MULTIDIMENSIONAL STICKY BROWNIAN MOTIONS AS LIMITS OF EXCLUSION PROCESSES

MIKLÓS Z. RÁCZ AND MYKHAYLO SHKOLNIKOV

Abstract. We study exclusion processes on the integer lattice in which particles change their velocities due to stickiness. Specifically, whenever two or more particles occupy adjacent sites, they stick together for an extended period of time and the entire particle system is slowed down until the "collision" is resolved. We show that under diffusive scaling of space and time such processes converge to what one might refer to as a sticky reflected Brownian motion in the wedge. The latter behaves as a Brownian motion with constant drift vector and diffusion matrix in the interior of the wedge, and reflects at the boundary of the wedge after spending an instant of time there. In particular, this leads to a natural multidimensional generalization of sticky Brownian motion on the half-line, which is of interest in both queueing theory and stochastic portfolio theory. For instance, this can model a market, which experiences a slowdown due to a major event (such as a court trial between some of the largest firms in the market) deciding about the new market leader.

1. Introduction

Stochastic processes with sticky points in the Markov process sense have been studied for more than half a century. Sticky Brownian motion on the half-line is the process evolving as a standard Brownian motion away from zero and reflecting at zero after spending an instant of time there—as opposed to a reflecting Brownian motion, which reflects instantaneously. This process was initially studied by Feller [10, 11], and Itô and McKean [18, 17] in a more general context, and was subsequently analyzed in more detail by several further authors [12, 1]. These papers show that sticky Brownian motion arises as a time change of a reflecting Brownian motion, and that it describes the scaling limit of random walks on the natural numbers whose jump rate at zero is significantly smaller than the jump rates at positive sites.

In stochastic analysis the stochastic differential equation (SDE)

$$\mathrm{d}S(t) = \mathbf{1}_{\{S(t)>0\}}\,\mathrm{d}B(t) + \eta\,\mathbf{1}_{\{S(t)=0\}}\,\mathrm{d}t \tag{1.1}$$

satisfied by sticky Brownian motion has drawn much attention, as it is an example of a SDE for which weak existence and uniqueness hold, but strong existence and pathwise uniqueness fail (see [6]). In fact, in [29] (see also the survey [8]) it is shown that a weak solution to (1.1) cannot be adapted to a cozy filtration, that is, a filtration generated by a finite or infinite-dimensional Brownian motion.

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The present study is motivated by the question of how one can define and analyze multidimensional analogues of (1.1) and whether solutions to the corresponding systems of SDEs arise as suitable scaling limits of interacting particle systems in analogy to the findings of [12] in the one-dimensional case. In [20, Section 3] it is shown that a large class of reflecting Brownian motions in the n-dimensional wedge

$$\mathcal{W} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_1 \leq x_2 \leq \ldots \leq x_n\}$$

arise as limits of certain exclusion processes with speed change under diffusive rescaling. As we show below, sticky Brownian motions with state space $\mathcal{W}$ can also be obtained as scaling limits of suitable exclusion processes with speed change.

1.1. **Exclusion processes with sticky particles.** To simplify the exposition, we next describe a simple class of particle systems which converge to sticky Brownian motions in $\mathcal{W}$ in the scaling limit, and postpone the description of the much wider class of particle systems that we can handle to section 3. We fix the number of particles $n \in \mathbb{N}$, and also rate parameters $a > 0$, $\Theta^L = \left(\theta^L_{i,j}\right)_{i\in[n],\, j\in[n-1]} \in [0,\infty)^{n\times(n-1)}$, and $\Theta^R = \left(\theta^R_{i,j}\right)_{i\in[n],\, j\in[n-1]} \in [0,\infty)^{n\times(n-1)}$ with the notation

$$[n] = \{1, 2, \ldots, n\}.$$

For a fixed value of the scaling parameter $M > 0$, the particles move on the rescaled lattice $\mathbb{Z}/\sqrt{M}$; to describe their motion we introduce the following Poisson processes, all of which are independent, and all of which have jump size $\frac{1}{\sqrt{M}}$. For $i \in [n]$, the Poisson processes P_i and Q_i have jump rates Ma, while for $i \in [n]$, $j \in [n-1]$, the Poisson processes $L_{i,j}$ and $R_{i,j}$ have jump rates $\sqrt{M}\theta^L_{i,j}$ and $\sqrt{M}\theta^R_{i,j}$, respectively. In addition, for notational convenience we introduce ghost particles at $\pm\infty$, namely: $X^M_0(\cdot) \equiv -\infty$ and $X^M_{n+1}(\cdot) \equiv \infty$. For any initial condition $X^M_1(0) < X^M_2(0) < \ldots < X^M_n(0)$ on $\mathbb{Z}/\sqrt{M}$ we can then define a particle system evolving on $\mathbb{Z}/\sqrt{M}$ in continuous time by setting

$$\begin{aligned} \mathrm{d}X^M_i(t) = {} & \mathbf{1}_{\left\{X^M_k(t)+\frac{1}{\sqrt{M}}<X^M_{k+1}(t),\, k\in[n-1]\right\}} \mathrm{d}\left(P_i(t) - Q_i(t)\right) \\ & + \sum_{j=1}^{n-1} \mathbf{1}_{\left\{X^M_i(t)+\frac{1}{\sqrt{M}}<X^M_{i+1}(t),\, X^M_j(t)+\frac{1}{\sqrt{M}}=X^M_{j+1}(t)\right\}} \mathrm{d}R_{i,j}(t) \\ & - \sum_{j=1}^{n-1} \mathbf{1}_{\left\{X^M_{i-1}(t)+\frac{1}{\sqrt{M}}<X^M_i(t),\, X^M_j(t)+\frac{1}{\sqrt{M}}=X^M_{j+1}(t)\right\}} \mathrm{d}L_{i,j}(t), \end{aligned} \tag{1.2}$$

for $i \in [n]$. Note that (1.2) guarantees that for any $t \geq 0$, the particle configuration $(X^M_1(t), X^M_2(t), \ldots, X^M_n(t))$ is an element of the discrete wedge

$$\mathcal{W}^M = \left\{x \in \left(\mathbb{Z}/\sqrt{M}\right)^n : \ x_k + \frac{1}{\sqrt{M}} \leq x_{k+1},\ k \in [n-1]\right\}.$$

Intuitively, when apart, the particles move independently on the rescaled lattice $\mathbb{Z}/\sqrt{M}$ according to the processes $P_i - Q_i$, $i = 1, 2, \ldots, n$ (in particular, with jump rates of order M); however, when two particles land on adjacent sites—an event we describe as a "collision"—the system experiences a *slowdown*: the particles change their jump rates to the ones of the processes $L_{i,j}$ and $R_{i,j}$, $i \in [n]$, $j \in [n-1]$, which

are of order $\sqrt{M}$. The interaction between adjacent particles can be described as *stickiness*, as it takes a long time (on the time scale Mt) until the collision is resolved and the particles return to jump rates of order M.

1.2. **Convergence to multidimensional sticky Brownian motions.** The described particle systems converge to a sticky Brownian motion in $\mathcal{W}$ under the following assumption. Define the *speed change matrix* $V = (v_{i,j})_{i\in[n],\, j\in[n-1]}$ by setting

$$v_{i,j} := \begin{cases} \theta^R_{i,j} - \theta^L_{i,j} & \text{if } j \neq i-1, i, \\ \theta^R_{i,i-1} & \text{if } j = i-1, \\ -\theta^L_{i,i} & \text{if } j = i, \end{cases} \tag{1.3}$$

and the *reflection matrix* $Q = (q_{j,j'})_{j,j'\in[n-1]}$ by setting $q_{j,j'} = v_{j+1,j'} - v_{j,j'}$. When there is a collision between particles j' and $j'+1$ and no other collisions, then the velocity of particle i is given by $v_{i,j'}$, and the velocity of gap j between particles j and $j+1$ is given by $q_{j,j'}$. Define also the $(n-1)\times(n-1)^2$ matrix $Q^{(2)} = \left(q^{(2)}_{i,(k,\ell)}\right)_{i,k,l=1}^{n-1}$ according to

$$q^{(2)}_{i,(k,\ell)} := \begin{cases} -\theta^L_{i,\ell} & \text{if } k = i-1, \ell \neq i-1, \\ \theta^L_{i+1,\ell} + \theta^R_{i,\ell} & \text{if } k = i, \ell \neq i, \\ -\theta^R_{i+1,\ell} & \text{if } k = i+1, \ell \neq i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $q_{\cdot,j}$ denote the jth column of Q, let $q^{(2)}_{\cdot,(k,\ell)}$ denote the column of $Q^{(2)}$ indexed by (k,ℓ), and let $\mathcal{I}^{(2)} \subseteq [n-1]^2$ denote the set of pairs of indices (k,ℓ) such that $q^{(2)}_{\cdot,(k,\ell)}$ is the zero vector. Note that $(k,k) \in \mathcal{I}^{(2)}$ for all $k \in [n-1]$.

Assumption 1. (a) Assume that the matrix Q is *completely-$\mathcal{S}$*, in the sense that there is a $\lambda \in [0,\infty)^{n-1}$ such that $Q\,\lambda \in (0,\infty)^{n-1}$ and the same property is shared by every principal submatrix of Q (see [28] for several equivalent definitions).

(b) Assume that the matrices Q and $Q^{(2)}$ (restricted to nonzero columns) are *"jointly completely-$\mathcal{S}$"*, in the following sense. For a vector $u \in \mathbb{R}^k$ and $J \subseteq [k]$, let $u^J \in \mathbb{R}^{|J|}$ denote the vector obtained from u by removing all coordinates of u whose index is not in J. We assume that for every $J \subseteq [n-1]$, $J \neq \emptyset$, there exists $\gamma = \gamma(J) \in (\mathbb{R}_+)^{|J|}$ such that $\gamma \cdot q^J_{\cdot,j} \geq 1$ for every $j \in J$ and $\gamma \cdot q^{(2),J}_{\cdot,(k,\ell)} \geq 1$ for every $k,\ell \in J$, $(k,\ell) \notin \mathcal{I}^{(2)}$.

Under Assumption 1—which we discuss in more detail below—we have the following convergence result.

Theorem 1. *Suppose that Assumption* 1 *holds, and also that the initial conditions* $\left\{\left(X^M_1(0), X^M_2(0), \ldots, X^M_n(0)\right), M > 0\right\}$ *are deterministic and converge to a limit* $(x_1, x_2, \ldots, x_n) \in \mathcal{W}$ *as* $M \to \infty$. *Then the laws of the paths of the particle systems* $\left\{\left(X^M_1(\cdot), X^M_2(\cdot), \ldots, X^M_n(\cdot)\right), M > 0\right\}$ *on* $D([0,\infty), \mathbb{R}^n)$ *(the space of càdlàg paths with values in* $\mathbb{R}^n$ *endowed with the topology of uniform convergence on compact sets)*

converge to the law of the unique weak solution of the system of SDEs

$$\mathrm{d}X_i(t) = \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\sqrt{2a}\,\mathrm{d}W_i(t) + \sum_{j=1}^{n-1}\mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\,v_{i,j}\,\mathrm{d}t, \tag{1.4}$$

$i \in [n]$, in $\mathcal{W}$ starting from $(x_1, x_2, \ldots, x_n)$. Here $(W_1, W_2, \ldots, W_n)$ is a standard Brownian motion in $\mathbb{R}^n$.

The solution to (1.4) *evolves as a Brownian motion when away from the boundary $\partial\mathcal{W}$ of $\mathcal{W}$, it does not spend a non-empty time interval on $\partial\mathcal{W}$, however it satisfies*

$$\mathbb{P}\left(\mathcal{L}\left(\{t \geq 0 :\ X(t) \in \partial\mathcal{W}\}\right) > 0\right) = 1,$$

where $\mathcal{L}$ is the Lebesgue measure on $[0, \infty)$.

We refer to the solution of (1.4) with $a = 1/2$ as *sticky Brownian motion* in $\mathcal{W}$ with *reflection matrix* V. We choose this terminology because the SDE (1.4) generalizes one-dimensional sticky Brownian motion as in [12, 1], and also because it is consistent with the terminology used in [28] and the references therein dealing with instantaneously reflecting Brownian motions.

Regarding our assumptions, Assumption 1(a) is a natural condition, which is necessary for the existence of the limiting stochastic process (see Theorem 3). Assumption 1(b) (which is stronger; it implies Assumption 1(a)), however, is a technical condition; it is readily satisfied in many natural situations, but it is not a necessary condition for the convergence result to hold. For instance, Assumption 1 is satisfied in the natural case when $\theta^L_{i,j} = \theta^R_{i,j} = \theta > 0$ for all $i \in [n]$, $j \in [n-1]$. See also [3], where essentially the same condition is required and used in the proof of [3, Theorem 7.7], and where it is shown that this condition is satisfied if the reflection matrix Q satisfies the Harrison-Reiman condition [13] and some additional conditions hold. On the other hand, consider the case when $\theta^L_{j,j} = \theta^R_{j+1,j} = \theta > 0$ for all $j \in [n-1]$, and $\theta^L_{i,j} = \theta^R_{i,j} = 0$ otherwise; in words, suppose that when a collision occurs, all particles not part of the collision "freeze", i.e., they cannot move until the collision is resolved. It is not hard to see that Assumption 1(b) cannot hold in this case, although Assumption 1(a) holds, and we expect that the convergence result holds as well.

In section 3 we prove a much stronger result than Theorem 1, allowing for non-exponential interarrival times between the jumps in the processes P_i, Q_i, $L_{i,j}$, and $R_{i,j}$, as well as for dependence between the latter processes (see Theorem 7). This then leads to the definition of a sticky Brownian motion in $\mathcal{W}$ whose components have unequal drift and diffusion coefficients. In addition, it is not hard to see from the proof that for each jump parameter $\theta^L_{i,j}$ or $\theta^R_{i,j}$ which is zero, we can choose the jump rate of the corresponding process $L_{i,j}$ or $R_{i,j}$ to be of order $o\big(\sqrt{M}\big)$ (not necessarily identically zero) for the result of Theorem 1 to still hold.

One of the main technical difficulties in the proof of Theorem 1 and its extension (Theorem 7 in section 3) is posed by the indicator function appearing in the diffusion matrix of the limiting process. This is in contrast to the main convergence result in [20], where the martingale part of the limiting process is a Brownian motion. Another major difference compared to the setting of [20] is that we consider a large class

of completely-$\mathcal{S}$ reflection matrices and are dealing with weak solutions of the limiting stochastic differential equation; whereas in [20] only a special class of reflection matrices is considered, allowing for a pathwise construction of the limiting object. Finally, we allow for dependence of interarrival times between jumps for different particles in Theorem 7 below, which is not addressed in [20].

1.3. **Applications.** We mention two potential areas of applications for the process in (1.4) and its extensions that appear in section 2. It is known that reflected Brownian motions in $\mathcal{W}$ give a class of tractable descriptive models for the logarithmic market capitalizations (that is, the logarithms of the total market values of stocks) of firms in a large equity market (see, e.g., [20]). These models lead to realistic capital distribution curves in the long-run and are also able to produce a realistic pattern of collisions. In the same spirit, one can think of (1.4) as a model for the logarithmic market capitalizations in an equity market in which the market experiences a slowdown whenever there is a possibility that two firms will exchange their ranks (described by a collision). For example, one can imagine a court trial between two firms, the result of which decides which firm becomes the market leader, leading to a slowdown of the market right before the time of the verdict as the market participants await the result of the trial. The question of whether real-world equity markets spend a positive amount of time in collisions (so that the logarithmic market capitalizations should be modeled by the solution of (1.4)) or the set of times spent in collisions has zero Lebesgue measure (so that a reflecting Brownian motion in $\mathcal{W}$ is a more appropriate model) is a challenging statistical problem which should be addressed in future research.

Another area of applications is the study of diffusion approximations of storage and queueing networks. It is well-known (see, e.g., the survey [31] and the references therein) that reflected Brownian motions in the orthant describe the heavy traffic limits of many queueing networks such as open queueing networks, single class networks and feedforward multiclass networks. Moreover, the paper [30] discusses a situation where a customer of a single server queueing network receives *exceptional service* when the server is idle before his arrival and standard service when the server is busy prior to his arrival. The results in [30], as well as their extensions to more general exceptional service policies in [22] and [23] show that the heavy traffic limits of such networks are described by sticky Brownian motions on the half-line. For further information on queueing networks with exceptional service mechanisms we refer the reader to [33], [12] and [23]. Similarly, in the setting of a multi-server queueing network one can think of a situation where the servers provide exceptional service to a customer if the server was idle prior to his arrival, and where such exceptional service slows down the entire queueing network, e.g., due to a commonly used resource. In view of the aforementioned results in the single-server case, we expect *sticky Brownian motions in the orthant* $(\mathbb{R}_+)^{n-1}$, given by the spacings processes

$$\left(X_2(\cdot) - X_1(\cdot), X_3(\cdot) - X_2(\cdot), \ldots, X_n(\cdot) - X_{n-1}(\cdot)\right),$$

to arise as heavy-traffic limits of multi-server queueing networks with appropriate exceptional service policies. We also anticipate the tools developed in this paper to appear at the heart of the proofs of the corresponding heavy-traffic limit theorems.

In the case that the exceptional service by one of the servers does not affect other servers, we expect the heavy-traffic limit to be given by a sticky Brownian motion with a *local* rather than a *global* slowdown (see subsection 1.4 for further discussion).

1.4. **Future directions.** A natural direction for future work is to study other types of sticky interaction between particles. Even in the class of exclusion processes in one dimension, there are avenues to be explored. For instance, the exclusion processes described by (1.2) experience a *global* slowdown when a collision occurs, whereas for some applications it would be interesting to consider particle systems with *local* slowdown. We believe that the techniques we develop in section 3 would carry over to such a setting with appropriate modifications; however, the difficulty of proving convergence of such processes to the appropriate continuous object comes from proving uniqueness for the limiting SDE. We expect the solution of this SDE to spend a positive amount of time on lower-dimensional faces of the wedge $\mathcal{W}$, making the analysis of the process more difficult.

1.5. **Outline.** The rest of the paper is structured as follows. Section 2 is devoted to the study of sticky Brownian motions in $\mathcal{W}$. In subsection 2.1 we give the proof of existence and uniqueness of the weak solution to a system of SDEs generalizing (1.4). Then, in subsection 2.2 we show that the solution is a Markov process and study the invariant distributions of a suitably normalized version thereof. Subsequently, section 3 deals with the convergence of exclusion processes to sticky Brownian motions in $\mathcal{W}$. First in section 3.1 we prove Theorem 1, and then in section 3.2 we state and prove our main result, namely a generalized version of Theorem 1, which deals with the convergence of exclusion processes with non-exponential and possibly dependent jump interarrival times to sticky Brownian motions in $\mathcal{W}$.

2. Multidimensional sticky Brownian motions

This section is devoted to the study of the system of SDEs

$$\mathrm{d}X_i\,(t) = \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}}\,(b_i\,\mathrm{d}t + \mathrm{d}W_i(t)) + \sum_{j=1}^{n-1} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}}\,v_{i,j}\,\mathrm{d}t, \tag{2.1}$$

$i \in [n]$, where b_i, $i \in [n]$, are real constants, $W = (W_1, W_2, \ldots, W_n)$ is an n-dimensional Brownian motion with zero drift vector and a strictly positive definite diffusion matrix $\mathfrak{C} = (\mathfrak{c}_{i,i'})_{i,i'\in[n]}$, $V = (v_{i,j})_{i\in[n],\,j\in[n-1]}$ is a matrix with real entries, and the initial conditions $X_i\,(0) = x_i$, $i \in [n]$, satisfy $(x_1, x_2, \ldots, x_n) \in \mathcal{W}$. We note that the diffusion matrix of the process X is both discontinuous and degenerate, so neither existence nor uniqueness of a weak solution to (2.1) can be obtained directly from the classical results in [27] or [2].

2.1. **Existence and uniqueness.** In this subsection, we show that Assumption 1(a) is necessary and sufficient for the existence and uniqueness of a weak solution to (2.1). Furthermore, even under Assumption 1(a) one cannot expect a strong solution to exist. Our proof relies on the classical results of [28] on the existence and uniqueness of semimartingale reflecting Brownian motions in an orthant; this connection highlights

the importance of Assumption 1(a). We first recall the main definition and the main result from [28].

Definition 1. [28, Definition 1.1] Let $\eta \in \mathbb{R}^d$, let Γ be a $d \times d$ non-degenerate covariance matrix, let R be a $d \times d$ matrix, and for $i \in [d]$, let $F_i = \left\{ \widetilde{x} \in (\mathbb{R}_+)^d : \widetilde{x}_i = 0 \right\}$. For $\widetilde{x} \in (\mathbb{R}_+)^d$, a *semimartingale reflecting Brownian motion (SRBM) in the orthant* $(\mathbb{R}_+)^d$ *associated with the data* (η, Γ, R) *that starts from* $\widetilde{x}$ is a continuous, $(\mathcal{F}_t)$-adapted, d-dimensional process $\widetilde{Z}$ defined on some filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_{\widetilde{x}}\right)$ such that under $P_{\widetilde{x}}$,

$$\widetilde{Z}(t) = \widetilde{X}(t) + R\widetilde{Y}(t) \quad \in (\mathbb{R}_+)^d \qquad \text{for all } t \geq 0,$$

where

(i) $\widetilde{X}$ is a d-dimensional Brownian motion with drift vector η and covariance matrix Γ such that $\left\{ \widetilde{X}(t) - \eta t, \mathcal{F}_t, t \geq 0 \right\}$ is a martingale and $\widetilde{X}(0) = \widetilde{x}$ $P_{\widetilde{x}}$-a.s.,

(ii) $\widetilde{Y}$ is an $(\mathcal{F}_t)$-adapted, d-dimensional process such that $P_{\widetilde{x}}$-a.s. for each $i \in [d]$, the ith component $\widetilde{Y}_i$ of $\widetilde{Y}$ satisfies
   (a) $\widetilde{Y}_i(0) = 0$,
   (b) $\widetilde{Y}_i$ is continuous and nondecreasing,
   (c) $\widetilde{Y}_i$ can increase only when $\widetilde{Z}$ is on the face F_i, i.e.,

$$\int_0^t \mathbf{1}_{\left\{ \widetilde{Z}(s) \in (\mathbb{R}_+)^d \setminus F_i \right\}} \, \mathrm{d}\widetilde{Y}_i(s) = 0$$

   for all $t \geq 0$.

$\widetilde{Y}$ is referred to as the "pushing" process of $\widetilde{Z}$.

Theorem 2. [28, Theorem 1.3 and Corollary 1.4] *There exists a SRBM in the orthant* $(\mathbb{R}_+)^d$ *with data* (η, Γ, R) *that starts from* $\widetilde{x} \in (\mathbb{R}_+)^d$ *if and only if* R *is completely-*$\mathcal{S}$*. Moreover, when it exists, the joint law of any SRBM, together with its associated pushing process, is unique.*

We are now ready to prove our result on the system of SDEs (2.1).

Theorem 3. *Under Assumption* 1*(a) there exists a unique weak solution to* (2.1)*. Moreover, if Assumption* 1*(a) does not hold, there is no weak solution to* (2.1)*.*

Proof. There are two key ideas in the proof. The first is to consider the process of spacings

$$\left(X_2(\cdot) - X_1(\cdot), X_3(\cdot) - X_2(\cdot), \ldots, X_n(\cdot) - X_{n-1}(\cdot)\right)$$

and the process $\sum_{i=1}^n X_i(\cdot)$, which together determine the process $X(\cdot)$. The second idea is to consider an appropriate (and naturally arising) time change.

Step 1. We start with the proof of weak existence. First, from Theorem 2 it follows that there exists a weak solution on a suitable filtered probability space $\left\{\Omega, (\mathcal{F}_t)_{t \geq 0}, P\right\}$ to the following system of SDEs:

$$\mathrm{d}\widehat{Z}_i(t) = (b_{i+1} - b_i)\,\mathrm{d}t + \mathrm{d}B_i(t) + \sum_{j=1}^{n-1} q_{i,j}\,\mathrm{d}\Lambda_j(t), \quad i \in [n-1],$$

with initial conditions $\widehat{Z}_i(0) = x_{i+1} - x_i$, $i \in [n-1]$, where $B = (B_1, B_2, \ldots, B_{n-1})$ is a Brownian motion with zero drift vector and diffusion matrix $A = (a_{i,i'})_{i,i' \in [n-1]}$ given by

$$a_{i,i'} = \mathfrak{c}_{i,i'} + \mathfrak{c}_{i+1,i'+1} - \mathfrak{c}_{i,i'+1} - \mathfrak{c}_{i+1,i'}, \tag{2.2}$$

the $\Lambda_j(\cdot)$, $j \in [n-1]$, are the semimartingale local times at zero of the processes $\widehat{Z}_j(\cdot)$, $j \in [n-1]$, respectively, and recall that $q_{i,j} = v_{i+1,j} - v_{i,j}$. Note that here we have used the fact that the matrix Q is completely-$\mathcal{S}$ (see Assumption 1(a)). Next, we can find (after extending the underlying probability space if necessary) a Brownian motion $\widehat{\beta} = \left(\widehat{\beta}_1, \widehat{\beta}_2, \ldots, \widehat{\beta}_n\right)$ with zero drift vector and diffusion matrix $\mathfrak{C}$ such that

$$B_i(\cdot) = \widehat{\beta}_{i+1}(\cdot) - \widehat{\beta}_i(\cdot), \quad i \in [n-1].$$

Therefore, we can define $\widehat{X} = (\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_n)$ as the unique process satisfying

$$\sum_{i=1}^{n} \widehat{X}_i(t) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \Big(b_i\, t + \widehat{\beta}_i(t) + \sum_{j=1}^{n-1} v_{i,j} \Lambda_j(t) \Big),$$

$$\left(\widehat{X}_2(t) - \widehat{X}_1(t), \ldots, \widehat{X}_n(t) - \widehat{X}_{n-1}(t)\right) = \left(\widehat{Z}_1(t), \ldots, \widehat{Z}_{n-1}(t)\right),$$

for all $t \geq 0$. Finally, we let

$$T(t) := t + \Lambda(t) := t + \sum_{j=1}^{n-1} \Lambda_j(t), \quad t \geq 0,$$

$$\tau(t) := \inf\{s \geq 0 : \ T(s) = t\}, \quad t \geq 0,$$

and set $X(\cdot) = \widehat{X}(\tau(\cdot))$. Then clearly

$$X_i(\cdot) - X_i(0) = b_i \tau(\cdot) + \widehat{\beta}_i(\tau(\cdot)) + \sum_{j=1}^{n-1} v_{i,j}\, \Lambda_j(\tau(\cdot)), \quad i \in [n]. \tag{2.3}$$

Moreover, we note that $\tau(\cdot)$, $\Lambda(\tau(\cdot))$ are nondecreasing functions, which induce nonnegative measures $\mathrm{d}\tau(\cdot)$, $\mathrm{d}\Lambda(\tau(\cdot))$ on $[0, \infty)$ satisfying

$$\mathrm{d}\tau(t) + \mathrm{d}\Lambda(\tau(t)) = \mathrm{d}t. \tag{2.4}$$

Therefore we have

$$\begin{aligned} \tau(\cdot) = \int_0^{\tau(\cdot)} \mathbf{1}_{\left\{\widehat{X}_1(t) < \widehat{X}_2(t) < \ldots < \widehat{X}_n(t)\right\}} \, \mathrm{d}t &= \int_0^{\cdot} \mathbf{1}_{\{X_1(t) < X_2(t) < \ldots < X_n(t)\}} \, \mathrm{d}\tau(t) \\ &= \int_0^{\cdot} \mathbf{1}_{\{X_1(t) < X_2(t) < \ldots < X_n(t)\}} \, \mathrm{d}t, \end{aligned}$$

which takes care of the first term on the right hand side of (2.3). In addition, the processes $\widehat{\beta}_i(\tau(\cdot))$, $i \in [n]$ are martingales with respect to the filtration $(\mathcal{F}_{\tau(t)})_{t \geq 0}$ with quadratic covariation processes given by

$$\mathfrak{c}_{i,i'}\, \tau(\cdot) = \mathfrak{c}_{i,i'} \int_0^{\cdot} \mathbf{1}_{\{X_1(t) < X_2(t) < \ldots < X_n(t)\}} \, \mathrm{d}t, \quad i, i' \in [n],$$

where we used the identity derived in the previous display. From the last computation we can conclude, in particular, that after extending the underlying probability space

if necessary, we can find a Brownian motion $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ with zero drift vector and diffusion matrix $\mathfrak{C}$ such that

$$\widehat{\beta}_i(\tau(\cdot)) = \int_0^{\cdot} \mathbf{1}_{\{X_1(t)<X_2(t)<\ldots<X_n(t)\}} \, \mathrm{d}\beta_i(t), \quad i \in [n].$$

Finally, we have

$$\begin{aligned}\Lambda_j(\tau(\cdot)) &= \int_0^{\cdot} \mathbf{1}_{\left\{\widehat{X}_j(\tau(t))=\widehat{X}_{j+1}(\tau(t))\right\}} \, \mathrm{d}\Lambda_j(\tau(t)) = \int_0^{\cdot} \mathbf{1}_{\left\{\widehat{X}_j(\tau(t))=\widehat{X}_{j+1}(\tau(t))\right\}} \, (\mathrm{d}t - \mathrm{d}\tau(t)) \\ &= \int_0^{\cdot} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}} \, \mathrm{d}t - \int_0^{\tau(\cdot)} \mathbf{1}_{\left\{\widehat{X}_j(t)=\widehat{X}_{j+1}(t)\right\}} \, \mathrm{d}t = \int_0^{\cdot} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}} \, \mathrm{d}t,\end{aligned}$$

for $j \in [n-1]$. Here the second identity is a consequence of (2.4) and the fact that the boundary local times $\Lambda_{j'}$, $j' \neq j$, do not charge the set $\{t : \widehat{X}_j(t) = \widehat{X}_{j+1}(t)\}$ (see the main result, Theorem 1, in [25]); and the fourth identity follows from the fact that the instantaneously reflecting Brownian motion $\widehat{Z}$ does not spend time on the boundary of the orthant $(\mathbb{R}_+)^{n-1}$ [28, Lemma 2.1]. All in all, we can now conclude that (X, β) is a weak solution to (2.1).

Step 2. We now turn to the proof of weak uniqueness. To this end, let (X, W) be any weak solution to (2.1). Define

$$\sigma(t) = \inf\left\{s \geq 0 : \int_0^s \mathbf{1}_{\{X_1(a)<X_2(a)<\ldots<X_n(a)\}} \, \mathrm{d}a = t\right\}, \quad t \geq 0,$$

and set $\widehat{X}(\cdot) = X(\sigma(\cdot))$. Using Lévy's characterization of Brownian motion one verifies that

$$\widehat{X}_i(t) = \widehat{X}_i(0) + b_i t + \widehat{W}_i(t) + \sum_{j=1}^{n-1} v_{i,j} L_j(t), \quad t \geq 0,$$

where $\widehat{W} = (\widehat{W}_1, \widehat{W}_2, \ldots, \widehat{W}_n)$ is a Brownian motion with zero drift vector and diffusion matrix $\mathfrak{C}$, and $\{L_j\}_{j\in[n-1]}$ are nondecreasing processes whose points of increase are contained in the sets

$$\left\{t \geq 0 : \widehat{X}_j(t) = \widehat{X}_{j+1}(t)\right\}, \quad j \in [n-1],$$

respectively. Moreover, the law of $\widehat{X}$ is uniquely determined by the joint law of

$$\left(\widehat{X}_2(\cdot) - \widehat{X}_1(\cdot), \widehat{X}_3(\cdot) - \widehat{X}_2(\cdot), \ldots, \widehat{X}_n(\cdot) - \widehat{X}_{n-1}(\cdot)\right) \quad \text{and} \quad \sum_{i=1}^n \widehat{X}_i(\cdot). \tag{2.5}$$

However, by the uniqueness result of Theorem 2 we can identify the first of the latter two processes as an instantaneously reflected Brownian motion in the orthant $(\mathbb{R}_+)^{n-1}$, so the joint law of that process and its boundary local times is uniquely determined. Moreover, the second process can be constructed by using the first process, its boundary local time processes and an additional independent one-dimensional standard Brownian motion, so the joint law of the processes in (2.5) is uniquely determined. Thus, the law of $\widehat{X}$ is uniquely determined as well. Finally, the law of X is also uniquely determined as one can verify that $X(\cdot) = \widehat{X}(\tau(\cdot))$, where τ is defined as in step 1 above.

Step 3. Suppose now that Assumption 1(a) does not hold. Then a weak solution of (2.1) cannot exist. Indeed, if (X, W) was such a weak solution, we could define the time change $\sigma(\cdot)$ as in step 2 above and let $\widehat{X}(\cdot) = X(\sigma(\cdot))$ as before. Then, the arguments in step 2 would show that the process of spacings

$$\left(\widehat{X}_2(\cdot) - \widehat{X}_1(\cdot), \widehat{X}_3(\cdot) - \widehat{X}_2(\cdot), \ldots, \widehat{X}_n(\cdot) - \widehat{X}_{n-1}(\cdot)\right)$$

is a reflecting Brownian motion in the orthant $(\mathbb{R}_+)^{n-1}$ in the sense of [28]. However, by Theorem 2 the latter process does not exist if the reflection matrix Q is not completely-$\mathcal{S}$. This is the desired contradiction. □

The following example shows that, even when Assumption 1(a) holds, one cannot expect a strong solution to (2.1) to exist.

Example 1. Consider the following specification of parameters: $n = 2$, $b_1 = b_2 = 0$, $\mathfrak{c}_{1,1} = \mathfrak{c}_{2,2} = 1$, $\mathfrak{c}_{1,2} = \mathfrak{c}_{2,1} = 0$, $v_{1,1} = -\frac{1}{2}$, $v_{2,1} = \frac{1}{2}$; that is, the system of SDEs is

$$\mathrm{d}X_1(t) = \mathbf{1}_{\{X_1(t)<X_2(t)\}}\,\mathrm{d}W_1(t) - \frac{1}{2}\,\mathbf{1}_{\{X_1(t)=X_2(t)\}}\,\mathrm{d}t, \tag{2.6}$$

$$\mathrm{d}X_2(t) = \mathbf{1}_{\{X_1(t)<X_2(t)\}}\,\mathrm{d}W_2(t) + \frac{1}{2}\,\mathbf{1}_{\{X_1(t)=X_2(t)\}}\,\mathrm{d}t, \tag{2.7}$$

with W_1 and W_2 being independent one-dimensional standard Brownian motions. We claim that this system does not admit a strong solution. It is well-known (see Theorem 3.2 in [5]) that strong existence and weak uniqueness together imply pathwise uniqueness, so it suffices to show that pathwise uniqueness does not hold for the system (2.6)-(2.7). To this end, we consider the SDE

$$\mathrm{d}Z(t) = \mathbf{1}_{\{Z(t)>0\}}\,\mathrm{d}\beta(t) + \mathbf{1}_{\{Z(t)=0\}}\,\mathrm{d}t, \tag{2.8}$$

where β is a Brownian motion with zero drift and diffusion coefficient 2. The main result in [6] shows that pathwise uniqueness does not hold for this equation. Therefore it suffices to argue that pathwise uniqueness for the system (2.6)-(2.7) would imply pathwise uniqueness for the equation (2.8). Indeed, let Z, Z' be two solutions of (2.8) on the same probability space and with respect to the same Brownian motion β. Extend the probability space so that it supports an independent Brownian motion W with zero drift and diffusion coefficient 2, and define S, S' according to

$$\mathrm{d}S(t) = \mathbf{1}_{\{Z(t)>0\}}\,\mathrm{d}W(t) \quad \text{and} \quad \mathrm{d}S'(t) = \mathbf{1}_{\{Z'(t)>0\}}\,\mathrm{d}W(t).$$

Finally, set

$$X_1 = \frac{S-Z}{2}, \quad X_2 = \frac{S+Z}{2} \quad \text{and} \quad X_1' = \frac{S'-Z'}{2}, \quad X_2' = \frac{S'+Z'}{2}.$$

Then both (X_1, X_2) and (X_1', X_2') are weak solutions of the system (2.6)-(2.7) with respect to the Brownian motion $((W-\beta)/2, (W+\beta)/2)$. Therefore if pathwise uniqueness did hold for the system (2.6)-(2.7), we would be able to conclude that $X_1 = X_1'$ and $X_2 = X_2'$ pathwise, and, hence, that $Z = Z'$ pathwise; in other words, the solution of (2.8) would be pathwise unique. This is the desired contradiction.

2.2. **Markov property and invariant measures.** Having established that the weak solution X of the system (2.1) exists and is unique (see Theorem 3), we can now proceed to study some of its properties. First, we remark that weak existence and uniqueness imply that the corresponding martingale problem is well-posed (see, e.g., Corollary 4.8 and Corollary 4.9 in chapter 5 of [21]). Therefore, by Theorem 6.2.2 in [27], the process X is Markovian. In addition, the relation $X(\cdot) = \widehat{X}(\tau(\cdot))$, where $\widehat{X}$ is an instantaneously reflecting Brownian motion in the wedge $\mathcal{W}$ with a nondegenerate diffusion matrix, shows that the process X has the Harris property (see, e.g., the appendix of [7]):

$$\forall\, x,\, y \in \mathcal{W},\ r > 0: \quad \mathbb{P}^x\left(|X(t) - y| < r \text{ for some } t \geq 0\right) > 0. \tag{2.9}$$

Moreover, the corresponding property is true for the process of spacings

$$Z(\cdot) = \left(X_2(\cdot) - X_1(\cdot), X_3(\cdot) - X_2(\cdot), \ldots, X_n(\cdot) - X_{n-1}(\cdot)\right).$$

Thus, Z has a unique invariant distribution provided that it is positive recurrent or, equivalently, if

$$\widehat{Z}(\cdot) = \left(\widehat{X}_2(\cdot) - \widehat{X}_1(\cdot), \widehat{X}_3(\cdot) - \widehat{X}_2(\cdot), \ldots, \widehat{X}_n(\cdot) - \widehat{X}_{n-1}(\cdot)\right)$$

is positive recurrent (see [24] and the references therein). By Proposition 2.8 in the dissertation [15], the latter is the case if and only if

$$Q^{-1}\left(b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1}\right)^T < 0 \tag{2.10}$$

componentwise. Here, the superscript T stands for the transpose of the vector under consideration. We summarize our findings in the next proposition.

Proposition 4. *The processes X and Z are Markovian. Both of them possess the Harris property. Moreover, the process Z has a unique invariant distribution if and only if the recurrence condition* (2.10) *is satisfied.*

For a wide class of coefficients the invariant distribution of the process Z can be given explicitly. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product in Euclidean space, i.e., for $x, y \in \mathbb{R}^d$, $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$, and let $F_i := \left\{z \in (\mathbb{R}_+)^{n-1} : z_i = 0\right\}$ denote the ith face of the orthant $(\mathbb{R}_+)^{n-1}$. With this notation we then have the following result.

Theorem 5. *Suppose that in addition to* (2.10) *the condition*

$$2A = QD + DQ^T \tag{2.11}$$

is satisfied, where A is given by (2.2) *and $D = \operatorname{diag}(A)$ (the diagonal matrix, whose diagonal elements coincide with those of A). Let*

$$\gamma = 2\, D^{-1} Q^{-1}\left(b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1}\right)^T \tag{2.12}$$

and write $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$. Then the invariant distribution of the process Z in the orthant $(\mathbb{R}_+)^{n-1}$ is given by

$$\frac{1}{C} e^{\langle\gamma, z\rangle}\left(\mathrm{d}z + \sum_{j=1}^{n-1} \frac{\sqrt{a_{j,j}}}{2} \mathbf{1}_{\{z \in F_j\}}\, \mathrm{d}z^j\right), \tag{2.13}$$

where $C = \frac{1 - \sum_{j=1}^{n-1} \sqrt{a_{j,j}}\gamma_j/2}{\prod_{j=1}^{n-1}(-\gamma_j)}$ is the appropriate normalization constant, and $\mathrm{d}z^j$, $j \in [n-1]$, are the Lebesgue boundary measures on the faces F_j, $j \in [n-1]$, respectively.

Proof. We first transform our process of interest Z in such a way as to make the diffusion matrix of the transformed process the identity; we refer to [16, Section 3.2.1] for similar computations. Let $U = (u_{i,j})_{i,j\in[n-1]}$ be an orthogonal matrix whose columns are the orthonormal eigenvectors of A, and let $G := U^T AU$, a diagonal matrix with the eigenvalues of A in its diagonal. Define the transformed process

$$\overline{Z}(\cdot) := G^{-1/2}U^T Z(\cdot).$$

This is a sticky Brownian motion in the cone

$$\overline{\mathcal{S}} := G^{-1/2}U^T(\mathbb{R}_+)^{n-1} = \left\{\overline{z} \in \mathbb{R}^{n-1} : UG^{1/2}\overline{z} \in (\mathbb{R}_+)^{n-1}\right\},$$

with drift vector $\overline{\mu} := G^{-1/2}U^T\mu$, where $\mu := (b_2 - b_1, \ldots, b_n - b_{n-1})^T$, identity diffusion matrix, and reflection matrix $\overline{Q} := G^{-1/2}U^T Q$. This transformed reflection matrix can be decomposed as

$$\overline{Q} = \left(\overline{\mathfrak{N}} + \overline{\mathfrak{T}}\right) D^{-1/2} \equiv \left(\overline{q}_{\cdot,1}, \ldots, \overline{q}_{\cdot,n-1}\right),$$

where

$$\begin{aligned}\overline{\mathfrak{N}} &:= G^{1/2}U^T D^{-1/2} \equiv \left(\overline{\mathfrak{n}}_{\cdot,1}, \ldots, \overline{\mathfrak{n}}_{\cdot,n-1}\right),\\ \overline{\mathfrak{T}} &:= G^{-1/2}U^T QD^{1/2} - \overline{\mathfrak{N}} \equiv \left(\overline{\mathfrak{t}}_{\cdot,1}, \ldots, \overline{\mathfrak{t}}_{\cdot,n-1}\right).\end{aligned}$$

The columns of $\overline{\mathfrak{N}}$ are unit vectors, since for all $j \in [n-1]$ we have

$$\sum_{i=1}^{n-1} \overline{\mathfrak{n}}_{i,j}^2 = \sum_{i=1}^{n-1}\left(\sqrt{g_{i,i}}u_{j,i}\frac{1}{\sqrt{a_{j,j}}}\right)^2 = \frac{1}{a_{j,j}}\sum_{i=1}^{n-1} g_{i,i}u_{j,i}^2 = 1, \tag{2.14}$$

where $G = (g_{i,j})_{i,j\in[n-1]}$ and in the last equality we used that $UGU^T = A$. Furthermore, the corresponding columns of $\overline{\mathfrak{N}}$ and $\overline{\mathfrak{T}}$ are orthogonal, since for every $i \in [n-1]$ we have

$$\begin{aligned}\overline{\mathfrak{n}}_{\cdot,i}^T\overline{\mathfrak{t}}_{\cdot,i} &= \sum_{j=1}^{n-1}\overline{\mathfrak{n}}_{j,i}\overline{\mathfrak{t}}_{j,i} = \sum_{j=1}^{n-1}\sqrt{g_{j,j}}u_{i,j}\frac{1}{\sqrt{a_{i,i}}}\sum_{k=1}^{n-1}\frac{1}{\sqrt{g_{j,j}}}u_{k,j}q_{k,i}\sqrt{a_{i,i}} - \sum_{j=1}^{n-1}\overline{\mathfrak{n}}_{j,i}^2\\ &= \sum_{k=1}^{n-1} q_{k,i}\sum_{j=1}^{n-1}u_{i,j}u_{k,j} - 1 = \sum_{k=1}^{n-1} q_{k,i}\mathbf{1}_{\{k=i\}} - 1 = q_{i,i} - 1 = 0,\end{aligned}$$

where we used (2.14), the fact that U is orthogonal, and that $\operatorname{diag}(Q) = I$, which follows from (2.11). In fact, $\overline{\mathfrak{n}}_{\cdot,i}$ is the inward unit normal to the ith face $\overline{F}_i := G^{-1/2}U^T F_i$ of the new state space $\overline{\mathcal{S}}$. To see this, let $w \in F_i$, and let $v := G^{-1/2}U^T w \in \overline{F}_i$. Then

$$\begin{aligned}\overline{\mathfrak{n}}_{\cdot,i}^T v &= \sum_{j=1}^{n-1}\overline{\mathfrak{n}}_{j,i}v_j = \sum_{j=1}^{n-1}\sqrt{g_{j,j}}u_{i,j}\frac{1}{\sqrt{a_{i,i}}}\sum_{k=1}^{n-1}\frac{1}{\sqrt{g_{j,j}}}u_{k,j}w_k\\ &= \frac{1}{\sqrt{a_{i,i}}}\sum_{k=1}^{n-1}w_k\sum_{j=1}^{n-1}u_{i,j}u_{k,j} = \frac{1}{\sqrt{a_{i,i}}}\sum_{k=1}^{n-1}w_k\mathbf{1}_{\{k=i\}} = \frac{1}{\sqrt{a_{i,i}}}w_i = 0,\end{aligned}$$

where the last equality is because $w \in F_i$. Thus the ith column $\overline{q}_{\cdot,i}$ of the new reflection matrix $\overline{Q}$ is decomposed into components that are normal and tangential to $\overline{F}_i$:

$$\overline{q}_{\cdot,i} = \frac{1}{\sqrt{a_{i,i}}} \left(\overline{\mathfrak{n}}_{\cdot,i} + \overline{\mathfrak{t}}_{\cdot,i} \right). \tag{2.15}$$

The advantage of this transformation is that the setup of the new process $\overline{Z}$ fits precisely into the framework of Harrison and Williams [14], who studied the stationary distribution of reflected Brownian motion with identity diffusion matrix in a convex polyhedral domain. Their main result is that the stationary distribution is of exponential form if the reflection matrix satisfies a certain skew symmetry condition, and they give explicit formulas for the exponent. The main difference between their setting and ours is that the process we study is sticky at the boundary of the domain, as opposed to reflecting instantaneously, as is the case in [14]. However, apart from taking care of this distinction at the boundary, the same methods and computations apply.

In particular, we can plug our expressions into the formulas of Harrison and Williams [14] to arrive at the skew symmetry condition for our process, and also to find the appropriate exponent. First, we have

$$\overline{\mathfrak{N}}^T \overline{\mathfrak{T}} + \overline{\mathfrak{T}}^T \overline{\mathfrak{N}} = D^{-1/2} \left(QD + DQ^T - 2A \right) D^{-1/2}. \tag{2.16}$$

The skew symmetry condition of Harrison and Williams (see [14, eq. (1.3)]) says that the left hand side of (2.16) is the zero matrix, which is the same as our condition (2.11). Second, plugging in to [14, eq. (4.9)], the $\overline{\gamma}$ arising in the exponent of the stationary distribution of exponential form for the transformed process $\overline{Z}$ should be

$$\overline{\gamma} = 2 \left(I - \left(\overline{\mathfrak{N}}^T \right)^{-1} \overline{\mathfrak{T}}^T \right)^{-1} \overline{\mu} = 2G^{1/2} U^T D^{-1} Q^{-1} \mu, \tag{2.17}$$

and thus the γ for the original process Z should be

$$\gamma = UG^{-1/2} \overline{\gamma} = 2D^{-1} Q^{-1} \mu,$$

just as in (2.12).

In the remainder of the proof, we go through the computations of [14] as applied to our setting. Let $\overline{L} := \frac{1}{2}\Delta + \overline{\mu} \cdot \nabla$, and for $j \in [n-1]$ let $\overline{\mathcal{D}}_j := \overline{q}_{\cdot,j} \cdot \nabla$. The generator $\overline{\mathcal{L}}$ of the sticky Brownian motion $\overline{Z}$ can then be written as

$$\overline{\mathcal{L}} = \overline{L} \mathbf{1}_{\{\overline{\mathcal{S}} \setminus \partial \overline{\mathcal{S}}\}} + \sum_{j=1}^{n-1} \overline{\mathcal{D}}_j \mathbf{1}_{\{\overline{F}_j\}}.$$

Let $\overline{p}(\overline{z}) := \exp(\langle \overline{\gamma}, \overline{z} \rangle)$ for $\overline{z} \in \overline{\mathcal{S}}$. In order to show that (2.13) is invariant for Z, we must show that for every $f \in C_c^\infty(\overline{\mathcal{S}})$ we have

$$\int_{\overline{\mathcal{S}}} \overline{p} \overline{L} f \, \mathrm{d}\overline{z} + \sum_{j=1}^{n-1} \frac{\sqrt{a_{j,j}}}{2} \int_{\overline{F}_j} \overline{p} \overline{\mathcal{D}}_j f \, \mathrm{d}\overline{z}^j = 0, \tag{2.18}$$

where for $j \in [n-1]$, $\mathrm{d}\overline{z}^j$ is the surface measure on the face $\overline{F}_j$. Define $\overline{L}^* := \frac{1}{2}\Delta - \overline{\mu}\cdot\nabla$. Using Green's second identity and the divergence theorem, we get that for every $f \in C_c^\infty\left(\overline{\mathcal{S}}\right)$ we have

$$\int_{\overline{\mathcal{S}}} \overline{p}\overline{L}f \,\mathrm{d}\overline{z} = \int_{\overline{\mathcal{S}}} f\overline{L}^*\overline{p}\,\mathrm{d}\overline{z} + \frac{1}{2}\sum_{j=1}^{n-1}\int_{\overline{F}_j}\left(f\frac{\partial\overline{p}}{\partial\overline{\mathfrak{n}}_{\cdot,j}} - \overline{p}\frac{\partial f}{\partial\overline{\mathfrak{n}}_{\cdot,j}} - 2\overline{\mu}\cdot\overline{\mathfrak{n}}_{\cdot,j}f\overline{p}\right)\mathrm{d}\overline{z}^j, \tag{2.19}$$

where we used $\partial/\partial\overline{\mathfrak{n}}_{\cdot,j} \equiv \overline{\mathfrak{n}}_{\cdot,j}\cdot\nabla$ to denote differentiation in the inward unit normal direction on the face $\overline{F}_j$. Now $\overline{L}^*\overline{p} = \left(\frac{1}{2}\left|\overline{\gamma}\right|^2 - \overline{\mu}\cdot\overline{\gamma}\right)\overline{p} = 0$, since using (2.17) we have that

$$\frac{1}{2}\left|\overline{\gamma}\right|^2 - \overline{\mu}\cdot\overline{\gamma} = \frac{1}{2}\left(\overline{\mathfrak{N}}^{-1}\overline{\gamma}\right)^T\overline{\mathfrak{N}}^T\overline{\mathfrak{T}}\left(\overline{\mathfrak{N}}^{-1}\overline{\gamma}\right),$$

which is zero, since $\overline{\mathfrak{N}}^T\overline{\mathfrak{T}}$ is skew symmetric due to (2.16) and our condition (2.11). Plugging (2.19) back into (2.18) and using (2.15) we get that showing (2.18) is equivalent to showing that for every $f \in C_c^\infty\left(\overline{\mathcal{S}}\right)$ we have

$$\sum_{j=1}^{n-1}\int_{\overline{F}_j}\left\{f\left(\left(\overline{\mathfrak{n}}_{\cdot,j} - \overline{\mathfrak{t}}_{\cdot,j}\right)\cdot\nabla\overline{p} - 2\overline{\mu}\cdot\overline{\mathfrak{n}}_{\cdot,j}\overline{p}\right) + \nabla\cdot\left(\overline{\mathfrak{t}}_{\cdot,j}\overline{p}f\right)\right\}\mathrm{d}\overline{z}^j = 0. \tag{2.20}$$

The relationship (2.17) between $\overline{\gamma}$ and $\overline{\mu}$ implies that $\overline{\gamma}^T\left(\overline{\mathfrak{N}} - \overline{\mathfrak{T}}\right) - 2\overline{\mu}^T\overline{\mathfrak{N}} = 0$, and thus for every $j \in [n-1]$ we have $\left(\overline{\mathfrak{n}}_{\cdot,j} - \overline{\mathfrak{t}}_{\cdot,j}\right)\cdot\nabla\overline{p} - 2\overline{\mu}\cdot\overline{\mathfrak{n}}_{\cdot,j}\overline{p} = 0$. Since $\overline{\mathfrak{t}}_{\cdot,j}$ is parallel to the face $\overline{F}_j$, the divergence in (2.20) is the same as the divergence taken in $\overline{F}_j$. Thus, by applying the divergence theorem on each face $\overline{F}_j$, it follows that showing (2.20) is equivalent to showing that for every $f \in C_c^\infty\left(\overline{\mathcal{S}}\right)$ we have

$$\sum_{j=1}^{n-1}\sum_{1\le k<j}\int_{\overline{F}_{j,k}}\left(\overline{\mathfrak{t}}_{\cdot,j}\cdot\overline{n}_{j,k} + \overline{\mathfrak{t}}_{\cdot,k}\cdot\overline{n}_{k,j}\right)\overline{p}f\,\mathrm{d}\overline{\sigma}_{j,k} = 0, \tag{2.21}$$

where $\overline{F}_{j,k} = \overline{F}_j \cap \overline{F}_k$, $\overline{\sigma}_{j,k}$ denotes $(n-3)$-dimensional surface measure on $\overline{F}_{j,k}$, and $\overline{n}_{j,k}$ denotes the unit vector that is normal to both $\overline{F}_{j,k}$ and $\overline{\mathfrak{n}}_{\cdot,j}$, and points into the interior of $\overline{F}_j$ from $\overline{F}_{j,k}$. In fact, $\overline{n}_{j,k}$ must lie in the two-dimensional space spanned by $\overline{\mathfrak{n}}_{\cdot,j}$ and $\overline{\mathfrak{n}}_{\cdot,k}$, and can be determined uniquely:

$$\overline{n}_{j,k} = \left(\overline{\mathfrak{n}}_{\cdot,k} - \overline{\mathfrak{n}}_{\cdot,j}^T\overline{\mathfrak{n}}_{\cdot,k}\overline{\mathfrak{n}}_{\cdot,j}\right)/\left(1 - \left(\overline{\mathfrak{n}}_{\cdot,j}^T\overline{\mathfrak{n}}_{\cdot,k}\right)^2\right)^{1/2}.$$

Consequently, since $\overline{\mathfrak{N}}^T\overline{\mathfrak{T}}$ is skew symmetric, we have $\overline{\mathfrak{t}}_{\cdot,j}\cdot\overline{n}_{j,k} + \overline{\mathfrak{t}}_{\cdot,k}\cdot\overline{n}_{k,j} = 0$ for all $1 \le k < j \le n-1$, showing that (2.21) indeed holds. □

3. Convergence and general setup

This section is divided into two parts. In the first part (subsection 3.1) we prove the convergence theorem (Theorem 1) as stated in the introduction. Then in the second part (subsection 3.2) we describe a much larger class of particle systems that converge to appropriate sticky Brownian motions in $\mathcal{W}$.

3.1. **Proof of the convergence theorem.** Given the uniqueness of a weak solution to the system of SDEs (1.4) as proved in Theorem 3, Theorem 1 is a consequence of Proposition 6 below. To state and obtain the latter, we study the following decomposition. For each $i \in [n]$ we can write

(3.1)
$$X_i^M(t) = X_i^M(0) + A_i^M(t) + \sum_{j=1}^{n-1} C_{i,j}^{R,M}(t) - \sum_{j=1}^{n-1} C_{i,j}^{L,M}(t) + \sum_{j=1}^{n-1} \Delta_{i,j}^{R,M}(t) - \sum_{j=1}^{n-1} \Delta_{i,j}^{L,M}(t),$$

where, for $j \in [n-1]$,

$$A_i^M(t) := \int_0^t \mathbf{1}_{\left\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),k\in[n-1]\right\}} \mathrm{d}\left(P_i(s) - Q_i(s)\right), \tag{3.2}$$

$$C_{i,j}^{R,M}(t) := \theta_{i,j}^R I_{i,j}^{R,M}(t) := \theta_{i,j}^R \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \mathrm{d}s, \tag{3.3}$$

$$\Delta_{i,j}^{R,M}(t) := \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \mathrm{d}\left(R_{i,j}(s) - \theta_{i,j}^R s\right), \tag{3.4}$$

and the processes $C_{i,j}^{L,M}$, $I_{i,j}^{L,M}$, and $\Delta_{i,j}^{L,M}$ are defined similarly to $C_{i,j}^{R,M}$, $I_{i,j}^{R,M}$, and $\Delta_{i,j}^{R,M}$, respectively. For $m \in \mathbb{N}$, let $D^m \equiv D([0,\infty),\mathbb{R}^m)$. We have the following convergence result.

Proposition 6. *Assume that Assumption 1 holds, and that the initial conditions $\left\{X^M(0), M > 0\right\}$ are deterministic and converge to a limit $x \in \mathcal{W}$ as $M \to \infty$. Then the family*

$$\left\{\left(X^M, A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M}\right), M > 0\right\} \tag{3.5}$$

is tight in D^{4n^2-2n}. Moreover, every limit point $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ satisfies the following for each $i \in [n]$:

$$X_i^\infty(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}} \sqrt{2a}\,\mathrm{d}W_i(s) + \sum_{j=1}^{n-1} v_{i,j} \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}} \mathrm{d}s, \tag{3.6}$$

$$A_i^\infty(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<X_2^\infty(s)<\cdots<X_n^\infty(s)\right\}} \sqrt{2a}\,\mathrm{d}W_i(s), \tag{3.7}$$

$$I_{i,j}^{L,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}} \mathrm{d}s, \qquad j \in [n-1] \setminus \{i-1\}, \tag{3.8}$$

$$I_{i,j}^{R,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}} \mathrm{d}s, \qquad j \in [n-1] \setminus \{i\}, \tag{3.9}$$

$$I_{i,i-1}^{L,\infty}(\cdot) = I_{i,i}^{R,\infty}(\cdot) = 0,$$
$$\Delta_{i,j}^{L,\infty}(\cdot) = \Delta_{i,j}^{R,\infty}(\cdot) = 0, \qquad j \in [n-1],$$

with a suitable n-dimensional standard Brownian motion $W = (W_1, \ldots W_n)$.

Proof. Step 1. The tightness of the family in (3.5) can be verified using the necessary and sufficient conditions of Corollary 3.7.4 in [9]. First, note that for $i \in [n]$ and $j \in [n-1]$, the processes $P_i(\cdot) - Q_i(\cdot)$, as well as $\left(M^{1/4}(R_{i,j}(t) - \theta_{i,j}^R t), t \geq 0\right)$

and $\left(M^{1/4}(L_{i,j}(t)-\theta^L_{i,j}t), t\geq 0\right)$ all converge to suitable one-dimensional Brownian motions in the limit $M\to\infty$. Therefore, the conditions of Corollary 3.7.4 in [9] hold for the corresponding families of processes indexed by $M>0$. One can then bound the indicator functions appearing in the integrands of the integrals in (3.2), (3.3), and (3.4) between 0 and 1 appropriately to show that the same conditions hold for the family $\left\{\left(A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M}\right), M>0\right\}$, which is thus tight in D^{4n^2-3n}. For example, for $i\in[n]$ and $t\geq 0$ we have that

$$\left|A^M_i(t)\right| \leq \int_0^t \operatorname{sgn}\left(P_i(s)-Q_i(s)\right)\,\mathrm{d}\left(P_i(s)-Q_i(s)\right).$$

The expression on the right hand side converges to $\sqrt{2a}\int_0^t \operatorname{sgn}(B(s))\,\mathrm{d}B(s)$ as $M\to\infty$, where B is a standard one-dimensional Brownian motion. By Tanaka's formula, this is equal to $\sqrt{2a}\left(|B(t)|-L(t)\right)$, where $L(\cdot)$ is the local time process at 0 of $B(\cdot)$. Consequently, the family of processes $\left\{A^M, M>0\right\}$ is tight. Verifying tightness of the other families of processes can be done similarly. In view of the decomposition (3.1), the first statement of the proposition now readily follows.

Step 2. Now fix a limit point $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ and to simplify notation assume that it is the limit of the whole family (3.5) as $M\to\infty$.

We start with a few simple observations about the limit point under consideration. Note first that, for any fixed $M>0$, the jumps of all components of $\left(X^M, A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M}\right)$ are bounded above in absolute value by $\frac{1}{\sqrt{M}}$, so all components of the limit point must have continuous paths. Moreover, for every fixed $t\geq 0$, the family $\left\{A^M(t), M>0\right\}$ is uniformly integrable due to the estimate

$$\begin{aligned}\mathbb{E}\left[A^M_i(t)^2\right] &= \mathbb{E}\left[\left[A^M_i\right](t)\right] = \mathbb{E}\left[\int_0^t \mathbf{1}_{\left\{X^M_k(s)+\frac{1}{\sqrt{M}}<X^M_{k+1}(s),\, k\in[n-1]\right\}}\,\mathrm{d}\left[P_i-Q_i\right](s)\right]\\ &= \mathbb{E}\left[\int_0^t \mathbf{1}_{\left\{X^M_k(s)+\frac{1}{\sqrt{M}}<X^M_{k+1}(s),\, k\in[n-1]\right\}}\frac{1}{\sqrt{M}}\left(\mathrm{d}P_i+\mathrm{d}Q_i\right)(s)\right] \leq 2\,a\,t, \quad i\in[n],\end{aligned}$$

where $[\cdot]$ denotes the quadratic variation process of a process with paths in D^1. This and the fact that A^M is a martingale for any fixed $M>0$ show that A^∞ is a martingale with respect to its own filtration (see, e.g., [19, Proposition IX.1.12]).

Next, we observe that, as limits of non-decreasing processes, $I^{L,\infty}_{i,j}$ and $I^{R,\infty}_{i,j}$ must be non-decreasing processes themselves for every $i\in[n]$, $j\in[n-1]$, and consequently they are also of finite variation. Furthermore, for all $i\in[n]$, $j\in[n-1]$, the quadratic variation processes of the martingales $\Delta^{L,M}_{i,j}$ and $\Delta^{R,M}_{i,j}$ satisfy

$$\forall\, t\geq 0: \quad \lim_{M\to\infty}\mathbb{E}\left[\left[\Delta^{L,M}_{i,j}\right](t)\right]=0 \quad\text{and}\quad \lim_{M\to\infty}\mathbb{E}\left[\left[\Delta^{R,M}_{i,j}\right](t)\right]=0.$$

Therefore, the distributional limits

$$\Delta^{L,\infty}_{i,j}\equiv\lim_{M\to\infty}\Delta^{L,M}_{i,j}, \quad \Delta^{R,\infty}_{i,j}\equiv\lim_{M\to\infty}\Delta^{R,M}_{i,j}$$

in D^1 exist and are identically equal to zero.

Finally, the two observations of the previous paragraph, together with the decomposition (3.1), show that for $i, i' \in [n]$ the quadratic covariation processes $\langle X_i^\infty, X_{i'}^\infty \rangle$ and $\langle A_i^\infty, A_{i'}^\infty \rangle$ are in fact equal. In particular, we have that $\langle X_i^\infty \rangle = \langle A_i^\infty \rangle$ for $i \in [n]$.

Step 3. To show (3.7), we study the quadratic covariation processes $\langle X_i^\infty, X_{i'}^\infty \rangle = \langle A_i^\infty, A_{i'}^\infty \rangle$, $i, i' \in [n]$. We first claim that $\langle A_i^\infty, A_{i'}^\infty \rangle = 0$ whenever $i \neq i'$. To this end, it suffices to show that for any such pair of indices $A_i^\infty(\cdot) A_{i'}^\infty(\cdot)$ is a martingale with respect to its own filtration. The latter is the limit in D^1 of the family of martingales $\left\{ A_i^M(\cdot) A_{i'}^M(\cdot), M > 0 \right\}$ by definition, so it is enough to prove that, for any fixed $t \geq 0$, the random variables $\left\{ A_i^M(t) A_{i'}^M(t), M > 0 \right\}$ are uniformly integrable. The latter is a consequence of the following chain of estimates:

$$
\begin{aligned}
\mathbb{E}\left[A_i^M(t)^2 A_{i'}^M(t)^2 \right] &= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \, \mathrm{d} A_{i'}^M(s)^2 \right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \, \mathrm{d} A_i^M(s)^2 \right] \\
&= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \, \mathrm{d}\left[A_{i'}^M \right](s) \right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \, \mathrm{d}\left[A_i^M \right](s) \right] \\
&\leq \mathbb{E}\left[\int_0^t A_i^M(s)^2 \, \mathrm{d}\left[P_{i'} - Q_{i'} \right](s) \right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \, \mathrm{d}\left[P_i - Q_i \right](s) \right] \\
&= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \, \frac{1}{\sqrt{M}} (\mathrm{d}P_{i'} + \mathrm{d}Q_{i'})(s) \right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \, \frac{1}{\sqrt{M}} (\mathrm{d}P_i + \mathrm{d}Q_i)(s) \right] \\
&= \mathbb{E}\left[\int_0^t A_i^M(s)^2 \, 2a \, \mathrm{d}s \right] + \mathbb{E}\left[\int_0^t A_{i'}^M(s)^2 \, 2a \, \mathrm{d}s \right] \\
&\leq 2a \int_0^t \left(\mathbb{E}\left[\left[P_i - Q_i \right](s) \right] + \mathbb{E}\left[\left[P_{i'} - Q_{i'} \right](s) \right] \right) \mathrm{d}s = 2a \int_0^t 4as \, \mathrm{d}s = 4a^2 t^2.
\end{aligned}
$$

We next aim to evaluate $\langle A_i^\infty \rangle$, $i \in [n]$. To this end, we first show that for any fixed $i \in [n]$ and $t \geq 0$, the random variables $\left\{ A_i^M(t)^2, M > 0 \right\}$ are uniformly integrable. By Itô's lemma for (not necessarily continuous) semimartingales we have that for any fixed $t \geq 0$,

$$
\begin{aligned}
A_i^M(t)^4 = &\int_0^t 4 \, A_i^M(s-)^3 \, \mathrm{d}A_i^M(s) + \frac{1}{2} \int_0^t 12 \, A_i^M(s-)^2 \, \mathrm{d}\left[A_i^M \right](s) \\
&+ \sum_{s \leq t} \Big\{ \left(A_i^M(s)^4 - A_i^M(s-)^4 \right) - 4 A_i^M(s-)^3 \left(A_i^M(s) - A_i^M(s-) \right) \\
&- 6 A_i^M(s-)^2 \left(A_i^M(s) - A_i^M(s-) \right)^2 \Big\}.
\end{aligned}
$$

Taking expectations on both sides and dropping the third line of the previous display we arrive at the following estimate:

$$
\begin{aligned}
\mathbb{E}\left[A_i^M(t)^4 \right] \leq &\mathbb{E}\left[\int_0^t 4 \, A_i^M(s-)^3 \, \mathrm{d}A_i^M(s) \right] + \mathbb{E}\left[\int_0^t 6 \, A_i^M(s-)^2 \, \mathrm{d}\left[A_i^M \right](s) \right] \\
&+ \mathbb{E}\left[\sum_{s \leq t} \left\{ \left(A_i^M(s)^4 - A_i^M(s-)^4 \right) - 4 A_i^M(s-)^3 \left(A_i^M(s) - A_i^M(s-) \right) \right\} \right].
\end{aligned}
$$

The first term on the right-hand side is equal to zero, since A_i^M is a martingale starting at zero. We can upper bound the second term on the right-hand side using the inequality $\mathrm{d}\left[A_i^M\right](s) \leq \frac{1}{\sqrt{M}}\mathrm{d}\left(P_i+Q_i\right)(s)$. Finally, we can upper bound the third expectation on the right-hand side using the identity $x^4-y^4-4y^3\left(x-y\right)=\left(x-y\right)^2\left(x^2+2xy+3y^2\right)$, the fact that the jumps of the process A_i^M are of size $\frac{1}{\sqrt{M}}$, and the fact that $A_i^M\left(s\right)-A_i^M\left(s-\right)\leq \mathrm{d}\left(P_i+Q_i\right)(s)$. We then arrive at the estimate

$$\mathbb{E}\left[A_i^M(t)^4\right]\leq \mathbb{E}\left[\int_0^t \left(3\,A_i^M(s)^2+11\,A_i^M\left(s-\right)^2\right)\,\frac{1}{\sqrt{M}}\,\mathrm{d}(P_i+Q_i)(s)\right],$$

and the right-hand side can now be bounded above by a constant depending only on a and t by arguing as above when estimating $\mathbb{E}\left[A_i^M(t)^2A_{i'}^M(t)^2\right]$. Thus indeed the random variables $\left\{A_i^M(t)^2, M>0\right\}$ are uniformly integrable. Putting this together with the fact that the functional

$$(\omega_1,\omega_2,\ldots,\omega_n)\mapsto \int_{t_1}^{t_2}\mathbf{1}_{\{\omega_1(s)<\cdots<\omega_n(s)\}}\,\mathrm{d}s$$

on D^n is lower semicontinuous and using the Portmanteau Theorem, we have for each $i\in[n]$ that

$$\begin{aligned}
&\mathbb{E}\left[G\left(A^\infty\right)\left(A_i^\infty\left(t_2\right)^2-A_i^\infty\left(t_1\right)^2-2a\int_{t_1}^{t_2}\mathbf{1}_{\{X_1^\infty(s)<\ldots<X_n^\infty(s)\}}\,\mathrm{d}s\right)\right]\\
&=\lim_{\epsilon\downarrow 0}\mathbb{E}\left[G\left(A^\infty\right)\left(A_i^\infty\left(t_2\right)^2-A_i^\infty\left(t_1\right)^2-2a\int_{t_1}^{t_2}\mathbf{1}_{\{X_k^\infty(s)+\epsilon<X_{k+1}^\infty(s),\,k\in[n-1]\}}\,\mathrm{d}s\right)\right]\\
&\geq\limsup_{M\to\infty}\mathbb{E}\left[G\left(A^M\right)\left(A_i^M\left(t_2\right)^2-A_i^M\left(t_1\right)^2-2a\int_{t_1}^{t_2}\mathbf{1}_{\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),k\in[n-1]\}}\mathrm{d}s\right)\right]\\
&=\limsup_{M\to\infty}\mathbb{E}\left[G\left(A^M\right)\left(A_i^M\left(t_2\right)^2-A_i^M\left(t_1\right)^2-\int_{t_1}^{t_2}\mathbf{1}_{\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),k\in[n-1]\}}\mathrm{d}[P_i-Q_i](s)\right)\right]\\
&=\limsup_{M\to\infty}\mathbb{E}\left[G\left(A^M\right)\left(A_i^M\left(t_2\right)^2-A_i^M\left(t_1\right)^2-\left[A_i^M\right](t_2)+\left[A_i^M\right](t_1)\right)\right]=0
\end{aligned}$$

for any nonnegative continuous bounded functional G on D^n measurable with respect to the σ-algebra generated by the coordinate mappings on $D^n\left([0,t_1]\right)$. Therefore, recalling from the end of step 2 of the proof that $\langle X_i^\infty\rangle=\langle A_i^\infty\rangle$, we conclude that

$$\forall\, 0\leq t_1<t_2:\quad \langle X_i^\infty\rangle\left(t_2\right)-\langle X_i^\infty\rangle\left(t_1\right)\geq 2a\int_{t_1}^{t_2}\mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}}\,\mathrm{d}s \tag{3.10}$$

holds with probability one. On the other hand,

$$\begin{aligned}
&\mathbb{E}\left[G\left(A^\infty\right)\left(A_i^\infty\left(t_2\right)^2-A_i^\infty\left(t_1\right)^2-2a(t_2-t_1)\right)\right]\\
&\quad=\lim_{M\to\infty}\mathbb{E}\left[G\left(A^M\right)\left(A_i^M\left(t_2\right)^2-A_i^M\left(t_1\right)^2-2a(t_2-t_1)\right)\right]\\
&\quad=\lim_{M\to\infty}\mathbb{E}\left[G\left(A^M\right)\left(A_i^M\left(t_2\right)^2-A_i^M\left(t_1\right)^2-[P_i-Q_i](t_2)+[P_i-Q_i](t_1)\right)\right]\leq 0
\end{aligned}$$

for any functional G on D^n as above. Hence,

$$\forall\, 0\leq t_1<t_2:\quad \langle X_i^\infty\rangle\left(t_2\right)-\langle X_i^\infty\rangle\left(t_1\right)\leq 2a\left(t_2-t_1\right) \tag{3.11}$$

must hold with probability 1.

In view of (3.11), we see that in order to improve (3.10) to an equality, it suffices to show that the measure $\mathrm{d}\langle X_i^\infty\rangle = \mathrm{d}\langle A_i^\infty\rangle$ assigns zero mass to the sets $\{t \geq 0 : X_j^\infty(t) = X_{j+1}^\infty(t)\}$, $j \in [n-1]$, with probability one. To this end, we first recall that for every $i \in [n]$ the square integrable martingale A_i^∞ is the limit in D^1 of the square integrable martingales $\left\{A_i^M, M > 0\right\}$, and the random variables $\left\{A_i^M(t)^2, M > 0\right\}$ are uniformly integrable for any fixed $t \geq 0$. Therefore $\langle A_i^\infty\rangle$ is the limit in D^1 of $\left\{\left[A_i^M\right], M > 0\right\}$, and so by the Portmanteau Theorem

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle A_i^\infty\rangle(t) - \langle A_{i'}^\infty\rangle(t)\right)^2\right] \leq \liminf_{M\to\infty} \mathbb{E}\left[\left(\left[A_i^M\right](t) - \left[A_{i'}^M\right](t)\right)^2\right] \\
&\leq \liminf_{M\to\infty} \mathbb{E}\Big[\Big(\int_0^t \mathbf{1}_{\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s), k\in[n-1]\}} \frac{1}{\sqrt{M}}\left(\mathrm{d}P_i + \mathrm{d}Q_i - \mathrm{d}P_{i'} - \mathrm{d}Q_{i'}\right)(s)\Big)^2\Big] \\
&= \liminf_{M\to\infty} \mathbb{E}\left[\int_0^t \mathbf{1}_{\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s), k\in[n-1]\}} \frac{1}{M}\,\mathrm{d}\left[P_i + Q_i - P_{i'} - Q_{i'}\right](s)\right] \\
&\leq \liminf_{M\to\infty} \frac{1}{M^{3/2}}\,\mathbb{E}[P_i(t) + Q_i(t) + P_{i'}(t) + Q_{i'}(t)] = 0
\end{aligned}$$

for any fixed $i, i' \in [n]$ and $t \geq 0$ with probability one. In view of the path continuity of the processes $\langle X_i^\infty\rangle$, $i \in [n]$, this implies

$$\langle X_1^\infty\rangle = \langle X_2^\infty\rangle = \ldots = \langle X_n^\infty\rangle$$

with probability one. To conclude the argument, we use the occupation time formula for continuous semimartingales (see, e.g., [26, Theorem VI.1.6]), which states that if Y is a continuous semimartingale and ϕ is a positive Borel function, then a.s. for every $t \geq 0$ we have

$$\int_0^t \phi\left(Y(s)\right)\,\mathrm{d}\langle Y\rangle(s) = \int_{-\infty}^\infty \phi(a)\,L^a(t)\,\mathrm{d}a,$$

where $L^a(\cdot)$ is the local time process at a of $Y(\cdot)$. In particular, the choice of $\phi(a) = \mathbf{1}_{\{a=0\}}$ gives that

$$\int_0^t \mathbf{1}_{\{Y(s)=0\}}\,\mathrm{d}\langle Y\rangle(s) = 0,$$

and now choosing $Y(\cdot) = X_{j+1}^\infty(\cdot) - X_j^\infty(\cdot)$ this implies that the measure

$$\mathrm{d}\left\langle X_{j+1}^\infty - X_j^\infty\right\rangle = \mathrm{d}\left\langle X_{j+1}^\infty\right\rangle + \mathrm{d}\left\langle X_j^\infty\right\rangle = 2\,\mathrm{d}\left\langle X_j^\infty\right\rangle = 2\,\mathrm{d}\langle X_i^\infty\rangle$$

assigns zero mass to the set $\{t \geq 0 :\ X_j^\infty(t) = X_{j+1}^\infty(t)\}$ with probability one. Hence, equality must hold in (3.10). The representation (3.7) with a suitable standard Brownian motion $W = (W_1, W_2, \ldots, W_n)$ now readily follows from the Martingale Representation Theorem in the form of Theorem 4.2 in chapter 3 of [21].

Step 4. We now turn to the proof of (3.6), (3.8), and (3.9). To this end, recalling the ghost particles $X_0^M(\cdot) \equiv -\infty$ and $X_{n+1}^M(\cdot) \equiv \infty$ introduced for notational

convenience, for any $M > 0$, $i \in \{0, 1, \ldots, n\}$, and $j \in [n-1]$ define

$$I_j^{M,1}(\cdot) := \int_0^{\cdot} \mathbf{1}_{\left\{X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \, \mathrm{d}s,$$
$$I_{i,j}^{M,2}(\cdot) := \int_0^{\cdot} \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}=X_{i+1}^M(s), X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \, \mathrm{d}s.$$

Then for any $i \in [n]$, $j \in [n-1]$, we have the decompositions

$$I_{i,j}^{L,M}(\cdot) := I_j^{M,1}(\cdot) - I_{i-1,j}^{M,2}(\cdot),$$
$$I_{i,j}^{R,M}(\cdot) := I_j^{M,1}(\cdot) - I_{i,j}^{M,2}(\cdot).$$

It is now easy to check that for each $i \in \{0, 1, \ldots, n\}$, $j \in [n-1]$, the families of processes $\left\{I_j^{M,1}, M > 0\right\}$ and $\left\{I_{i,j}^{M,2}, M > 0\right\}$ satisfy the tightness criterion of Corollary 3.7.4 in [9]. So, after passing to a subsequence if necessary, we obtain the existence of suitable limits $I_j^{\infty,1}$ and $I_{i,j}^{\infty,2}$, respectively; for notational convenience we assume that the full families of processes converge jointly to the respective limit points.

The limiting processes inherit many properties of the prelimit processes. First, clearly the limits are non-decreasing processes and inherit the property that for every $i \in \{0, 1, \ldots, n\}$,

$$\forall\, 0 \le t_1 < t_2 : \quad I_{i,j}^{\infty,2}(t_2) - I_{i,j}^{\infty,2}(t_1) \le I_j^{\infty,1}(t_2) - I_j^{\infty,1}(t_1). \tag{3.12}$$

Second, the prelimit processes satisfy

$$\int_0^{\infty} \mathbf{1}_{\left\{X_j^M(t)+\frac{1}{\sqrt{M}}<X_{j+1}^M(t)\right\}} \, \mathrm{d}I_j^{M,1}(t) = 0, \tag{3.13}$$
$$\int_0^{\infty} \left(\mathbf{1}_{\left\{X_i^M(t)+\frac{1}{\sqrt{M}}<X_{i+1}^M(t)\right\}} + \mathbf{1}_{\left\{X_j^M(t)+\frac{1}{\sqrt{M}}<X_{j+1}^M(t)\right\}} \right) \mathrm{d}I_{i,j}^{M,2}(t) = 0,$$

and from these we have that the limiting processes satisfy

$$\int_0^{\infty} \mathbf{1}_{\left\{X_j^{\infty}(t)<X_{j+1}^{\infty}(t)\right\}} \, \mathrm{d}I_j^{\infty,1}(t) = 0, \tag{3.14}$$

$$\int_0^{\infty} \left(\mathbf{1}_{\left\{X_i^{\infty}(t)<X_{i+1}^{\infty}(t)\right\}} + \mathbf{1}_{\left\{X_j^{\infty}(t)<X_{j+1}^{\infty}(t)\right\}} \right) \mathrm{d}I_{i,j}^{\infty,2}(t) = 0. \tag{3.15}$$

These properties can be shown by arguing as in the second half of the proof of Theorem 4.1 in [32] (see also the proof of Proposition 9 in [20]); we provide a sketch on how to obtain (3.14) from (3.13), and (3.15) follows similarly. We can first use the Skorokhod representation theorem [9, Theorem 3.1.8] and the fact that the limiting processes $\left(X^{\infty}, I^{\infty,1}, I^{\infty,2}, I^{L,\infty}, I^{R,\infty}\right)$ are a.s. continuous to replace the sequence of processes $\left\{\left(X^M, I^{M,1}, I^{M,2}, I^{L,M}, I^{R,M}\right), M > 0\right\}$ by one that has the same distribution and which a.s. converges uniformly on compact time intervals. Let $\{f_m\}_{m\ge1}$ be a sequence of continuous functions such that for every m, $f_m : \mathbb{R} \to [0,1]$, $f_m(x) = 0$ for $x \le 1/m$ and $f_m(x) = 1$ for $x \ge 2/m$. By passing to the $m \to \infty$ limit, in order to show (3.14) it suffices to show that for each $t \ge 0$, $j \in [n-1]$, and $m \ge 1$, a.s.

$$\int_0^t f_m\left(X_{j+1}^{\infty}(s) - X_j^{\infty}(s)\right) \mathrm{d}I_j^{\infty,1}(s) = 0. \tag{3.16}$$

To do this, fix $j \in [n-1]$, $m \geq 1$, and $t \geq 0$. For $M > m^2$, (3.13) implies that a.s.

$$\int_0^t f_m\left(X_{j+1}^M(s) - X_j^M(s)\right) \mathrm{d}I_j^{M,1}(s) = 0,$$

and thus to show (3.16) it suffices to show that a.s.

(3.17)
$$\int_0^t f_m\left(X_{j+1}^M(s) - X_j^M(s)\right) \mathrm{d}I_j^{M,1}(s) \to \int_0^t f_m\left(X_{j+1}^\infty(s) - X_j^\infty(s)\right) \mathrm{d}I_j^{\infty,1}(s)$$

as $M \to \infty$. The almost sure convergence assumed above implies that a.s. as $M \to \infty$, $X_{j+1}^M(\cdot) - X_j^M(\cdot) \to X_{j+1}^\infty(\cdot) - X_j^\infty(\cdot)$ uniformly on compacts, and since f_m is uniformly continuous, we have that a.s. as $M \to \infty$, $f_m\left(X_{j+1}^M(\cdot) - X_j^M(\cdot)\right) \to f_m\left(X_{j+1}^\infty(\cdot) - X_j^\infty(\cdot)\right)$ uniformly on compacts. We also have that a.s. as $M \to \infty$, $I_j^{M,1}(\cdot) \to I_j^{\infty,1}(\cdot)$ uniformly on compacts, and the remaining details of showing (3.17) are as in the end of the proof of Theorem 4.1 in [32].

Next, we define the time change

$$\sigma(t) = \inf\left\{s \geq 0 : \int_0^s \mathbf{1}_{\{X_1^\infty(r) < X_2^\infty(r) < \ldots < X_n^\infty(r)\}} \mathrm{d}r = t\right\}, \quad t \geq 0$$

and let $\widehat{X}^\infty(\cdot) = X^\infty(\sigma(\cdot))$, $\widehat{I}^{\infty,1}(\cdot) = I^{\infty,1}(\sigma(\cdot))$, and $\widehat{I}^{\infty,2}(\cdot) = I^{\infty,2}(\sigma(\cdot))$. Using Lévy's characterization of Brownian motion, we conclude that the components of $\widehat{X}^\infty$ admit the decomposition

$$\widehat{X}_i^\infty(\cdot) = \widehat{X}_i^\infty(0) + \sqrt{2a}\widehat{W}_i(\cdot) + \sum_{j=1}^{n-1} v_{i,j}\, \widehat{I}_j^{\infty,1}(\cdot) + \sum_{\substack{j=1\\ j\neq i-1}}^{n-1} \theta_{i,j}^L\, \widehat{I}_{i-1,j}^{\infty,2}(\cdot) - \sum_{\substack{j=1\\ j\neq i}}^{n-1} \theta_{i,j}^R\, \widehat{I}_{i,j}^{\infty,2}(\cdot)$$

with $\widehat{W} = (\widehat{W}_1, \widehat{W}_2, \ldots, \widehat{W}_n)$ being a suitable standard Brownian motion. As we shall show shortly, for every $i \in [n]$ we have

$$\widehat{I}_{i,j}^{\infty,2}(\cdot) \equiv 0, \qquad j \in [n-1] \setminus \{i\}, \tag{3.18}$$

and thus the decomposition simplifies to

$$\widehat{X}_i^\infty(\cdot) = \widehat{X}_i^\infty(0) + \sqrt{2a}\widehat{W}_i(\cdot) + \sum_{j=1}^{n-1} v_{i,j}\, \widehat{I}_j^{\infty,1}(\cdot).$$

We can then identify the process of spacings

$$\left(\widehat{X}_2^\infty(\cdot) - \widehat{X}_1^\infty(\cdot),\ \widehat{X}_3^\infty(\cdot) - \widehat{X}_2^\infty(\cdot),\ \ldots,\ \widehat{X}_n^\infty(\cdot) - \widehat{X}_{n-1}^\infty(\cdot)\right)$$

as a reflected Brownian motion in the orthant $(\mathbb{R}_+)^{n-1}$ with reflection matrix Q (recall from section 1.2 that $q_{j,j'} = v_{j+1,j'} - v_{j,j'}$ for $j, j' \in [n-1]$), and the processes $\widehat{I}_j^{\infty,1}(\cdot)$, $j \in [n-1]$, with its boundary local times. At this point one can argue as in step 2 in the proof of Theorem 3 to obtain the representations (3.6), (3.8), and (3.9).

Thus what is left is to show (3.18). For $j \in [n-1]$ let $\widehat{Z}_j(\cdot) = \widehat{X}_{j+1}^\infty(\cdot) - \widehat{X}_j^\infty(\cdot)$, thus $\widehat{Z}(\cdot) = \left(\widehat{Z}_1(\cdot), \ldots, \widehat{Z}_{n-1}(\cdot)\right)$ is the process of spacings. By (3.15), (3.18) reduces

to showing that

$$\int_0^\infty \mathbf{1}_{\left\{\widehat{Z}_i(t)=\widehat{Z}_j(t)=0\right\}} \, \mathrm{d}\widehat{I}_{i,j}^{\infty,2}(t) = 0. \tag{3.19}$$

This can be done by generalizing the proof of Theorem 1 in [25], along the lines of [3, Theorem 7.7] and [20, Lemma 1], and, in particular, it uses the Lyapunov functions constructed in the proof of Lemma 4 in [25]. Here we provide a sketch of the proof, and refer to [25], [3], and [20] for details. This is the only point in our proof where we use Assumption 1(b).

First we introduce some notation to simplify the representation of $\widehat{Z}$. For $i \in [n-1]$, let $\widehat{B}_i(\cdot) := \sqrt{2a}\left(\widehat{W}_{i+1}(\cdot) - \widehat{W}_i(\cdot)\right)$; then $\widehat{B} := \left(\widehat{B}_1, \ldots, \widehat{B}_{n-1}\right)$ is a Brownian motion with mean zero and diffusion matrix $A = (a_{i,j})_{i,j=1}^{n-1}$ given by

$$a_{i,j} := \begin{cases} 4a & \text{if } i = j, \\ -2a & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the following we think of $\widehat{I}^{\infty,2}$ as an $(\mathbb{R}_+)^{(n-1)^2}$-valued process whose components are indexed by ordered pairs (i,j), $i,j \in [n-1]$, and the component indexed by (i,j) is $\widehat{I}_{i,j}^{\infty,2}(\cdot)$. Recalling the definition of the matrix $Q^{(2)}$ from section 1.2, we can write $\widehat{Z}$ as

$$\widehat{Z}(\cdot) = \widehat{Z}(0) + \widehat{B}(\cdot) + Q\widehat{I}^{\infty,1}(\cdot) + Q^{(2)}\widehat{I}^{\infty,2}(\cdot).$$

Then by Itô's formula, for any function f that is twice continuously differentiable in some domain containing $(\mathbb{R}_+)^{n-1}$ we have that a.s. for all $t \geq 0$:

$$\begin{aligned} f\left(\widehat{Z}(t)\right) - f\left(\widehat{Z}(0)\right) = &\int_0^t \nabla f\left(\widehat{Z}(s)\right) \mathrm{d}\widehat{B}(s) + \sum_{j=1}^{n-1} \int_0^t q_{\cdot,j} \cdot \nabla f\left(\widehat{Z}(s)\right) \mathrm{d}\widehat{I}_j^{\infty,1}(s) \\ &+ \sum_{k,\ell=1}^{n-1} \int_0^t q_{\cdot,(k,\ell)}^{(2)} \cdot \nabla f\left(\widehat{Z}(s)\right) \mathrm{d}\widehat{I}_{k,\ell}^{\infty,2}(s) + \int_0^t Lf\left(\widehat{Z}(s)\right) ds, \end{aligned}$$

where recall that $q_{\cdot,j}$ is the jth column of Q, $q_{\cdot,(k,\ell)}^{(2)}$ is the column of $Q^{(2)}$ corresponding to index (k,ℓ), and

$$L = \frac{1}{2} \sum_{i,j=1}^{n-1} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.$$

We apply Itô's formula to an appropriately defined family of functions, just as in [25]. Let $\gamma = \gamma([n-1]) \in (\mathbb{R}_+)^{n-1}$ be the vector guaranteed by Assumption 1(b) for $J = [n-1]$. Let $\delta := Q^T\gamma$; by assumption $\delta \in [1,\infty)^{n-1}$. Define $\alpha = A\gamma$. For each $x \in (\mathbb{R}_+)^{n-1}$ and $r \in (0,1)$, let $d^2(x,r) := (x + r\alpha)^T A^{-1}(x + r\alpha)$. Then, for each $\varepsilon \in (0,1)$, define

$$\phi_\varepsilon(x) := \begin{cases} \frac{1}{2-(n-1)} \int_\varepsilon^1 r^{(n-1)-2} \left(d^2(x,r)\right)^{\frac{2-(n-1)}{2}} \mathrm{d}r & \text{if } n-1 \geq 3, \\ \frac{1}{2} \int_\varepsilon^1 \ln\left(d^2(x,r)\right) \mathrm{d}r & \text{if } n-1 = 2. \end{cases} \tag{3.20}$$

For each $\varepsilon \in (0,1)$, ϕ_ε is twice continuously differentiable in some domain containing $(\mathbb{R}_+)^{n-1}$, and on each compact subset of $(\mathbb{R}_+)^{n-1}$ it is bounded, uniformly in ε. Moreover, we have that $L\phi_\varepsilon = 0$ in some domain containing $(\mathbb{R}_+)^{n-1}$, due to the fact that the integrands in (3.20) are L-harmonic functions of $x \in \mathbb{R}^{n-1} \setminus \{-r\alpha\}$. Now, with $\|\cdot\|$ denoting the Euclidean norm in $\mathbb{R}^{n-1}$, define for each $m \in \mathbb{N}$ the stopping time

$$\tau_m := \inf\left\{t \geq 0 : \left\|\widehat{Z}(t)\right\| \geq m \text{ or } \widehat{I}_j^{\infty,1}(t) \geq m \text{ for some } j\right\} \wedge m.$$

Applying Itô's formula to the function ϕ_ε and the stopping time τ_m, we get that a.s.

$$\phi_\varepsilon\left(\widehat{Z}(\tau_m)\right) - \phi_\varepsilon\left(\widehat{Z}(0)\right) = \int_0^{\tau_m} \nabla\phi_\varepsilon\left(\widehat{Z}(s)\right) \mathrm{d}\widehat{B}(s) \tag{3.21}$$
$$+ \sum_{j=1}^{n-1} \int_0^{\tau_m} q_{\cdot,j} \cdot \nabla\phi_\varepsilon\left(\widehat{Z}(s)\right) \mathrm{d}\widehat{I}_j^{\infty,1}(s) + \sum_{k,\ell=1}^{n-1} \int_0^{\tau_m} q^{(2)}_{\cdot,(k,\ell)} \cdot \nabla\phi_\varepsilon\left(\widehat{Z}(s)\right) \mathrm{d}\widehat{I}_{k,\ell}^{\infty,2}(s).$$

Since $\widehat{B}$ has no drift and ϕ_ε and its first derivatives are bounded on each compact subset of $(\mathbb{R}_+)^{n-1}$, the definition of the stopping time τ_m implies that the stochastic integral with respect to $\mathrm{d}\widehat{B}$ in (3.21) has zero expectation. To bound the other terms on the right hand side of (3.21) it is necessary to bound the directional derivatives of ϕ_ε; this is exactly what is done in [25, p. 93-95]. In particular, the results of [25] give two bounds. First, for every $j \in [n-1]$ there exists a constant $\widehat{c}_j < \infty$ such that for all $x \in (\mathbb{R}_+)^{n-1}$ and all $\varepsilon \in (0,1)$,

$$q_{\cdot,j} \cdot \nabla\phi_\varepsilon(x) \geq -\widehat{c}_j.$$

Here the constant $\hat{c}_j$ depends on A, Q, γ, and δ, but does not depend on x nor ε (see [25, eq. (24)]). Next, for $j \in [n-1]$ let $\beta_j = \delta_j / \|A^{-1}q_{\cdot,j}\|$. Then for every $j \in [n-1]$ there exists a constant $c_j > 0$ such that for all $x \in (\mathbb{R}_+)^{n-1}$ satisfying $\|x\| < \varepsilon\beta_j$,

$$q_{\cdot,j} \cdot \nabla\phi_\varepsilon(x) \geq -c_j(\ln\varepsilon + 1). \tag{3.22}$$

Here the constant c_j depends on A, Q, γ, δ, and β_j, but does not depend on x nor ε. Note that for ε small the term on the right hand side of (3.22) is large and positive. Furthermore, due to Assumption 1(b) and the choice of γ, the same arguments as in [25, p. 93-95] can be repeated to bound the directional derivatives $q^{(2)}_{\cdot,(k,\ell)} \cdot \nabla\phi_\varepsilon$. In particular, for every $(k,\ell) \notin \mathcal{I}^{(2)}$ there exists a constant $\widehat{c}_{(k,\ell)} < \infty$ such that for all $x \in (\mathbb{R}_+)^{n-1}$ and all $\varepsilon \in (0,1)$,

$$q^{(2)}_{\cdot,(k,\ell)} \cdot \nabla\phi_\varepsilon(x) \geq -\widehat{c}_{(k,\ell)}.$$

Here the constant $\widehat{c}_{(k,\ell)}$ depends on $A, Q, Q^{(2)}, \gamma$, and δ, but does not depend on x nor ε. If $(k,\ell) \in \mathcal{I}^{(2)}$ then by definition $q^{(2)}_{\cdot,(k,\ell)}$ is the zero vector, and thus $q^{(2)}_{\cdot,(k,\ell)} \cdot \nabla\phi_\varepsilon = 0$.

Plugging these bounds into (3.21) and taking expectation we get that

(3.23)
$$\mathbb{E}\left[\phi_\varepsilon\left(\widehat{Z}\left(\tau_m\right)\right)-\phi_\varepsilon\left(\widehat{Z}\left(0\right)\right)\right]\geq-\left(\ln\varepsilon+1\right)\sum_{j=1}^{n-1}c_j\mathbb{E}\left[\int_0^{\tau_m}\mathbf{1}_{\left\{\left\|\widehat{Z}(s)\right\|<\varepsilon\beta_j\right\}}\,\mathrm{d}\widehat{I}_j^{\infty,1}\left(s\right)\right]$$
$$-\sum_{j=1}^{n-1}\widehat{c}_j\mathbb{E}\left[\widehat{I}_j^{\infty,1}\left(\tau_m\right)\right]-\sum_{(k,\ell)\notin\mathcal{I}^{(2)}}\widehat{c}_{(k,\ell)}\mathbb{E}\left[\widehat{I}_{k,\ell}^{\infty,2}\left(\tau_m\right)\right].$$

The left hand side of (3.23) is bounded as $\varepsilon\to0$ since ϕ_ε is uniformly bounded on compact subsets of $\left(\mathbb{R}_+\right)^{n-1}$, while the last two terms in (3.23) are independent of ε. So dividing (3.23) by $-\left(\ln\varepsilon+1\right)$ and letting $\varepsilon\to0$ we get that

$$\lim_{\varepsilon\to0}\sum_{j=1}^{n-1}c_j\mathbb{E}\left[\int_0^{\tau_m}\mathbf{1}_{\left\{\left\|\widehat{Z}(s)\right\|<\varepsilon\beta_j\right\}}\,\mathrm{d}\widehat{I}_j^{\infty,1}\left(s\right)\right]\leq0.$$

Each term in the sum above is nonnegative and $c_j>0$, so by Fatou's lemma it follows that

$$\int_0^{\tau_m}\mathbf{1}_{\left\{\widehat{Z}_{j'}(s)=0,j'\in[n-1]\right\}}\,\mathrm{d}\widehat{I}_j^{\infty,1}\left(s\right)=0$$

for every $j\in[n-1]$ a.s. By letting $m\to\infty$ we have that

$$\int_0^{\infty}\mathbf{1}_{\left\{\widehat{Z}_{j'}(s)=0,j'\in[n-1]\right\}}\,\mathrm{d}\widehat{I}_j^{\infty,1}\left(s\right)=0$$

for every $j\in[n-1]$ a.s. Finally, by using the backward induction argument of [25, Lemma 5], it follows that with probability one, for all $j\in[n-1]$ and $J\subseteq[n-1]$ such that $|J|\geq2$ we have that

$$\int_0^{\infty}\mathbf{1}_{\left\{\widehat{Z}_{j'}(s)=0,j'\in J\right\}}\,\mathrm{d}\widehat{I}_j^{\infty,1}\left(s\right)=0.$$

Together with (3.12), this implies (3.19). □

3.2. **General setup.** In this last subsection, we introduce a much more general class of particle systems which converge to appropriate multidimensional sticky Brownian motions in the sense of Theorem 1. We now allow for non-exponential interarrival times between the jumps of the particles and for dependence between the arrival times of the jumps for different particles.

To define this more general class of particle systems, we introduce the following parameters: $n\in\mathbb{N}$ for the number of particles as before; $a>0$; λ_i^L and λ_i^R for $i\in[n]$; $c_{i,i'}^{L,L}$, $c_{i,i'}^{L,R}$, and $c_{i,i'}^{R,R}$ for $i,i'\in[n]$; and finally $\theta_{i,j}^L$ and $\theta_{i,j}^R$ for $i\in[n]$, $j\in[n-1]$. We fix a value $M>0$ of the scaling parameter. The random variables and processes we define next all depend on M, but for the sake of readability we mostly do not denote this dependence explicitly.

We let $\left\{u^L(k),k\in\mathbb{N}\right\}$ and $\left\{u^R(k),k\in\mathbb{N}\right\}$ be two independent sequences of i.i.d. random vectors with values in $(0,\infty)^n$ (the interarrival times between jumps to the left and to the right when there are no collisions), and for $i\in[n]$, $j\in[n-1]$, let $\left\{w_{i,j}^L\left(k\right),k\in\mathbb{N}\right\}$ and $\left\{w_{i,j}^R\left(k\right),k\in\mathbb{N}\right\}$ be two independent families of i.i.d. random

variables taking values in $(0,\infty)$ (the interarrival times between jumps to the left and to the right when there is a collision). We assume that

$$\mathbb{E}[u_i^L(1)] = \left(a + \frac{\lambda_i^L}{\sqrt{M}}\right)^{-1}, \quad \mathbb{E}[u_i^R(1)] = \left(a + \frac{\lambda_i^R}{\sqrt{M}}\right)^{-1},$$
$$\operatorname{cov}(u_i^L(1), u_{i'}^L(1)) = c_{i,i'}^{L,L},\ \operatorname{cov}(u_i^L(1), u_{i'}^R(1)) = c_{i,i'}^{L,R},\ \operatorname{cov}(u_i^R(1), u_{i'}^R(1)) = c_{i,i'}^{R,R},$$
$$\mathbb{E}[w_{i,j}^L(1)] = \left(\theta_{i,j}^L\right)^{-1}, \quad \mathbb{E}[w_{i,j}^R(1)] = \left(\theta_{i,j}^R\right)^{-1}.$$

Next, define the corresponding partial sum processes

$$U_i^L(0) = 0, \quad U_i^L(\ell) = \sum_{k=1}^{\ell} u_i^L(k), \quad U_i^R(0) = 0, \quad U_i^R(\ell) = \sum_{k=1}^{\ell} u_i^R(k),$$
$$W_{i,j}^L(0) = 0, \quad W_{i,j}^L(\ell) = \sum_{k=1}^{\ell} w_{i,j}^L(k), \quad W_{i,j}^R(0) = 0, \quad W_{i,j}^R(\ell) = \sum_{k=1}^{\ell} w_{i,j}^R(k)$$

for all $i \in [n]$, $j \in [n-1]$, and also the corresponding renewal processes

$$S_i^L(t) = \max\left\{k \geq 0 :\ U_i^L(k) \leq t\right\}, \quad S_i^R(t) = \max\left\{k \geq 0 :\ U_i^R(k) \leq t\right\},$$
$$T_{i,j}^L(t) = \max\left\{k \geq 0 :\ W_{i,j}^L(k) \leq t\right\}, \quad T_{i,j}^R(t) = \max\left\{k \geq 0 :\ W_{i,j}^R(k) \leq t\right\}.$$

We now define the particle system for any fixed value of the scaling parameter $M > 0$ according to

$$\begin{aligned}
\mathrm{d}X_i^M(t) = {} & \frac{1}{\sqrt{M}}\, \mathbf{1}_{\left\{X_k^M(t) + \frac{1}{\sqrt{M}} < X_{k+1}^M(t),\, k \in [n-1]\right\}}\, \mathrm{d}\left(S_i^R(M\,t) - S_i^L(M\,t)\right) \\
& + \frac{1}{\sqrt{M}} \sum_{j=1}^{n-1} \mathbf{1}_{\left\{X_i^M(t) + \frac{1}{\sqrt{M}} < X_{i+1}^M(t),\, X_j^M(t) + \frac{1}{\sqrt{M}} = X_{j+1}^M(t)\right\}}\, \mathrm{d}T_{i,j}^R\left(\sqrt{M}\,t\right) \\
& - \frac{1}{\sqrt{M}} \sum_{j=1}^{n-1} \mathbf{1}_{\left\{X_{i-1}^M(t) + \frac{1}{\sqrt{M}} < X_i^M(t),\, X_j^M(t) + \frac{1}{\sqrt{M}} = X_{j+1}^M(t)\right\}}\, \mathrm{d}T_{i,j}^L\left(\sqrt{M}\,t\right),
\end{aligned} \tag{3.24}$$

for $i \in [n]$. Note that the particle configuration $\left(X_1^M(t), X_2^M(t), \ldots, X_n^M(t)\right)$ is an element of the discrete wedge $\mathcal{W}^M$ for any $t \geq 0$.

Intuitively, this general particle system behaves as follows. When apart, the particles jump on the rescaled lattice $\mathbb{Z}/\sqrt{M}$ with jump rates of order M; these jumps are governed by the renewal processes S_i^L and S_i^R, $i \in [n]$, and thus the movements of the particles are not necessarily independent, and not necessarily governed by Poisson processes. However, when a collision occurs (i.e., two particles are on adjacent sites), then the system experiences a slowdown, with the particles moving with jump rates of order $\sqrt{M}$; these jumps are governed by the renewal processes $T_{i,j}^L$ and $T_{i,j}^R$, $i \in [n]$, $j \in [n-1]$, and thus the movements of the particles are independent, but not necessarily governed by Poisson processes.

The particle system (3.24) indeed generalizes (1.2), as the following parameter specifications show. If $\lambda_i^L = \lambda_i^R = 0$ for $i \in [n]$, $c_{i,i'}^{L,L} = c_{i,i'}^{L,R} = c_{i,i'}^{R,R} = 0$ whenever $i \neq i'$, $c_{i,i}^{L,L} = c_{i,i}^{R,R} = a^{-2}$ and $c_{i,i}^{L,R} = 0$ for $i \in [n]$, and all interarrival times above

are independent exponential random variables with appropriate means, then (3.24) reduces to (1.2).

For the extension of our convergence theorem to particle systems as in (3.24), we need the following moment assumption on the interarrival times between jumps. This assumption is needed in order to have uniform integrability of the appropriate sequences of random variables.

Assumption 2. Assume that there exists $\delta > 0$ such that

$$\sup_{M>0} \max_{i\in[n]} \left(\mathbb{E}\left[u_i^L (1)^{2+\delta} \right] + \mathbb{E}\left[u_i^R (1)^{2+\delta} \right] \right) < \infty,$$
$$\sup_{M>0} \max_{i\in[n],\, j\in[n-1]} \left(\mathbb{E}\left[w_{i,j}^L (1)^{2+\delta} \right] + \mathbb{E}\left[w_{i,j}^R (1)^{2+\delta} \right] \right) < \infty.$$

Under Assumption 2 we have the following convergence result, which generalizes Theorem 1.

Theorem 7. *Suppose that Assumptions* 1 *and* 2 *hold, and that the initial conditions* $\left\{ X^M (0), M > 0 \right\}$ *are deterministic and converge to a limit* $x \in \mathcal{W}$ *as* $M \to \infty$. *Then the laws of the paths of the particle systems* $\left\{ X^M (\cdot), M > 0 \right\}$ *defined in* (3.24) *converge in* $D([0,\infty), \mathbb{R}^n)$ *to the law of the unique weak solution of the system of SDEs*

(3.25)
$$\mathrm{d}X_i (t) = \mathbf{1}_{\{X_1(t)<\ldots<X_n(t)\}} \left(\left(\lambda_i^R - \lambda_i^L \right) \mathrm{d}t + a^{3/2} \mathrm{d}W_i (t) \right) + \sum_{j=1}^{n-1} \mathbf{1}_{\{X_j(t)=X_{j+1}(t)\}} v_{i,j} \, \mathrm{d}t$$

for $i \in [n]$, *taking values in* $\mathcal{W}$ *and starting from* x. *Here,* $W = (W_1, W_2, \ldots, W_n)$ *is a Brownian motion in* $\mathbb{R}^n$ *with zero drift vector and diffusion matrix given by*

$$\mathfrak{C} = (\mathfrak{c}_{i,i'}) = (c_{i,i'}^{L,L} + c_{i,i'}^{L,R} + c_{i',i}^{L,R} + c_{i,i'}^{R,R});$$

and $v_{i,j}$ *is as in* (1.3).

The existence and uniqueness of a weak solution to the system of SDEs (3.25) is proven in Theorem 3, so Theorem 7 is a consequence of Proposition 8 below, which is the appropriate generalization of Proposition 6 in subsection 3.1.

As in subsection 3.1, we need to study an appropriate decomposition of the particle dynamics. For each $i \in [n]$, we write

$$X_i^M (t) = X_i^M (0) + A_i^M (t) + \sum_{j=1}^{n-1} C_{i,j}^{R,M} (t) - \sum_{j=1}^{n-1} C_{i,j}^{L,M} (t) + \sum_{j=1}^{n-1} \Delta_{i,j}^{R,M} (t) - \sum_{j=1}^{n-1} \Delta_{i,j}^{L,M} (t),$$

where now

$$
\begin{aligned}
A_i^M(t) &:= \frac{1}{\sqrt{M}} \int_0^t \mathbf{1}_{\left\{X_k^M(s)+\frac{1}{\sqrt{M}}<X_{k+1}^M(s),\, k\in[n-1]\right\}} \,\mathrm{d}\left(S_i^R(M\,s) - S_i^L(M\,s)\right),\\
C_{i,j}^{R,M}(t) &:= \theta_{i,j}^R\, I_{i,j}^{R,M} := \theta_{i,j}^R \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),\, X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \,\mathrm{d}s,\\
\Delta_{i,j}^{R,M}(t) &:= \frac{1}{\sqrt{M}} \int_0^t \mathbf{1}_{\left\{X_i^M(s)+\frac{1}{\sqrt{M}}<X_{i+1}^M(s),\, X_j^M(s)+\frac{1}{\sqrt{M}}=X_{j+1}^M(s)\right\}} \,\mathrm{d}\overline{T}_{i,j}^R\left(\sqrt{M}\,s\right),\\
\overline{T}_{i,j}^R(t) &:= T_{i,j}^R(t) - \theta_{i,j}^R\, t,
\end{aligned}
$$

and the processes $C_{i,j}^{L,M}$, $I_{i,j}^{L,M}$, $\Delta_{i,j}^{L,M}$, and $\overline{T}_{i,j}^L$ are defined similarly to $C_{i,j}^{R,M}$, $I_{i,j}^{R,M}$, $\Delta_{i,j}^{R,M}$, and $\overline{T}_{i,j}^R$, respectively. The following proposition is the appropriate generalization of Proposition 6 to the present framework.

Proposition 8. *Suppose that Assumptions* 1 *and* 2 *hold, and that the initial conditions* $\left\{X^M(0), M>0\right\}$ *are deterministic and converge to a limit* $x\in\mathcal{W}$ *as* $M\to\infty$. *Then the family*

$$
\left\{\left(X^M, A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M}\right),\ M>0\right\} \tag{3.26}
$$

is tight in D^{4n^2-2n}. *Moreover, every limit point* $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ *satisfies the following for each* $i\in[n]$:

$$
\begin{aligned}
X_i^\infty(\cdot) = &\int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}} \left(\left(\lambda_i^R-\lambda_i^L\right)\mathrm{d}s + a^{3/2}\,\mathrm{d}W_i(s)\right)\\
&+\sum_{j=1}^{n-1} v_{i,j}\int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s,
\end{aligned} \tag{3.27}
$$

$$
A_i^\infty(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}} \left(\left(\lambda_i^R-\lambda_i^L\right)\mathrm{d}s + a^{3/2}\,\mathrm{d}W_i(s)\right), \tag{3.28}
$$

$$
I_{i,j}^{L,\infty}(\cdot) = \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \qquad j\in[n-1]\setminus\{i-1\}, \tag{3.29}
$$

$$
\begin{aligned}
I_{i,j}^{R,\infty}(\cdot) &= \int_0^\cdot \mathbf{1}_{\left\{X_j^\infty(s)=X_{j+1}^\infty(s)\right\}}\,\mathrm{d}s, \qquad j\in[n-1]\setminus\{i\},\\
I_{i,i-1}^{L,\infty}(\cdot) &= I_{i,i}^{R,\infty}(\cdot) = 0,\\
\Delta_{i,j}^{L,\infty}(\cdot) &= \Delta_{i,j}^{R,\infty}(\cdot) = 0, \qquad j\in[n-1],
\end{aligned} \tag{3.30}
$$

with a Brownian motion $W=(W_1, W_2, \ldots, W_n)$ *as in the statement of Theorem* 7.

Proof. One can proceed as in the proof of Proposition 6, so we only explain the arguments which are different. First, note that Theorem 14.6 in [4] and its proof extend to the case of the multidimensional renewal processes

$$
\left\{S_i^L\right\}_{i\in[n]},\quad \left\{S_i^R\right\}_{i\in[n]},\quad \left\{T_{i,j}^L\right\}_{i\in[n],j\in[n-1]},\quad \left\{T_{i,j}^R\right\}_{i\in[n],j\in[n-1]},
$$

yielding the joint convergence of

$$\begin{aligned} &\left\{\left(M^{-1/2}(S_i^R(M\,t)-S_i^L(M\,t)), t\geq 0\right)\right\}_{i\in[n]}, \\ &\left\{\left(M^{-1/4}\,\overline{T}_{i,j}^L(\sqrt{M}\,t), t\geq 0\right)\right\}_{i\in[n],j\in[n-1]}, \\ \text{and}\quad &\left\{\left(M^{-1/4}\,\overline{T}_{i,j}^R(\sqrt{M}\,t), t\geq 0\right)\right\}_{i\in[n],j\in[n-1]} \end{aligned}$$

to appropriate Brownian motions. The rest of step 1 and step 2 in the proof of Proposition 6 carry over to the present setting in a straightforward manner.

Now, one needs to show that every limit point $\left(X^\infty, A^\infty, I^{L,\infty}, I^{R,\infty}, \Delta^{L,\infty}, \Delta^{R,\infty}\right)$ satisfies

$$\langle X_i^\infty, X_{i'}^\infty\rangle(\cdot) = \langle A_i^\infty, A_{i'}^\infty\rangle(\cdot) = a^3\,\mathfrak{c}_{\mathrm{i},\mathrm{i}'}\int_0^{\cdot}\mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}}\,\mathrm{d}s.$$

To this end, one can first proceed as in step 3 in the proof of Proposition 6 to show that

$$\mathrm{d}\,\langle X_i^\infty, X_{i'}^\infty\rangle = \frac{\mathfrak{c}_{i,i'}}{\mathfrak{c}_{j,j}}\,\mathrm{d}\left\langle X_j^\infty\right\rangle, \quad i,i'\in[n],\ j\in[n]. \tag{3.31}$$

Next, one can invoke the Portmanteau Theorem as before to conclude that for all $i\in[n]$ and $0\leq t_1<t_2$:

$$a^3\,\mathfrak{c}_{\mathrm{i},\mathrm{i}}\int_{t_1}^{t_2}\mathbf{1}_{\left\{X_1^\infty(s)<\ldots<X_n^\infty(s)\right\}}\,\mathrm{d}s \leq \langle X_i^\infty\rangle(t_2)-\langle X_i^\infty\rangle(t_1)\leq a^3\,\mathfrak{c}_{\mathrm{i},\mathrm{i}}\,(t_2-t_1). \tag{3.32}$$

Moreover, since the measures $\mathrm{d}\left\langle X_j^\infty\right\rangle$, $j\in[n-1]$, assign zero mass to the sets $\{t\geq 0:\ X_j^\infty(t)=X_{j+1}^\infty(t)\}$, $j\in[n-1]$, respectively, (3.31) and (3.32) suffice to identify all quadratic covariation processes $\langle X_i^\infty, X_{i'}^\infty\rangle$, $i,i'\in[n]$. Similarly, one can identify the bounded variation parts of the processes A_i^∞, $i\in[n]$, as multiples of the quadratic variation processes $\langle X_i^\infty\rangle$, $i\in[n]$, respectively. The rest of the proof can be carried out by following the arguments in step 4 in the proof of Proposition 6. □

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University of California, Berkeley, Department of Statistics, Berkeley, CA 94720
E-mail address: racz@stat.berkeley.edu

University of California, Berkeley, Department of Statistics, Berkeley, CA 94720
E-mail address: mshkolni@gmail.com