A class of non-ergodic interacting particle systems with unique invariant measure

Benedikt Jahnel * and Christof Külske †

October 16, 2013

Abstract

We consider a class of discrete q-state spin models defined in terms of a translation-invariant quasilocal specification with discrete clock-rotation invariance which have extremal Gibbs measures μ'_φ labelled by the uncountably many values of φ in the one-dimensional sphere (introduced by van Enter, Opoku, Kuelske [8].) In the present paper we construct an associated Markov jump process with quasilocal rates whose semigroup $(S_t)_{t\geqslant 0}$ acts by a continuous rotation $S_t(\mu'_\varphi) = \mu'_{\varphi+t}$.

As a consequence our construction provides examples of interacting particle systems with unique translation-invariant invariant measure, which is not long-time limit of all starting measures, answering an old question (compare Liggett [24] question four chapter one.) The construction of this particle system is inspired by recent conjectures of Maes and Shlosman about the intermediate temperature regime of the nearest-neighbor clock model. We define our generator of the interacting particle system as a (non-commuting) sum of the rotation part and a Glauber part.

Technically the paper rests on the control of the spread of weak non-localities and relative entropy-methods, both in equilibrium and dynamically, based on Dobrushin-uniqueness bounds for conditional measures.

AMS 2000 subject classification: 82B20, 82C22, 60K35.

Keywords: Interacting particle systems, non-equilibrium, non-ergodicity, discretization, Gibbs measures, XY-model, clock model.

*Ruhr-Universität Bochum, Fakultät für Mathematik, D44801 Bochum, Germany, Benedikt.Jahnel@ruhr-uni-bochum.de, http://http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Kuelske/jahnel.html

†Ruhr-Universität Bochum, Fakultät für Mathematik, D44801 Bochum, Germany, Christof.Kuelske@ruhr-uni-bochum.de, http://www.ruhr-uni-bochum.de/ffm/Lehrsttuehle/Kuelske/kuelske.html /∼kuelske/

1 Introduction

Consider an interacting particle system (IPS) on the infinite d-dimensional integer lattice with finite local state space and quasilocal rates. Such an IPS is a Markov process in continuous time where particles (or spins) which sit on the lattice sites taking one of finitely many spin values are updated after exponential waiting times to take new states with probabilities which depend in an (essentially) local way on the states of the neighboring particles. Assume that these updating rules are lattice translation-invariant. Such infinite-volume processes may possess multiple equilibria (time-invariant measures.) Indeed, any Gibbsian potential (Hamiltonian) for a discrete-spin model allows to prescribe rates defining a Glauber dynamics for which the corresponding Gibbs measures are time-invariant and moreover reversible. Consequently, if there is a phase-transition (meaning that there is non-uniqueness of the Gibbs measures for this Hamiltonian), the set of time-invariant measures has more than one point (see [24].) To prove on the other hand that for a Glauber dynamics there are no time-invariant measures other than Gibbs measures is more intricate, and in general dimensions this statement is only known to be true if one assumes all measures to be lattice-translation invariant (see [19, 24] and compare Proposition 1.4.)

To pose our problem let us start now from any lattice translation-invariant IPS without assumptions on reversibility. Consider a lattice translation-invariant measure which is invariant under the IPS dynamics. Suppose there is only one such measure. Is it true that the dynamics is necessarily ergodic? The notion of ergodicity for an IPS means that for any starting measure the time-evolved measures converge to the unique invariant measure.

This is an old question which was picked up again in a recent very interesting paper by Maes and Shlosman [25] about dynamics of clock models (see [14, 15, 2] and [26].) In their paper the authors conjecture that this may not be the case and suggest a mechanism producing time-periodic behavior of rotating infinite-volume states. The concrete model they suggest to analyze is the discrete rotator model with standard scalarproduct nearest-neighbor interactions at intermediate temperatures, and a non-reversible time-evolution. Non-ergodicity could appear because if one uses one of the Gibbs measures as the initial measure, the discrete rotators would keep rotating coherently and so the starting distribution would be repeated periodically under the dynamics. While these conjectures seemed plausible, at the same time no simple proof based on their heuristics in their model seemed possible.

To see naively how periodicity can create non-ergodicity think of the example of a two-state discrete time Markov chain with transition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This chain has the unique invariant distribution $(\frac{1}{2}, \frac{1}{2})$, but obviously never forgets its initial condition. The same phenomenon of a unique stationary measure which does not attract all starting measures occurs for a Markov chain if the state space is finite and transition graph is bipartite.

Can such a periodic behavior with unique invariant measure persist for Markov processes with time-simultaneous updating of all spins with local rules on the infinite-lattice? Yes, and an example for non-ergodicity of discrete-time, parallel updating PCA (probabilistic cellular automaton), was only recently given in [3]. However, the issue of existence of a non-ergodic IPS which interests us here can not immediately be reduced to that of a non-ergodic PCA. Indeed, continuous Markovian time-evolutions in comparison with discrete time-evolution have a tendency to wash out synchronization and forget initial conditions. (Clearly the continuous time version of the simple two-state Markov chain example mentioned above **is** ergodic.)

In the present paper we construct a dynamics for a q-state particle system (q possibly large but finite) which does the job: It has a unique translation-invariant invariant measure for which the dynamics is not ergodic. Our construction is inspired by the conjectures of Maes and Shlosman (and different from [3]) which we put to a situation where they can be proved.

In order to do this we will relate an IPS to a hidden system of continuous S^1-valued spins via a discretization transformation which acts on each local state space. This will allow us to carry over knowledge about phase transitions in the continuous system to the discrete system we want to analyze. Technically it builds on earlier works of [8, 23] about the preservation of the Gibbs property under such discretization transformations. While these results concern properties of equilibrium measures the main new idea of the present paper is the definition of an associated non-reversible dynamics. This dynamics is chosen in such a way that it preserves the set of equilibrium measures. It does not (unlike a Glauber dynamics) preserve the individual equilibrium measures but rotates the lattice translation-invariant equilibrium measures into each other periodically. In this way a periodic orbit of measures is constructed. That such a dynamics can be realized by means of a generator with quasilocal jump rates is one main result of this paper; that this dynamics has a unique time-invariant translation-invariant measure is another main result.

The interest in the study of rotation dynamics also has an independent source which comes from biological applications like interacting neurons or collective behavior of animals. Usually the models studied in this context are of mean-field type like the famous Kuramoto model. This is natural from the perspective of many applications and also has the technical advantage of reducing all relevant questions to questions about (paths of) empirical distributions which makes them more tractable than lattice systems. In these models one usually studies S^1-valued spins under diffusive time-evolutions which contain a mean-field coupling that tends to synchronize the rotators. Often these models contain additional sources of quenched randomness (modelling individual rotation frequencies) which lead to a non-reversible character and a periodic orbit which is deformed in a way which depends on fluctuations of the realizations of the rotation frequencies. The relevant questions starting with existence of synchronized rotating states and their finer properties have been very successfully studied in particular in the Kuramoto model [1, 4, 17].

Viewed in this light our construction of a lattice dynamics hints at the existence of synchronization phenomena also on the lattice, even for discrete local spaces. It would be interesting to know more about the domain of attraction of the periodic orbit whose existence we prove, but we don't tackle this furthergoing issue in this paper where we only analyze properties **on** the cycle. Let us mention in this context that our construction of a rotation dynamics to implement the Maes-Shlosman mechanism of non-ergodic behavior can be carried over to a mean-field setup. We perform the construction of such a dynamics and the analysis of its properties in the related paper [20]. In that paper synchronization for discrete rotators is actually proved, and a Lyapunov function is constructed to prove attractivity of the cycle of rotating Gibbs measures.

1.1 Main result

To construct our IPS we have to introduce a continuous-spin model first which will be given in terms of a Gibbsian specification for an absolutely summable Hamiltonian (energy function) acting on continuous spins. The particle dynamics will be related to this model in a further step. To define this continuous-spin model we consider an S^1-rotation invariant and translation-invariant Gibbsian specification γ^Φ on the lattice $G = \mathbb{Z}^d$, with local state space $S^1 = [0, 2\pi)$. Let this specification $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda \subset G}$ be given in the standard way by an absolutely summable, S^1-invariant and translation-invariant potential $\Phi = (\Phi_A)_{A \subset G, A \text{ finite}}$, w.r.t to the Lebesgue measure λ on the spheres. This means that the Gibbsian specification is given by the family of probability kernels

$$\gamma^\Phi_\Lambda(B|\eta) = \frac{\int 1_B(\omega_\Lambda \eta_{\Lambda^c}) \exp(-H_\Lambda(\omega_\Lambda \eta_{\Lambda^c}))\lambda^{\otimes \Lambda}(d\omega_\Lambda)}{\int \exp(-H_\Lambda(\omega_\Lambda \eta_{\Lambda^c}))\lambda^{\otimes \Lambda}(d\omega_\Lambda)} \tag{1}$$

for finite $\Lambda \subset G$ and Hamiltonian $H_\Lambda = \sum_{A \cap \Lambda \neq \emptyset} \Phi_A$ applied to a measurable set $B \subset (S^1)^G$ and a boundary condition $\eta \in (S^1)^G$ (for details on Gibbsian specifications see [16].) We use notation $\Lambda^c := G \backslash \Lambda$. H_Λ also has to be differentiable under variation at a single site and these partial derivatives have to be uniformly bounded. A standard example of such a model is provided by the nearest-neighbor scalarproduct interaction rotator model with Hamiltonian

$$H_\Lambda(\omega_\Lambda \eta_{\Lambda^c}) = -\beta \sum_{i,j \in \Lambda : <i,j>} \cos(\omega_i - \omega_j) - \beta \sum_{i \in \Lambda, j \in \Lambda^c : <i,j>} \cos(\omega_i - \eta_j). \tag{2}$$

Denote by $\mathcal{G}(\gamma^\Phi)$ the simplex of the Gibbs measures corresponding to this specification, which are the probability measures μ on $(S^1)^G$ which satisfy the DLR-equation $\int \mu(d\eta)\gamma^\Phi_\Lambda(B|\eta) = \mu(B)$ for all finite Λ. Denote by $\mathcal{G}_\theta(\gamma^\Phi)$ the lattice translation-invariant Gibbs measures.

We will make as an assumption on the class of potentials (Hamiltonians) we discuss moreover that it has a continuous symmetry breaking in the following sense. Assume that the extremal translation-invariant Gibbs measures can be obtained as weak limits with homogeneous boundary conditions, i.e. with $\eta_\varphi \in (S^1)^G$ defined as $(\eta_\varphi)_i = \varphi$ for all $i \in G$ and $\varphi \in S^1$ we have

$$\text{ex}\, \mathcal{G}_\theta(\gamma^\Phi) = \{\mu_\varphi | \mu_\varphi = \lim_{\Lambda \nearrow G} \gamma^\Phi_\Lambda(\cdot|\eta_\varphi), \varphi \in S^1\}.$$

We further assume that different boundary conditions η_φ yield different measures so that there is a unique labelling of states μ_φ by the angles φ in the sphere S^1. It is a non-trivial proven fact that this assumption is true in the case of the standard rotator model (2) in $d = 3$ for λ-a.a. temperatures in the low-temperature region as discussed in [25, 13, 28].

We will now describe the discretization transformation which maps the continuous-spin model to a discrete-spin (or particle) model on which then the dynamics will be constructed in the following step.

Denote by T the local coarse-graining with equal arcs, i.e. $T : [0, 2\pi) \mapsto \{1, \dots, q\}$ where $T(\varphi) := k$ iff $2\pi(k-1)/q \leqslant \varphi < 2\pi k/q$. Extend this map to infinite-volume configurations by performing it sitewise. We will refer to the image space $\{1, \dots, q\}^G$ as the coarse-grained layer. In particular we will consider images of infinite-volume measures under T.

We will need to choose the parameter of this discretization $q \geqslant q_0(\Phi)$ large enough so that the image measures are again Gibbs measures for a discrete specification on the coarse-grained layer. That this is always possible (even for large interactions) follows from our earlier investigations [23, 8]. More precisely, let us assume that the condition from Theorem 2.1 of [8] is fulfilled (ensuring a regime where the Dobrushin uniqueness condition holds for the so-called constrained first-layer models - the Dobrushin condition is a weak dependence condition implying uniqueness and locality properties.) Note, as in our notation the usual temperature parameter β is incorporated into Φ, for β tending to infinity so does $q_0(\Phi)$.

We are now ready to describe our definition of a dynamics on the coarse-grained layer in terms of a generator which plays well together with the discretization transformation T just introduced. This dynamics has two parts, a reversible part and a non-reversible part. We begin with the more interesting non-reversible part and define a Markov process with state space $\{1, \dots, q\}^G$ in terms of the generator

$$(L\psi)(\omega') := \sum_{i \in G} c_L(\omega', (\omega')^i)(\psi((\omega')^i) - \psi(\omega')) \tag{3}$$

acting on sufficiently smooth observables ψ. The jump rates are given in terms of certain expectations of conditional infinite-volume measures which naturally arise in the course of the discretization transformation.

The choice of these rates may not seem intuitive at this stage, but they can be obtained heuristically from a straightforward computation, as we will explain at a later stage, namely (23). Let us at this stage just describe their definition which is

$$\begin{aligned} c_L(\omega', (\omega')^i) &:= \frac{\int \mu_{G\setminus i}[\omega'_{G\setminus i}](d\omega_{G\setminus i}) e^{-H_i(2\pi\omega'_i/q, \omega_{G\setminus i})}}{\int \mu_{G\setminus i}[\omega'_{G\setminus i}](d\omega_{G\setminus i}) \int \lambda(d\omega_i) e^{-H_i(\omega_i, \omega_{G\setminus i})} 1_{T(\omega_i)=\omega'_i}} \\ &= \frac{\mu_{G\setminus i}[\omega'_{G\setminus i}](e^{-H_i(\omega'_i|^r, \cdot_{i^c})})}{\mu_{G\setminus i}[\omega'_{G\setminus i}](\lambda^i(e^{-H_i} 1_{\omega'_i}))} \end{aligned} \tag{4}$$

where we have written the expression on the first line for clarity and the second line is the short notation we will continue to use. Further we used the following notation: $(\omega')^i$ is the discrete configuration which coincides with ω' except at the site i where it is increased by the amount of one unit (modulo q.) The continuous spin value $\omega'_i|^r := 2\pi\omega'_i/q \in S^1$ is the right endpoint of the interval in continuous single-spin space at the site i prescribed by ω'_i. (In other words, in the definition of the rate to jump up at site i from ω'_i to $\omega'_i + 1$, the Hamiltonian appearing under the integral gets evaluated right at the continuous-spin boundary $\omega'_i|^r$ between the segments of S^1 labelled by ω'_i and $\omega'_i + 1$.) Finally, the measure $\mu_{G\backslash i}[\omega'_{G\backslash i}]$ is the unique continuous-spin Gibbs measure for a system on the smaller volume $G\backslash i$ with conditional specification obtained by deleting all interactions with i and constrained to take values $\omega_{G\backslash i}$ with discretization images $T(\omega_{G\backslash i}) = \omega'_{G\backslash i}$. For more details and precise definition of $\mu_{G\backslash i}[\omega'_{G\backslash i}]$ in terms of formulae see Section 2, namely (11). Note that these constrained Gibbs measures are well-defined and well-behaved for sufficiently fine discretization $q \geqslant q_0(\Phi)$, see [8, 23] and Section 2. For general background on constrained Gibbs measures in the context of preservation of Gibbsianness see [6, 11] and [22].

From the definition of the rates (4) it is clear that the corresponding dynamics will be irreversible since jumps are only possible in one direction. Note that these rates depend on the original continuous-spin Hamiltonian in two places, namely in the H_i and in the $\mu_{G\backslash i}$.

Having defined the non-reversible part of our dynamics, we next consider a Glauber-type generator K on the same space $\{1, \dots, q\}^G$ by putting

$$(K\psi)(\omega') := \sum_{i\in G}\Big[c_K(\omega', (\omega')^i)\big(\psi((\omega')^i) - \psi(\omega')\big) + c_K(\omega', (\omega')^{i-})\big(\psi((\omega')^{i-}) - \psi(\omega')\big)\Big] \tag{5}$$

with $(\omega')^{i-}$ being the discrete configuration which coincides with ω' except at the site i where it is decreased by the amount of one unit. We choose the rates to go up and down respectively such that they satisfy

$$\frac{c_K(\omega', (\omega')^i)}{c_K((\omega')^i, \omega')} = \frac{\mu_{G\backslash i}[\omega'_{G\backslash i}](\lambda^i(e^{-H_i}1_{(\omega'_i)^i}))}{\mu_{G\backslash i}[\omega'_{G\backslash i}](\lambda^i(e^{-H_i}1_{\omega'_i}))}. \tag{6}$$

(For clarity of notation let us spell out that e.g. the denominator on the r.h.s means $\mu_{G\backslash i}[\omega'_{G\backslash i}](\lambda^i(e^{-H_i}1_{\omega'_i})) = \int \mu_{G\backslash i}[\omega'_{G\backslash i}](d\omega_{G\backslash i}) \int \lambda(d\omega_i) e^{-H_i(\omega_i, \omega_{G\backslash i})} 1_{T(\omega_i)=\omega'_i}$.) A possible choice of K is obtained by identifying numerators (resp. denominators) on r.h.s and l.h.s of (6).

Having defined the two generators L and K we are finally in the position to formulate our main result. We have the following theorem.

Theorem 1.1 *Consider a translation-invariant, rotation-invariant and continuously differentiable potential Φ which satisfies the decay assumption*

$$\sum_{A\ni 0}\sum_{k\in G} e^{\varepsilon|k|}\delta_k(\Phi_A) < \infty \tag{7}$$

for some $\varepsilon > 0$ where $\delta_k(\Phi_A) = \sup_{\omega,\bar\omega:\omega_{k^c}=\bar\omega_{k^c}} |\Phi_A(\omega) - \Phi_A(\bar\omega)|$ denotes the variation at the site k. Assume fine enough discretization $q \geqslant q_0(\Phi)$ and let $\alpha > 0$ be arbitrary.

1. *Then the generator $L + \alpha K$ gives rise to a well-defined IPS with quasilocal rates.*

2. *The class of translation-invariant measures which are invariant under the associated Markov semigroup $(S_t^{L+\alpha K})_{t\geqslant 0}$ consists of a single element.*

3. *There are translation-invariant measures which do not converge under the dynamics to the unique invariant measure.*

Note that any finite-range potential or exponentially decaying pair-potential satisfies (7). We further note that the requirements on the potential can be relaxed. For example one could replace exponential decay by polynomial decay of sufficiently high order as will become clear from the proof. The conditions will be presented whenever they get used for the first time.

1.2 Idea of Proof

The proof relies on the fact that the discretization transformation T preserves the Gibbsian structure of the continuous and discrete-spin system if we assume fine enough discretization $q \geqslant q_0(\Phi)$, in the following sense.

First, to talk about the correspondence between the continuous and the discrete system we need to make explicit the relevant Gibbsian specification for the latter. To do so define a family of kernels $\gamma' = (\gamma'_\Lambda)_{\Lambda\subset G, \Lambda \text{ finite}}$ for the discretized model by

$$\gamma'_\Lambda(\omega'_\Lambda|\omega'_{G\setminus\Lambda}) = \frac{\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega'_\Lambda}))}{\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}))} \tag{8}$$

where in analogy to the explanation for $\mu_{G\setminus i}[\omega'_{G\setminus i}]$ given before, $\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}]$ is the unique continuous-spin Gibbs measure for the continuous specification on the volume $G\setminus\Lambda$, not interacting with Λ and conditioned to a discrete configuration $\omega'_{G\setminus\Lambda} \in \{1, \dots, q\}^{G\setminus\Lambda}$. This γ' indeed is a quasilocal specification and the discretized Gibbs measures will be Gibbs for this γ'. For details see Section 2.

Further, the infinite-volume discretization map T is injective when applied to the set of translation-invariant extremal Gibbs states in the continuum model (ex $\mathcal{G}_\theta(\gamma^\Phi)$.) More precisely we have the following theorem.

Theorem 1.2 *T is a bijection from ex $\mathcal{G}_\theta(\gamma^\Phi)$ to ex $\mathcal{G}_\theta(\gamma')$ with inverse given by the kernel $\mu_G[\omega'](d\omega)$.*

Here $\mu_G[\omega'](d\omega)$ is the unique conditional continuous-spin Gibbs measure on the whole volume G (see (11).) It is important to understand that this kernel gets us back from a discrete-spin Gibbs measure to a continuous-spin Gibbs measure in a way which **does not depend** on the choice of the initial measure. This is crucial for the possibility to construct a rotation generator L with the desired properties, as we will see.

The fact that $T\mu := \mu \circ T^{-1}$ is Gibbs for γ' when μ is Gibbs for γ^Φ, is already proved in [8, 23] and based on the uniform Dobrushin condition on the coarse-graining. The part that each translation-invariant discrete Gibbs measure has a discretization preimage in the continuous Gibbs measures is new and uses the Gibbs variational principle which involves considerations of relative entropy densities (see [16].)

The following step of the proof presents the main new structure of our paper. We show that rotation on the level of discrete extremal Gibbs states $\mu'_\varphi = T\mu_\varphi$ can be realized by the rotation dynamics with generator L with quasilocal jump rates as defined above. This can be formulated as follows.

Theorem 1.3 *1. The semigroup $(S^L_t)_{t\geqslant 0}$ associated to L is well-defined.*

2. $S^L_t(T\mu_\varphi) = T\mu_{\varphi+t}$ for all $\mu_\varphi \in ex\, \mathcal{G}_\theta(\gamma^\Phi)$ and $t \geqslant 0$.

The theorem expresses that a discretization of a deterministic rotation of the continuous-spin model can be represented as a stochastic time evolution after discretization. The heuristic reason why this works and the heuristic route to the identification of such a suitable L is explained in formula (23): The idea is to compute the time derivative $\frac{d}{dt}_{|t=0}(T\mu_{\varphi+t})(f)$ for indicator functions f, and to identify the appearing terms as $(T\mu_\varphi)(Lf)$. During this computation one makes explicit the kernel from discrete to continuous variables of Theorem 1.2, uses its properties, and the rates defining L pop out. If we already knew that the trajectory $t \mapsto T\mu_{\varphi+t}$ can be realized in terms of a semigroup, this would identify its generator. A difficulty in the actual proof is that we do not know this a priori, and more arguments are needed. This involves the definition of weighted triple-norms (weighted sums of variations of observables) to control the weak non-localities which are present in the rates and the spreading of these under the action of the dynamics.

Rephrasing the result in a group theoretical language we can say $(t, \mu_\varphi) \mapsto \mu_{\varphi+t}$ is an S^1-action on the extremal translation-invariant Gibbs measures ex $\mathcal{G}_\theta(\gamma^\Phi)$ and $(t, \mu'_\varphi) \mapsto \mu'_{\varphi+t}$ is an S^1-action on ex $\mathcal{G}_\theta(\gamma')$. The second statement of the theorem then says that T is an equivariant map (i.e. a group-action preserving map.)

Let us now turn to the discussion of the reversible generator K. Having defined the discretized local specification $\gamma' = (\gamma'_\Lambda)_{\Lambda\subset G}$ we note that the generator K defined above plays the role of a corresponding Glauber dynamics. To understand the final arguments providing us with a unique translation-invariant invariant measure for the joint dynamics and understand better this Glauber part of the dynamics we prove the following intermediate result.

Proposition 1.4 *1. The semigroup $(S^K_t)_{t\geqslant 0}$ associated to K is well-defined.*

2. The translation-invariant measures which are invariant under the Glauber dynamics $(S^K_t)_{t\geqslant 0}$ are precisely the discrete Gibbs measures $\mathcal{G}_\theta(\gamma')$.

To see that invariance under this dynamics implies Gibbsianness we use an adaptation of the relative entropy arguments exposed in Liggett ("Holley's argument") [19, 24] from the Ising lattice gas context to our situation. The standard idea here is to exploit the form of the time derivatives of relative entropy densities of the time-evolved measure relative to a suitable finite-volume version of a Gibbs measure. Putting these to zero, along with translation-invariance and estimation of boundary terms, produces a single-site DLR equation implying that the invariant measures are Gibbs for γ'.

The technical treatment of this beautiful argument will have to be substantially modified in view of the new terms arising from the joint dynamics corresponding to $L+\alpha K$ which we want to consider finally. The result is the following proposition which is essential for the proof of the main theorem.

Proposition 1.5 *Let $\alpha > 0$.*

1. *The semigroup $(S_t^{L+\alpha K})_{t\geqslant 0}$ associated to $L + \alpha K$ is well-defined.*
2. *$S_t^{L+\alpha K}(T\mu_\varphi) = T\mu_{\varphi+t}$ for all $\mu_\varphi \in ex\, \mathcal{G}_\theta(\gamma^\Phi)$ and $t \geqslant 0$.*
3. *The translation-invariant measures which are invariant under the joint dynamics $(S_t^{L+\alpha K})_{t\geqslant 0}$ must necessarily be elements of the discrete Gibbs measures $\mathcal{G}_\theta(\gamma')$.*

For the proof we use that the Glauber part leaves the discrete Gibbs measures invariant. Let us point out some of the issues which come into play. A bit of care needs to be taken for the second statement since the rotation part L and the Glauber part K do not commute. However, one can follow the same line of arguments as for the proof of Theorem (1.3) part 2, using weighted triple-norms, to control the weak non-localities of L and K. The idea of the third part is this: To see that invariance under joint dynamics implies Gibbsianess we would like to use again relative entropy arguments as in the proof of Proposition (1.4) part 2, but note that we now have to deal with a sum of two terms each corresponding to L and K. For the new part corresponding to L we apply the arguments to a finite-volume open boundary version of L as well as of the measure in the second slot of the relative entropy. The correction term is only of boundary-order. The bulk terms have a good sign by a finite-volume argument since the modified L is attractive to the modified measure. Together we arrive at the desired single-site DLR equation.

Combining the second and the third part of Proposition (1.5) we conclude:

Corollary 1.6 *Let $\alpha > 0$. Then the only translation-invariant measure which is invariant under the joint dynamics $(S_t^{L+\alpha K})_{t\geqslant 0}$ is the measure $\frac{1}{2\pi}\int d\varphi T\mu_\varphi$.*

Finally, together with part 2 of Proposition (1.5) which shows that there is no relaxation of the pure measure μ'_φ under $(S_t^{L+\alpha K})_{t\geqslant 0}$ we arrive at the proof of Theorem 1.1.

1.3 Extensions

Theorem 1.2 stays true also for models where for every angle there are more than one Gibbs measures. This could occur for potentials with highly non-convex shapes [10]. The well-definedness of the rotation semigroup is untouched and one has:

Theorem 1.7 *The map $T : ex\ \mathcal{G}_\theta(\gamma^\Phi) \mapsto ex\ \mathcal{G}_\theta(\gamma')$ is an equivariant bijection for the S^1-actions on continuous and discrete-spin Gibbs measures.*

The equivariance property says $S_t^{L+\alpha K}(T\mu) = TR_t\mu$ for all $\alpha \geqslant 0$, where $R_t\mu$ is the measure obtained by joint rotation of the realizations of the measure μ by an angle t. The conclusions of Theorem (1.2) and Theorem (1.1) part 1 and part 3 apply. Theorem (1.1) part 2 (the uniqueness of the invariant measure) does not apply because Corollary (1.6) does not apply since the symmetrization over the angles will produce more then one invariant measure.

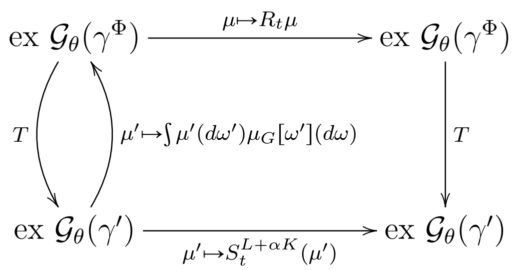

Figure 1: Equivariance property of the bijective discretization map T for the deterministic rotation action $(R_t)_{t\geqslant 0}$ and the action of the IPS $(S_t^{L+\alpha K})_{t\geqslant 0}$.

The remainder of the paper contains the following: In Section 2 we prove Theorem 1.2 using the variational principle. For this we need to present generalities and facts on discretizations and recall criteria on the preservation of Gibbsianess. In Section 3 we consider the rotation dynamics and prove Theorem 1.3. In Section 4 we consider the Glauber dynamics and prove Proposition 1.4. In Section 5 we consider the joint dynamics and prove the main Proposition 1.5.

1.4 Acknowledgement

We would like to dedicate this paper to A. van Enter on the occasion of his 60th birthday. We thank C. Maes and A. van Enter for stimulating discussions at the workshop 'Dynamical Gibbs-non-Gibbs Transitions' at Eurandom, and an anonymous referee for useful comments. This work is supported by the Sonderforschungsbereich SFB | TR12-Symmetries and Universality in Mesoscopic Systems.

2 Discretizations

In the present section we will give a self-contained presentation of properties of the discretization map T which maps continuous-spin Gibbs measures to discrete-spin Gibbs measures. We will already obtain in this section the 'vertical' parts of the commutating diagram of Figure 1, i.e. those parts not involving dynamics. These generalities about local discretizations we are going to present are easily explained in a setup which is broader than that of S^1-valued spins on an integer lattice.

Take an underlying site space G, a local spin-space S equipped with a σ-algebra and the configuration space $\Omega = S^G$ carrying the product-σ-algebra. S^1 will often serve as an example for the local state space, but one can also consider subsets of the Euclidean space of any finite dimension or finite-dimensional manifolds. We will refer to this space as the continuous spin-space. Consider a Gibbsian potential $\Phi = (\Phi_A)_{A \subset G, A \text{ finite}}$ which is absolutely summable. Write for the Hamiltonian in the finite volume $\Lambda \subset G$, $H_\Lambda = \sum_{A \cap \Lambda \neq \varnothing} \Phi_A$ and let $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda \subset G, \Lambda \text{ finite}}$ be the associated Gibbsian specification with a priori measure λ. We denote by $\mathcal{G}(\gamma^\Phi)$ the corresponding Gibbs measures, defined by the DLR equation and by $\mathcal{G}_\theta(\gamma^\Phi)$ the translation-invariant Gibbs measures. Together we call this the first-layer system.

Let $S = \bigcup_{s'=1}^q S_{s'}$ be a disjoint decomposition of the local state space into sets of positive λ-measure. As in [8, 23] the map $T(s) := s'$ for $S_{s'} \ni s$ defines a deterministic transformation on S, called the discretization map. The space $\Omega' := \{1, \dots, q\}^G$ will be referred to as the discrete or coarse-grained configuration space. It is convenient to use a notation which identifies the label $s' \in \{1, ..., q\}$ with the measurable subset of S described by it and write $1_{s'}(s) = 1$ iff $T(s) = s'$.

Lemma 2.1 *For each fixed discrete spin variable $\omega' \in \Omega'$ define a family of kernels on the continuous spin-space by constraining the continuous spins to ω' and putting, for each finite $\Lambda \subset G$, and bounded measurable observable φ,*

$$\gamma^{\omega'}_\Lambda(\varphi|\omega_{\Lambda^c}) := \frac{\gamma^\Phi_\Lambda(\varphi 1_{\omega'_\Lambda}|\omega_{\Lambda^c})}{\gamma^\Phi_\Lambda(1_{\omega'_\Lambda}|\omega_{\Lambda^c})}. \tag{9}$$

Then this family defines a Gibbsian specification $\gamma^{\omega'}$ on $\Omega^{\omega'} = \times_{i \in G} S_{\omega'_i}$ in the sense of [16, 6].

It will be useful to sometimes indicate measurability of functions w.r.t sub-σ-algebras in the following way: We write $f(\omega'_\Lambda)$ equivalently to $f(\omega')$ if f evaluates ω' only inside the volume Λ. For example in case of (9) we write $\gamma^{\omega'_\Lambda}_\Lambda$ for $\gamma^{\omega'}_\Lambda$.

Proof: One verifies the defining properties of a specification which need to be fulfilled to be a useful candidate system of conditional probabilities of an infinite-volume measure. To begin with, from the compatibility property of γ^Φ follows the compatibility property of $\gamma^{\omega'}$ for each fixed ω'. The quasilocality of γ^Φ implies that of $\gamma^{\omega'}$ for all ω'. Since γ^Φ is proper it is easy to see that $\gamma^{\omega'}$

is proper, where properness means for all finite $\Lambda \subset G$ and $A \subset \Omega$ measurable and dependent only on sites in Λ^c we have $\gamma_\Lambda^\Phi(A|\cdot) = 1_A$. Finally the property of non-nullness on the constrained first-layer local spin-spaces (uniform boundedness of local probabilities from below) follows from the positive measure of the sets in the decomposition and the absolute summability of Φ. □

We will need to choose the discretization fine enough such that there is only one Gibbs measure which is compatible with this specification, for any ω'. One way to see that this is always possible and implement this requirement is to use Dobrushin uniqueness theory. From general results of the theory also further information about the unique Gibbs measure follows and we will make use of this later. We define a uniform Dobrushin matrix $\bar{C} = (\bar{C}_{ij})_{i,j\in G}$ which is associated to the family of specifications $\gamma^{\omega'}$, indexed by ω', by letting their entries be

$$\bar{C}_{ij} := \sup_{\omega'} \sup_{\substack{\omega,\tilde{\omega}: \\ \omega_{j^c}=\tilde{\omega}_{j^c}, T(\omega)=T(\tilde{\omega})=\omega'}} \|\gamma^{\omega'}(\cdot|_i|\omega) - \gamma^{\omega'}(\cdot|_i|\tilde{\omega})\|_i \tag{10}$$

where $\|\cdot\|_i$ is the total variational distance at site i between the marginal distributions at site i (for details see [8, 16].) Notice that we used another supremum over the discrete configurations and hence the corresponding Dobrushin constant $\bar{c} := \sup_i \sum_j \bar{C}_{ij}$ is uniform in ω'.

We will always suppose that the discretization is fine enough such that $\bar{c} < 1$. (Later we will even suppose a slightly stronger exponential decay property that will appear in Lemma 3.4.)

Then it follows from the theory of Dobrushin uniqueness (see Theorem 8.23 in [16]) that, for any fixed ω' the specification $\gamma^{\omega'}$ has a unique Gibbs measure. Moreover, for each finite or infinite $V \subset G$ there is a kernel from coarse-grained configurations ω' (inside V) and boundary conditions of first-layer configurations ω outside V, namely $\gamma_V^{\omega'}(\cdot|\omega)$, which has the infinite-volume compatibility property $\gamma_V^{\omega'}\gamma_W^{\omega'} = \gamma_V^{\omega'}$, between all (and not only finite) subsets of G.

For the unique first-layer Gibbs measure for given discretized variable ω' we use the notation

$$\mu_G[\omega'](d\omega) := \gamma_G^{\omega'}(d\omega). \tag{11}$$

We note that $\mu[\cdot](d\omega)$ is a probability kernel from Ω' to Ω, since it is also measurable as a function of the coarse-grained configuration.

We report the result of [8] which gives a criterion for the fineness of the discretization in our main example, the standard nearest-neighbor model (the planar rotor or XY-model), with Hamiltonian as given in (2): For $q \geqslant q(\beta)$ large enough such that $2d\beta(\sin\frac{\pi}{q})^2 < 1$ we have $\bar{c} < 1$. Notice similar criteria are immediate for high-dimensional rotators, for details see [8].

There is no obstacle to use this theory also for even more general models to which the hypothesis of Theorem 1.1 apply. We report the bound on the matrix elements of the Dobrushin matrix as given in [8] which takes the form

$$\bar{C}_{ij} \leqslant \sup_{s'} \operatorname{diam}_{ij} S_{s'}/4 \tag{12}$$

with a family of metrics $(d_{ij})_{j\in G\setminus i}$ on the local spin-space at site $i \in G$ which are generated by variations of the energy as follows

$$d_{ij}(\sigma_i, \tau_i) := \sup_{\substack{\zeta,\bar\zeta: \\ \zeta_{j^c}=\bar\zeta_{j^c}; T(\zeta_j)=T(\bar\zeta_j)}} \Big| H_i(\sigma_i\zeta_{i^c}) - H_i(\sigma_i\bar\zeta_{i^c}) - \Big(H_i(\tau_i\zeta_{i^c}) - H_i(\tau_i\bar\zeta_{i^c})\Big)\Big|$$

and $\mathrm{diam}_{ij}(S_{s'}) := \sup_{s,\tilde s\in S_{s'}} d_{ij}(s,\tilde s)$.

Using the above criterion we suppose from now on that potential and discretization are chosen such that we are conditionally uniformly in the Dobrushin regime $\bar c < 1$. We note that to each quasilocal continuous-spin observable f there is naturally associated a discrete-spin observable $f'(\omega') := \mu_G[\omega'](f)$ which is easily seen to be quasilocal as well (but on Ω') using Dobrushin uniqueness techniques. Denoting by $\mathcal{F}'$ the σ-algebra over Ω generated by the infinite-volume coarse-graining map T we have that f' is a regular version of the conditional expectation $\mu(f|\mathcal{F}')(\omega')$ for every Gibbs measure $\mu \in \mathcal{G}(\gamma^\Phi)$, independently of its choice.

Lemma 2.2 *For a continuous-spin Gibbs measure μ denote its discretization image by $\mu' = T\mu$. Then the measures μ and μ' are close in the sense that $\mu(f) = \mu'(f')$ for all continuous-spin observables f, and moreover differences between corresponding correlations obey the estimate*

$$|\Big(\mu(fg) - \mu(f)\mu(g)\Big) - \Big(\mu'(f'g') - \mu'(f')\mu'(g')\Big)| \leqslant \frac{1}{4}\sum_{i,j\in G} \delta_i(f)\delta_j(g)\bar D_{ij}. \quad (13)$$

with the matrix $(\bar D_{ij})_{i,j\in G} := \sum_{n\geqslant 0} \bar C^n$, and $g' = \mu(g|\mathcal{F}')$.

Proof: To see that (13) holds, write

$$\begin{aligned}&\mu(fg) - \mu(f)\mu(g) = \mu(\mu(fg|\mathcal{F}')) - \mu(f)\mu(g) \\ &= \mu'(\mu(fg|\mathcal{F}') - \mu(f|\mathcal{F}')\mu(g|\mathcal{F}')) + \mu'(\mu(f|\mathcal{F}')\mu(g|\mathcal{F}')) - \mu'(\mu(f|\mathcal{F}'))\mu'(\mu(g|\mathcal{F}')).\end{aligned}$$

Further the standard estimate (see Proposition 8.34 in [16]) in the Dobrushin uniqueness regime yields

$$\sup_{\omega'} |\mu_G[\omega'](fg) - \mu_G[\omega'](f)\mu_G[\omega'](g)| \leqslant \frac{1}{4}\sum_{i,j\in G}\delta_i(f)\delta_j(g)\bar D_{ij} \quad (14)$$

which proves (13). □

On the lattice this statement can be used to see that power law decay of correlations for a continuous-spin observable f (as it can occur in the standard rotor model in space dimension 2) carries over to power law decay between correlations in the associated observable f' when the discretization is fine enough, since in that case the matrix elements of D are decaying exponentially fast.

It is clear that the map from μ to $\mu' := T\mu$ is injective when viewed on the (not necessarily translation-invariant) Gibbs measures of the continuous-spin system: Indeed, we can restore an initial Gibbs measure μ from its coarse-grained image via $\mu(\varphi) = \int \mu'(d\omega')\mu_G[\omega'](\varphi)$ where $\mu_G[\omega'](\varphi)$ does not depend on μ. Hence different μ's must have different images μ'.

Next recall the definition of the specification γ' for the coarse-grained system (see also [23]) given in (8), that we will sometimes also call the second-layer system. We have the following Lemma.

Lemma 2.3 *In the uniform Dobrushin regime, the discretization image of any continuous-spin Gibbs measure is Gibbs for the specification* γ'.

Proof: This is shown by standard arguments which we include for convenience of the reader. Any conditional probability with finite-volume conditioning can be written as

$$\begin{aligned}\mu'(\omega'_{\Lambda'}|\omega'_{\Lambda\setminus\Lambda'}) &= \frac{\int \mu(d\omega_{\Lambda^c})\gamma_\Lambda(1_{\omega'_{\Lambda'}}1_{\omega'_{\Lambda\setminus\Lambda'}}|\omega_{\Lambda^c})}{\int \mu(d\omega_{\Lambda^c})\gamma_\Lambda(1_{\omega'_{\Lambda\setminus\Lambda'}}|\omega_{\Lambda^c})}\\ &= \frac{\int \mu(d\omega_{\Lambda^c})(\gamma^{\omega'_{\Lambda\setminus\Lambda'}}|_{G\setminus\Lambda'})_{\Lambda\setminus\Lambda'}(\lambda^{\Lambda'}(e^{-H_{\Lambda'}}1_{\omega'_{\Lambda'}}))|\omega_{\Lambda^c})}{\int \mu(d\omega_{\Lambda^c})(\gamma^{\omega'_{\Lambda\setminus\Lambda'}}|_{G\setminus\Lambda'})_{\Lambda\setminus\Lambda'}(\lambda^{\Lambda'}(e^{-H_{\Lambda'}}))|\omega_{\Lambda^c})}\end{aligned} \tag{15}$$

where $\mu'(\omega'_{\Lambda'}) = \mu(1_{\omega'_{\Lambda'}})$ and $\gamma^{\omega'}|_{G\setminus\Lambda'}$ denotes the specification on $\Omega_{G\setminus\Lambda'}^{\omega'_{G\setminus\Lambda'}} = \times_{i\in G\setminus\Lambda'}S_{\omega'_i}$ obtained by putting all potentials Φ_A with $A \cap \Lambda' \neq \emptyset$ equal to zero.

Then, by martingale convergence, $\mu'(\omega'_{\Lambda'}|\omega'_{\Lambda\setminus\Lambda'})$ converges as Λ tends to G in the a.s.- and L^1-sense to $\mu(1_{\omega'_{\Lambda'}}|\mathcal{F}'_{G\setminus\Lambda'})(\omega_{G\setminus\Lambda'})$ where $1_{\omega'_{\Lambda\setminus\Lambda'}}(\omega_{G\setminus\Lambda'}) = 1$ for all $\Lambda \supset \Lambda'$ and $\mathcal{F}'_{G\setminus\Lambda'}$ is the σ-algebra over Ω generated by the coarse-graining map T applied only in the infinite-volume $G\setminus\Lambda'$.

On the other hand, for any finite Λ', there is convergence uniformly in the integration variable ω under the μ-integrals since the conditional specification is in the uniform Dobrushin regime and we have

$$\begin{aligned}\gamma'_{\Lambda'}(\omega'_{\Lambda'}|\omega'_{G\setminus\Lambda'}) &= \frac{\lim_{\Lambda\uparrow G}(\gamma^{\omega'_{G\setminus\Lambda'}}|_{G\setminus\Lambda'})_{\Lambda\setminus\Lambda'}(\lambda^{\Lambda'}(e^{-H_{\Lambda'}}1_{\omega'_{\Lambda'}})|\omega_{G\setminus\Lambda})}{\lim_{\Lambda\uparrow G}(\gamma^{\omega'_{G\setminus\Lambda'}}|_{G\setminus\Lambda'})_{\Lambda\setminus\Lambda'}(\lambda^{\Lambda'}(e^{-H_{\Lambda'}})|\omega_{G\setminus\Lambda})}\\ &= \frac{\mu_{G\setminus\Lambda'}[\omega'_{G\setminus\Lambda'}](\lambda^{\Lambda'}(e^{-H_{\Lambda'}}1_{\omega'_{\Lambda'}}))}{\mu_{G\setminus\Lambda'}[\omega'_{G\setminus\Lambda'}](\lambda^{\Lambda'}(e^{-H_{\Lambda'}}))}.\end{aligned} \tag{16}$$

The limiting measure in the last line is the unique Gibbs measure of the specification restricted to $G\setminus\Lambda'$ with open boundary conditions and this proves (8). $\square$

It is easy to see using the standard Dobrushin estimates that the specification γ' built with these kernels is quasilocal.

Now we are in the position to discuss new results which are related to the proof of the bijectivity of the map T. To start, note that we also have that the influence of variations of the boundary condition outside Λ' on probabilities inside Λ' has the estimate, uniformly in the configuration $\omega'_{\Lambda'}$,

$$\log \frac{\gamma'_{\Lambda'}(\omega'_{\Lambda'}|\omega'_{G\setminus\Lambda'})}{\gamma'_{\Lambda'}(\omega'_{\Lambda'}|\bar{\omega}'_{G\setminus\Lambda'})} \leqslant 4 \sum_{A\cap\Lambda'\neq\emptyset, A\cap\Lambda'^c\neq\emptyset} \|\Phi_A\|. \tag{17}$$

Further note, that for summable potentials and Λ' being cubes on the lattice, the r.h.s is bounded by a constant times the length of the boundary of Λ', in other words $\log \frac{d\gamma'_{\Lambda'}(\cdot|\omega'_{G\setminus\Lambda'})}{d\gamma'_{\Lambda'}(\cdot|\bar{\omega}'_{G\setminus\Lambda'})} = O(|\partial\Lambda'|)$, where $|\cdot|$ denotes the cardinality.

Let us now restrict to the lattice case, i.e. $G = \mathbb{Z}^d$ and discuss the relative entropy density. The following lemma should be seen as a generalization of the contractivity of the relative entropy (density) between two measures (see Lemma 3.3 in [6]) under strictly local transforms to transforms which are not strictly but "sufficiently" local.

Lemma 2.4 *Let $\mu'_1, \mu'_2 \in \mathcal{G}(\gamma')$ for some specification for which $\log \frac{d\gamma'_\Lambda(\cdot|_\Lambda|\omega'_1)}{d\gamma'_\Lambda(\cdot|_\Lambda|\omega'_2)}$ is of the order $o(|\Lambda|)$ for cubes. Take a kernel $\mu_G[\omega'](d\omega)$ where $\log \frac{d\mu_G[\omega'](\cdot|_\Lambda)}{d\mu_G[\bar{\omega}'](\cdot|_\Lambda)}$ is also of the order $o(|\Lambda|)$ uniformly in all configurations ω' and $\bar{\omega}'$ which coincide on Λ. Then the relative entropy density between the mapped measures equals zero, i.e.*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} H\Big(\int \mu'_1(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda \Big| \int \mu'_2(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda\Big) = 0 \tag{18}$$

along cubes.

Proof: We need to estimate the relative entropy H in a volume Λ where $\Lambda \subset \mathbb{Z}^d$ is a finite cube appearing in the formula above, which is

$$\int \mu'_1(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda \Big(\log \frac{d\int \mu'_1(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda}{d\int \mu'_2(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda}\Big). \tag{19}$$

Using the DLR equation for the integrand as well as the conditions on the Radon Nikodym derivatives we find

$$\begin{aligned}\log \frac{d\int \mu'_1(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda}{d\int \mu'_2(d\tilde{\omega}')\mu_G[\tilde{\omega}']|_\Lambda} &= \log \frac{\int \mu'_1(d\tilde{\omega}'_1)\frac{d\mu_G[\tilde{\omega}'_1]|_\Lambda}{d\lambda^\Lambda}}{\int \mu'_2(d\tilde{\omega}'_2)\frac{d\mu_G[\tilde{\omega}'_2]|_\Lambda}{d\lambda^\Lambda}} \\ &\leqslant \sup_{\omega'_1,\omega'_2} \log \frac{\int(\gamma'_\Lambda)|_\Lambda(d\tilde{\omega}'_1|\omega'_1)\frac{d\mu_G[(\tilde{\omega}'_1)_\Lambda(\omega'_1)_{\Lambda^c}]|_\Lambda}{d\lambda^\Lambda}}{\int(\gamma'_\Lambda)|_\Lambda(d\tilde{\omega}'_2|\omega'_2)\frac{d\mu_G[(\tilde{\omega}'_2)_\Lambda(\omega'_2)_{\Lambda^c}]|_\Lambda}{d\lambda^\Lambda}} = o(|\Lambda|)\end{aligned} \tag{20}$$

where the estimate in the last line uses the two assumptions in the hypothesis. Hence the relative entropy density as the limit of the relative entropy devided by the volumes of a cofinal sequence of cubes is equal to zero. □

Applying the lemma and using now the Gibbs variational principle in the form of Theorem 15.37 of [16] our desired result, stating that every discrete Gibbs measure has a continuous preimage, follows:

Proposition 2.5 *Let $\mu' \in \mathcal{G}_\theta(\gamma')$, then $\mu(d\omega) := \int \mu'(d\omega')\mu_G[\omega'](d\omega) \in \mathcal{G}_\theta(\gamma^\Phi)$.*

Proof: Let $\mu_0 \in \mathcal{G}_\theta(\gamma^\Phi)$ be a Gibbs measure for the original system and $\mu'_0 := T\mu_0$ its coarse-grained image. We want to use the preceding lemma, i.e. justify the conditions and therefore conclude that the relative entropy density between the two translation-invariant measures is zero. Hence, by the variational principle applied to the original system, also $\mu \in \mathcal{G}_\theta(\gamma^\Phi)$.

Indeed, (17) asserts the condition of Lemma 2.4 for the coarse-grained specification γ'. Also we have for $\omega', \bar\omega'$ coinciding on Λ

$$\begin{aligned}\log \frac{d\mu_G[\omega']|_\Lambda}{d\mu_G[\bar\omega']|_\Lambda} &= \log \frac{\int \mu_G[\omega'](d\tilde\omega_1)\frac{d(\gamma_\Lambda^{\omega'})|_\Lambda}{d\lambda^\Lambda}(\cdot|\tilde\omega_1)}{\int \mu_G[\bar\omega'](d\tilde\omega_2)\frac{d(\gamma_\Lambda^{\omega'})|_\Lambda}{d\lambda^\Lambda}(\cdot|\tilde\omega_2)} \\ &\leqslant \sup_{\tilde\omega_1,\tilde\omega_2} \log \frac{d\gamma_\Lambda^{\omega'}(\cdot|\tilde\omega_1)}{d\gamma_\Lambda^{\omega'}(\cdot|\tilde\omega_2)} \leqslant 4 \sum_{A\cap\Lambda'\neq\varnothing, A\cap\Lambda'^c\neq\varnothing} \|\Phi_A\| = o(|\Lambda|). \qquad \square\end{aligned}$$

Together with the injectivity of T this means that the map from the translation-invariant Gibbs measures of the original system $\mathcal{G}_\theta(\gamma^\Phi)$ to the translation-invariant measures for the coarse-grained configuration $\mathcal{G}_\theta(\gamma')$ is one-to-one.

Remark 2.6 *This one-to-one correspondence also holds for the extremals: If μ is tail-trivial, then so is $T\mu$ since the tail-σ-algebra of discrete events is contained in the tail-σ-algebra of all events, $\mathcal{T}' \subset \mathcal{T}$. In particular ex $\mathcal{G}(\gamma') \supset T(\text{ex}\,\mathcal{G}(\gamma^\Phi))$. To see that also $\mu \in \text{ex}\,\mathcal{G}(\gamma^\Phi)$ for $T\mu \in \text{ex}\,\mathcal{G}(\gamma')$ one can use the fact that the mapping T is affine: Let us assume $T\mu \in \text{ex}\,\mathcal{G}(\gamma')$ and $\mu = s\mu_1 + (1-s)\mu_2$ for $s \in [0,1]$ and $\mu_1, \mu_2 \in \mathcal{G}(\gamma^\Phi)$. Then we have $T\mu = sT\mu_1 + (1-s)T\mu_2$ and hence $T\mu = T\mu_1 = T\mu_2$ since $T\mu$ is extremal. But that means $\mu = \mu_1 = \mu_2$ and thus $\mu \in \text{ex}\,\mathcal{G}(\gamma^\Phi)$.*

It is interesting to note that the proof of the preceding remark also follows from the fact that tail-triviality is preserved under the kernel (even not assuming initial Gibbs measures.) This property explains the "essentially local" nature of the transformation T from the perspective of the tail events.

Proposition 2.7 *Assume that μ' is any probability measure (not necessarily Gibbs) on Ω' which is trivial on $\mathcal{T}'$. Then $\mu(d\omega) := \int \mu'(d\omega')\mu_G[\omega'](d\omega)$ is trivial on the tail-σ-algebra $\mathcal{T}$.*

Proof: We assume that also $\sup_j \sum_i \bar{C}_{ij} < 1$ which is guaranteed in the fine-discretization regime ensured by our criteria.

If $A \in \mathcal{T}$ then $\mu_G[\omega'](A)$ is $\mathcal{T}'$-measurable. To see this, suppose that W is a finite subset of G, that V contains W and that A is in $\mathcal{T}_V$, the σ-algebra of events not depending on spins inside V. Assuming that A is a cylinder at first we have

$$\sup_{\omega',\bar\omega':\omega'_{W^c}=\bar\omega'_{W^c}} (\mu_G[\omega'](A) - \mu_G[\bar\omega'](A)) \leqslant \sum_{i\in\text{supp}(A), j\in W} \bar{D}_{ij} \leqslant \sum_{i\in V^c, j\in W} \bar{D}_{ij}. \qquad (21)$$

Next we note that this inequality also holds by approximation of probabilities of general events by cylinders, (by a semiring-approximation argument) for all $A \in \mathcal{T}_V$. Since $A \in \mathcal{T}$ is in any $\mathcal{T}_V$ we may let $V \nearrow G$ and obtain that

$$\sup_{\omega', \bar{\omega}' : \omega'_{W^c} = \bar{\omega}'_{W^c}} (\mu_G[\omega'](A) - \mu_G[\bar{\omega}'](A)) \leqslant 0. \tag{22}$$

Since W was arbitrary this is the tail-measurability.

Further we note that $\mu_G[\omega'](A) \in \{0, 1\}$ for each fixed ω' and $A \in \mathcal{T}$ since the original measure constrained to coarse-grained configurations is in the Dobrushin uniqueness regime, hence tail-trivial. So $\mu_G[\omega'](A) = 1_{A'}(\omega')$ for some $A' \in \mathcal{T}'$ and this implies $\mu(A) = \int \mu'(d\omega')\mu_G[\omega'](A) = \mu'(A') \in \{0, 1\}$ by tail-triviality of μ'. □

3 Continuous rotations for discrete-spin models

After the preparations of the last section we turn now to the discussion of the rotation dynamics. Let us specialize to a translation-invariant S^1-model and look at the Markov process given in (3) with rates given in (4).

Intuitively the choice of the rates can be understood as follows: Consider the single-site discrete observable $f(\sigma') = 1_a(\sigma'_i)$ with $a \in \{1, \ldots, q\}$ fixed and let μ_φ be an extremal translation-invariant Gibbs measure of the $d \geqslant 3$ XY-Model labelled by the angle φ. Then we have

$$\begin{aligned}
\frac{d}{dt}_{|t=0}(T\mu_{\varphi+t})(1_a(\sigma'_i)) &= \frac{d}{dt}_{|t=0}\mu_\varphi(\sigma_i \in (a|^l - t, a|^r - t)) \\
&= \frac{d}{dt}_{|t=0}\int (T\mu_\varphi)(d\omega')\mu_G[\omega'](\sigma_i \in (a|^l - t, a|^r - t)) \\
&= \frac{d}{dt}_{|t=0}\int (T\mu_\varphi)(d\omega')\int_{a|^l - t}^{a|^r - t} \frac{d\mu_G[\omega']|_i}{d\lambda}(s)ds \\
&= \frac{d}{dt}_{|t=0}\int (T\mu_\varphi)(d\omega')\int_{a|^l - t}^{a|^r - t} \frac{\mu_{G\setminus i}[\omega'_{G\setminus i}](e^{-H_i(s, \cdot_{i^c})})}{\mu_{G\setminus i}[\omega'_{G\setminus i}](\lambda^i(e^{-H_i}1_a))}ds \\
&= \int (T\mu_\varphi)(d\omega')\Big(\frac{\mu_{G\setminus i}[\omega'_{G\setminus i}](e^{-H_i(a|^l, \cdot_{i^c})})}{\mu_{G\setminus i}[\omega'_{G\setminus i}](\lambda^i(e^{-H_i}1_a))} - \frac{\mu_{G\setminus i}[\omega'_{G\setminus i}](e^{-H_i(a|^r, \cdot_{i^c})})}{\mu_{G\setminus i}[\omega'_{G\setminus i}](\lambda^i(e^{-H_i}1_a))}\Big) \\
&= \int (T\mu_\varphi)(d\omega')c_L(\omega', (\omega')^i)(1_a((\omega')^i) - 1_a(\omega')) = (T\mu_\varphi)(Lf)
\end{aligned} \tag{23}$$

where in the first line we wrote $a|^l := 2\pi(a-1)/q$ (resp. $a|^r := 2\pi a/q$) to indicate the left (resp. right) endpoint of a. In the second line we used Theorem 1.2. In the third line we rewrote the constrained Gibbs measure as a marginal density (at site i) w.r.t the Lebesgue measure λ on the sphere, which as indicated in the fourth line can again be re-expressed by seperating the part of the potential interacting with the site i.

3.1 Well-definedness of the rotation generator

In this subsection we prove Theorem 1.3 part 1. We use methodology of Liggett [24] via the Hille-Yosida Theorem to prove well-definedness. Let us start with an overview on function spaces we need in the investigation of the dynamics.

Definition 3.1 *Let us fix the following notations. We write*

1. $\mathcal{L}' := \{f : \Omega' \to \mathbb{R} : f \text{ is local}\}$ *for the local functions.*
2. $C(\Omega') = \overline{\mathcal{L}'}_{\|\cdot\|}$ *equivalently for the space of continuous functions on the compact configuration-space* Ω' *which, since* q *is finite, coincides with the space of bounded quasilocal functions which is just the* $\|\cdot\|$*-completion of the local functions. Here* $\|\cdot\|$ *denotes the uniform norm.*
3. $D(\Omega') := \{f \in C(\Omega') : |||f||| := \sum_{i\in G} \delta_i(f) < \infty\}$ *for the core functions.*
4. $\overline{\mathcal{L}'}_{|||\cdot|||}$ *for the triple-norm completion of the local functions.*
5. $D_{p(\varrho)}(\Omega') := \{g \in C(\Omega') : |||g|||_{p(\varrho)} := \sum_{i\in G} p(\varrho(i,0))\delta_i(g) < \infty\}$ *for the space of weighted triple-normed functions, where* ϱ *is an increasing, translation-invariant semi-metric on the site space and* $p : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *any weight-function.*

Let us clarify the relations between those spaces and specialize to p being either an exponential function with some factor $\varepsilon > 0$ or a monomial function with power $m \in \mathbb{N}$. Let the semi-metric just be the euclidean metric $|\cdot|$ on an ordering of G. We have

$$\mathcal{L}' \subset D_{e^{\varepsilon|\cdot|}}(\Omega') \subset D_{|\cdot|^m}(\Omega') \subset D_{|\cdot|^1}(\Omega') \subset \overline{\mathcal{L}'}_{|||\cdot|||} \subset D(\Omega') \subset C(\Omega'). \tag{24}$$

Notice, all those spaces are dense in $C(\Omega')$ with respect to the $\|\cdot\|$-norm. All inclusions should be clear except $D_{|\cdot|^1}(\Omega') \subset \overline{\mathcal{L}'}_{|||\cdot|||}$.

Proposition 3.2 $D_{|\cdot|^1}(\Omega') \subset \overline{\mathcal{L}'}_{|||\cdot|||}$.

Proof: Let $f \in D_{|\cdot|^1}(\Omega')$ for an ordering $o : G \to \mathbb{N}$. Define $\Lambda_i := \{j \in G : o(j) \leqslant o(i)\}$ an exhausting sequence of finite volumes, then $\sum_{i\in G} |\Lambda_i|\delta_i(f) = \sum_{i\geqslant 0} i\delta_{o^{-1}(i)}(f) < \infty$ and $\sum_{i\geqslant n} i\delta_{o^{-1}(i)}(f) \to 0$ for $n \to \infty$. Let $\eta \in \Omega'$ be fixed and define a sequence of local functions $f_n(\omega) := f(\omega_{\Lambda_n}\eta_{\Lambda_n^c})$, then we have

$$\begin{aligned} |||f - f_n||| &= \sum_{i\in\Lambda_n} \delta_i(f - f_n) + \sum_{i\in\Lambda_n^c} \delta_i(f) \leqslant 2n\|f - f_n\| + \sum_{i\in\Lambda_n^c} \delta_i(f) \\ &\leqslant 2n \sum_{i\in\Lambda_n^c} \delta_i(f) + \sum_{i\in\Lambda_n^c} \delta_i(f) \leqslant 2 \sum_{i>n} i\delta_{o^{-1}(i)}(f) \to 0 \text{ for } n \to \infty. \end{aligned} \tag{25}$$

Hence $f \in \overline{\mathcal{L}'}_{|||\cdot|||}$ and $D_{|\cdot|^1}(\Omega') \subset \overline{\mathcal{L}'}_{|||\cdot|||}$. □

In the sequel, we will drop the notation $o^{-1}(i)$ and just write $\sum_{i\geqslant 0} i\delta_i(f)$.

Let us check the criteria for well-definedness proposed in [24]. Note, the jump rates are uniformly bounded since we assumed the potential to be absolutely summable and translation-invariant and the coarse-graining to be finite. Further the rates have to be of bounded variation, i.e.

Lemma 3.3 $\sup_{i\in G}\sum_{j\neq i}\delta_j(c_L(\cdot,\cdot^i))<\infty$ *if* $\sup_{i\in G}\sum_{A\ni i}|||\Phi_A|||<\infty$.

Proof: This follows from the Dobrushin comparison theorem (see [16] Theorem 8.20.), indeed

$$\begin{aligned}\delta_j(c_L(\cdot,\cdot^i)) \leqslant{}& Ce^{\|H_i\|}\sup_{\substack{\omega'=\tilde\omega'\\ \text{off } j}}|\mu_{G\backslash i}[\omega'_{G\backslash i}](e^{-H_i(\omega_i'|^r,\cdot_{i^c})})-\mu_{G\backslash i}[\tilde\omega'_{G\backslash i}](e^{-H_i(\tilde\omega_i'|^r,\cdot_{i^c})})|\\ &+Ce^{3\|H_i\|}\sup_{\substack{\omega'=\tilde\omega'\\ \text{off } j}}|\mu_{G\backslash i}[\omega'_{G\backslash i}](\lambda^i(e^{-H_i}1_{\omega_i'}))-\mu_{G\backslash i}[\tilde\omega'_{G\backslash i}](\lambda^i(e^{-H_i}1_{\tilde\omega_i'}))|\end{aligned}$$

therefore it suffices to look at the Gibbs measures $\mu_{G\backslash i}[\omega'_{G\backslash i}](\cdot)$ and $\mu_{G\backslash i}[\tilde\omega'_{G\backslash i}](\cdot)$ on $(S^1)^{G\backslash i}$ applied to the quasilocal functions $\psi_1^{\omega_i'}(\cdot):=e^{-H_i(\omega_i'|^r,\cdot_{i^c})}$ and $\psi_2^{\omega_i'}(\cdot):=\lambda^i(e^{-H_i(\cdot,\cdot_{i^c})}1_{\omega_i'})$. For any fixed first-layer boundary condition $\omega\in\Omega$ the measure $\mu_{G\backslash i}[\omega'_{G\backslash i}](\cdot)$ is uniquely specified by the specification

$$\gamma^{\omega'_{G\backslash i}}:=\Big((\gamma^{\omega'_{G\backslash i}}|_{G\backslash i})_{\Lambda\backslash i}(\cdot|\omega_{\Lambda^c\backslash i})\Big)_{\Lambda\subset G\backslash i} \tag{26}$$

Λ being finite subsets of $G\backslash i$. We have for $\omega'_{G\backslash j}=\tilde\omega'_{G\backslash j}$

$$\|(\gamma^{\omega'_{G\backslash i}}|_{G\backslash i})_{l\backslash i}(\cdot|\omega_{l^c\backslash i})-(\gamma^{\tilde\omega'_{G\backslash i}}|_{G\backslash i})_{l\backslash i}(\cdot|\omega_{l^c\backslash i})\|_l\leqslant 1_{l=j}. \tag{27}$$

Hence for $\omega'_{G\backslash j}=\tilde\omega'_{G\backslash j}$ and $\psi^{\omega_i'}\in\{\psi_1^{\omega_i'},\psi_2^{\omega_i'}\}$ the comparison theorem gives us

$$|\mu_{G\backslash i}[\omega'_{G\backslash i}](\psi^{\omega_i'})-\mu_{G\backslash i}[\tilde\omega'_{G\backslash i}](\psi^{\omega_i'})|\leqslant\sum_{k\neq i}\delta_k(\psi^{\omega_i'})D_{kj}(\gamma^{\omega'_{G\backslash i}})\leqslant\sum_{k\neq i}\delta_k(\psi^{\omega_i'})\bar D_{kj}$$

where we used the fact that the specifications $\gamma^{\omega'_{G\backslash i}}$ are in the Dobrushin region uniformly in the constraint ω'. Since $\bar c:=\sup_i\sum_j\bar C_{ij}<1$ we have $\sum_{j\in G}\bar D_{kj}<\infty$ for all $k\in G$ and can therefore conclude

$$\sup_{i\in G}\sup_{\omega_i'\in S'}\sum_{j\neq i}\sum_{k\neq i}\delta_k(\psi^{\omega_i'})\bar D_{kj}\leqslant C\sum_{k\in G}\sup_{i\in G}\sup_{\omega_i'\in S'}\delta_k(\psi^{\omega_i'})\leqslant C\sup_{i\in G}\sum_{k\in G}\delta_k(\psi^i)$$

with $\psi^i\in\{\psi_1^i,\psi_2^i\}$ and $\psi_1^i(\cdot):=e^{-H_i(\cdot,\cdot_{i^c})}$ and $\psi_2^i(\cdot):=\lambda^i(e^{-H_i(\cdot,\cdot_{i^c})})$. In case ψ^i is a local function, uniformly bounded in i (for instance in the XY-Model), the sum is finite and thus less than infinity. In the general case were the ψ^i are coming from an uniformly bounded Hamiltonian which is only quasilocal the summability is not guaranteed. But if we stipulate $\sup_{i\in G}\sum_{A\ni i}|||\Phi_A|||<\infty$ we have for ψ_1^i and ψ_2^i

$$\sum_{k\in G}\delta_k(\psi_2^i)\leqslant C\sum_{k\in G}\delta_k(\psi_1^i)\leqslant Ce^{\|H_0\|}\sum_{k\in G}\sum_{A\ni ik}\delta_k(\Phi_A)=Ce^{\|H_0\|}\sum_{A\ni i}|||\Phi_A|||<\infty$$

where we used $|e^x - e^y| \leqslant |x-y| e^{\max(|x|,|y|)}$. $\square$

Note that in particular $c_L(\cdot, \cdot^i) \in D(\Omega') \subset \mathcal{L}'_{\|\cdot\|}$ for all $i \in G$ and thus the rates are quasilocal.

Later we will need even stronger regularity of the rates in the following sense.

Lemma 3.4 *Suppose* $\sup_{i\in G} \sum_{A\ni i} \sum_{k\in G} e^{\varrho(i,k)} \delta_k(\Phi_A) < \infty$ *and*

$$\bar{c}_\varrho := \sup_{i\in G} \sum_{j\neq i} e^{\varrho(i,j)} \bar{C}_{ij} < 1 \tag{28}$$

then $\sup_{i\in G} \sum_{j\neq i} e^{\varrho(i,j)} \delta_j(c_L(\cdot, \cdot^i)) < \infty$.

Notice, the first condition given in the above lemma is independent of the hidden temperature parameter β and with $\varrho(i,k) := \varepsilon|i-k|$ corresponds to condition (7) in Theorem 1.1. The condition (28) is the requirement on the finess of discretization $q \geqslant q_0(\Phi)$ formulated in Theorem 1.1.

Proof: As a consequence of the exponential decay condition on the Dobrushin matrix (28) (for a translation-invariant semi-metric ϱ on G) we have $\sup_{i\in G} \sum_{j\in G} e^{\varrho(i,j)} \bar{D}_{ij} \leqslant \frac{1}{1-\bar{c}_\varrho}$ and by the triangle inequality

$$\begin{aligned} \sup_{i\in G} \sum_{j\neq i} e^{\varrho(i,j)} \delta_j(c_L(\cdot, \cdot^i)) &\leqslant \sup_{i\in G} \sum_{j\in G\setminus i} \sum_{k\in G\setminus i} e^{\varrho(i,j)} \delta_k(\psi^i) \bar{D}_{kj} \\ &\leqslant \frac{1}{1-\bar{c}_\varrho} \sup_{i\in G} \sum_{k\in G\setminus i} e^{\varrho(i,k)} \delta_k(\psi^i). \end{aligned} \tag{29}$$

But for ψ_1^i and ψ_2^i using the same arguments as in the proof of Lemma 3.3

$$\sum_{k\in G} e^{\varrho(i,k)} \delta_k(\psi_2^i) \leqslant C \sum_{k\in G} e^{\varrho(i,k)} \delta_k(\psi_1^i) \leqslant C e^{2K} \sum_{A\ni i} \sum_{k\in G} e^{\varrho(i,k)} \delta_k(\Phi_A) < \infty. \tag{30}$$

$\square$

Instead of imposing an exponential decay property of the Dobrushin matrix one could just consider polynomial weights $p(\varrho(i,j))$ which would admit Hamiltonians with polynomial dependence. In fact for our purposes that would be sufficient.

After these preparations we are in the position to use Theorem 3.9. of [24] and assert: 1. The closure $\overline{L}$ of L is a Markov generator of a Markov semiproup $(S_t^L)_{t\geqslant 0}$ connected to the generator via the Hille-Yosida Theorem. $D(\Omega')$ is a core for $\overline{L}$. 2. For observables $f \in D(\Omega')$ we can control the oscillation of $S_t f$ at any site $i \in G$ via

$$\delta_i(S_t f) \leqslant [e^{t\Gamma} \delta.(f)](i)$$

where $\Gamma : l_1 \to l_1$, $[\Gamma \delta.(f)](i) := \sum_{j\neq i} \delta_i(c_L(\cdot, \cdot^j)) \delta_j(f)$ is a bounded operator with $\|\Gamma\| =: M$. In particular for $f \in D(\Omega')$ we have $|||S_t f||| \leqslant e^{tM} |||f|||$ and thus $S_t f \in D(\Omega')$.

3.2 Rotation property of the generator

The goal of this subsection is to verify Theorem 1.3 part 2. We use the following strategy:

1. We verify the rotation property for infinitesimal times by comparing the generator to the derivative on the level of the probability density. We do this directly on local observables.

2. In order to get from infinitesimal to finite time, we consider the associated semigroup $(S_t^L)_{t\geqslant 0}$ and use Taylor expansion. To match the first-order terms it is necessary to verify the infinitesimal rotation for local functions propagated by S_t^L. Those functions are no longer strictly local but lie in a larger space, namely $\mathcal{L}'_{|||\cdot|||}$. Since later we need (and will verify) the stronger result $S_t^L f \in D_{p(\varrho)}(\Omega')$ for local f and weight-function $p(x) = x^2$, at this point we just assume $S_t^L f \in \mathcal{L}'_{|||\cdot|||}$.

3. The two second-order error terms need to be estimated. As for the first one we can use the contraction property of the semigroup. For the other one we compute the second derivative of the measure again on the level of the probability density and local observables. It turns out the desired upper bound exists as long as the observable lies in a space of weighted triple-normed functions.

4. By assuming exponential decay of the Dobrushin matrix (see (28)) the rates of the generator are elements of this space even for arbitrary polynomial weights. One can think of these spaces as containing functions with a certain degree of locality. The amount of non-locality the semigroup injects into a local function is controlled by the degree of locality of the rates. This can simplest be captured by looking at the operator Γ mentioned above. We can show under these assumptions that local observables propagated by the semigroup stay in the space of weighted triple-normed functions.

Let us start with an infinitesimal rotation and show $\mu'_{(t+s)_{\text{mod } 2\pi}}(f) = \mu'_t(S_s^L f)$ for all $t \in [0, 2\pi)$, $s > 0$, $\mu'_\varphi = T\mu_\varphi \in \text{ex}\,\mathcal{G}(\gamma')$ and local observables f on Ω'. Since the coarse-graining is finite it suffices to use $f = 1_{a_\Lambda}$ for finite Λ and $a_\Lambda \in \{1, \dots, q\}^\Lambda$. Write $\rho_\Lambda = d\gamma_\Lambda^\Phi / d\lambda^\Lambda$ for the Lebesgue density of the local specification in Λ. Then we can proceed similary to the intuitive calculations done in (23) and write

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} \mu'_{t+\varepsilon}(1_{a_\Lambda}) = \int \mu_t(d\omega) \frac{d}{d\varepsilon}_{|\varepsilon=0} \Big(\prod_{i\in\Lambda} \int_{a_i|^l-\varepsilon}^{a_i|^r-\varepsilon}\Big) d\varphi_\Lambda \rho_\Lambda(\varphi_\Lambda, \omega)$$
$$= \sum_{j\in\Lambda} \int \mu_t(d\omega) \Big(\prod_{i\in\Lambda\setminus j} \int_{a_i|^l}^{a_i|^r}\Big) d\varphi_{\Lambda\setminus j} \Big(\rho_\Lambda(a_j|^l, \varphi_{\Lambda\setminus j}, \omega) - \rho_\Lambda(a_j|^r, \varphi_{\Lambda\setminus j}, \omega)\Big)$$

since μ_t admits γ_Λ for all $t \in [0, 2\pi)$. On the other hand

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} \mu'_t S^L_\varepsilon(1_{a_\Lambda}) = \mu'_t(L 1_{a_\Lambda})$$
$$= \sum_{j\in\Lambda}\Big(\sum_{\omega':\omega'^j_\Lambda = a_\Lambda} c_L(\omega', (\omega')^j)\mu'_t(\omega') - \sum_{\omega':\omega'_\Lambda = a_\Lambda} c_L(\omega', (\omega')^j)\mu'_t(\omega')\Big).$$

Looking at the individual summands we find

$$\sum_{\omega':\omega'_\Lambda = a_\Lambda} c_L(\omega', (\omega')^j)\mu'_t(\omega') = \int \mu'_t(d\omega') 1_{a_\Lambda}(\omega') \frac{\mu_{G\setminus j}[\omega'_{G\setminus j}](e^{-H_j(\omega'_j|^r, \cdot_{j^c})})}{\mu_{G\setminus j}[\omega'_{G\setminus j}](\lambda^j(e^{-H_j} 1_{\omega'_j}))}$$
$$= \int \mu'_t(d\omega') 1_{a_\Lambda}(\omega') \frac{\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}](\lambda^{\Lambda\setminus j}(e^{-H_\Lambda(a_j|^r, \cdot_{\Lambda\setminus j}, \cdot_{\Lambda^c})} 1_{a_{\Lambda\setminus j}}))}{\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda} 1_{a_\Lambda}))}$$
$$= \int \mu'_t(d\omega') 1_{a_\Lambda}(\omega') \mu_G[\omega'_G](\frac{\lambda^{\Lambda\setminus j}(e^{-H_\Lambda(a_j|^r, \cdot_{\Lambda\setminus j}, \cdot_{\Lambda^c})} 1_{a_{\Lambda\setminus j}})}{\lambda^\Lambda(e^{-H_\Lambda} 1_{a_\Lambda})})$$
$$= \int \mu'_t(d\omega') 1_{a_\Lambda}(\omega') \mu_G[\omega'_G](\frac{1}{\gamma_\Lambda(1_{a_\Lambda}|\cdot)}(\prod_{i\in\Lambda\setminus j} \int_{a_i|^l}^{a_i|^r}) d\varphi_{\Lambda\setminus j}\big(\rho_\Lambda(a_j|^r, \varphi_{\Lambda\setminus j}, \cdot)\big))$$
$$= \int \mu'_t(d\omega') \mu_G[\omega'_G](\frac{1_{a_\Lambda}}{\gamma_\Lambda(1_{a_\Lambda}|\cdot)}(\prod_{i\in\Lambda\setminus j} \int_{a_i|^l}^{a_i|^r}) d\varphi_{\Lambda\setminus j}\big(\rho_\Lambda(a_j|^r, \varphi_{\Lambda\setminus j}, \cdot)\big))$$
$$= \int \mu_t(d\omega)(\prod_{i\in\Lambda\setminus j} \int_{a_i|^l}^{a_i|^r}) d\varphi_{\Lambda\setminus j}\big(\rho_\Lambda(a_j|^r, \varphi_{\Lambda\setminus j}, \omega)\big)$$

where we used the DLR equation in the second last line and the fact that

$$\frac{\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}](\varphi(\cdot_{\Lambda^c})\lambda^\Lambda(e^{-H_\Lambda} 1_{\omega'_\Lambda}))}{\mu_{G\setminus\Lambda}[\omega'_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\omega'_\Lambda}))} = \mu_G[\omega'_G](\varphi(\cdot_{\Lambda^c})). \tag{31}$$

We proceed similarly for the other summand. Thus we have $\frac{d}{d\varepsilon}_{|\varepsilon=0}\mu'_{t+\varepsilon}(f) = \mu'_t(Lf)$ for all local observables. Later we want to apply $S^L_t f$ and will show $S^L_t f \in \mathcal{L}'_{|||\cdot|||}$ if f is local. So let us prove the following proposition.

Proposition 3.5 *If $f \in \mathcal{L}'_{|||\cdot|||}$ then $\frac{d}{d\varepsilon}_{|\varepsilon=0}\mu'_{t+\varepsilon}(f) = \mu'_t(Lf)$.*

Proof: Assume $(f_n)_{n\in\mathbb{N}}$ to be a sequence of local functions such that $|||f - f_n||| \to 0$ for $n \to \infty$. Then we have according to Proposition 3.2 of [24]

$$|\mu'_t(Lf) - \mu'_t(Lf_n)| \leqslant \|Lf - Lf_n\| \leqslant C|||f - f_n||| \xrightarrow{n\to\infty} 0. \tag{32}$$

On the other hand with $g := f - f_n$ and $o : G \to \mathbb{N}$ an ordering of G we have

$$\mu'_{t+\varepsilon}(g) - \mu'_t(g) = \int \mu_t(d\tilde\omega)(g(T(\tilde\omega - \varepsilon 1_G)) - g(T(\tilde\omega)))$$
$$= \int \mu_t(d\tilde\omega) \sum_{j\in G}(g(T(\tilde\omega - \varepsilon 1_{\{0,..,o(j)\}})) - g(T(\tilde\omega - \varepsilon 1_{\{0,..,(o(j)-1)\}})))$$
$$\leqslant \sum_{j\in G} \delta_j(g)\mu_t(\{\tilde\omega : T(\tilde\omega_j - \varepsilon) = T(\tilde\omega_j) - 1\})$$

where we use a telescopic sum in the second line. Further we have with $A_j := \{\tilde{\omega} : T(\tilde{\omega}_j - \varepsilon) = T(\tilde{\omega}_j) - 1\} = \{\tilde{\omega} : \tilde{\omega}_j \in [a|^l, a|^l + \varepsilon] \text{ for some } a \in \{1, \ldots, q\}\}$,

$$\mu_t(A_j) \leqslant \sup_{\omega \in \Omega} \gamma_j(A_j|\omega) \leqslant \frac{\varepsilon q e^{\|H_j\|}}{2\pi e^{-\|H_j\|}} \leqslant \bar{K}\varepsilon \tag{33}$$

uniformly in t and j, hence $\frac{1}{\varepsilon}|\mu'_{t+\varepsilon}(f - f_n) - \mu'_t(f - f_n)| \leqslant \bar{K}\|\!|f - f_n\|\!| \xrightarrow{n\to\infty} 0$ and we can conclude

$$|\frac{d}{d\varepsilon}_{|\varepsilon=0} \mu'_{t+\varepsilon}(f) - \mu'_t(Lf)| \leqslant |\frac{d}{d\varepsilon}_{|\varepsilon=0} \mu'_{t+\varepsilon}(f - f_n)| + |\mu'_t(L(f - f_n))| \xrightarrow{n\to\infty} 0.$$

□

Assume for the moment $S^L_t f \in \mathcal{L}'_{\|\!|\cdot\|\!|}$ for local f. In order to verify the rotation property for finite times we use the following iteration procedure. Let f be local, $k \in \mathbb{N}$, $t \in [0, 2\pi)$, $s > 0$ and $\varepsilon := s/k$. On the one hand

$$\begin{aligned} \mu'_t(S^L_s f) &= \mu'_t(S^L_\varepsilon S^L_{s-\varepsilon} f) = \mu'_t((1 + \varepsilon L + S^L_\varepsilon - (1 + \varepsilon L))S^L_{s-\varepsilon} f) \\ &= \mu'_t(g) + \varepsilon\mu'_t(Lg) + \mu'_t((S^L_\varepsilon - (1 + \varepsilon L))g) \end{aligned} \tag{34}$$

where we set $g := S^L_{s-\varepsilon} f$. On the other hand we can use Taylor expansion in Lagrange form and write

$$\begin{aligned} \mu'_{t+\varepsilon}(g) &= \mu'_t(g) + \varepsilon \frac{d}{d\varepsilon}_{|\varepsilon=0} \mu'_{t+\varepsilon}(g) + \frac{\varepsilon^2}{2} \frac{d^2}{d\hat{\varepsilon}^2}_{|\hat{\varepsilon}\in[0,\varepsilon]} \mu'_{t+\hat{\varepsilon}}(g) \\ &= \mu'_t(g) + \varepsilon\mu'_t(Lg) + \frac{\varepsilon^2}{2} \frac{d^2}{d\hat{\varepsilon}^2}_{|\hat{\varepsilon}\in[0,\varepsilon]} \mu'_{t+\hat{\varepsilon}}(g). \end{aligned} \tag{35}$$

By iteration

$$\begin{aligned} \mu'_t(S^L_s f) - \mu'_{t+s}(f) = &\sum_{l=0}^{k-1} \mu'_{t+l\varepsilon}((S^L_\varepsilon - (1 + \varepsilon L))S^L_{s-(l+1)\varepsilon} f) \\ &- \frac{\varepsilon^2}{2} \sum_{l=0}^{k-1} \frac{d^2}{d\hat{\varepsilon}^2}_{|\hat{\varepsilon}\in[0,\varepsilon]} \mu'_{t+l\varepsilon+\hat{\varepsilon}}(S^L_{s-(l+1)\varepsilon} f) \end{aligned} \tag{36}$$

where the error terms should go to zero as k tends to infinity. Let us look at the first error term on the r.h.s of (36) and use the uniform continuity of the Markov semigroup, we have

$$\begin{aligned} \sum_{l=0}^{k-1} \mu'_{t+l\varepsilon}((S^L_\varepsilon - (1 + \varepsilon L))S^L_{s-(l+1)\varepsilon} f) &\leqslant \varepsilon \sum_{l=0}^{k-1} \|\frac{S^L_\varepsilon S^L_{s-(l+1)\varepsilon} f - S^L_{s-(l+1)\varepsilon} f}{\varepsilon} - LS^L_{s-(l+1)\varepsilon} f\| \\ &\leqslant \varepsilon \sum_{l=0}^{k-1} \|\frac{S^L_\varepsilon f - f}{\varepsilon} - Lf\| = s\|\frac{S^L_\varepsilon f - f}{\varepsilon} - Lf\| \end{aligned}$$

where the r.h.s goes to zero as ε goes to zero since the semigroup is generated by L and f in the domain of L. In particular this is true for core observables of L.

Let us check the second error term on the r.h.s of (36). Let $t'(l) \in [t + l\varepsilon, t + (l+1)\varepsilon]$ then it suffices to find a constant $C(s, f)$ such that

$$\frac{d^2}{d\hat{\varepsilon}^2}_{|\hat{\varepsilon}=0} \mu'_{t'(l)+\hat{\varepsilon}}(S^L_{s-(l+1)\varepsilon} f) \leqslant C(s, f) \tag{37}$$

for all l, since then we have $\varepsilon^2 \sum_{l=0}^{k-1} \frac{d^2}{d\hat{\varepsilon}^2}_{|\hat{\varepsilon}=0} \mu'_{t'(l)+\hat{\varepsilon}}(S^L_{s-(l+1)\varepsilon} f) \leqslant \frac{s^2}{k} C(s, f) \xrightarrow{k\to\infty} 0$.

Consider the second derivative when we apply the extremal Gibbs measure at first to a local indicator function 1_{a_Λ}. Then we have $\frac{d^2}{d\varepsilon^2}_{|\varepsilon=0} \mu'_{t+\varepsilon}(1_{a_\Lambda}) =$

$$\begin{aligned}
&\sum_{j\in\Lambda} \int \mu_t(d\omega) \frac{d}{d\varepsilon} \Big(\prod_{i\in\Lambda\backslash j} \int_{a_i|^l-\varepsilon}^{a_i|^r-\varepsilon} \Big) d\varphi_{\Lambda\backslash j} \Big(\rho_\Lambda(a_j|^l - \varepsilon, \varphi_{\Lambda\backslash j}, \omega) - \rho_\Lambda(a_j|^r - \varepsilon, \varphi_{\Lambda\backslash j}, \omega) \Big) \\
&= \sum_{j\in\Lambda} \int \mu_t(d\omega) \Bigg[\Big(\prod_{i\in\Lambda\backslash j} \int_{a_i|^l}^{a_i|^r} \Big) d\varphi_{\Lambda\backslash j} \Big(\rho_\Lambda(a_j|^r, \varphi_{\Lambda\backslash j}, \omega) \frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^r} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j}, \omega) \\
&\qquad - \rho_\Lambda(a_j|^l, \varphi_{\Lambda\backslash j}, \omega) \frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^l} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j}, \omega) \Big) \\
&\quad + \sum_{k\in\Lambda\backslash j} \Big(\prod_{i\in\Lambda\backslash\{j,k\}} \int_{a_i|^l}^{a_i|^r} \Big) d\varphi_{\Lambda\backslash j} \Big(\rho_\Lambda(a_j|^l, a_k|^l, \varphi_{\Lambda\backslash\{j,k\}}, \omega) - \rho_\Lambda(a_j|^l, a_k|^r, \varphi_{\Lambda\backslash\{j,k\}}, \omega) \\
&\qquad - \rho_\Lambda(a_j|^r, a_k|^l, \varphi_{\Lambda\backslash\{j,k\}}, \omega) + \rho_\Lambda(a_j|^r, a_k|^r, \varphi_{\Lambda\backslash\{j,k\}}, \omega) \Big) \Bigg] \\
&=: \sum_{j\in\Lambda} \int \mu_t(d\omega) \big[A(j, a_\Lambda, \omega) + \sum_{k\in\Lambda\backslash j} B(j, k, a_\Lambda, \omega) \big]
\end{aligned}$$

where, as we see from the formular, the Hamiltonian of the first-layer system needs to be differentiable as a function on S^1. Let us assume these partial derivatives are also uniformly bounded with $K' := \sup_{i\in G} \sup_{\omega\in\Omega} \|\frac{d}{d\varepsilon} H_i(\varepsilon, \omega_{i^c})\| < \infty$. Then we have $A(j, a_\Lambda, \omega) =$

$$\begin{aligned}
&\Big(\prod_{i\in\Lambda\backslash j} \int_{a_i|^l}^{a_i|^r} \Big) d\varphi_{\Lambda\backslash j} \Big(\frac{e^{-H_\Lambda(a_j|^r, \varphi_{\Lambda\backslash j}, \omega_{\Lambda^c})}}{\int d\varphi_\Lambda e^{-H_\Lambda(\varphi_\Lambda, \omega_{\Lambda^c})}} \Big(\frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^r} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j}, \omega) - \frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^l} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j} \omega) \Big) \\
&\qquad + \frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^l} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j}, \omega) \frac{e^{-H_\Lambda(a_j|^r, \varphi_{\Lambda\backslash j}, \omega_{\Lambda^c})} - e^{-H_\Lambda(a_j|^l, \varphi_{\Lambda\backslash j}, \omega_{\Lambda^c})}}{\int d\varphi_\Lambda e^{-H_\Lambda(\varphi_\Lambda, \omega_{\Lambda^c})}} \Big)
\end{aligned}$$

where $\frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^r} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j}, \omega) - \frac{d}{d\varepsilon_j}_{|\varepsilon_j = a_j|^l} H_\Lambda(\varepsilon_j, \varphi_{\Lambda\backslash j} \omega) \leqslant \delta_j(\frac{d}{d\varepsilon_j} H_j) \leqslant 2K'$ and $e^{-H_\Lambda(a_j|^r, \varphi_{\Lambda\backslash j}, \omega_{\Lambda^c})} - e^{-H_\Lambda(a_j|^l, \varphi_{\Lambda\backslash j}, \omega_{\Lambda^c})} \leqslant 2e^K e^{-\sum_{A\cap\Lambda\neq\varnothing, j\notin A} \Phi_A(\varphi_{\Lambda\backslash j}, \omega_{\Lambda^c})}$. Thus

$$A(j,a_\Lambda,\omega) \leqslant 2K'e^{2K}2\pi \frac{(\prod_{i\in\Lambda\setminus j}\int_{a_i|^l}^{a_i|^r})d\varphi_{\Lambda\setminus j} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ j\notin A}}\Phi_A(\varphi_{\Lambda\setminus j},\omega_{\Lambda^c})}}{\int d\varphi_{\Lambda\setminus j} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ j\notin A}}\Phi_A(\varphi_{\Lambda\setminus j},\omega_{\Lambda^c})}}$$
$$+ 2e^{2K}(K' + (|\Lambda|-1)K)2\pi \frac{(\prod_{i\in\Lambda\setminus j}\int_{a_i|^l}^{a_i|^r})d\varphi_{\Lambda\setminus j} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ j\notin A}}\Phi_A(\varphi_{\Lambda\setminus j},\omega_{\Lambda^c})}}{\int d\varphi_{\Lambda\setminus j} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ j\notin A}}\Phi_A(\varphi_{\Lambda\setminus j},\omega_{\Lambda^c})}}.$$

Let us check the second term. We can write

$$B(j,k,a_\Lambda,\omega) \leqslant 4e^{4K}4\pi^2 \frac{(\prod_{i\in\Lambda\setminus\{j,k\}}\int_{a_i|^l}^{a_i|^r})d\varphi_{\Lambda\setminus\{j,k\}} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ \{j,k\}\not\subset A}}\Phi_A(\varphi_{\Lambda\setminus\{j,k\}},\omega_{\Lambda^c})}}{\int d\varphi_{\Lambda\setminus\{j,k\}} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ \{j,k\}\not\subset A}}\Phi_A(\varphi_{\Lambda\setminus\{j,k\}},\omega_{\Lambda^c})}}.$$

For convenience set $\check{K} := \max\{K,K'\}$ and $\bar{K} := \max\{4\pi\check{K}e^{2\check{K}}, 8\pi^2 e^{4\check{K}}\}$. Also we want to adopt a notation we introduced earlier namely

$$\gamma_{\Lambda\setminus j}|_{j^c}(1_{a_{\Lambda\setminus j}}|\omega_{\Lambda^c}) = \frac{(\prod_{i\in\Lambda\setminus j}\int_{a_i|^l}^{a_i|^r})d\varphi_{\Lambda\setminus j} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ j\notin A}}\Phi_A(\varphi_{\Lambda\setminus j},\omega_{\Lambda^c})}}{\int d\varphi_{\Lambda\setminus j} e^{-\sum_{\substack{A\cap\Lambda\neq\varnothing\\ j\notin A}}\Phi_A(\varphi_{\Lambda\setminus j},\omega_{\Lambda^c})}}.$$

Before we combine these estimates, let us apply the measure to a general local functions h on the coarse-grained space with support Λ. h can be written as $h(\omega') = \sum_{a_\Lambda\in\{1,\dots,q\}^\Lambda}\kappa_{a_\Lambda}1_{a_\Lambda}(\omega')$ with $\|h\| = \sup_{a_\Lambda}|\kappa_{a_\Lambda}|$. Hence

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}_{|\varepsilon=0}\mu'_{t+\varepsilon}(h) &\leqslant \|h\|\sum_{j\in\Lambda}\int\mu_t(d\omega)\Big[q\bar{K}(|\Lambda|+1)\sum_{a_{\Lambda\setminus j}\in\{1,\dots,q\}^{\Lambda\setminus j}}\gamma_{\Lambda\setminus j}|_{j^c}(1_{a_{\Lambda\setminus j}}|\omega_{\Lambda^c})\\ &\quad+\sum_{k\in\Lambda\setminus j}q^2\bar{K}\sum_{a_{\Lambda\setminus\{j,k\}}\in\{1,\dots,q\}^{\Lambda\setminus\{j,k\}}}\gamma_{\Lambda\setminus\{j,k\}}|_{\{j,k\}^c}(1_{a_{\Lambda\setminus\{j,k\}}}|\omega_{\Lambda^c})\Big]\\ &\leqslant \|h\||\Lambda|(q\bar{K}(|\Lambda|+1) + q^2\bar{K}(|\Lambda|-1)) \leqslant \hat{K}|\Lambda|^2\|h\|.\end{aligned}$$

For a general quasilocal function f one can write again a telescopic sum using an ordering of G and a generic configuration η'

$$\begin{aligned}f(\omega') &= f(\omega'_1,\eta'_{\{1\}^c}) + (f(\omega'_1,\omega'_2,\eta'_{\{1,2\}^c}) - f(\omega'_1,\eta'_{\{1\}^c}))\\ &\quad+\sum_{n\geqslant 3}(f(\omega'_{\{1,..,n\}},\eta'_{\{1,\dots,n\}^c}) - f(\omega'_{\{1,..,n-1\}},\eta'_{\{1,\dots,n-1\}^c})).\end{aligned} \tag{38}$$

Let us define $g_n(\omega') := \big(f(\omega'_{\{1,..,n\}},\eta'_{\{1,\dots,n\}^c}) - f(\omega'_{\{1,..,n-1\}},\eta'_{\{1,\dots,n-1\}^c})\big) \in \mathcal{F}_{\{1,..,n\}}$. In particular $\|g_n\| \leqslant \delta_n(f)$. Hence we can write

$$\frac{d^2}{d\varepsilon^2}_{|\varepsilon=0}\mu'_{t+\varepsilon}(f) \leqslant \|f\|\hat{K} + \sum_{n\geqslant 2}\|g_n\|\hat{K}n^2 \leqslant 2\hat{K}\sum_{n\geqslant 1}n^2\delta_n(f). \tag{39}$$

Thus in order to have (37) it suffices to show $S_t^L f \in D_{|\cdot|^2}(\Omega')$ for local f. To do that, let us use the exponential decay property of the Dobrushin matrix introduced in (28) and the exponentially decaying Hamiltonian, i.e. by Lemma 3.4 assume the model to satisfy $\sup_{i\in G}\sum_{j\neq i} e^{\varrho(i,j)}\delta_j(c_L(\cdot,\cdot^i)) =: \check{M}_\varrho < \infty$ for some translation-invariant increasing semi-metric ϱ in G. With this we can prove the following proposition.

Proposition 3.6 *Let f be a local observable on Ω' and $(S_t^L)_{t\geqslant 0}$ associated to the rotation generator L. For all polynomials p on $\mathbb{R}_0^+$ we have $S_t^L f \in D_{p(\varrho)}(\Omega')$.*

Proof: Let us consider only the monomials x^n. It suffices to look at $n = 2^m$ for some $m \in \mathbb{N}$. We know from Theorem 3.9. in [24]

$$\sum_{i\geqslant 0} \varrho(i,0)^m \delta_i(S_t^L f) \leqslant \sum_{i\geqslant 0} \varrho(i,0)^m [e^{t\Gamma}\delta.(f)](i)$$

where $[\Gamma\delta.(f)](i) := \sum_{j\neq i}\delta_i(c_L(\cdot,\cdot^j))\delta_j(f)$. There exists a constant $K_{m,\varrho}$ such that for fixed $j, m \in \mathbb{N}$ we have $\varrho(i,j)^m \leqslant K_{m,\varrho} e^{\varrho(i,j)}$. Of course local $f \in D_{\varrho^m}(\Omega')$ for all $m \in \mathbb{N}$ and also for exponential weight. Under the above condition on the jump rates, the operator Γ is bounded as well in the exponential weighted triple-norm with norm $\check{M}$, indeed

$$\begin{aligned} |||\Gamma|||_{e^\varrho} &= \sup_{|||v|||_{e^\varrho}\leqslant 1} \frac{|||\Gamma v|||_{e^\varrho}}{|||v|||_{e^\varrho}} = \sup_{|||v|||_{e^\varrho}\leqslant 1} \frac{\sum_{i\geqslant 0}\sum_{j\neq i} e^{\varrho(i,0)}\delta_i(c_L(\cdot,\cdot^j))v_j}{\sum_{j\geqslant 0} e^{\varrho(j,0)}v_j} \\ &\leqslant \sup_{|||v|||_{e^\varrho}\leqslant 1} \frac{\check{M}\sum_{j\geqslant 0} e^{\varrho(j,0)}v_j}{\sum_{j\geqslant 0} e^{\varrho(j,0)}v_j} = \check{M}. \end{aligned} \tag{40}$$

Then we can write

$$\begin{aligned} \sum_{i\geqslant 0} \varrho(i,0)^m [e^{t\Gamma}\delta.(f)](i) &\leqslant K_{m,\varrho}\sum_{i\geqslant 0} e^{\varrho(i,0)}[e^{t\Gamma}\delta.(f)](i) = K_{m,\varrho}|||e^{t\Gamma}\delta.(f)|||_{e^\varrho} \\ &\leqslant K_{m,\varrho}|||e^{t\Gamma}|||_{e^\varrho}|||\delta.(f)|||_{e^\varrho} \leqslant K_{m,\varrho} e^{t|||\Gamma|||_{e^\varrho}}|||\delta.(f)|||_{e^\varrho}. \end{aligned}$$

□

In particular for local f, we have $S_{s-\varepsilon}^L f \in D_{p(|\cdot|)}(\Omega') \subset \mathcal{L}'_{|||\cdot|||}$ for all polynomial and even exponential weights p. In other words, we can control the diffusion of the semi-group applied to a local function by looking at the decay property of the conditional Dobrushin matrix as well as of the first-layer Hamiltonian. In particular if those are well behaved (which is the case for the XY-model with some slightly refined coarse-graining) the second order terms in the Taylor expansion are controlled. We can conclude $\mu'_{t+\varepsilon} = S_\varepsilon^L(\mu'_t)$ for all extremal Gibbs measures labelled by $t \in S^1$ and $\varepsilon > 0$.

4 Reversible dynamics for discrete-spin models

The infinite-volume dynamics K given in (5) with rates satisfying (6) is reversible. By expressing the r.h.s of (6) in terms of the specification γ' it is clear that K has detailed balance with respect to γ'.

These rates are bounded (by boundedness of H_j), translation-invariant (by the translation-covariance of the $\mu_{G\setminus j}[\omega'_{G\setminus j}]$ in the conditional Dobrushin regime) and of exponentially decaying influence (however not strictly local.) The rates are even uniformly bounded and bounded in the triple-norm by the same arguments as used for the rotation dynamics, so Proposition (1.4) part 1 is true.

In the next subsection we adapt a line of arguments presented for $q = 2$ in [24] for general finite q.

4.1 Translation-invariant invariant measures are Gibbs measures

Let us put ourselves in dimension $d \geqslant 3$. In the right temperature region there are multiple Gibbs measures for the XY model, ferromagnetically ordered on S^1.

Since in the following subsections we will only deal with second-layer configurations it is convenient to suppress the primes and write $c_K(\omega, \omega^i)$ for the up-flip at site $i \in G$ and $c_K(\omega, \omega^{i-})$ for the down-flip. Assume the rates to be defined as in (6), in particular for the corresponding process second-layer Gibbs measures are invariant w.r.t K.

We now show, invariant measures w.r.t K that are also translation-invariant are second-layer Gibbs measures. This is precisely part 2 of Proposition (1.4). We use Holleys argument [19]. Recall the definition of the second-layer specification and define the local relative entropy

$$H_\Lambda(\nu|\gamma'_\Lambda(\cdot|\zeta)) := \sum_{\omega\in\{1,\dots,q\}^\Lambda} \nu(1_\omega) \log \frac{\nu(1_\omega)}{\gamma'_\Lambda(1_\omega|\zeta)} \tag{41}$$

where $\Lambda \subset \mathbb{Z}^d$ is finite, $\nu \in \mathcal{P}(\Omega')$ and $\zeta \in \Omega'$ an arbitrary but fixed boundary condition. Let $(S_t^K)_{t\geqslant 0}$ be the semigroup for the generator K and define $\nu_t := S_t^K(\nu)$. Let us compute $\frac{d}{dt}_{|t=0} H_\Lambda(\nu_t|\gamma'_\Lambda(\cdot|\zeta))$ in two steps

$$\frac{d}{dt}_{|t=0} \sum_{\omega\in\{1,\dots,q\}^\Lambda} \nu_t(1_\omega)\log\nu_t(1_\omega) = \sum_\omega [1+\log\nu(1_\omega)] \int K 1_\omega d\nu$$

$$= \sum_{\omega,i\in\Lambda} \log\nu(1_\omega) \int \nu(d\eta)[c_K(\eta,\eta^i)(1_\omega(\eta^i)-1_\omega(\eta)) + c_K(\eta,\eta^{i-})(1_\omega(\eta^{i-})-1_\omega(\eta))]$$

$$= \sum_{\omega,i\in\Lambda} \big[\Gamma(\omega,i^+)\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} + \Gamma(\omega,i^-)\log\frac{\nu(1_{\omega^{i-}})}{\nu(1_\omega)}\big]$$

where we wrote $\Gamma(\omega, i^\pm) := \int \nu(d\eta) c_K(\eta,\eta^{i\pm}) 1_\omega(\eta)$ for the outflows of 1_ω in the direction $i^\pm$.

Note, if ν is invariant w.r.t K, then $\nu(1_{\omega_\Lambda}) > 0$ for all $\omega_\Lambda \in \{1,...,q\}^\Lambda$, indeed

$$\begin{aligned}0 &= \int K 1_{\omega_\Lambda} d\nu \\ &= \sum_{i\in\Lambda}\int d\nu(d\eta)[c_K(\eta,\eta^i)(1_{\omega_\Lambda^{i-}}(\eta) - 1_{\omega_\Lambda}(\eta)) + c_K(\eta,\eta^{i-})(1_{\omega_\Lambda^i}(\eta) - 1_{\omega_\Lambda}(\eta))].\end{aligned} \tag{42}$$

Since all flip-rates are positive $\nu(1_{\omega_\Lambda}) = 0$ would imply $\nu(1_{\omega_\Lambda^i}) = 0 = \nu(1_{\omega_\Lambda^{i-}})$ for all $i \in \Lambda$ and thus by iteration $\nu(1_{\eta_\Lambda}) = 0$ for all $\eta_\Lambda \in \{1,...,q\}^\Lambda$ which is a contradiction to ν being a probability measure.

Let us look at the second summand of $\frac{d}{dt}_{|t=0} H_\Lambda(\nu_t|\gamma'_\Lambda(\cdot|\zeta))$. Since the normalizing constant in the specification is independent of ω_Λ and $\sum_{\omega_\Lambda}\frac{d}{dt}_{|t=0}\int \nu_t(d\omega) = 0$, we can directly compute

$$\begin{aligned}&\frac{d}{dt}_{|t=0}\int \nu_t(d\omega)\log \mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda}))\\ &\quad= \sum_{\omega_\Lambda\in\{1,...,q\}^\Lambda}\int \nu(d\eta_{\Lambda^c})\nu(\omega_\Lambda|\eta_{\Lambda^c})K\log\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\cdot\Lambda}))(\omega_\Lambda\eta_{\Lambda^c})\\ &\quad= \sum_{\omega_\Lambda,i\in\Lambda}\Big[\log\frac{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda^i}))}{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda}))}\int\nu(d\eta)1_{\omega_\Lambda}(\eta)c_K(\eta,\eta^i)\\ &\qquad\qquad+\log\frac{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda^{i-}}))}{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda}))}\int\nu(d\eta)1_{\omega_\Lambda}(\eta)c_K(\eta,\eta^{i-})\Big]\\ &\quad= \sum_{\omega_\Lambda,i\in\Lambda}\big[V(\omega,i^+)\Gamma(\omega,i^+) + V(\omega,i^\cdot)\Gamma(\omega,i^-)\big]\end{aligned} \tag{43}$$

where we defined $V(\omega,i^\pm) := \log\frac{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda^{i\pm}}))}{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda}))}$. Notice we have

$$\frac{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda^i}))}{\mu_{G\setminus\Lambda}[\zeta_{G\setminus\Lambda}](\lambda^\Lambda(e^{-H_\Lambda}1_{\omega_\Lambda}))} = \frac{\mu_{G\setminus i}[\zeta_{G\setminus\Lambda}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i^i}))}{\mu_{G\setminus i}[\zeta_{G\setminus\Lambda}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i}))} = \frac{c_K(\omega_\Lambda\zeta_{\Lambda^c},\omega_\Lambda^i\zeta_{\Lambda^c})}{c_K(\omega_\Lambda^i\zeta_{\Lambda^c},\omega_\Lambda\zeta_{\Lambda^c})}.$$

Combining the two summands we have

$$\begin{aligned}&\frac{d}{dt}H_\Lambda(\nu|\gamma'_\Lambda(\cdot|\zeta))_{|t=0} =\\ &\sum_{\omega,i\in\Lambda}\Big[\Gamma(\omega,i^+)\big(\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} - V(\omega,i^+)\big) + \Gamma(\omega,i^-)\big(\log\frac{\nu(1_{\omega^{i-}})}{\nu(1_\omega)} - V(\omega,i^-)\big)\Big].\end{aligned} \tag{44}$$

Since $\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} - V(\omega,i^+) = -(\log\frac{\nu(1_\omega)}{\nu(1_{\omega^i})} - V(\omega^i,i^-))$ we can write

$$\begin{aligned}2\frac{d}{dt}_{|t=0}H_\Lambda(\nu_t|\gamma'_\Lambda(\cdot|\zeta)) = \sum_{\omega,i\in\Lambda}&\Big[(\Gamma(\omega,i^+) - \Gamma(\omega^i,i^-))\big(\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} - V(\omega,i^+)\big)\\ &+ (\Gamma(\omega,i^-) - \Gamma(\omega^{i-},i^+))\big(\log\frac{\nu(1_{\omega^{i-}})}{\nu(1_\omega)} - V(\omega,i^-)\big)\Big].\end{aligned} \tag{45}$$

Adding zeros we have

$$2\frac{d}{dt}H_\Lambda(\nu|\gamma'_\Lambda(\cdot|\zeta))_{|t=0} =$$

$$-\sum_{\omega,i\in\Lambda}\big[(\Gamma(\omega,i^+)-\Gamma(\omega^i,i^-))\log\frac{\Gamma(\omega,i^+)}{\Gamma(\omega^i,i^-)}+(\Gamma(\omega,i^-)-\Gamma(\omega^{i-},i^+))\log\frac{\Gamma(\omega,i^-)}{\Gamma(\omega^{i-},i^+)}\big]$$

$$+\sum_{\omega,i\in\Lambda}\big[(\Gamma(\omega,i^+)-\Gamma(\omega^i,i^-))\times\Big[\log\frac{\Gamma(\omega,i^+)}{\nu(1_\omega)}-\log\frac{\Gamma(\omega^i,i^-)}{\nu(1_{\omega^i})}-V(\omega,i^+)\Big]$$

$$+(\Gamma(\omega,i^-)-\Gamma(\omega^{i-},i^+))\times\Big[\log\frac{\Gamma(\omega,i^-)}{\nu(1_\omega)}-\log\frac{\Gamma(\omega^{i-},i^+)}{\nu(1_{\omega^{i-}})}-V(\omega,i^-)\Big]\big].$$

If ν is invariant w.r.t K it follows

$$\sum_{\omega,i\in\Lambda}\big[(\Gamma(\omega,i^+)-\Gamma(\omega^i,i^-))\log\frac{\Gamma(\omega,i^+)}{\Gamma(\omega^i,i^-)}+(\Gamma(\omega,i^-)-\Gamma(\omega^{i-},i^+))\log\frac{\Gamma(\omega,i^-)}{\Gamma(\omega^{i-},i^+)}\big]$$

$$=\sum_{\omega,i\in\Lambda}\big[(\Gamma(\omega,i^+)-\Gamma(\omega^i,i^-))\Big[\log\frac{\Gamma(\omega,i^+)}{\nu(1_\omega)}-\log\frac{\Gamma(\omega^i,i^-)}{\nu(1_{\omega^i})}-V(\omega,i^+)\Big]$$

$$+(\Gamma(\omega,i^-)-\Gamma(\omega^{i-},i^+))\Big[\log\frac{\Gamma(\omega,i^-)}{\nu(1_\omega)}-\log\frac{\Gamma(\omega^{i-},i^+)}{\nu(1_{\omega^{i-}})}-V(\omega,i^-)\Big]\big]$$

where the left hand side is non-negative. We want to exploit properties of the d-dimensional lattice in order to show the r.h.s of the last equation goes to zero for $\Lambda\nearrow G$. Let us define

$$\kappa_\Lambda(i^\pm):=\sum_\omega(\Gamma(\omega,i^\pm)-\Gamma(\omega^{i\pm},i^\mp))\log\frac{\Gamma(\omega,i^\pm)}{\Gamma(\omega^{i\pm},i^\mp)}$$

$$\beta_\Lambda(i^\pm):=\sum_\omega|\Gamma(\omega,i^\pm)-\Gamma(\omega^{i\pm},i^\mp)|$$

$$\vartheta_\Lambda(i):=\sum_{j\notin\Lambda}\sup_{\eta_{j^c}=\tilde\eta_{j^c}}\frac{|c_K(\eta,\eta^i)-c_K(\tilde\eta,\tilde\eta^i)|}{c_K(\eta,\eta^i)}+\sum_{j\notin\Lambda}\sup_{\eta_{j^c}=\tilde\eta_{j^c}}\frac{|c_K(\eta,\eta^{i-})-c_K(\tilde\eta,\tilde\eta^{i-})|}{c_K(\eta,\eta^{i-})}.\tag{46}$$

We estimate $-V(\omega,i^+)+\log\frac{\Gamma(\omega,i^+)}{\nu(1_\omega)}-\log\frac{\Gamma(\omega^i,i^-)}{\nu(1_{\omega^i})}=$

$$\log\frac{\int\nu(d\eta)1_{\omega_\Lambda}(\eta)\frac{c_K(\omega_\Lambda\eta_{\Lambda^c},\omega^i_\Lambda\eta_{\Lambda^c})}{c_K(\omega_\Lambda\zeta_{\Lambda^c},\omega^i_\Lambda\zeta_{\Lambda^c})}}{\nu(1_{\omega_\Lambda})}-\log\frac{\int\nu(d\eta)1_{\omega^i_\Lambda}(\eta)\frac{c_K(\omega^i_\Lambda\eta_{\Lambda^c},\omega_\Lambda\eta_{\Lambda^c})}{c_K(\omega^i_\Lambda\zeta_{\Lambda^c},\omega_\Lambda\zeta_{\Lambda^c})}}{\nu(1_{\omega^i_\Lambda})}$$

$$\leqslant\sup\{\log\frac{c_K(\eta_1,\eta_1^i)}{c_K(\eta_2,\eta_2^i)}:\eta_1=\eta_2\text{ on }\Lambda\}+\sup\{\log\frac{c_K(\eta_1,\eta_1^{i-})}{c_K(\eta_2,\eta_2^{i-})}:\eta_1=\eta_2\text{ on }\Lambda\}.$$

Using $\log a\leqslant a-1$ and expressing the oscillation on Λ^c via single-point oscillations we arrive at $\vartheta_\Lambda(i)$. Similarly we get for the second summand:

$$-V(\omega,i^-)+\log\frac{\Gamma(\omega,i^-)}{\nu(1_\omega)}-\log\frac{\Gamma(\omega^{i-},i^+)}{\nu(1_{\omega^{i-}})}$$

$$\leqslant\sup\{\log\frac{c_K(\eta_1,\eta_1^{i-})}{c_K(\eta_2,\eta_2^{i-})}:\eta_1=\eta_2\text{ on }\Lambda\}+\sup\{\log\frac{c_K(\eta_1,\eta_1^i)}{c_K(\eta_2,\eta_2^i)}:\eta_1=\eta_2\text{ on }\Lambda\}\leqslant\vartheta_\Lambda(i).$$

Hence

$$\sum_{i\in\Lambda}[\kappa_\Lambda(i^+)+\kappa_\Lambda(i^-)] \leqslant \sum_{i\in\Lambda}[(\beta_\Lambda(i^+)+\beta_\Lambda(i^-))\vartheta_\Lambda(i)].$$

Notice that $\vartheta_\Lambda(i) \to 0$ for all $i \in G$ as $\Lambda \nearrow G$ since our flip-rates are quasilocal and summable, indeed by the well-definedness we have for all $i \in G$

$$\sum_{j\notin\Lambda}\sup_{\eta_{j^c}=\tilde{\eta}_{j^c}}\frac{|c_K(\eta,\eta^i)-c_K(\tilde{\eta},\tilde{\eta}^i)|}{c_K(\eta,\eta^i)} \leqslant \frac{2\pi e^{\|H_0\|}}{\min_{l\in\{1,\dots,q\}}\lambda(l)}\sup_{i\in G}\sum_{j\in G}\sup_{\eta_{j^c}=\tilde{\eta}_{j^c}}|c_K(\eta,\eta^i)-c_K(\tilde{\eta},\tilde{\eta}^i)|$$
$$=: A\times B^+ < \infty.$$

Notice also that $\kappa_{\Lambda_1}(i^\pm) \leqslant \kappa_{\Lambda_2}(i^\pm)$ if $i \in \Lambda_1 \subset \Lambda_2$. Indeed if we look at the subadditive function $\varphi(x,y) = (x-y)\log\frac{x}{y}$ for $x,y>0$ and use

$$\Gamma_{\Lambda_1}(\omega_1,i^\pm) = \sum_{\omega_2=\omega_1 \text{ on } \Lambda_1}\Gamma_{\Lambda_2}(\omega_2,i^\pm)$$

we have

$$\begin{aligned}\kappa_{\Lambda_1}(i^\pm) &= \sum_{\omega_1\in\{1,\dots,q\}^{\Lambda_1}}\varphi[\Gamma_{\Lambda_1}(\omega_1,i^\pm),\Gamma_{\Lambda_1}(\omega_1^{i\pm},i^\mp)] \\ &\leqslant \sum_{\omega_2\in\{1,\dots,q\}^{\Lambda_2}}\varphi[\Gamma_{\Lambda_2}(\omega_2,i^\pm),\Gamma_{\Lambda_2}(\omega_2^{i\pm},i^\mp)] = \kappa_{\Lambda_2}(i^\pm).\end{aligned} \tag{47}$$

We are now in the position to finish the proof of Proposition (1.4) part 2. This is a standard argument from [24] using translation-invariance and explicit control over boundary terms, applied to the q-state model.

Theorem 4.1 *Suppose that $G=\mathbb{Z}^d$ and the Glauber dynamics flip-rates*

$$\frac{c_K(\omega',(\omega')^i)}{c_K((\omega')^i,\omega')} = \frac{\mu_{G\setminus i}[\omega'_{G\setminus i}](\lambda^i(e^{-H_i}1_{(\omega'_i)^i}))}{\mu_{G\setminus i}[\omega'_{G\setminus i}](\lambda^i(e^{-H_i}1_{\omega'_i}))} \tag{48}$$

are defined for a translation-invariant first-layer potential H. Then a measure that is translation invariant and invariant w.r.t K must be Gibbs for γ'.

Proof: Let ν be invariant w.r.t K and translation-invariant. Denote by Λ_n cubes in $\mathbb{Z}^d$ of side length n. Then we have

$$\frac{1}{(kn)^d}\sum_{i\in\Lambda_{kn}}[\kappa_{\Lambda_{kn}}(i^+)+\kappa_{\Lambda_{kn}}(i^-)] \geqslant \frac{1}{n^d}\sum_{i\in\Lambda_n}[\kappa_{\Lambda_n}(i^+)+\kappa_{\Lambda_n}(i^-)]. \tag{49}$$

On the other hand $\beta_\Lambda(i^+)$ and $\beta_\Lambda(i^-)$ are uniformly bounded and

$$\begin{aligned}\frac{1}{n^d}\sum_{i\in\Lambda_n}\vartheta_{\Lambda_n}(i) &\leqslant \frac{A}{n^d}\sum_{i\in\Lambda_n}\sum_{j\notin\Lambda_n}[\delta_j(c_K(\cdot,\cdot^i))+\delta_j(c_K(\cdot,\cdot^{i-}))] \\ &= \frac{A}{n^d}\sum_{i\in\Lambda_n}\sum_{j\notin\Lambda_n}[\delta_{j-i}(c_K(\cdot,\cdot^0))+\delta_{j-i}(c_K(\cdot,\cdot^{0-}))] \\ &= A\sum_{l\in\mathbb{Z}^d}[\delta_l(c_K(\cdot,\cdot^0))+\delta_l(c_K(\cdot,\cdot^{0-}))]\frac{\#\{i\in\Lambda_n : i+l\notin\Lambda_n\}}{n^d}.\end{aligned} \tag{50}$$

This tends to zero since the oscillations are bounded by $B^{\pm}$ and the fact that an increasing strip of boundary of cubes goes to infinity slower than the volume. Together we can write

$$\begin{aligned} \frac{1}{n^d}\sum_{i\in\Lambda_n}[\kappa_{\Lambda_n}(i^+)+\kappa_{\Lambda_n}(i^-)] &\leqslant \frac{1}{n^d}\sum_{i\in\Lambda_n}[(\beta_{\Lambda_n}(i^+)+\beta_{\Lambda_n}(i^-))\vartheta_{\Lambda_n}(i)] \\ &\leqslant C\frac{1}{n^d}\sum_{i\in\Lambda_n}\vartheta_{\Lambda_n}(i)\to 0 \text{ for } n\to\infty \end{aligned} \tag{51}$$

and hence by the non-negativity of $\kappa_\Lambda(i^\pm)$ we have $\kappa_{\Lambda_n}(i^+)=\kappa_{\Lambda_n}(i^-)=0$ for all $i\in\Lambda_n$. By the subadditivity argument $\kappa_\Lambda(i^+)=\kappa_\Lambda(i^-)=0$ for all Λ and $i\in\Lambda$. Thus for all finite Λ, $\omega\in\{1,...,q\}^\Lambda$ and $i\in\Lambda$

$$\begin{aligned} 0 &= \Gamma(\omega,i^+)-\Gamma(\omega^i,i^-) \\ &= \int\nu(d\eta)[\nu(\omega_i|\omega_{\Lambda\backslash i}\eta_{\Lambda^c})c_K(\omega_\Lambda\eta_{\Lambda^c},\omega^i_\Lambda\eta_{\Lambda^c})-\nu(\omega^i_i|\omega_{\Lambda\backslash i}\eta_{\Lambda^c})c_K(\omega^i_\Lambda\eta_{\Lambda^c},\omega_\Lambda\eta_{\Lambda^c})]. \end{aligned}$$

So ν-a.s. we have

$$\frac{\nu(\omega^i_i|\omega_{\Lambda\backslash i}\eta_{\Lambda^c})}{\nu(\omega_i|\omega_{\Lambda\backslash i}\eta_{\Lambda^c})}=\frac{c_K(\omega_\Lambda\eta_{\Lambda^c},\omega^i_\Lambda\eta_{\Lambda^c})}{c_K(\omega^i_\Lambda\eta_{\Lambda^c},\omega_\Lambda\eta_{\Lambda^c})}=\frac{\mu_{G\backslash i}[\eta_{\Lambda^c}\omega_{\Lambda\backslash i}](\lambda(e^{-H_i}1_{\omega^i_i}))}{\mu_{G\backslash i}[\eta_{\Lambda^c}\omega_{\Lambda\backslash i}](\lambda(e^{-H_i}1_{\omega_i}))}.$$

Since we compare discrete measures on sites $i\in G$ it follows by the remark below, $\nu(\omega_i|\omega_{i^c})=\gamma'_i(\omega_i|\omega_{i^c})$ ν-almost everywhere and thus $\nu\in\mathcal{G}(\gamma')$. □

Remark 4.2 *Let $(a_1,...a_q)$ and $(b_1,...,b_q)$ be probability vectors with $\frac{a_k}{a_{k+1}}=\frac{b_k}{b_{k+1}}$ for all $k\in\{1,...,q\}$. Then we have $\frac{a_l}{a_k}=\frac{b_l}{b_k}$ for all $k,l\in\{1,...,q\}$ and thus*

$$a_l=\frac{a_l}{\sum_{k=1}^q a_k}=\frac{1}{1+\sum_{k\neq l}\frac{a_k}{a_l}}=\frac{1}{1+\sum_{k\neq l}\frac{b_k}{b_l}}=b_l. \tag{52}$$

5 Joint dynamics

Let us now consider the joint dynamics $L+\alpha K$ for $\alpha>0$. Of course well-definedness (Proposition (1.5) part 1) follows directly from the fact, that the individual rates of L and K are well-defined.

As a warning we note, the generators L and K do not commute (except in the limit $q\to\infty$.) To see this we apply $LK-KL$ to the local observable $\psi:=1_{\eta_\Lambda}$ for a finite $\Lambda\subset G$. Evaluated for instance at $\omega_\Lambda=\eta_\Lambda$, we find the expression

$$\begin{aligned} &\sum_{i\in\Lambda}\Big(c_L(\omega,\omega^i)c_K(\omega^i,\omega)-c_K(\omega,\omega^{i-})c_L(\omega^{i-},\omega)\Big) \\ &\quad=\sum_{i\in\Lambda}\Big(\mu_{G\backslash i}[\omega_{G\backslash i}](e^{-H_i((\omega_i)^l,\cdot_{i^c})})-\mu_{G\backslash i}[\omega_{G\backslash i}](e^{-H_i((\omega_i^{i-})^l,\cdot_{i^c})})\Big). \end{aligned} \tag{53}$$

This does not vanish in general and thus in general the commutator is not zero. But if we consider the limit of the coarse-graining, i.e. letting q the number of discrete states go to infinity, we approach a commutative setting. This result reflects the continuum situation in the Maes and Shlosman program [25].

As a consequence $S_t^{L+\alpha K} \neq S_t^L S_t^{\alpha K}$ and it is not immediate that the joint dynamics also rotates the discrete Gibbs measures in the sense of Proposition (1.5) part 2. To see that this is nevertheless true one has to follow the same arguments as in section 3.2 and notice $|||\Gamma^{joint}|||_{e^\varrho} < \infty$.

5.1 The invariant measure for the joint dynamics

In this subsection we show Proposition (1.5) part 3 and Corollary (1.6). First let us verify that indeed the symmetrically mixed measure is invariant and in the set of Gibbs measures this is the only one. Finally we prove that measures that are invariant under the joint dynamics must be Gibbs.

The mixture of all translation-invariant extremal Gibbs measures μ'_t

$$\mu'_* := \frac{1}{2\pi}\int_0^{2\pi} \mu'_t dt$$

is invariant for the rotation dynamics and hence for the joint dynamics $L + \alpha K$. Indeed, let $(S_t^L)_{t\geqslant 0}$ be the semigroup for L and f a quasilocal observable we have

$$\begin{aligned}\int S_t^L f(\eta)\mu'_*(d\eta) &= \frac{1}{2\pi}\int_0^{2\pi}\int S_t^L f(\eta)\mu'_s(d\eta)ds \\ &= \int f(\eta)\frac{1}{2\pi}\int_0^{2\pi}\mu'_{s+t}(d\eta)ds = \int f(\eta)\mu'_*(d\eta).\end{aligned} \tag{54}$$

Proposition 5.1 *There are no translation-invariant invariant Gibbs measures for the rotation dynamics other then μ'_*.*

Proof: We know from Theorem 7.26 of [16], every Gibbs measure $\mu' \in \mathcal{G}_\theta(\gamma')$ has a unique representation

$$\mu' = \int_{\text{ex } \mathcal{G}_\theta(\gamma')} \bar{\mu} w_{\mu'}(d\bar{\mu})$$

where $w_{\mu'} \in \mathcal{P}(\text{ex } \mathcal{G}_\theta(\gamma'), \sigma(\text{ex } \mathcal{G}_\theta(\gamma')))$ and $\sigma(\mathcal{P})$ is the so called evaluation σ-algebra. Since the Gibbs measures can be labelled as described above, there is a bijection

$$b : \text{ex } \mathcal{G}_\theta(\gamma') \to [0, 2\pi) = S^1 \qquad \mu' \mapsto \arg(e_{\hat{m}'_0}(\mu')/m_\beta)$$

where b is $(\sigma(\text{ex } \mathcal{G}_\theta(\gamma')), \mathcal{B}([0, 2\pi)))$ measurable and arg denotes the argument of a number in S^1. Indeed since $\hat{m}'_0$ is bounded and measurable, so is $e_{\hat{m}'_0} : \mu' \mapsto \mu'(\hat{m}'_0) \in \mathbb{R}^2$ and thus $b(\mu') = \arg(e_{\hat{m}'_0}(\mu')/m_\beta)$ is a composition of measurable functions. Hence we can consider image measures $v_{\mu'}$ of $w_{\mu'}$ under b.

On the other hand for all local coarse-grained sets $A' \in \mathcal{F}'$ the mapping

$$c_{A'} : [0, 2\pi) \to [0, 1] \qquad t \mapsto \mu'_t(A') = \mu_t(A) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \gamma_\Lambda(A|\omega_t)$$

where $A := T^{-1}(A')$ and ω_t the homogeneous boundary condition as described in the introduction, is Borel-measurable as a composition of measurable maps. We also used the measurability of $t \mapsto \omega_t$. Hence this is true for all $A' \in \mathcal{F}'$.

By the transformation theorem for measurable maps we have for all $A' \in \mathcal{F}'$

$$\begin{aligned}\mu'(A') &= \int_{\text{ex } \mathcal{G}_\theta(\gamma')} \bar{\mu}(A') w_{\mu'}(d\bar{\mu}) = \int_{\text{ex } \mathcal{G}_\theta(\gamma')} c_{A'}(b(\bar{\mu})) w_{\mu'}(d\bar{\mu}) \\ &= \int_0^{2\pi} c_{A'}(t) w_{\mu'}(b^{-1}(dt)) = \int_0^{2\pi} c_{A'}(t) v_{\mu'}(dt) = \int_0^{2\pi} \mu'_t(A') v_{\mu'}(dt).\end{aligned} \tag{55}$$

By looking at tail-measurable interval sets

$$A_{[0,u)} := \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \arg(\frac{\omega_j}{m_\beta}) = [0,u)\}$$

and $\varphi_{[0,u)}(\omega') := \mu_G[\omega'](A_{[0,u)})$ we see that $v_{\mu'}$ has to be a translation-invariant Borel-measure, indeed

$$\begin{aligned}v_{\mu'}([0,u)) &= \int_0^{2\pi} \mu'_t(\varphi_{[0,u)}) v_{\mu'}(dt) = \mu'(\varphi_{[0,u)}) = \mu' S_s^L(\varphi_{[0,u)}) \\ &= \int_0^{2\pi} \mu'_{t+s}(\varphi_{[0,u)}) v_{\mu'}(dt) = v_{\mu'}([-s, u-s))\end{aligned} \tag{56}$$

for all $s \in [0, 2\pi)$. Since $\{[0,u) : u \in [0,2\pi)\}$ is a generator for the Borel-σ-algebra and $v_{\mu'}$ is a probability measure we have $v_{\mu'}(dt) = \frac{1}{2\pi}\lambda(dt)$. □

Since $S_s^{L+\alpha K}(\mu'_t) = \mu'_{t+s} = S_s^L(\mu'_t)$ we can conclude μ'_* is the only translation-invariant measure that is also invariant w.r.t the joint dynamics. The next proposition proves Proposition (1.5) part 3.

Proposition 5.2 *Every translation-invariant measure that is invariant for the joint dynamics $L + \alpha K$ with $\alpha > 0$ is a Gibbs measure.*

Proof: Let $\Lambda \subset \mathbb{Z}^d$ be a finite set and $\zeta \in \Omega'$ an arbitrary but fixed boundary condition for the second-layer specification, i.e. consider the coarse-grained measure $\gamma'_\Lambda(\omega|\zeta) = \frac{\mu_{\Lambda^c}[\zeta_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\omega_\Lambda}))}{\mu_{\Lambda^c}[\zeta_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda}))}$ on $\{1, ..., q\}^\Lambda$. Our strategy for the proof is to again look at the derivative of the local relative entropy $H_\Lambda(\nu|\gamma'_\Lambda(\cdot|\zeta))$ for ν translation-invariant and invariant w.r.t the joint dynamics. We have seen in case of the Glauber dynamics how to verify Gibbsianness for invariant measures by estimating certain terms in the derivative of the local relative entropy. Those term are only of the order of the boundary $|\partial\Lambda|$. This allowed us to prove the DLR equality for the invariant measure. A crucial ingredience is the translation-invariance of both, the model as well as the invariant measure.

Essentially we follow the same line of arguments here, taking special care of the contribution of the rotation. We look at an approximating local open boundary rotation dynamics and show its relative entropy is decreasing. This means the approximating rotation only "helps" the Glauber dynamics argument. The error we make by using the approximation instead of the infinite-volume rotation dynamics is only of boundary order and thus again increases slower than the volume.

Since the time-derivative of the local relative entropy is additive as a sum of the two terms corresponding to the two generators K and L, we can calculate separately for the Glauber and for the rotation dynamics. We write $\nu_{t,L}$ (resp. $\nu_{t,K}$, $\nu_{t,L+\alpha K}$) for the measure ν propagated only by the rotation (resp. by the Glauber dynamics, by the joint dynamics.)

Let us compute for the rotation $\frac{d}{dt}_{|t=0} H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta))$ with $\nu = \nu_0$. Again we do this in two steps. Similarly to the computations done in (4.1) we find

$$\frac{d}{dt}_{|t=0} \sum_{\omega\in\{1,\dots,q\}^\Lambda} \nu_{t,L}(1_\omega) \log \nu_{t,L}(1_\omega) = \sum_{\omega,i\in\Lambda} \Gamma_L(\omega, i^+) \log \frac{\nu(1_{\omega^i})}{\nu(1_\omega)} \tag{57}$$

where we again wrote $\Gamma_L(\omega, i^+) := \int \nu(d\eta) c_L(\eta, \eta^i) 1_\omega(\eta)$ for the outflows of 1_ω in the direction i^+. For the other summand of $\frac{d}{dt}_{|t=0} H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta))$ we have

$$\frac{d}{dt}_{|t=0} \int \nu_{t,L}(d\omega) \log \mu_{\Lambda^c}[\zeta_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\omega_\Lambda})) = \sum_{\omega,i\in\Lambda} V^\zeta(\omega, i^+) \Gamma_L(\omega, i^+) \tag{58}$$

where we again defined $V^\zeta(\omega, i^+) := \log \frac{\mu_{\Lambda^c}[\zeta_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\omega^i_\Lambda}))}{\mu_{\Lambda^c}[\zeta_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} 1_{\omega_\Lambda}))}$. Together we have

$$\frac{d}{dt}_{|t=0} H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta)) = \sum_{\omega,i\in\Lambda} \Gamma_L(\omega, i^+) \big(\log \frac{\nu(1_{\omega^i})}{\nu(1_\omega)} - V^\zeta(\omega, i^+)\big). \tag{59}$$

We define the approximating local generator $\tilde L_\Lambda$ via the following open boundary rates

$$c_{\tilde L_\Lambda}(\eta, \eta^i) := \begin{cases} \frac{\lambda^{\Lambda\setminus i}(e^{-\tilde H_\Lambda(\eta_i|^r,\cdot)} 1_{\eta_{\Lambda\setminus i}})}{\lambda^\Lambda(e^{-\tilde H_\Lambda} 1_{\eta_\Lambda})}, & \text{if } i \in \Lambda \\ 0, & \text{if } i \in \Lambda^c \end{cases}$$

where Λ is a fixed finite volume and $\tilde H_\Lambda := \sum_{A\subset\Lambda} \Phi_A$ is the open boundary Hamiltonian for Λ in the first-layer model. Let $(S_t^{\tilde L_\Lambda})_{t\geqslant 0}$ be the corresponding semigroup. Since we assume the underlying first-layer potential to be rotation-invariant, the open boundary measure $\tilde\gamma_\Lambda(\omega_\Lambda) := \lambda^\Lambda(e^{-\tilde H_\Lambda} 1_{\omega_\Lambda})/\lambda^\Lambda(e^{-\tilde H_\Lambda})$ on $\{1,\dots,q\}^\Lambda$ is invariant for $\tilde L_\Lambda$. Indeed for all $\omega_\Lambda \in \{1,\dots,q\}^\Lambda$ we have

$$\begin{aligned} \tilde\gamma_\Lambda(\tilde L_\Lambda(1_{\omega_\Lambda})) &= \frac{\sum_{i\in\Lambda} [\lambda^{\Lambda\setminus i}(e^{-\tilde H_\Lambda(\omega_i|^l,\cdot)} 1_{\omega_{\Lambda\setminus i}}) - \lambda^{\Lambda\setminus i}(e^{-\tilde H_\Lambda(\omega_i|^r,\cdot)} 1_{\omega_{\Lambda\setminus i}})]}{\lambda^\Lambda(e^{-\tilde H_\Lambda})} \\ &= \frac{d}{d\varepsilon}_{|\varepsilon=0} \tilde\gamma_\Lambda(1_{\omega_\Lambda+\varepsilon}) = \frac{d}{d\varepsilon}_{|\varepsilon=0} \tilde\gamma_\Lambda(1_{\omega_\Lambda}) = 0. \end{aligned} \tag{60}$$

We can employ a standard argument for the decrease of relative entropy in finite volume in order to determine the sign of $\frac{d}{dt}_{|t=0} H_\Lambda(\nu_{t,\tilde L_\Lambda}|\tilde\gamma_\Lambda)$. Indeed if we use the convex function $\psi(x) = x\log x + x - 1$ the relative entropy reads

$$\begin{aligned} H_\Lambda(\nu_{t,\tilde L_\Lambda}|\tilde\gamma_\Lambda) &= \sum_\omega \tilde\gamma_\Lambda(1_\omega)\psi(\frac{\nu_{t,\tilde L_\Lambda}(1_\omega)}{\tilde\gamma_\Lambda(1_\omega)}) \\ &= \sum_\omega \tilde\gamma_\Lambda(1_\omega)\psi(\frac{1}{\tilde\gamma_\Lambda(1_\omega)}\sum_\eta S_t^{\tilde L_\Lambda}(1_\omega)(\eta)\frac{\nu(\eta)}{\tilde\gamma_\Lambda(\eta)}\tilde\gamma_\Lambda(\eta)) \end{aligned} \tag{61}$$

where $\frac{S_t^{\tilde L_\Lambda}(1_\omega)(\eta)}{\tilde\gamma(1_\omega)}\tilde\gamma(d\eta) = \frac{1_\omega(\eta)}{\tilde\gamma(1_\omega)}\tilde\gamma(d\eta)$ is a probability measure. Hence we can use Jensen's inequality and obtain

$$\begin{aligned} H_\Lambda(\nu_{t,\tilde L_\Lambda}|\tilde\gamma_\Lambda) &\leqslant \sum_\omega \tilde\gamma(1_\omega)\frac{1}{\tilde\gamma(1_\omega)}\sum_\eta S_t^{\tilde L_\Lambda}(1_\omega)(\eta)\psi(\frac{\nu(\eta)}{\tilde\gamma_\Lambda(\eta)})\tilde\gamma_\Lambda(\eta) \\ &= \sum_\omega \psi(\frac{\nu(\omega)}{\tilde\gamma_\Lambda(\omega)})\tilde\gamma_\Lambda(\omega) = H_\Lambda(\nu|\tilde\gamma_\Lambda) \end{aligned} \tag{62}$$

with equality iff $\nu_{t,\tilde L_\Lambda} = \tilde\gamma_\Lambda$. Thus the derivative must be non-positive

$$\begin{aligned} 0 \geqslant \frac{d}{dt}_{|t=0} H_\Lambda(\nu_{t,\tilde L_\Lambda}|\tilde\gamma_\Lambda) &= \sum_{\omega, i\in\Lambda} \nu_t(1_\omega)c_{\tilde L_\Lambda}(\omega,\omega^i)(\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} - \log\frac{\tilde\gamma_\Lambda(\omega^i)}{\tilde\gamma_\Lambda(\omega)}) \\ &=: \sum_{\omega, i\in\Lambda} \Gamma_{\tilde L_\Lambda}(\omega,i^+)\big(\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} - V_{\tilde L_\Lambda}(\omega,i^+)\big). \end{aligned} \tag{63}$$

We are going to show $|\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta)) - \frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,\tilde L_\Lambda}|\tilde\gamma_\Lambda)| = o(|\Lambda|)$. Let us start with the following estimate

$$\begin{aligned} &\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta) - \frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,\tilde L_\Lambda}|\tilde\gamma_\Lambda) = \sum_{\omega,i\in\Lambda}[\Gamma_L(\omega,i^+) - \Gamma_{\tilde L_\Lambda}(\omega,i^+)]\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)} \\ &\quad + \sum_{\omega,i\in\Lambda}\big[[V_{\tilde L_\Lambda}(\omega,i^+) - V^\zeta(\omega,i^+)]\Gamma_L(\omega,i^+) - [\Gamma_L(\omega,i^+) - \Gamma_{\tilde L_\Lambda}(\omega,i^+)]V_{\tilde L_\Lambda}(\omega,i^+)\big] \\ &\leqslant \sum_{\omega,i\in\Lambda} A(\omega,i^+)\nu(1_\omega)|\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)}| + \sum_{\omega,i\in\Lambda}\big[B(\omega,i^+)\Gamma_L(\omega,i^+) - A(\omega,i^+)\nu(1_\omega)|V_{\tilde L_\Lambda}(\omega,i^+)|\big] \end{aligned} \tag{64}$$

where we defined $B(\omega,i^+) := |V_{\tilde L_\Lambda}(\omega,i^+) - V^\zeta(\omega,i^+)|$ and used the following estimate and definition

$$\begin{aligned} \Gamma_L(\omega,i^+) - \tilde\Gamma_{\tilde L_\Lambda}(\omega,i^+) &= \int \nu(d\eta)c_L(\eta,\eta^i)1_\omega(\eta) - \nu(1_\omega)c_{\tilde L_\Lambda}(\omega,\omega^i) \\ &= \int \nu(d\eta_{\Lambda^c})\nu(1_\omega|\eta_{\Lambda^c})[c_L(\omega_\Lambda\eta_{\Lambda^c},\omega_\Lambda^i\eta_{\Lambda^c}) - c_{\tilde L_\Lambda}(\omega,\omega^i)] \\ &\leqslant \sup_{\eta_{\Lambda^c}}|\frac{\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](e^{-H_i(\omega_i^r,\cdot_{i^c})})}{\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i(\cdot)}1_{\omega_i}))} - \frac{\lambda^{\Lambda\setminus i}(e^{-\tilde H_\Lambda(\omega_i^r,\cdot_{\Lambda\setminus i})}1_{\omega_{\Lambda\setminus i}})}{\lambda^\Lambda(e^{-\tilde H_\Lambda(\cdot_\Lambda)}1_{\omega_\Lambda})}|\nu(1_\omega) \\ &=: A(\omega,i^+)\nu(1_\omega). \end{aligned}$$

We first verify $\sup_\omega\sum_{i\in\Lambda}A(\omega,i^+) = o(|\Lambda|)$ and $\sup_\omega\sum_{i\in\Lambda}B(\omega,i^+) = o(|\Lambda|)$. We do this in the two following lemmas.

Lemma 5.3 $\sup_\omega \sum_{i\in\Lambda} A(\omega, i^+) = o(|\Lambda|)$.

Proof: In order to see cancellations we define for a given second-layer boundary condition inside Λ, namely ω_Λ, and open boundary conditions outside Λ, the conditional first-layer probability measures on $(S^1)^{G\backslash i}$

$$\tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}](\varphi) := \frac{\lambda^{i^c}(\varphi e^{-\sum_{i\notin A\subset\Lambda}\Phi_A} 1_{\omega_{\Lambda\backslash i}})}{\lambda^{i^c}(e^{-\sum_{i\notin A\subset\Lambda}\Phi_A} 1_{\omega_{\Lambda\backslash i}})}. \tag{65}$$

In particular $\frac{\tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}](e^{-\tilde{H}_i(\omega_i|^r,\cdot_{\Lambda\backslash i})})}{\tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}](\lambda(e^{-\tilde{H}_i(\cdot_\Lambda)}1_{\omega_i}))} = \frac{\lambda^{\Lambda\backslash i}(e^{-\tilde{H}_\Lambda(\omega_i|^r,\cdot_{\Lambda\backslash i})}1_{\omega_{\Lambda\backslash i}})}{\lambda^\Lambda(e^{-\tilde{H}_\Lambda(\cdot_\Lambda)}1_{\omega_\Lambda})}$. These fractions again give rise to a specification $\tilde{\gamma}$ on the second layer when we look at subvolumes, keeping the Λ fixed.

In essence we want to exploit the Dobrushin comparison theorem. Since we can bound every term by some constant times $e^{\pm\|H_i\|} = e^{\pm\|H_i\|} := e^K$ it suffices to estimate the distance of the conditional first-layer Gibbs measures $\mu_{G\backslash i}[\eta_{\Lambda^c}\omega_{\Lambda\backslash i}]$ and $\tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}]$ applied to the quasilocal functions

$$\begin{aligned} &\psi_1^{\omega_i}(\cdot) := e^{-H_i(\omega_i|^r,\cdot_{i^c})} && \psi_2^{\omega_i}(\cdot) := \lambda(e^{-H_i(\cdot,\cdot_{i^c})}1_{\omega_i}) && \text{and} \\ &\tilde{\psi}_1^{\omega_i}(\cdot) := e^{-\tilde{H}_i(\omega_i|^r,\cdot_{i^c})} && \tilde{\psi}_2^{\omega_i}(\cdot) := \lambda(e^{-\tilde{H}_i(\cdot,\cdot_{i^c})}1_{\omega_i}). \end{aligned} \tag{66}$$

(Notice that we have done computations of the same flavor in the section about well-definedness of the rotation dynamics.) For any fixed first-layer boundary condition $w \in \Omega$ the measure $\mu_{G\backslash i}[\eta_{\Lambda^c}\omega_{\Lambda\backslash i}]$ is uniquely admitted by the specification

$$\gamma^{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}}|_{i^c} := \Big((\gamma^{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}}|_{i^c})_\Delta(\cdot|w_{\Delta^c\backslash i})\Big)_{\Delta\subset i^c} \tag{67}$$

and $\tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}]$ is admitted by $\tilde{\gamma}^{\omega_{\Lambda\backslash i}}|_{i^c} := \Big((\tilde{\gamma}^{\omega_{\Lambda\backslash i}}|_{i^c})_\Delta(\cdot|w_{\Delta^c\backslash i})\Big)_{\Delta\subset i^c}$, Δ being finite subsets of i^c. The total variational distance between the two specifications on the site $l \neq i$ can be estimated by

$$\begin{aligned} b_l &:= \sup_{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}, w_{l^c\backslash i}} \|(\gamma^{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}}|_{i^c})_l(\cdot|w_{l^c\backslash i}) - (\tilde{\gamma}^{\omega_{\Lambda\backslash i}}|_{i^c})_l(\cdot|w_{l^c\backslash i})\|_l \\ &= \sup_{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}, w_{l^c\backslash i}, B\in\mathcal{S}^1} \Big| \frac{\lambda(1_B 1_{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}} e^{-\sum_{i\notin A\ni l}\Phi_A(\cdot,w_{l^c\backslash i})})}{\lambda(1_{\eta_{\Lambda^c}\omega_{\Lambda\backslash i}} e^{-\sum_{i\notin A\ni l}\Phi_A(\cdot,w_{l^c\backslash i})})} - \frac{\lambda(1_B 1_{\omega_{\Lambda\backslash i}} e^{-\sum_{i\notin A\ni l, A\subset\Lambda}\Phi_A(\cdot,w_{l^c\backslash i})})}{\lambda(1_{\omega_{\Lambda\backslash i}} e^{-\sum_{i\notin A\ni l, A\subset\Lambda}\Phi_A(\cdot,w_{l^c\backslash i})})} \Big| \\ &\leqslant \begin{cases} 1, & \text{if } l \in \Lambda^c \\ K\sum_{l\in A\not\subset\Lambda}\|\Phi_A\|, & \text{if } l \in \Lambda\backslash i \end{cases} \end{aligned}$$

where K some constant and again we used $|e^x - e^y| \leqslant |x-y|e^{\max(|x|,|y|)}$. Notice, for any fixed l when Λ tends to $\mathbb{Z}^d$, because of the absolute summability of the Hamiltonian, this goes to zero. Further we want to estimate

$$\begin{aligned} &\sup_{\eta_{\Lambda^c}\omega_\Lambda} \big|\mu_{G\backslash i}[\eta_{\Lambda^c}\omega_{\Lambda\backslash i}](\psi_1^{\omega_i}) - \tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}](\tilde{\psi}_1^{\omega_i})\big| \qquad \text{and} \\ &\sup_{\eta_{\Lambda^c}\omega_\Lambda} \big|\mu_{G\backslash i}[\eta_{\Lambda^c}\omega_{\Lambda\backslash i}](\psi_2^{\omega_i}) - \tilde{\mu}_{G\backslash i}[\omega_{\Lambda\backslash i}](\tilde{\psi}_2^{\omega_i})\big|. \end{aligned} \tag{68}$$

We do this for both terms simultaneously by just writing ψ instead of ψ_1, ψ_2.

$$\begin{aligned}
&\sup_{\eta_{\Lambda^c}\omega_\Lambda} \left|\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\psi^{\omega_i}) - \tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i})\right| \\
&\leqslant \sup_{\eta_{\Lambda^c}\omega_\Lambda} \left|\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i}) - \tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i})\right| \\
&\qquad + \sup_{\eta_{\Lambda^c}\omega_\Lambda} \left|\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\psi^{\omega_i}) - \mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i})\right|.
\end{aligned} \tag{69}$$

For the second part in (69) we have

$$\sup_{\eta_{\Lambda^c},\omega_\Lambda} \left|\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\psi^{\omega_i}) - \mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i})\right| \leqslant \sup_{\omega_i} \|\psi^{\omega_i} - \tilde{\psi}^{\omega_i}\| \leqslant K \sum_{i\in A\not\subset\Lambda} \|\Phi_A\|$$

which tends to zero as $\Lambda \nearrow \mathbb{Z}^d$ by the absolute summability of the potential. In particular there exists a radius $r \in \mathbb{N}$ such that $\sup_{i\in\Lambda_{n-r}} \sum_{i\in A\not\subset\Lambda_n} \|\Phi_A\| < \varepsilon$ for all centered cubes Λ_n such that $n - r \geqslant 0$. Hence

$$\frac{1}{|\Lambda_n|} \sum_{i\in\Lambda_n} \sum_{i\in A\not\subset\Lambda_n} \|\Phi_A\| < \varepsilon + \|H_0\| \frac{|\Lambda_n \setminus \Lambda_{n-r}|}{|\Lambda_n|} \tag{70}$$

where the r.h.s becomes arbitrarily small as $n \to \infty$.

Let us look at the first part of (69) and use the Dobrushin comparison theorem, which states

$$\begin{aligned}
\sup_{\eta_{\Lambda^c}\omega_\Lambda} \left|\mu_{G\setminus i}[\eta_{\Lambda^c}\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i}) - \tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\tilde{\psi}^{\omega_i})\right| &\leqslant \sup_{\eta_{\Lambda^c}\omega_\Lambda} \sum_{k\neq i, l\neq i} \delta_k(\tilde{\psi}^{\omega_i}) D_{kl}(\gamma^{\eta_{\Lambda^c}\omega_{\Lambda\setminus i}}) b_l \\
&\leqslant \sup_{\omega_i} \sum_{k\in\Lambda\setminus i} \sum_{l\in\Lambda^c} \delta_k(\tilde{\psi}^{\omega_i}) \overline{D}_{kl} + \sup_{\omega_i} \sum_{k\in\Lambda\setminus i} \sum_{l\in\Lambda\setminus i} \delta_k(\tilde{\psi}^{\omega_i}) \overline{D}_{kl} b_l.
\end{aligned} \tag{71}$$

As for the second term on the r.h.s of (71) we have

$$\begin{aligned}
\sum_{i\in\Lambda} \sum_{k\in\Lambda\setminus i} \sum_{l\in\Lambda\setminus i} \sup_{\omega_i} \delta_k(\tilde{\psi}^{\omega_i}) \overline{D}_{kl} b_l &\leqslant (\sum_{l\in\Lambda\setminus i} b_l)(\sup_l \sum_{k\in\Lambda\setminus i} \overline{D}_{kl})(\sup_k \sum_{i\in\Lambda} \sup_{\omega_i} \delta_k(\tilde{\psi}^{\omega_i})) \\
&\leqslant K \sum_{l\in\Lambda\setminus i} b_l = o(|\Lambda|).
\end{aligned}$$

Indeed we have for all k

$$\sum_{i\in\Lambda} \sup_{\omega_i} \delta_k(\tilde{\psi}^{\omega_i}) \leqslant e^C \sum_{i\in\Lambda} \sum_{\{i,k\}\subset A\subset\Lambda} \delta_k(\Phi_A) \leqslant e^C \sum_{0\in A} |||\Phi_A||| < \infty \tag{72}$$

and also $\sum_{k\in\Lambda\setminus i} \overline{D}_{kl} \leqslant \sum_k \overline{D}_{kl} = \sum_k \overline{D}_{0,l-k} = \sum_j \overline{D}_{0,j} < \infty$ for all l. Finally

$$\frac{1}{|\Lambda|} \sum_{l\in\Lambda\setminus i} b_l \leqslant \frac{K}{|\Lambda|} \sum_{l\in\Lambda} \sum_{l\in A\not\subset\Lambda} \|\Phi_A\| \to 0 \text{ as } \Lambda \nearrow \mathbb{Z}^d$$

by the Cesàro argument as in (70). Let us consider for the first term on the r.h.s of (71)

$$\sum_{i\in\Lambda}\sum_{k\in\Lambda\setminus i}\sum_{l\in\Lambda^c}\sup_{\omega_i}\delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl}=\sum_{i\in\Lambda}\sum_{l\in\Lambda^c}\sum_{k\in\Lambda\setminus i}\sup_{\omega_i}\delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl}. \tag{73}$$

Notice we assume the model to have the exponential decay property (28) with increasing translation invariant semi-metric ϱ on G and again summability of the potential in the triple-norm. Thus for all i and l by the triangle inequality

$$\sum_{k\in\Lambda\setminus i}e^{\varrho(i,l)}\sup_{\omega_i}\delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl}\leqslant(\sup_{i\in\Lambda}\sum_k e^{\varrho(i,k)}\sup_{\omega_i}\delta_k(\tilde{\psi}^{\omega_i}))(\sup_{l,k}e^{\varrho(k,l)}\overline{D}_{kl})\leqslant\tilde{C}. \tag{74}$$

Hence we can write

$$\sum_{i\in\Lambda}\sum_{l\in\Lambda^c}\sum_{k\in\Lambda\setminus i}\sup_{\omega_i}\delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl}=\tilde{C}\sum_{j\in\mathbb{Z}^d}e^{-\varrho(0,j)}\#\{i\in\Lambda: i+j\notin\Lambda\} \tag{75}$$

which again tends to infinity slower than $|\Lambda|$. □

Lemma 5.4 $\sup_\omega\sum_{i\in\Lambda}B(\omega,i^+)=o(|\Lambda|)$.

Proof: For the next error term in (64) we have

$$\begin{aligned}B(\omega,i^+)&=|\log\frac{\tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\lambda(e^{-\tilde{H}_i}1_{\omega_i^i}))}{\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i^i}))}-\log\frac{\tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\lambda(e^{-\tilde{H}_i}1_{\omega_i}))}{\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i}))}|\\&\leqslant\frac{|\tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\lambda(e^{-\tilde{H}_i}1_{\omega_i^i}))-\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i^i}))|}{\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i^i}))}\\&\quad+\frac{|\tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\lambda(e^{-\tilde{H}_i}1_{\omega_i}))-\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i}))|}{\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i}))}.\end{aligned} \tag{76}$$

where we assumed $\tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\lambda(e^{-\tilde{H}_i}1_{\omega_i^i}))\geqslant\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i^i}))$ and $\tilde{\mu}_{G\setminus i}[\omega_{\Lambda\setminus i}](\lambda(e^{-\tilde{H}_i}1_{\omega_i}))\geqslant\mu_{G\setminus i}[\zeta_{\Lambda^c}\omega_{\Lambda\setminus i}](\lambda(e^{-H_i}1_{\omega_i}))$. In that case as well as in all other cases we can follow the exact same arguments as before and get

$$\frac{1}{|\Lambda|}\sup_\omega\sum_{i\in\Lambda}B(\omega,i^+)\to 0\text{ as }\Lambda\nearrow\mathbb{Z}^d. \tag{77}$$

□

Given the fact that $\sup_i\sum_\omega\Gamma_L(\omega,i^+)<\infty$ and $\sup_{i\in\Lambda}\sum_\omega\nu(1_\omega)|V_{\tilde{L}_\Lambda}(\omega,i^+)|\leqslant K\sup_{i\in\Lambda}\sum_\omega\nu(1_\omega)=K<\infty$ we have by now verified that the second summand in the last line of (64) is indeed $o(|\Lambda|)$. The first summand in the last line of (64) requires some extra care. We prepare by writing

$$\sum_{\omega,i\in\Lambda}A(\omega,i^+)\nu(1_\omega)|\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)}|\leqslant\sum_{i\in\Lambda}\sup_\omega A(\omega,i^+)\sum_\omega\nu(1_\omega)|\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)}|. \tag{78}$$

The next step is then to show boundary order of the r.h.s of (78), in other words to show the following lemma.

Lemma 5.5 $\sum_{i\in\Lambda}\sup_\omega A(\omega,i^+)\sum_\omega \nu(1_\omega)|\log\frac{\nu(1_{\omega^i})}{\nu(1_\omega)}| = o(|\Lambda|)$.

Notice, since the rates are bounded from below away from zero and bounded from above, i.e $e^{-2\|H_0\|}\leqslant c_L(\omega,\omega^i)\leqslant \tilde{k}e^{2\|H_0\|}$, $e^{-\|H_0\|}\leqslant c_K(\omega,\omega^i)\leqslant \tilde{k}e^{\|H_0\|}$, $e^{-\|H_0\|}\leqslant c_K(\omega,\omega^{i-})\leqslant \tilde{k}e^{\|H_0\|}$ and ν is invariant, i.e. $0=\int(L+\alpha K)1_{\omega_\Lambda}d\nu$ we have for all $\omega\in\{1,...,q\}^\Lambda$ (after separation of the terms in this equation proportional to $\nu(1_\omega)$ from $\nu(1_{\omega^i})$ and $\nu(1_{\omega^{-i}})$)

$$\tilde{K}\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}[\frac{\nu(1_{\omega^{i-}})}{\nu(1_\omega)}+\frac{\nu(1_{\omega^i})}{\nu(1_\omega)}]\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\frac{\nu(1_\omega)}{\nu(1_{\omega^i})}=\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\frac{\nu(\omega_i|\omega_{\Lambda\setminus i})}{\nu(\omega_i^i|\omega_{\Lambda\setminus i})}. \tag{79}$$

To control possibly small arguments of the logarithm, we need to bound ν-probabilities from below. For this the following lemma will be useful.

Lemma 5.6 *Let $\nu\in\mathcal{P}(\Omega')$ be translation-invariant and invariant for the joint dynamics. There exists a constant $\hat{K}$ such that for all finite sets Δ we have*

$$\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_0^0|\omega_{\Delta\setminus 0})}}<\hat{K}.$$

Proof of Lemma 5.6: By the Jensen inequality, it suffices to show this for centered cubes Δ. Let us consider the ν-expectation of our essential estimate (79) and apply Jensen's inequality to obtain

$$\begin{aligned}\tilde{K}&\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\int\nu(d\omega)\frac{\nu(\omega_i|\omega_{\Lambda\setminus i})}{\nu(\omega_i^i|\omega_{\Lambda\setminus i})}\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sum_{k=1}^{q}[\int\nu(d\omega)\frac{\nu(k|\omega_{\Lambda\setminus i})}{\sqrt{\nu((k+1)|\omega_{\Lambda\setminus i})}}]^2\\&\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\min_l\int\nu(d\omega)\frac{\nu(l|\omega_{\Lambda\setminus i})}{\sqrt{\nu((l+1)|\omega_{\Lambda\setminus i})}}\sum_{k=1}^{q}\int\nu(d\omega)\frac{\nu(k|\omega_{\Lambda\setminus i})}{\sqrt{\nu((k+1)|\omega_{\Lambda\setminus i})}}\\&\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\min_l\int\nu(d\omega)\nu(l|\omega_{\Lambda\setminus i})\sum_{k=1}^{q}\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{\Lambda\setminus i})}}\\&\geqslant\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\min_l\nu(l)\sum_{k=1}^{q}\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{\Lambda\setminus i})}}\geqslant\frac{\varepsilon_0}{|\Lambda|}\sum_{i\in\Lambda}\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{\Lambda\setminus i})}}\end{aligned}$$

where we used $\min_l\nu(l)\geqslant\varepsilon_0>0$.

Remark 5.7 *In fact take $\Lambda=\{0\}$ in (79) then we have $\frac{\nu(k)}{\nu(k+1)}\leqslant\tilde{K}$ and hence $\nu(k)=\frac{1}{\sum_{l=1}^q\frac{\nu(l)}{\nu(k)}}\geqslant\frac{1}{1+\tilde{K}+\tilde{K}^2+\cdots+\tilde{K}^{(q-1)}}=\frac{1}{\sum_{l=0}^{q-1}\tilde{K}^l}=:\varepsilon_0$.*

Consider $\Lambda_n:=[-n,n]^d$ and $m:=n-\lfloor n^\kappa\rfloor$ with $\kappa\in(0,1)$. Then the above inequality can be further estimated by

$$\frac{\tilde{K}}{\varepsilon_0}\geqslant\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_m}\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{\Lambda_n\setminus i})}}\geqslant\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_m}\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{(\Delta+i)\setminus i})}}$$

where Δ is the largest centered cube such that for all $i \in \Lambda_m$ we have $\Delta + i \subset \Lambda$. We used the conditional Jensen inequality in the last line. Because of translation-invariance we have $\frac{\tilde{K}}{\varepsilon_0} \geqslant \frac{|\Lambda_m|}{|\Lambda_n|} \int \nu(d\omega) \frac{1}{\sqrt{\nu(\omega_0^0|\omega_{\Delta\setminus 0})}}$. Since $\frac{|\Lambda_m|}{|\Lambda_n|} \to 1$ for $n \to \infty$ and $\Lambda_n \setminus \Lambda_m$ allows Δ to become arbitrarily large, the result of Lemma (5.6) follows. □

Proof of Lemma 5.5: Consider centered cubes Λ_n of side-length $2n+1$ and write $\partial_r(\Lambda_n) := \{i \in \Lambda : d(i, \Lambda^c) = r\}$ where $d(.,.)$ is the uniform norm. We show for $i \in \partial_r(\Lambda_n)$

$$\sup_{\omega} A(\omega, i^+) \leqslant f(r) \tag{80}$$

with $\lim_{r\to\infty} f(r) = 0$. Indeed, let us look at (69) again. We can estimate the second part by

$$\sup_{i\in\partial_r(\Lambda_n)} \sup_{\omega_i} \|\psi^{\omega_i} - \tilde{\psi}^{\omega_i}\| \leqslant \sup_{i\in\partial_r(\Lambda_n)} K \sum_{i\in A\not\subset\Lambda_n} \|\Phi_A\| \leqslant K \sum_{0\in A\not\subset\Lambda_r} \|\Phi_A\| \tag{81}$$

which goes to zero as r tends to infinity. For the first part of (69) we have

$$\begin{aligned} \sup_{\eta_{\Lambda_n^c}\omega_\Lambda} \left|\mu_{G\setminus i}[\eta_{\Lambda_n^c}\omega_{\Lambda_n\setminus i}](\tilde{\psi}^{\omega_i}) - \tilde{\mu}_{G\setminus i}[\omega_{\Lambda_n\setminus i}](\tilde{\psi}^{\omega_i})\right| &\leqslant \sup_{\omega_i} \sum_{k\neq i, l\neq i} \delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl} b_l \\ \leqslant \sup_{\omega_i} \sum_{k\in\Lambda_n\setminus i} \sum_{l\in\Lambda_n^c} \delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl} &+ \sup_{\omega_i} \sum_{k\in\Lambda_n\setminus i} \sum_{l\in\Lambda_n\setminus i} \delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl} b_l. \end{aligned} \tag{82}$$

By looking at (74) we notice for $i \in \partial_r(\Lambda_n)$

$$\sup_{\omega_i} \sum_{k\in\Lambda_n\setminus i} \sum_{l\in\Lambda_n^c} \delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl} \leqslant \tilde{C} \sum_{l\in\Lambda_n^c} e^{-\varrho(i,l)} \leqslant \tilde{C} \sum_{l\in(\Lambda_n - i)^c} e^{-\varrho(0,l)} \leqslant \tilde{C} \sum_{l\in\Lambda_r^c} e^{-\varrho(0,l)}$$

which goes to zero as r tends to infinity. Let us define $n_l := d(l, \Lambda_n^c)$ for the distance between the site l and Λ_n^c. For the other summand in (82) we have

$$\sup_{\omega_i} \sum_{k\in\Lambda_n\setminus i} \sum_{l\in\Lambda_n\setminus i} \delta_k(\tilde{\psi}^{\omega_i})\overline{D}_{kl} b_l \leqslant \sum_k \sup_{\omega_0} \delta_k(\tilde{\psi}^{\omega_0}) \sum_l \overline{D}_{0,l} \sum_{0\in A\not\subset\Lambda_{n_{i+k+l}}} \|\Phi_A\|.$$

Notice if $n_i \to \infty$ then $n_{i+l+k} \to \infty$ for every fixed l, k. In particular since $\sum_l \overline{D}_{0,l} < \infty$ we have for $n_i \to \infty$

$$\sum_l \overline{D}_{0,l} \sum_{0\in A\not\subset\Lambda_{n_{l+k+l}}} \|\Phi_A\| \to 0$$

by the dominated convergence theorem. Similarly to (72) we have

$$\sum_k \delta_k(\tilde{\psi}^{\omega_0}) \leqslant e^C \sum_k \sum_{\{i,k\}\subset A} \delta_k(\Phi_A) \leqslant e^C \sum_{0\in A} |||\Phi_A||| < \infty \tag{83}$$

and thus for $n_i \to \infty$ we can conclude again with the dominated convergence theorem

$$\sum_k \sup_{\omega_0} \delta_k(\tilde{\psi}^{\omega_0}) \sum_l \overline{D}_{0,l} \sum_{0\in A\not\subset\Lambda_{n_{l+k+l}}} \|\Phi_A\| \to 0.$$

Since $-\log(x) < \frac{1}{\sqrt{x}} < \frac{1}{x}$ on $(0,1]$, we finally have

$$\begin{aligned}
&\sum_{r=1}^{n}\sum_{i\in\partial_r(\Lambda_n)}\sup_{\omega}A(\omega,i^+)\int\nu(d\omega)|\log\frac{\nu(\omega_i^i|\omega_{\Lambda_n\setminus i})}{\nu(\omega_i|\omega_{\Lambda_n\setminus i})}| \\
&\leqslant\sum_{r=1}^{n}f(r)\sum_{i\in\partial_r(\Lambda_n)}[\int-\log\nu(\omega_i|\omega_{\Lambda_n\setminus i})\nu(d\omega)+\int-\log\nu(\omega_i^i|\omega_{\Lambda_n\setminus i})\nu(d\omega)] \\
&\leqslant\sum_{r=1}^{n}f(r)\sum_{i\in\partial_r(\Lambda_n)}[\int\nu(d\omega)\frac{1}{\nu(\omega_i|\omega_{\Lambda\setminus i})}+\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{\Lambda_n\setminus i})}}] \\
&=\sum_{r=1}^{n}f(r)\sum_{i\in\partial_r(\Lambda_n)}[\sum_{k=1}^{q}\int\nu(d\omega)\frac{\nu(k|\omega_{\Lambda\setminus i})}{\nu(k|\omega_{\Lambda\setminus i})}+\int\nu(d\omega)\frac{1}{\sqrt{\nu(\omega_i^i|\omega_{\Lambda_n\setminus i})}}] \\
&\leqslant\sum_{r=1}^{n}f(r)\sum_{i\in\partial_r(\Lambda_n)}[q+\hat{K}]=K\sum_{r=1}^{n}f(r)|\partial_r(\Lambda_n)|
\end{aligned}\tag{84}$$

where we used the last lemma in the last line. Since $f(r)\to 0$ for $r\to\infty$ there exists a $R\in\mathbb{N}$ such that for all $r\geqslant R$ we have $f(r)<\varepsilon$, hence for large n

$$\frac{1}{|\Lambda_n|}\sum_{r=1}^{n}f(r)|\partial_r(\Lambda_n)|=\frac{1}{|\Lambda_n|}(\sum_{r=R+1}^{n}f(r)|\partial_r(\Lambda_n)|+\sum_{r=1}^{R}f(r)|\partial_r(\Lambda_n)|)\leqslant\varepsilon+K\frac{|\partial(\Lambda_n)|}{|\Lambda_n|}$$

where the second summand goes to zero as n tends to infinity. □

Together we see the combined error caused by the finite-volume approximation vanishes from the point of view of difference between time derivatives of relative entropy densities, i.e. $\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta))-\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,\tilde{L}_\Lambda}|\tilde{\gamma}_\Lambda)=o(|\Lambda|)$. For a translation-invariant measure ν, that is also invariant w.r.t the joint dynamics we have

$$0=\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L+\alpha K}|\gamma'_\Lambda(\cdot|\zeta))=\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta))+\alpha\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,K}|\gamma'_\Lambda(\cdot|\zeta)).$$

Hence with the notation given in (46) we can write

$$\begin{aligned}
\alpha\sum_{i\in\Lambda}[\kappa_{\Lambda,K}(i^+)+\kappa_{\Lambda,K}(i^-)]&\leqslant\alpha\sum_{i\in\Lambda}[(\beta_{\Lambda,K}(i^+)+\beta_{\Lambda,K}(i^-))\vartheta_{\Lambda,K}(i)]+2\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|(\zeta)) \\
&\leqslant\alpha\hat{C}\sum_{i\in\Lambda}\sum_{j\notin\Lambda}\delta_j(c_K(\cdot,\cdot^i))+2\tilde{C}|\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,L}|\gamma'_\Lambda(\cdot|\zeta))-\frac{d}{dt}_{|t=0}H_\Lambda(\nu_{t,\tilde{L}_\Lambda}|\tilde{\gamma}_\Lambda)| \\
&\leqslant\alpha\hat{C}o(|\Lambda|)+2\tilde{C}\tilde{K}o(|\Lambda|)=o(|\Lambda|)
\end{aligned}$$

where in the second line we dropped the contribution of the finite-volume part since it is only negative.

But this estimate implies the single-site DLR equation and thus ν must be Gibbs. This finishes the proof of Proposition 5.2. □

References

[1] J.A. Acebrón, L.L. Bonilla, C.J. Pérez Vicente, F. Ritort, R. Spigler: The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys. , 137185 (2005)

[2] A. F. Brito, J.A. Redinz, J.A. Plascak: Two-dimensional XY and clock models studied via the dynamics generated by rough surfaces, Phys. Rev. E 81, 031130 (2010)

[3] P. Chassaing, J. Mairesse: A non-ergodic probabilistic cellular automaton with a unique invariant measure. Stochastic Process. Appl., vol. 121, Issue 11, 2474-2487 (2011)

[4] F. Collet, P. Dai Pra: The role of disorder in the dynamics of critical fluctuations of mean field models. Electron. J. Probab. 17, no. 26, (2012)

[5] N. Crawford: On Random Field Induced Ordering in the Classical XY Model, J. Stat. Phys. 142, 11-42 (2011)

[6] A.C.D. van Enter, R. Fernández, A.D. Sokal: Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, J. Stat. Phys. 72, 879-1167 (1993)

[7] A.C.D. van Enter, C. Külske, A.A. Opoku, W.M. Ruszel: Gibbs-non-Gibbs properties for n-vector lattice and mean-field models, Braz. J. Prob. Stat. 24, 226-255 (2010)

[8] A.C.D. van Enter, C. Külske, A.A. Opoku: Discrete approximations to vector spin models, J. Phys. A: Math. Theor. 44 (2011)

[9] A.C.D. van Enter, W.M. Ruszel: Gibbsianness vs. Non-Gibbsianness of time-evolved planar rotor models, Stoch. Proc. Appl. 119, 1866–1888 (2009)

[10] A.C.D. van Enter, S.B. Shlosman: Provable first-order transitions for liquid crystal and lattice gauge models with continuous symmetries, Comm. Math. Phys. 255, 2132 (2005)

[11] R. Fernández: Gibbsianness and non-Gibbsianness in lattice random fields, Les Houches, LXXXIII, (2005)

[12] J. Fröhlich, C.-E. Pfister: Spin waves, vortices, and the structure of equilibrium states in the classical XY model, Comm. Math. Phys. 89, 303327 (1983)

[13] J. Fröhlich, B. Simon, T. Spencer: Infrared bounds, phase transitions and continuous symmetry breaking, Comm. Math. Phys. 50, 7995 (1976)

[14] J. Fröhlich, T. Spencer: The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas, Comm. Math.Phys. 81, 527–602 (1981)

[15] J. Fröhlich, T. Spencer: Massless phases and symmetry restoration in Abelian Gauge symmetries and spin systems, Comm. Math. Phys. 83, 411–454 (1982)

[16] H.-O. Georgii: Gibbs measures and phase transitions, New York: De Gruyter (2011)

[17] G. Giacomin, K. Pakdaman, X. Pellegrin, C. Poquet: Transitions in active rotator systems: Invariant hyperbolic manifold approach, SIAM J. Math. Analysis 44(6), 4165-4194 (2012)

[18] O. Häggström, C. Külske: Gibbs properties of the fuzzy Potts model on trees and in mean field, Markov Proc. Rel. Fields 10 No. 3, 477-506 (2004)

[19] R. Holley: Free energy in a Markovian model of a lattice spin system, Commun. Math. Phys. 23, 8799 (1971)

[20] B. Jahnel, C. Külske: Synchronization for discrete mean-field rotators, arXiv:1308.1260 (2013)

[21] O.K. Kozlov: Gibbs description of a system of random variables, Probl. Info. Trans. 10, 258-265 (1974)

[22] C. Külske, A. Le Ny, F. Redig, Relative entropy and variational properties of generalized Gibbsian measures, Ann.Probab.32 No. 2, 1691-1726 (2004)

[23] C. Külske, A.A. Opoku: The Posterior metric and the Goodness of Gibbsianness for transforms of Gibbs measures, Electron. J. Probab. 1307–1344 (2008)

[24] T. Liggett: Interacting Particle Systems, New York: Springer-Verlag (1985)

[25] C. Maes, S.B. Shlosman: Rotating states in driven clock- and XY-models, J. Stat Phys. 144, 12381246 (2011)

[26] C.M. Newman, L.S. Schulman: Asymptotic symmetry: Enhancement and stability, Phys. Rev. B 26, 3910–3914 (1982)

[27] A.A. Opoku: On Gibbs properties of transforms of lattice and mean-field systems, Groningen thesis (2009)

[28] C.-E. Pfister: Translation invariant equilibrium states of ferromagnetic abelian lattice systems, Comm. Math. Phys. 86, 375390 (1982)

[29] W.M. Ruszel: Gibbs and non-Gibbs aspects of continuous spin models, Groningen thesis, (2010)