

where p is a number in the unit interval $[0,1]$ and f_+ and f_- are themselves probability density functions that are linear combinations of m^+ and m^- exponentials respectively, with real-valued weights $a_1^+, \dots, a_{m^+}$ and $a_1^-, \dots, a_{m^-}^-$ and strictly positive parameters $\alpha_1^+, \dots, \alpha_{m^+}^+$ and $\alpha_1^-, \dots, \alpha_{m^-}^-$. To ensure that the functions f_+ and f_- are PDFs the parameters $\{a_k^\pm, k = 1, \dots, m^\pm\}$ need to satisfy certain restrictions; necessary and sufficient conditions for f_+ and f_- to be PDFs are

$$p_1^\pm > 0, \qquad \sum_{k=1}^{m^\pm} p_k^\pm \alpha_k^\pm \geq 0 \qquad \text{and} \qquad \sum_{k=1}^{l} p_k^\pm \alpha_k^\pm \geq 0 \quad \forall l = 1, ..., m^\pm,$$

respectively (see Bartholomew [4]).

The characteristic exponent of the Lévy process $X - X_0$ is given by

$$\Psi(\theta) = -\frac{\sigma^2}{2}\theta^2 + \mathtt{i}\eta\theta + p\sum_{k=1}^{m^+} a_k^+ \frac{\mathtt{i}\theta}{\alpha_k^+ - \mathtt{i}\theta} - (1-p)\sum_{j=1}^{m^-} a_j^- \frac{\theta\mathtt{i}}{\alpha_j^- + \theta\mathtt{i}}.$$

As the function Ψ is rational, it admits an analytical continuation to the complex plane omitting the finite set $\{-\mathtt{i}\alpha_1^+, \dots, -\mathtt{i}\alpha_{m^+}^+, \mathtt{i}\alpha_1^-, \dots, \mathtt{i}\alpha_{m^-}^-\}$, which is again denoted by Ψ. The mixed-exponential Lévy process satisfies Assumption 2.5 precisely if the parameters satisfy the restiction

$$\psi'(0) = \eta + p\sum_{k=1}^{m^+} \frac{a_k^+}{\alpha_k^+} - (1-p)\sum_{k=1}^{m^-} \frac{a_k^-}{\alpha_k^-} < 0. \tag{6.2}$$

The Wiener-Hopf factors associated to X are given by (from Lewis & Mordecki [28])

$$\Psi^+(q,\theta) = \frac{1}{\left(1 - \frac{\mathtt{i}\theta}{\rho_0^+(q)}\right)} \prod_{k=1}^{m^+} \frac{\left(1 - \frac{\mathtt{i}\theta}{\alpha_k^+}\right)}{\left(1 - \frac{\mathtt{i}\theta}{\rho_k^+(q)}\right)}, \qquad \Psi^-(q,\theta) = \frac{1}{\left(1 - \frac{\mathtt{i}\theta}{\rho_0^-(q)}\right)} \prod_{k=1}^{m^-} \frac{\left(1 + \frac{\mathtt{i}\theta}{\alpha_k^-}\right)}{\left(1 - \frac{\mathtt{i}\theta}{\rho_k^-(q)}\right)}, \tag{6.3}$$

for $q > 0$, where $\rho_k^+(q), k = 0, \dots, m^+$, and $\rho_j^-(q), j = 0, \dots, m^-$, are the roots of the Cramér-Lundberg equation

$$\Psi(-\mathtt{i}\theta) - q = 0 \tag{6.4}$$

with positive and negative real parts, respectively (where multiple roots are listed as many times as their multiplicity). By analytical continuation and continuous extension it follows that the expressions in (6.3) remains valid for $q \in [-\lambda^*, 0]$. The quasi-invariant distributions are expressed in terms of these ingredients as follows:

Proposition 6.1. *For any $\lambda \in (0, \lambda^*]$*

$$\widehat{\mu}_\lambda(\theta) = \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda) + \theta} \cdot \frac{\rho_0^+(-\lambda)}{\rho_0^+(-\lambda) + \theta} \prod_{j=1}^{m^+} \frac{\left(1 + \frac{\theta}{\alpha_j^+}\right)}{\left(1 + \frac{\theta}{\rho_j^+(-\lambda)}\right)}. \tag{6.5}$$

is the Laplace transform of the λ-invariant probability distribution μ_λ of $\{X_t, t < \tau_0^X\}$, where $\rho_k^+(-\lambda), k = 0, \dots, m^+$, denote the roots ρ of $\Psi(-\mathtt{i}\rho) = -\lambda$ with $\Re(\rho) > \bar{\phi}(-\lambda)$.

In Appendix A we present a self-contained verification of the λ-invariance of μ_λ using residue calculus.

Remark 6.2. In the case that the roots $\rho_k^+(-\lambda)$ are all distinct the probability measure μ_λ is a mixed-exponential distribution that can be obtained from the Laplace transform $\widehat{\mu}_\lambda$ by partial fraction decomposition and termwise inversion:

$$\mu_\lambda(\mathrm{d}x) = \mathbf{1}_{\mathbb{R}^+}(x) \cdot m_\lambda(x)\mathrm{d}x, \qquad m_\lambda(x) = A_0^- \bar{\phi}(-\lambda)\mathrm{e}^{-\bar{\phi}(-\lambda)x} + \sum_{k=0}^{m_+} A_k^+ \rho_k^+(-\lambda)\mathrm{e}^{-\rho_k^+(-\lambda)x}, \tag{6.6}$$

Here, the constants A_k^+, $k = 0, \ldots, m_+$, and $A_0^- := A_{-1}^+$ are given by

$$A_k^+ = \left(1 - \frac{\rho_k^+(-\lambda)}{\alpha_k^+}\right) \cdot \prod_{j=-1, j\neq k}^{m^+} \frac{\left(1 - \frac{\rho_k^+(-\lambda)}{\alpha_j^+}\right)}{\left(1 - \frac{\rho_k^+(-\lambda)}{\rho_j^+(-\lambda)}\right)}, \tag{6.7}$$

where $\rho_{-1}^+(-\lambda) := \bar{\phi}(-\lambda)$ and the constants α_{-1}^+ and α_0^+ are to be taken equal to $+\infty$ (so that the factors $(1 + \rho_k^+(-\lambda)/\alpha_0^+)$ and $(1 + \rho_k^+(-\lambda)/\alpha_{-1}^+)$ in the product are equal to 1).

Remark 6.3. The class of mixed-exponential Lévy processes is dense in the class of all Lévy processes (in the sense of weak convergence in the Skorokhod topology J_1 on the Skorokhod space $D(\mathbb{R})$), which can be seen as follows. It is well known (see e.g. Jacod & Shiryaev [21, Cor. VII.3.6]) that a sequence of Lévy processes converges weakly precisely if the values at time $t = 1$ converge in distribution. The corresponding infinitely divisible distributions may be approximated arbitrarily closely by a sequence of compound Poisson distributions $\mathrm{CP}(F_n, \ell_n)$ where the distributions F_n may be chosen to be double-mixed exponential distributions as the latter form a dense class in the sense of weak-covergence in the set of all probability distributions on the real line (see Botta & Harris [8]).

7. Application to credit-risk modeling

With the results on inverse first-passage time problems in hand, we next turn to an application of these results to the problem of counterparty risk valuation. As noted in the introduction, in the structural approach that was initially proposed by Black & Cox [6] the time of default of a firm is defined as the first epoch that the value of the firm falls below the value of its debt, which in the setting of [6] is equal to the first-hitting time of a geometric Brownian motion to some level. Subsequent studies such as [1, 19] present stylized 'default barrier models' for the time of default as the epoch of first-passage of a stochastic process over a default-barrier.

A Credit Default Swap (CDS) is a commonly traded financial contract that provides insurance against the event that a specific company defaults on its financial obligations. An important problem for a financial institution is to ensure that the model-values of traded credit derivatives (such as the CDS) that are recorded in its books are consistent with market quotes. In a default-barrier model for the value of the CDS one is led to the inverse problem of identifying the boundary that will equate model- and market-values.

Apart from featuring in the valuation of credit derivatives such as the CDS, the credit risk of a company may also affect the value of other assets in the portfolio, especially in the cases where the company in question acts as counterparty in a trade. The quantification of this type of risk, named *counterparty risk*, requires the joint modeling of asset values and the risk of default of the company in question (see Cesari *et al.* [11] for background on counterparty risk). Various aspects of the modeling of counterparty risk in default barrier models have been investigated for instance in [7, 10, 16, 29, 30, 33]; in these papers the model and market quotes are matched by calibration of the model parameters. Next we present an explicit example of the valuation of a call option under counterparty risk in a default-barrier model that is *by construction* consistent with a given risk-neutral probability of default, using the solution to the RIFPT problem given in Corollary 2.7.

7.1. **Valuation of a call option under counterparty risk.** This problem involves three entities, a company A, whose stock price is denoted by S_t, a bank B and the bank's counterparty C. The problem under consideration is the fair valuation of the counterparty risk to B resulting from a transaction in which C has sold to B a European call option on the stock of company A. We consider the situation where only C is default risky while A and B are free of default risk—in the finance literature the call option is in this case referred to as a *vulnerable* call option

(first labelled such by Johnson & Stultz [24]; see also Jarrow & Turnbull [23] for an application to zero-coupon bond valuation). Then B, as the owner of the call option, is exposed to counterparty risk, namely the potential loss that is incurred if its counterparty C goes into default before the maturity T of the call option, and fails to deliver the pay-off of the call option. If τ denotes the epoch of default of C then the fair value π of the potential loss of the holder of the option (discounted to time 0 at the risk-free rate r) and the so-called *expected positive exposure* P_t are given by

$$\Pi = E[V_\tau \mathbf{1}_{\{\tau \le T\}}], \tag{7.1}$$

$$P_t = E[V_\tau | \tau = t], \qquad t \in [0, T], \tag{7.2}$$

where V_τ denotes the value at time τ of a T-maturity call-option with strike K on the value of stock, discounted to time 0:

$$V_\tau = \mathrm{e}^{-r\tau} E[\mathrm{e}^{-r(T-\tau)}(S_T - K)^+ | \mathcal{F}_\tau]. \tag{7.3}$$

The conditional expectation in (7.2) is understood as the regular version of the conditional expectation $E[V_\tau|\tau]$ (under Assumption 7.1(iii) below this conditional expectation can just be defined in the usual way for continuous random variables). We will phrase the model in terms of two independent Lévy processes X and Z satisfying Assumption 2.5. Throughout this section we work under the following additional assumptions:

Assumption 7.1. (i) We have $\underline{\theta}_X < -1$, $\overline{\theta}_X > 1 + \alpha$, $\overline{\theta}_Z > 1 + \alpha$ for some $\alpha > 0$.

(ii) The CDF H has a continuous density h, and satisfies $\overline{H}(T) > 0$ and $\lambda_X^* > -\log \overline{H}(T)/H(T)$, where λ_X^* denotes the maximizer in (2.7).

(iii) For any $x > 0$ there exists a collection of measures $\{p_{t,x}(\mathrm{d}y), t \in \mathbb{R}^+\}$ on $(\mathbb{R}_-, \mathcal{B}(\mathbb{R}_-))$ satisfying $p_{t,x}(\mathrm{d}y)\mathrm{d}t = P(X_{\tau_{-x}^X} \in \mathrm{d}y, \tau_{-x}^X \in \mathrm{d}t)$.

Let the credit-worthiness of the counterparty C be described by the *distance-to-default* Y, in the sense that the default of C occurs at the first moment that the process Y falls below the level 0. We assume that the process Y is given in terms of X by

$$Y_t = Y_0 + X_{I(t)}, \qquad I(t) = I_{\mu_{\lambda^0}^X}(t) = T \cdot \frac{\log \overline{H}(t)}{\log \overline{H}(T)}, \qquad t \in [0, T], \tag{7.4}$$

$$Y_0 \sim \mu_{\lambda^0}^X, \qquad \lambda^0 = -T^{-1} \cdot \log \overline{H}(T), \tag{7.5}$$

where, as before, Y_0 is independent of X and $\mu_{\lambda^0}^X$ denotes the λ^0-invariant distribution of $\{X_t, t < \tau_0^X\}$. Here we have chosen λ^0 so as to normalise the ratio $I(T)/T$ to unity. Note that the CDF of the first-passage time τ_0^Y of the process Y defined in (7.4) is given by H (in view of Corollary 2.7 and Assumption 7.1(ii)).

In the case that the price process S is independent of credit index process Y we note that the expectation in (7.1) is just equal to the integral of the expectation $E[V_t]$ against the measure $H(\mathrm{d}t)$. Next we consider an instance of the complementary case that S and Y are dependent. More specifically, we assume that S is given by

$$\begin{cases} S_t = S_0 \exp\left\{(r-d)t + L_t - \kappa_{t,I(t)}(-\mathtt{i})\right\}, & t \in [0,T], \qquad S_0 > 0, \\ L_t = \rho X_{I(t)} + Z_t, & \rho \in [-1, 1], \\ \kappa_{t_1,t_2}(u) = \Psi_Z(u)t_1 + \Psi_X(u\rho)t_2, & \Im(u) \in [-1-\alpha, 0], \end{cases} \tag{7.6}$$

where Ψ_Z and Ψ_X denote the characteristic exponents of the Lévy processes X and Z and r and d denote the risk-free rate and the dividend yield, respectively. The degree of dependence between the stock price process S and the credit index process Y is controlled by the parameter ρ. Note that κ_t has been specified such that the discounted stock-price process with reinvested dividends $\mathrm{e}^{-rt}[\mathrm{e}^{dt}S_t]$ is a martingale. In the following result a semi-analytical expression is derived for π and $P(t)$ in terms of an inverse Fourier-transform $\mathcal{F}_\xi^{-1}$ and an inverse Laplace-transform $\mathcal{L}_q^{-1}$ with respect to ξ and q, respectively.

Proposition 7.2. *The values π and P_t, $t \in [0,T]$, are given by*

$$\Pi = \int_0^T N(t) \frac{h(t)}{\lambda^0 \overline{H}(t)} \mathrm{d}t, \qquad P_t = \frac{N(t)}{\lambda^0 \overline{H}(t)}, \tag{7.7}$$

$$N(t) = \mathrm{e}^{rT} \mathcal{F}_\xi^{-1} \left(\frac{D_{t,T}(u)\, C_t(u)}{u(u-1)} \right)(k), \qquad u = 1 + \alpha + \mathtt{i}\xi, \tag{7.8}$$

$$C_t(u) = (\lambda^0 \overline{H}(t))^{-1} \exp\{(r-t)tu - \kappa_{t,I(t)}(-\mathtt{i})u + \Psi_Z(-\mathtt{i}u)t\} \times \mathcal{L}_q^{-1}(f_{\rho u}(q))(t) \tag{7.9}$$

$$D_{t,T}(u) = E[\mathrm{e}^{u(L_T - L_t)}] = \exp\{\kappa_{T,I(T)}(-\mathtt{i}u) - \kappa_{t,I(t)}(-\mathtt{i}u)\} \tag{7.10}$$

with $k = \log K/c'$, $c' = \exp(-rT + (r-d)(T-t) - \kappa_T(-\mathtt{i}) + \kappa_t(-\mathtt{i}))$.

The proof relies on the following auxiliary result:

Lemma 7.3. *For any u with $\Re(u) \in [0, \overline{\theta}_X)$ and $t \in [0,T]$ we have, with $\tau = \tau_0^Y$,*

$$E\left[\mathrm{e}^{uX_{I(\tau)}} \middle| \tau = t \right] = \frac{1}{\lambda^0 \overline{H}(t)} \mathcal{L}_q^{-1}\left(f_u(q)\right)\right)(I(t)), \qquad f_u(q) = \int_{\mathbb{R}^+} \mu_{\lambda^0}^X(\mathrm{d}x) E\left[\mathrm{e}^{uX_{\tau_{-x}^X} - q\tau_{-x}^X} \right], \tag{7.11}$$

where $f_u(q)$ is given in terms of the Wiener-Hopf factor Ψ^- of X in (4.14). In particular, for u satisfying in addition $\Re(u) \in [0, \overline{\theta}_Z \wedge \overline{\theta}_X/\rho)$ we have

$$E[S_\tau^u | \tau = t] = \frac{S_0^u}{\lambda^0 \overline{H}(t)} \cdot \exp\{(r-d)tu - \kappa_t(-\mathtt{i})u + \Psi_Z(-\mathtt{i}u)t\} \cdot (\mathcal{L}_q^{-1} f_{\rho u}(q))(t). \tag{7.12}$$

Proof of Lemma 7.3: The spatial homogeneity of the Lévy process X and the definition of the stopping time τ yield $P(X_{I(\tau)} \in \mathrm{d}y, \tau \in \mathrm{d}t) = \int \mu(\mathrm{d}x) P(X_{\tau_{-x}^X} \in \mathrm{d}y, \tau_{-x}^X \in \mathrm{d}I(t))$. Since the CDF of τ_0^Y is given by H, it follows thus by Bayes' lemma that the conditional expectation in the lhs of (7.11) can be expressed as

$$E[\mathrm{e}^{uX_{I(\tau)}} | \tau = t] = \frac{1}{h(t)} \int_{\mathbb{R}^+} \mu_{\lambda^0}^X(\mathrm{d}x) \int_{\mathbb{R}} \mathrm{e}^{ux} p_{I(t),x}(\mathrm{d}y) I'(t). \tag{7.13}$$

The form of the derivative $I'(t) = h(t)/[\lambda^0 \overline{H}(t)]$ then implies that the rhs of (7.13) and (7.11) are equal. The identity in (7.12) follows now as a direct consequence of the form of S in given in (7.6) and the independence of Z and τ. □

Proof of Proposition 7.2. Note first that the form of π is obtained by integrating P_t against $h(t)$ over the interval $[0,T]$ and performing the change of variables $u = I(t)$ and using the observation $I'(t) = h(t)/[\lambda \overline{H}(t)]$.

The independence of the increments of $\log S$ implies

$$P_t = E[G_{\tau,S_\tau}(k) | \tau = t], \qquad G_{t,s}(k) = s' \mathrm{e}^{-rT} \cdot E[(\mathrm{e}^{L_T - L_t} - \mathrm{e}^k)^+],$$
$$s' = sc', \qquad c' = \exp((r-d)(T-t) - \kappa_T(-\mathtt{i}) + \kappa_t(-\mathtt{i})), \qquad k = \log(K/s').$$

By a standard Fourier-transform argument it can be shown that $G_{t,s}(k)$ admits an explicit integral representation in terms of the characteristic exponents of X and Z. More specifically, since the dampened function $k \mapsto \exp(\alpha k) \cdot G_{t,s}(k)$ and its Fourier transform are integrable, the Fourier Inversion Theorem implies

$$G_{t,s}(k) = [\mathcal{F}_\xi^{-1}(G_{t,s}^\wedge)](k), \qquad G_{t,s}^\wedge(\xi) = s' \cdot D_{t,T}(1 + \alpha + \mathtt{i}\xi), \qquad \xi \in \mathbb{R}, \tag{7.14}$$

where $D_{t,T}(1+\alpha+\mathtt{i}\xi)$ is given in (7.10). By an interchange of the expectation and integration (justified by Fubini's theorem) we find that P_t, $t \in [0,T]$, is equal to

$$P_t = \frac{c'}{2\pi} \int_{\mathbb{R}} E[S_\tau^{1+\alpha+\mathtt{i}\xi} | \tau = t] \cdot \left(\frac{K}{c'} \right)^{-\alpha - \mathtt{i}\xi} D_{t,T}(\xi) \mathrm{d}\xi. \tag{7.15}$$

The expression for P_t in (7.8) then follows by inserting the expression in (7.12) in Lemma 7.3. □

7.2. **Extensions.** We end this section with a brief description of a number of possible extensions and related problems in the current model setting. Firstly, we mention that, in addition to the case of the call option that was considered above, it is of interest to value the counterparty risk for other instances of commonly traded securities in foreign exchange, fixed income, equity or commodity markets, such as swap contracts which are contracts involving regular payments of both parties that entered into the contract. Secondly, we recall that in the setting above it was assumed that parties A (the company that issued the stock) and B (the bank) were free of default risk. The case where two of three or all three parties are subject to default is a natural extension that is applicable in many situations. Such an extension may still be treated in the current setting deploying the solution of the multi-dimensional IFPT in Theorem 3.1. Finally, especially of interest to financial market practitioners will be the development of an efficient numerical implementation of the model. In the interest of brevity, these questions are left for future research.

Appendix A. Proof of quasi-invariance by residue calculus

In this section we provide an alternative proof of the $\lambda-$invariance of the probability measure μ_λ based on an application of Cauchy's Residue Theorem to the integral identity (4.15), restated here for convenience:

$$\widehat{\mu}(\theta)\cdot\frac{q}{q+\lambda}=\Psi^+(q,\mathtt{i}\theta)\cdot\frac{1}{2\pi\mathtt{i}}\int_{a-\mathtt{i}\infty}^{a+\mathtt{i}\infty}\widehat{\mu}(-u)\Psi^-(q,-\mathtt{i}u)\frac{\mathrm{d}u}{u+\theta},\qquad q>0. \tag{A.1}$$

According to Proposition 4.4, the identity in (A.1) is satisfied by any λ-invariant distribution of $\{X_t, t<\tau_0^X\}$. We show below that μ_λ satisfies the identity in (A.1) for any fixed $q>0$. For the ease of presentation we restrict to the case that both the roots ρ of the equation $\Psi(-\mathtt{i}\rho)=-\lambda$ and those of the equation $\Psi(-\mathtt{i}\rho)=q$ are distinct; the case of multiple roots can be dealt with by similar arguments.

Let us describe the form of the integrand of the Bromwich integral in (A.1) in the case of a mixed-exponential Lévy process and $\mu=\mu_\lambda$. Since the positive Wiener-Hopf factor and the function $\widehat{\mu}_\lambda(\theta)$ are both rational (*cf.* (2.10) and (6.3)) also the function $f:\mathbb{C}^+\to\mathbb{C}$ given by

$$f(u)=f_{\theta,\lambda,q}(u)=\frac{\Psi^+(q,\mathtt{i}\theta)\widehat{\mu}_\lambda(-u)\Psi^-(q,-\mathtt{i}u)}{u+\theta} \tag{A.2}$$

is rational, for any triplet (θ,λ,q) with $\theta\in(\underline{\theta},\overline{\theta})$, $\lambda\in(0,\lambda^*]$ and $q>0$. Moreover, the collection of poles of f is finite and given by $\mathcal{P}^+\cup\mathcal{P}^-$ with

$$\mathcal{P}^+=\{\bar{\phi}(-\lambda)\}\cup\{\rho_k^+(-\lambda);k=0,\dots,m^+\}\subset\mathbb{C}^{++},\ \mathcal{P}^-=\{-\theta,\rho_j^-(q),j=0,\dots,m^-\}\subset\mathbb{C}^{--}, \tag{A.3}$$

where we denote $\mathbb{C}^{--}:=\{u\in\mathbb{C}:\Re(u)<a\}$ and $\mathbb{C}^{++}:=\{u\in\mathbb{C}:\Re(u)>a\}$ where a is some fixed arbitrary number in the interval $(0,\bar{\phi}(-\lambda))$.

Denote by $\mathcal{C}_T^+$ the contour with clockwise orientation consisting of the segment $\mathcal{I}_T=\{u\in\mathbb{C}:\Im(u)\in[-T,T],\Re(u)=a\}$ and the semi-circle that joins $a-\mathtt{i}T$ and $a+\mathtt{i}T$ such that $\mathcal{C}_T^+$ is contained in the set $\{u\in\mathbb{C}:\Re(u)\geq a\}$. For T sufficiently large, the contour $\mathcal{C}_T^+$ encloses all the poles in the set $\mathcal{P}^+$. Next we evaluate the contour integral of f over the curve $\mathcal{C}_T^+$.

Lemma A.1. *Assume that all the elements of the sets $\mathcal{P}^+$ and $\mathcal{P}^-$ are distinct. Then, for any $T>0$ sufficiently large, and any $q,\theta\in\mathbb{R}^+\backslash\{0\}$ and $\lambda\in(0,\lambda^*]$ we have*

$$I_o^+(T):=\oint_{\mathcal{C}_T^+}f=\frac{q}{q+\lambda}\widehat{\mu}_\lambda(\theta), \tag{A.4}$$

where f is given in (A.2)*. Furthermore, we have*

$$\frac{1}{2\pi\mathtt{i}}\int_{a-\mathtt{i}\infty}^{a+\mathtt{i}\infty}f(u)\mathrm{d}u=\frac{q}{q+\lambda}\widehat{\mu}_\lambda(\theta),\qquad a\in(0,\bar{\phi}(-\lambda)). \tag{A.5}$$

In particular, for any $\lambda\in(0,\lambda^*]$, μ_λ satisfies the identity in (A.1)*.*

Remark A.2. The identity in (A.4) is also valid if one replaces $\mathcal{C}_T^+$ by the contour $\mathcal{C}_T^-$ consisting of the segment $\mathcal{I}_T$ and the semi-circle that joins $a-\mathtt{i}T$ and $a+\mathtt{i}T$ such that $\mathcal{C}_T^-$ is contained in the set $\{u\in\mathbb{C}:\Re(u)\leq a\}$ (see Figure 2). This fact is shown by arguments that are analogous the ones given below in the proof of Lemma A.1.

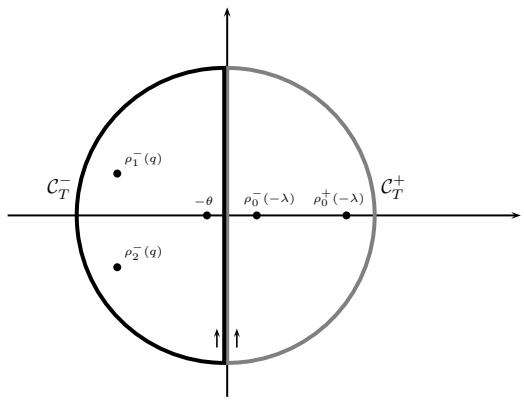

FIGURE 2. Pictured is the complex plane with an example of the two contours $\mathcal{C}_T^+$ (grey) and $\mathcal{C}_T^-$ (black) and the poles in $\mathcal{P}^+$ and $\mathcal{P}^-$. The contour $\mathcal{C}_T^+$ has a clockwise orientation and encloses the poles $p \in \mathcal{P}^+$ while the contour $\mathcal{C}_T^-$ has anti-clockwise orientation and encloses the poles $p \in \mathcal{P}^-$.

Proof of Lemma A.1. By Cauchy's Residue Theorem the integral $I_o^+(T)$ of the function f over the curve $\mathcal{C}_T^+$ is for all T sufficiently large equal to

$$I_o^+(T) = \frac{1}{2\pi \mathtt{i}} \sum_{p \in \mathcal{P}^+} n(\mathcal{C}_T^+, p) \mathrm{Res}_p(f) \tag{A.6}$$

where $\mathrm{Res}_p(f)$ denotes the residue of the function f at the pole p and, for any $p \in \mathbb{C}$ and any curve $\Gamma : [0, 2\pi] \to \mathbb{C}$, $n(\Gamma, p)$ denotes the winding number of Γ around p. Note that we have $n(\mathcal{C}_T^+, p) = -1$ for any $p \in \mathcal{P}^+$ (see Figure 2).

Since by assumption the poles are all distinct we find that the residues at the poles $p \in \mathcal{P}^+$ satisfy

$$\mathrm{Res}_p(f) = 2\pi \mathtt{i} \cdot \lim_{s \to p} (s - p) f(s), \quad p \in \mathcal{P}^+. \tag{A.7}$$

Inserting the explicit form of f into (A.7) we find by straightforward algebra

$$\Psi^+(q, \mathtt{i}\theta)^{-1} \frac{\mathrm{Res}_p(f)}{2\pi \mathtt{i}} = -A^+(p) \cdot \frac{p}{p + \theta}, \qquad p \in \mathcal{P}^+, \qquad \text{with} \tag{A.8}$$

$$A^+(p) = \Psi^-(q, -\mathtt{i}p) \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda) - p} \frac{\prod_{k=1}^{m^+} \left(1 - \frac{p}{\alpha_k^+}\right)}{\prod_{k=0, k \neq j}^{m^+} \left(1 - \frac{p}{\rho_k^+(-\lambda)}\right)}. \tag{A.9}$$

By using these explicit expressions we next verify that the following key-identity holds true:

$$\frac{1}{2\pi \mathtt{i}} \sum_{p \in \mathcal{P}^+} (-1) \cdot \frac{\mathrm{Res}_p(f)}{\Psi^+(q, \mathtt{i}\theta)} = R(\theta) := \frac{q}{q + \lambda} \frac{\widehat{\mu}_\lambda(\theta)}{\Psi^+(q, \mathtt{i}\theta)}. \tag{A.10}$$

This key-identity immediately follows from (A.8) and the following partial-fraction decomposition of $R(\theta)$:

$$R(\theta) = \frac{q}{q + \lambda} \left[\sum_{j=0}^{m^+} A^+(\rho_j^+(-\lambda)) \frac{\rho_j^+(-\lambda)}{\rho_j^+(-\lambda) + \theta} + A^+(\bar{\phi}(-\lambda)) \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda) + \theta} \right], \tag{A.11}$$

where the coefficients $A^+(\bar{\phi}(-\lambda))$ and $A^+(\rho_j^+(-\lambda))$, $j = 0, \ldots, m^+$ are given by (A.9).

We next show that (A.11) holds in two steps.

(a) As first step we record the relation

$$\Psi^-(q, -\mathtt{i}\rho_j^+(-\lambda)) = \frac{q}{q + \lambda} \Psi^+(q, -\mathtt{i}\rho_j^+(-\lambda))^{-1}, \qquad \lambda \in (0, \lambda^*]. \tag{A.12}$$

To see why this holds true, note that, for any $q > 0$ and $\lambda \in (0, \lambda^*]$, it follows by analytical extension that the Wiener-Hopf identities in (4.3) remains valid for any $\theta \in \mathbb{C}$ except some finite set (namely, the sets of roots ρ of the equation $\Psi(\rho) = q$). Substituting $\theta \to -\mathtt{i}p$ ($p \in \mathcal{P}^+$) in (4.3) and using that by definition $\Psi(-\mathtt{i}\rho_j^+(-\lambda)) = -\lambda$ we obtain the relation (A.12).

(b) Inserting the explicit forms of $\Psi^+(q, \mathtt{i}\theta)$ and $\widehat{\mu}_\lambda(\theta)$ (given in (6.3) and (6.5)) into (A.10) we find

$$R(\theta) = \frac{q}{q+\lambda} \cdot \frac{\bar{\phi}(-\lambda)}{\bar{\phi}(-\lambda)+\theta} \prod_{k=0}^{m^+} \frac{1+\frac{\theta}{\rho_k^+(q)}}{1+\frac{\theta}{\rho_k^+(-\lambda)}}.$$

It is a matter of algebra to verify that $R(\theta)$ admits a partial-fraction decomposition of the form (A.11) for some coefficients $A^+(\bar{\phi}(-\lambda))$ and $A^+(\rho_j^+(-\lambda))$, $j = 0, \ldots, m^+$. Furthermore, by deploying the identity on the lhs of (A.12), it is easy to show that these coefficients are equal to the expression given in (A.8), so that (A.11) is established.

Combining (A.6) and (A.10) shows that for all T sufficiently large we have

$$I_o^+(T) = \frac{q}{q+\lambda}\widehat{\mu}_\lambda(\theta).$$

Finally, we note that the integral $I_c^+(T)$ over the semi-circles only (that is, over $\mathcal{C}_T^+ \backslash \mathcal{I}_T$) tends to zero as $T \to \infty$, since the length of the semi-circles $\mathcal{C}_c^+(T)$ is proportional to T while f we have the bound $\max_{u \in \mathcal{C}_c^+(T)} |f(u)| \leq C^+/T^2$ for some constant $C^+ > 0$. Thus, we conclude that $I_o^+(T)$ converges to the rhs of (A.5) as T tends to infinity, and the proof is complete. □

References

[1] Avellaneda, M. and Zhu, J. (2001) Modeling the distance-to-default of a firm. *Risk* 14, 125–129.

[2] Alili, L. and Kyprianou, A. E. (2005) Some remarks on first-passage of Lévy process, the American put and pasting principles. *Annals of Applied Probability* 15, pp. 2062–2080.

[3] Applebaum, D. (2004) *Lévy processes and Stochastic Calculus*. Cambridge University Press.

[4] Bartholomew, D.J. (1969), Sufficient conditions for a mixture of exponentials to be a probability density function. *Annals of Mathematical Statistics*, 40, 2183–2188.

[5] Bertoin, J. (1996) *Lévy Processes*, Cambridge University Press.

[6] Black, F. and Cox, J. C. (1976), Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance* 31, 315–376.

[7] Blanchet-Scalliet, C., and Patras, F. (2008) Counterparty risk valuation for CDS. `arXiv:0807.0309`

[8] Botta, R.F. and Harris, C.M. (1986) Approximation with generalized hyper-exponential distributions: Weak convergence results. *Queueing Systems* 1, 169–190.

[9] Brigo, D. and Mercurio, F. (2006) *Interest Rate Models: Theory and Practice*. Springer.

[10] Brigo, D., and Tarenghi, M. (2005) Credit default swap calibration and counterparty risk valuation with a scenario based first passage model, `SSRN.com/abstract=683160.`

[11] Cesari, G., Aquilina, J., Charpillon, N., Filipovic, Z., Lee, G., Manda, I. (2009) *Modeling, Pricing, and Hedging Counterparty Credit Exposure. A Technical Guide*. Springer.

[12] Chen, X., Cheng, L., Chadam, J., and Saunders, D. (2011) Existence and uniqueness of solutions to the inverse boundary crossing problem for diffusions. *Annals of Applied Probability* 21, 1663–1693.

[13] Cheng, L., Chen, X., Chadam, J., and Saunders, D. (2006) Analysis of an inverse first passage problem from risk management. *SIAM Journal of Mathematical Analysis* 38, 845–873.

[14] Cont, R. and Tankov, P. (2004) *Financial Modeling with Jump Processes*. Chapman & Hall/CRC.

[15] Davis, M.H.A. and Pistorius, M.R. (2010) Quantification of counterparty risk via Bessel bridges. `SSRN.com/abstract=1722604.`

[16] Ettinger, B., Evans, S. N., and Hening, A. (2014) Killed Brownian motion with a prescribed lifetime distribution and models of default. *Annals of Applied Probability* 24, 1–33.

[17] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol II. Wiley.

[18] Huang, H. and Tian, W. (2006) Constructing default boundaries. *Bankers, Markets & Investors* 80, 21–28.

[19] Hull, J. and White, A. (2001) Valuing credit default swaps II: Default correlations. *Journal of Derivatives* 8, 12–22.

[20] Jackson, K., Kreinin, A., Zhang W. (2009) Randomization in the first hitting time problem. *Statistics and Probability Letters* 79, 2422–2428.

[21] Jacod, J. and Shiryaev, A.N. (1987) *Limit Theorems for Stochastic Processes*, Springer.

[22] Jaimungal, S., Kreinin, A. and Valov, A. (2009) Integral equations and the first passage time of Brownian motions. `arXiv:0902.2569`

[23] Jarrow, R. A. and Turnbull, S. (1995) Pricing derivatives on financial securities subject to credit risk. *Journal of Finance* 50, 53–85.

[24] Johnson, H. and Stultz, R. (1987) The pricing of options with default risk. *Journal of Finance* 42, 267–280.

[25] Kuznetsov, A. (2010) An analytical proof of the Pecherskii-Rogozin identity and the Wiener-Hopf factorisation. *Theory of Probability and its Applications* 55, 432–443.

[26] Kalbfleisch, J. D. and Prentice, R. L. (2002) *The Statistical Analysis of Failure Time Data*, Wiley.

[27] Kyprianou, A.E. (2006) *Introductory Lectures on Fluctuations of Lévy Processes, with Applications*, Springer.

[28] Lewis, A.L. and Mordecki, E. (2008) Wiener-Hopf factorisation for Lévy processes having positive jumps with rational transforms. *Journal of Applied Probability* 6, 118–134

[29] Lipton, A. and Sepp, A. (2009) Credit value adjustment for credit default swaps via the structural default model. *Journal of Credit Risk*, 5, 123–146.

[30] Lipton, A. and Savescu, I. (2013) CDSs, CVA and DVA - a structural approach. *Risk* April 2013, 56–61.

[31] Longstaff, F. and Schwartz, E. (1995) A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance* 50, 789–819.

[32] Merton, R.C. (1974) On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance* 29, 449–470.

[33] Overbeck, L. and Schmidt, W. (2005) Modeling default dependence with threshold models. *Journal of Derivatives* 12, 10–19.

[34] Pecherskii, E.A. and Rogozin, B.A. (1969) The combined distributions of the random variables connected with the fluctuations of a process with independent increments. *Teor. Verojatnost. i Primenen.* **14**, 431–444.

[35] Peskir, G. (2002) On integral equations arising in the first-passage problem for Brownian motion. *Journal of Integral Equations and Applications* 14, 397–423.

[36] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press.

[37] Song, J.-S. and Zipkin, P. (2011) An approximation for the inverse first passage problem. *Advances in Applied Probability* 43, 264–275.

[38] Zucca, C. and Sacerdote, L. (2009) On the inverse first-passage time problem for a Wiener process. *Annals of Applied Probability* 19(4), 1319–1346.

Imperial College London, Department of Mathematics
E-mail address: {mark.davis, m.pistorius}@imperial.ac.uk