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ON AN INTEGRAL EQUATION FOR THE FREE-BOUNDARY OF STOCHASTIC, IRREVERSIBLE INVESTMENT PROBLEMS

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Abstract In this paper we derive a new handy integral equation for the free-boundary of infinite time horizon, continuous time, stochastic, irreversible investment problems with uncertainty modeled as a one-dimensional, regular diffusion X. The new integral equation allows to explicitly find the free-boundary $b(\cdot)$ in some so far unsolved cases, as when the operating profit function is not multiplicatively separable and X is a three-dimensional Bessel process or a CEV process. Our result follows from purely probabilistic arguments. Indeed, we first show that $b(X(t)) = l^*(t)$, with l^* the unique optional solution of a representation problem in the spirit of Bank-El Karoui [5]; then, thanks to such an identification and the fact that l^* uniquely solves a backward stochastic equation, we find the integral problem for the free-boundary.

1. Introduction. In this paper we find a new integral equation for the free-boundary $b(\cdot)$ arising in infinite time horizon, continuous time, stochastic, irreversible investment problems of the form

$$\sup_{\nu} \mathbb{E}\Big\{ \int_0^\infty e^{-rt}\, \pi(X^x(t), y+\nu(t))\, dt - \int_0^\infty e^{-rt} d\nu(t) \Big\}, \tag{1}$$

with X^x regular, one-dimensional diffusion modeling market uncertainty. The integral problem for $b(\cdot)$ is derived by means of purely probabilistic arguments. After having completely characterized the solution of the singular control problem (1) by some first order conditions for optimality and in

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terms of the *base capacity* process l^*, unique optional solution of a representation problem à la Bank-El Karoui [5], we show that $l^*(t) = b(X^x(t))$. Such an identification, the strong Markov property and a beautiful result in [17] on the joint law of a regular, one-dimensional diffusion and its running supremum both stopped at an independent exponentially distributed random time, lead to the integral equation for $b(\cdot)$

$$\psi_r(x) \int_x^{\overline{x}} \left(\int_{\underline{x}}^z \pi_c(y, b(z)) \psi_r(y)\, m(dy) \right) \frac{s(dz)}{\psi_r^2(z)} = 1. \tag{2}$$

Here $\pi_c(x, c)$ is the instantaneous marginal profit function, $\underline{x}$ and $\overline{x}$ the endpoints of the domain of X^x, r the discount factor, $\mathcal{G}$ the infinitesimal generator associated to X^x, $\psi_r(x)$ the increasing solution to the ordinary differential equation $\mathcal{G}u = ru$, and $m(dx)$ and $s(dx)$ the speed measure and the scale function measure of X^x, respectively. The rather simple structure of equation (2) allows to explicitly find the free-boundary even in some non-trivial settings; that is, for example, the case of X^x given by a three-dimensional Bessel process and Cobb-Douglas or CES (constant elasticity of substitution) profits. Such a result appears here for the first time.

The connection between irreversible investment problems under uncertainty, optimal stopping and free-boundary problems is well known in the economic and mathematical literature (cf., e.g., the monography by Dixit and Pyndick [22]). From the mathematical point of view, a problem of optimal irreversible investment may be modeled as a 'monotone follower' problem; that is, a problem in which control strategies are nondecreasing stochastic processes, not necessarily absolutely continuous with respect to the Lebesgue measure as functions of the time. Work on 'monotone follower' problems and their application to Economics started with the early papers by Karatzas, Karatzas and Shreve, El Karoui and Karatzas (cf. [29], [30] and [23]), among others. These authors studied the problem of optimally minimizing expected costs when the controlled diffusion is a Brownian motion starting at $x \in \mathbb{R}$ tracked by a nondecreasing process, i.e. the monotone follower. By relying on purely probabilistic arguments, they showed that one may associate to such a singular stochastic control problem a suitable optimal stopping problem whose value function v is related to the value function V of the original control problem by $v = \frac{\partial}{\partial x} V$. Moreover, the optimal stopping time τ^* is such that $\tau^* = \inf\{t \geq 0 : \nu^*(t) > 0\}$, with ν^* the optimal singular control. Later on, this kind of link has been established also for more complicated dynamics of the controlled diffusion; that is the case, for example, of a geometric Brownian motion [2], or of a quite general controlled Itô diffusion (see [9] and [10], among others).

Usually (see [12], [13], [32], [34], [36] and [37], among others) the optimal irreversible investment policy consists in waiting until the shadow value of installed capital is below the marginal cost of investment; on the other hand, the times at which the shadow value of installed capital equals the marginal cost of investment are optimal times to invest. It follows that from the mathematical point of view one must find the region in which it is profitable to invest immediately (the so called 'action region') and the region in which, instead, it is optimal to wait (the so called 'no-action region' or 'continuation region'). The boundary between these two regions is the free-boundary of the optimal stopping problem naturally associated to the singular control one. The optimal investment is then the least effort to keep the controlled process inside the closure of the 'continuation region'; that is, in a diffusive setting, the local time of the optimally controlled diffusion at the free-boundary.

In the last decade many papers addressed singular stochastic control problems by means of a first order conditions approach (cf., e.g., [3], [6], [14], [15], [37] and [39]), not necessarily relying on any Markovian or diffusive structure. The solution of the optimization problem is indeed related to that of a representation problem for optional processes (cf. [5]): the optimal policy consists in keeping at time t the state variable always above the lower bound $l^*(t)$, unique optional solution of a stochastic backward equation à la Bank-El Karoui [5]. Clearly such a policy acts like the optimal control of singular stochastic control problems as the original monotone follower problem (see, e.g., [29] and [30]) or, more generally, irreversible investment problems (cf. [2], [13], [32] and [34], among others). Therefore, in a diffusive setting, the signal process l^* and the free-boundary $b(\cdot)$ arising in singular stochastic control problems must be linked. In [14] the authors studied a continuous time, singular stochastic irreversible investment problem over a finite time horizon and they showed that for a production capacity given by a controlled geometric Brownian motion with deterministic, time-dependent coefficients one has $l^*(t) = b(t)$.

In this paper we aim to understand the meaning of the process l^* for the whole class of infinite time horizon, irreversible investment problems of type (1). By means of a first order conditions approach we first find the optimal investment policy in terms of the 'base capacity' process l^* (cf. [37], Definition 3.1), unique optional solution of a representation problem in the spirit of Bank-El Karoui [5]. That completely solves control problem (1). The policy to invest just enough to keep the production capacity above $l^*(t)$ turns out to be the optimal investment strategy at time t. The base capacity process defines therefore a desirable value of capacity that the controller aims to maintain. We show indeed that $l^*(t) = b(X^x(t))$, where $b(\cdot)$ is the

free-boundary of the optimal stopping problem

$$v(x,y) = \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}\pi_c(X^x(s),y)\,ds + e^{-r\tau}\Big\} \tag{3}$$

associated to (1) (cf., e.g., [2], Lemma 2). Such an identification, together with the fact that l^* uniquely solves a backward stochastic equation (see (16) below), yields a new integral equation for the free-boundary (cf. (2) and also our Theorem 3.11 below). Our equation does not rely on Itô's formula and does not require any smooth-fit property or a priori continuity of $b(\cdot)$ to be applied. In this sense it differs from that one could derive from the local time-space calculus of Peskir for semimartingales on continuous surfaces [35] (such approach has been used in the context of stochastic, irreversible investment problems in [13] and, more recently, in [19] for a reversible, stochastic investment problem). Notice that for multiplicatively separable profit functions (i.e. $\pi(x,c) = f(x)g(c)$, as in the Cobb-Douglas case) problem (3) may be easily reduced to the linearly parameter-dependent optimal stopping problem $\sup_{\tau\geq 0}\mathbb{E}\{e^{-r\tau}(u(X^x(\tau))-k)\}$ completely solved in [8] for a regular, one-dimensional diffusion X (take $u(x) := \mathbb{E}\{\int_0^\infty e^{-rs}f(X^x(s))ds\}$ and $k := 1/g'(y)$ as a real parameter to obtain by the strong Markov property $v(x,y) = g'(y)[u(x) - \sup_{\tau\geq 0}\mathbb{E}\{e^{-r\tau}(u(X^x(\tau))-k)\}]$). In [8] the free-boundary in the (x,k)-plane is obtained in terms of the infimum of an auxiliary function of one variable that can be determined from the Laplace transforms of the level passage times of X. However, our integral equation (2) is derived for very general concave profit functions and can be analytically solved even in non-separable cases, as when the profit is of CES type (see Section 4.2 below). This represents one of the main novelties of this work.

The paper is organized as follows. Section 2 introduces the optimal control problem. In Section 3 we find the optimal investment strategy, we identify the link between the 'base capacity' process and the free-boundary and we derive the integral equation for the latter one. Finally, in Section 4 we discuss some relevant examples, as the case in which the economic shock X^x is a geometric Brownian motion, a three-dimensional Bessel process or a CEV process and the profits are Cobb-Douglas or CES.

2. The Optimal Investment Problem. On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\{\mathcal{F}_t, t\geq 0\}$ the filtration generated by an exogenous Brownian motion $\{W(t), t\geq 0\}$ and augmented by $\mathbb{P}$-null sets, consider the optimal irreversible investment problem of a firm. The uncertain status of the economy is represented by the one-dimensional, time-homogeneous

diffusion $\{X^x(t), t \geq 0\}$ with state space $\mathcal{I} \subseteq \mathbb{R}$, satisfying the stochastic differential equation (SDE)

$$\begin{cases} dX^x(t) = \mu(X^x(t))dt + \sigma(X^x(t))dW(t) \\ X^x(0) = x, \end{cases} \tag{4}$$

for some Borel functions $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto (0, +\infty)$. We assume that μ and σ fulfill

$$\begin{cases} |\mu(x) - \mu(y)| \leq K|x - y|, \\ |\sigma(x) - \sigma(y)| \leq h(|x - y|), \end{cases} \tag{5}$$

for every $x, y \in \mathcal{I}$, and for some $K > 0$ and $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ strictly increasing, such that $h(0) = 0$ and

$$\int_{(0,\epsilon)} \frac{du}{h^2(u)} = \infty, \quad \text{for every } \epsilon > 0. \tag{6}$$

Hence pathwise uniqueness holds for the SDE (4) by the Yamada-Watanabe Theorem (cf. [31], Proposition 5.2.13 and Remark 5.3.3, among others); Moreover, from (5) and (6)

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)} \, dy < +\infty, \quad \text{for some } \epsilon > 0, \tag{7}$$

for every $x \in \text{int}(\mathcal{I})$. Local integrability condition (7) implies that (4) has a weak solution (up to a possible explosion time) that is unique in the sense of probability law (cf. [31], Section 5.5C). Therefore, (4) has a unique strong solution (possibly up to an explosion time) due to [31], Corollary 5.3.23. Also, it follows from (7) that the diffusion process X^x is regular in $\mathcal{I}$, i.e. X^x reaches y with positive probability starting at x, for any x and y in $\mathcal{I}$. Hence the state space $\mathcal{I}$ cannot be decomposed into smaller sets from which X^x could not exit (see, e.g., [38], Chapter VII). We shall denote by $m(dx)$, $s(dx)$, $\mathcal{G}$ and $\mathbb{P}_x$ the speed measure, the scale function measure, the infinitesimal generator and the probability measure such that $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot|X(0) = x)$, $x \in \mathcal{I}$, respectively. Notice that, under (7), $m(dx)$ and $s(dx)$ are well defined, and there always exist two linearly independent, positive solutions of the ordinary differential equation $\mathcal{G}u = \beta u$, $\beta > 0$ (cf. [25]). These functions are uniquely defined up to multiplication, if one of them is required to be strictly increasing and the other to be strictly decreasing. Finally, throughout this paper we assume that $\mathcal{I}$ is an interval with endpoints $-\infty \leq \underline{x} < \overline{x} \leq +\infty$.

The firm's manager aims to increase the production capacity

$$C^{y,\nu}(t) = y + \nu(t), \qquad C^{y,\nu}(0) = y \geq 0, \tag{8}$$

by optimally choosing an irreversible investment plan $\nu \in \mathcal{S}_o$, where

$$\mathcal{S}_o := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, \text{ nondecreasing, left-continuous, adapted such that } \nu(0) = 0, \ \mathbb{P} - \text{a.s.}\}$$

is the non empty, convex set of irreversible investment processes. The firm makes profit at rate $\pi(x, c)$ when its own capacity is c and the status of the economy is x, and the firm's manager discounts revenues and costs at positive constant rate r. As for the operating profit function $\pi : \mathcal{I} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ we make the following

Assumption 2.1.

1. *The mapping $c \mapsto \pi(x, c)$ is strictly increasing and strictly concave with continuous derivative $\pi_c(x, c) := \frac{\partial}{\partial c}\pi(x, c)$ on $\mathcal{I} \times (0, \infty)$ satisfying*
$$\lim_{c \to 0} \pi_c(x, c) = \infty, \qquad \lim_{c \to \infty} \pi_c(x, c) = \kappa,$$
for some $0 \leq \kappa < \infty$.
2. *The process $(\omega, t) \mapsto \pi_c(X^x(\omega, t), y)$ is $\mathbb{P} \otimes e^{-rt}dt$ integrable for any $y > 0$.*

Remark 2.2. *Notice that when $\kappa = 0$ we fall into the classical Inada conditions which are satisfied, for example, by a Cobb-Douglas operating profit. In the case of a CES profit function of the form $\pi(x, c) = (x^{\frac{1}{n}} + c^{\frac{1}{n}})^n$, $n \geq 2$, (see Section 4.2 below) one has instead $\kappa = 1$.*

The optimal investment problem is then

$$V(x, y) := \sup_{\nu \in \mathcal{S}_o} \mathcal{J}_{x,y}(\nu), \tag{9}$$

where the profit functional $\mathcal{J}_{x,y}(\nu)$, net of investment costs, is defined as

$$\mathcal{J}_{x,y}(\nu) = \mathbb{E}\Big\{ \int_0^\infty e^{-rt}\, \pi(X^x(t), C^{y,\nu}(t))\, dt - \int_0^\infty e^{-rt} d\nu(t) \Big\}. \tag{10}$$

Under Assumption 2.1, $\mathcal{J}_{x,y}$ is well defined but potentially infinite. Since $\pi(x, \cdot)$ is strictly concave, $\mathcal{S}_o$ is convex and $C^{y,\nu}$ is affine in ν, then, if an optimal solution ν^* to (9) does exist, it is unique. Under further minor requirements the existence of a solution to (9) is a well known result (see, e.g., [37], Theorem 2.3, for an existence proof in a not necessarily Markovian framework).

3. The Optimal Solution and the Integral Equation for the Free-Boundary. A problem similar to (9) (with depreciation in the capacity dynamics) has been completely solved by Riedel and Su in [37], or (in the case of a time-dependent, stochastic finite fuel) by Bank in [6]. By means of a first order conditions approach and without relying on any Markovian or diffusive assumption, these authors show that it is optimal to keep the production capacity always above a desirable lower value of capacity, the *base capacity* process (see [37], Definition 3.1), which is the unique optional solution of a stochastic backward equation in the spirit of Bank-El Karoui [5]. In this Section we aim to understand the meaning of the base capacity process l^* in our setting.

Following [6], [15] or [37] (among others), we start by deriving first order conditions for optimality and by finding the solution of (9) in terms of a base capacity process. Then, as a main new result, we identify the link between l^* and the free-boundary of the optimal stopping problem naturally associated to the original singular control one (9) and we determine an integral equation for the latter one.

Let $\mathcal{T}$ denote the set of all $(\mathcal{F}_t)$-stopping times $\tau \geq 0$ a.s. and notice that we may associate to $\mathcal{J}_{x,y}(\nu)$ its supergradient as the unique optional process defined by

$$\nabla \mathcal{J}_{x,y}(\nu)(\tau) := \mathbb{E}\Big\{ \int_\tau^\infty e^{-rs} \pi_c(X^x(s), C^{y,\nu}(s))\, ds \Big| \mathcal{F}_\tau \Big\} - e^{-r\tau}, \tag{11}$$

for any $\tau \in \mathcal{T}$.

Remark 3.1. *Following [3], Remark 3.1, among others, the quantity $\nabla \mathcal{J}_{x,y}(\nu)(t)$ may be interpreted as the marginal expected profit resulting from an additional infinitesimal investment at time t when the investment plan is ν. Mathematically, $\nabla \mathcal{J}_{x,y}(\nu)$ is the Riesz representation of the profit gradient at ν. More precisely, define $\nabla \mathcal{J}_{x,y}(\nu)$ as the optional projection of the product-measurable process*

$$\Phi(\omega, t) := \int_t^\infty e^{-rs}\, \pi_c(X^x(\omega, s), C^{y,\nu}(\omega, s))\, ds \, - \, e^{-rt}, \tag{12}$$

for $\omega \in \Omega$ and $t \geq 0$. Hence $\nabla \mathcal{J}_{x,y}(\nu)$ is uniquely determined up to $\mathbb{P}$-indistinguishability and it holds

$$\mathbb{E}\Big\{ \int_0^\infty \nabla \mathcal{J}_{x,y}(\nu)(t) d\nu(t) \Big\} = \mathbb{E}\Big\{ \int_0^\infty \Phi(t) d\nu(t) \Big\}$$

for all admissible ν (cf. [27], Theorem 1.33).

THEOREM 3.2. *Under Assumption 2.1, a control $\nu^* \in \mathcal{S}_o$ is the unique optimal investment strategy for problem (9) if and only if the following first order conditions for optimality*

$$\begin{cases} \nabla \mathcal{J}_{x,y}(\nu^*)(\tau) \leq 0, \quad a.s.\ \forall \tau \in \mathcal{T}, \\ \mathbb{E}\Big\{ \int_0^\infty \nabla \mathcal{J}_{x,y}(\nu^*)(t) d\nu^*(t) \Big\} = 0, \end{cases} \tag{13}$$

hold true.

PROOF. Sufficiency follows from concavity of $\pi(x, \cdot)$ (see, e.g., [6]), whereas for necessity see [39], Proposition 3.2. □

Although the first order conditions (13) completely characterize the optimal investment plan ν^*, they are not always binding and thus they cannot be directly applied to determine ν^*. Nevertheless, the optimal control may be obtained in terms of the solution of a suitable Bank-El Karoui's representation problem [5] related to (13).

For a fixed $T \leq +\infty$, the Bank-El Karoui Representation Theorem (cf. [5], Theorem 3 and Remark 2.1) states that, given

- an optional process $Y = \{Y(t), t \in [0, T]\}$ of class (D), lower-semicontinuous in expectation with $Y(T) = 0$,
- a nonnegative, atomless optional random Borel measure $\mu(\omega, dt)$ on $[0, T]$,
- $f(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ such that $f(\omega, t, \cdot) : \mathbb{R} \mapsto \mathbb{R}$ is continuous, strictly decreasing from $+\infty$ to $-\infty$, and the stochastic process $f(\cdot, \cdot, x) : \Omega \times [0, T] \mapsto \mathbb{R}$ is progressively measurable and integrable with respect to $d\mathbb{P} \otimes \mu(\omega, dt)$,

then there exists an optional process $\xi = \{\xi(t), t \in [0, T]\}$ taking values in $\mathbb{R} \cup \{-\infty\}$ such that for all $\tau \in \mathcal{T}$,

$$f(t, \sup_{\tau \leq u < t} \xi(u)) \mathbb{1}_{(\tau, T]}(t) \in L^1 \left(d\mathbb{P} \otimes \mu(\omega, dt) \right)$$

and

$$\mathbb{E}\Big\{ \int_{(\tau, T]} f(s, \sup_{\tau \leq u < s} \xi(u))\, \mu(ds) \,\Big|\, \mathcal{F}_\tau \Big\} = X(\tau). \tag{14}$$

In [5], Lemma 4.1 (see also [4], Remark 1.4-(ii)), a real valued process ξ is considered upper right-continuous on $[0, T)$ if, for each t, $\xi(t) = \limsup_{s \searrow t} \xi(s)$

with

$$\limsup_{s\searrow t}\xi(s) := \lim_{\epsilon\downarrow 0}\sup_{s\in[t,(t+\epsilon)\wedge T]}\xi(s). \tag{15}$$

Then, by [5], Theorem 1, any progressively measurable, upper right-continuous solution ξ to (14) is uniquely determined up to optional sections on $[0,T)$ in the sense that

$$\xi(\tau) = \operatorname*{ess\,inf}_{\tau<\sigma\leq T}\Xi_{\tau,\sigma}, \qquad \tau\in[0,T),$$

where $\Xi_{\tau,\sigma}$ is the unique (up to a $\mathbb{P}$-null set) $\mathcal{F}_\tau$-measurable random variable satisfying

$$\mathbb{E}\{X(\tau)-X(\sigma)|\mathcal{F}_\tau\} = \mathbb{E}\Big\{\int_{(\tau,\sigma]} f(t,\Xi_{\tau,\sigma})\,\mu(dt)\Big|\mathcal{F}_\tau\Big\}.$$

With κ as in Assumption 2.1, from now one we make the following

Assumption 3.3. *$r>\kappa$,*

and the following result holds.

Proposition 3.4. *Under Assumptions 2.1 and 3.3 there exists a unique (up to indistinguishability) strictly positive optional solution l^* to the backward stochastic equation*

$$\mathbb{E}\Big\{\int_\tau^\infty e^{-rs}\pi_c(X^x(s),\sup_{\tau\leq u<s} l^*(u))\,ds\Big|\mathcal{F}_\tau\Big\} = e^{-r\tau}, \quad \tau\in\mathcal{T}. \tag{16}$$

Moreover, the process l^* has upper right-continuous paths.

Proof. Take κ as in Assumption 2.1, apply the Bank-El Karoui Representation Theorem with $T=+\infty$ to

$$Y(\omega,t) := e^{-rt}, \qquad \mu(\omega,dt) := e^{-rt}dt \tag{17}$$

and

$$f(\omega,t,l) := \begin{cases} \pi_c\left(X(\omega,t),-\frac{1}{l}\right), & \text{for } l<0,\\ -l+\kappa\,, & \text{for } l\geq 0,\end{cases} \tag{18}$$

and define

$$\Xi^l(t) := \operatorname*{ess\,inf}_{\tau\geq t}\mathbb{E}\Big\{\int_t^\tau f(s,l)\mu(ds)+Y(\tau)\Big|\mathcal{F}_t\Big\}, \qquad l\in\mathbb{R},\ \ t\geq 0. \tag{19}$$

Then, the optional process (cf. [5], eq. (23) and Lemma 4.13)

$$\xi^*(t) = \sup\{l \in \mathbb{R} : \ \Xi^l(t) = Y(t)\}, \qquad t \geq 0, \tag{20}$$

solves the representation problem

$$\mathbb{E}\Big\{ \int_\tau^\infty e^{-rs} f(s, \sup_{\tau \leq u < s} \xi^*(u))\, ds \,\Big|\, \mathcal{F}_\tau \Big\} = e^{-r\tau}, \quad \tau \in \mathcal{T}. \tag{21}$$

If now ξ^* has upper right-continuous paths and it is strictly negative, then the strictly positive, upper right-continuous process $l^*(t) = -\frac{1}{\xi^*(t)}$ solves

$$\begin{aligned} e^{-r\tau} &= \mathbb{E}\Big\{ \int_\tau^\infty e^{-rs} \pi_c\Big(X^x(s), \frac{1}{-\sup_{\tau \leq u < s}(-\frac{1}{l^*(u)})}\Big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\} \\ &= \mathbb{E}\Big\{ \int_\tau^\infty e^{-rs} \pi_c\Big(X^x(s), \frac{1}{\inf_{\tau \leq u < s}(\frac{1}{l^*(u)})}\Big)\, ds \,\Big|\, \mathcal{F}_\tau \Big\} \\ &= \mathbb{E}\Big\{ \int_\tau^\infty e^{-rs} \pi_c(X^x(s), \sup_{\tau \leq u < s} l^*(u))\, ds \,\Big|\, \mathcal{F}_\tau \Big\}, \end{aligned}$$

for any $\tau \in \mathcal{T}$; i.e. l^* solves (16), thanks to (18) and (21). Moreover, ξ^* (and hence l^*) is unique up to optional sections by [5], Theorem 1, as it is optional and upper right-continuous. Therefore it is unique up to indistinguishability by Meyer's optional section theorem (see, e.g., [20], Theorem IV.86).

To complete the proof, we must show that $\xi^*(t)$ is indeed upper right-continuous and strictly negative. We start by proving its upper right-continuity. To accomplish that we only need to prove that ξ^* has upper semi right-continuous sample paths, i.e.

$$\limsup_{s \searrow t} \xi^*(s) \leq \xi^*(t), \tag{22}$$

since

$$\limsup_{s \searrow t} \xi^*(s) \geq \xi^*(t)$$

by definition (cf. (15)). Thanks to [21], Proposition 2, (cf. also [7], proof of Theorem 1) it suffices to show that $\lim_{n\to\infty} \xi^*(\tau_n) \leq \xi^*(\tau)$, for any sequence of stopping times $\{\tau_n\}_{n\geq 1}$ such that $\tau_n \downarrow \tau$ and for which there exists a.s. $\zeta := \lim_{n\to\infty} \xi^*(\tau_n)$. Recall Ξ^l of (19), with Y, μ and f as in (17) and (18), and also that $\xi^*(t) = \sup\{l \in \mathbb{R} : \ \Xi^l(t) = Y(t)\}$ (cf. [5], eq. (23)). Now, given $\epsilon > 0$, for $\{\tau_n\}_{n\geq 1}$ as above we have

$$\Xi^{\zeta-\epsilon}(\tau) = \lim_{n\to\infty} \Xi^{\zeta-\epsilon}(\tau_n) = Y(\tau),$$

where we have used right-continuity of $t \mapsto \Xi^l(t)$, the fact that $l \mapsto \Xi^l(t)$ is a continuous, decreasing mapping (cf. [5], Lemma 4.12) and the threshold representation of ξ^*. Hence $\zeta - \epsilon \leq \xi^*(\tau)$ and ξ^* is upper right-continuous because $\epsilon > 0$ was arbitrary. Finally, we now show that ξ^* is strictly negative. Define

$$\sigma := \inf\{t \geq 0 : \xi^*(t) \geq 0\},$$

then for $\omega \in \{\omega : \sigma(\omega) < +\infty\}$, the upper semi right-continuity of ξ^* implies $\xi^*(\sigma) \geq 0$ and thus $\sup_{\sigma \leq u < s} \xi^*(u) \geq 0$ for all $s > \sigma$. Therefore, (21) with $\tau = \sigma$, i.e.

$$e^{-r\sigma} = \mathbb{E}\Big\{ \int_\sigma^\infty e^{-rs} \Big[- \sup_{\sigma \leq u < s} \xi^*(u) + \kappa \Big] \, ds \,\Big|\, \mathcal{F}_\sigma \Big\}, \tag{23}$$

or equivalently

$$\Big(\frac{r-\kappa}{r}\Big) e^{-r\sigma} = -\mathbb{E}\Big\{ \int_\sigma^\infty e^{-rs} \sup_{\sigma \leq u < s} \xi^*(u) \, ds \,\Big|\, \mathcal{F}_\sigma \Big\},$$

is not possible for $\omega \in \{\omega : \sigma(\omega) < +\infty\}$ since the right-hand side of (23) is nonpositive, whereas the left-hand side is always strictly positive due to Assumption 3.3. It follows that $\sigma = +\infty$ a.s. and hence $\xi^*(t) < 0$ for all $t \geq 0$ a.s. □

Proposition 3.5. *Under Assumptions 2.1 and 3.3, the unique optimal irreversible investment process for problem (9) is given by*

$$\nu^*(t) = (\sup_{0 \leq s < t} l^*(s) - y) \vee 0, \quad t > 0, \qquad \nu^*(0) = 0, \tag{24}$$

where l^* is the unique optional upper right-continuous solution to (16).

Proof. See, e.g., [37], Theorem 3.2. □

In the literature on stochastic, irreversible investment problems (cf. [2], [12], [13] and [14], among others), or more generally on singular stochastic control problems of monotone follower type (see, e.g., [6], [23], [30]), it is well known that to a monotone control problem one may associate a suitable optimal stopping problem whose optimal solution, τ^*, is related to the optimal control, ν^*, by the simple relation $\tau^* = \inf\{t \geq 0 : \nu^*(t) > 0\}$. Economically, it means that a firm's manager has to decide how to optimally invest or, equivalently, when to profitably exercise the investment option. Indeed, if we introduce for any $\nu \in \mathcal{S}_o$ the level passage times

$\tau^\nu(q) := \inf\{t \geq 0 : \nu(t) > q\}$, $q \geq 0$, then for every $x \in \mathcal{I}$ and $y \geq 0$ we may write (cf., e.g., [2], Lemma 2)

$$\begin{aligned}
\mathcal{J}_{x,y}(\nu) - \mathcal{J}_{x,y}(0) &= \int_y^\infty \mathbb{E}\Big\{ \int_{\tau^\nu(z-y)}^\infty e^{-rs}\pi_c(X^x(s), z)\, ds - e^{-r\tau^\nu(z-y)} \Big\}\, dz \\
&\leq \int_y^\infty \sup_{\tau\geq 0} \mathbb{E}\Big\{ \int_\tau^\infty e^{-rs}\pi_c(X^x(s), z) ds - e^{-r\tau} \Big\} dz \\
&= \int_y^\infty \mathbb{E}\Big\{ \int_0^\infty e^{-rs}\pi_c(X^x(s), z)\, ds \Big\}\, dz \\
&\qquad - \int_y^\infty \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}\pi_c(X^x(s), z)\, ds + e^{-r\tau} \Big\}\, dz.
\end{aligned}$$

Therefore, if a process $\nu^* \in \mathcal{S}_o$ is such that its level passage times are optimal for the previous optimal stopping problems, then ν^* must be optimal for problem (9). Hence

$$v(x, y) := \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}\pi_c(X^x(s), y)\, ds + e^{-r\tau} \Big\} \tag{25}$$

is the optimal timing problem naturally associated to the optimal investment problem (9). Notice that $v(x, y) \leq 1$, for all $x \in \mathcal{I}$ and $y > 0$, and that the mapping $y \mapsto v(x, y)$ is nonincreasing for any $x \in \mathcal{I}$, because $\pi(x, \cdot)$ is strictly concave. We may now define the continuation region

$$\mathcal{C} := \{(x, y) \in \mathcal{I} \times (0, \infty) : v(x, y) < 1\} \tag{26}$$

and the stopping region

$$\mathcal{S} := \{(x, y) \in \mathcal{I} \times (0, \infty) : v(x, y) = 1\}. \tag{27}$$

Intuitively $\mathcal{S}$ is the region in which it is optimal to invest immediately, whereas $\mathcal{C}$ is the region in which it is profitable to delay the investment option. The nonincreasing property of $y \mapsto v(x, y)$ implies that $\mathcal{S}$ is below $\mathcal{C}$ and therefore that

$$b(x) := \sup\{y > 0 : \, v(x, y) = 1\}, \qquad x \in \mathcal{I}, \tag{28}$$

is the boundary between these two regions, i.e. the free-boundary.

Assumption 3.6. *The mapping $x \mapsto \pi_c(x, c)$ is nondecreasing for any $c \in (0, \infty)$.*

Notice that, if π were twice continuously differentiable, then Assumption 3.6 would mean that π is supermodular. In [37], Section 5, supermodularity of the profit function has been used to derive comparative statics results for the base capacity process l^*. It is easy to see that Cobb-Douglas and CES profit functions are supermodular on $(0,\infty)\times(0,\infty)$. Condition 3.6 has also a reasonable economic meaning (see also the discussion in [33], p. 844, in the context of a stochastic, reversible investment problem). Indeed, if the process X models the uncertain status of the market as, e.g., the price of or the demand for the produced good, then it seems natural to imagine that marginal profits are positively affected by improving market conditions.

PROPOSITION 3.7. *Under Assumptions 2.1 and 3.6, $x \mapsto v(x,y)$ is nondecreasing for any $y>0$.*

PROOF. For $y>0$, take $x_1>x_2$, $x_1,x_2\in\mathcal{I}$, let $\tau^*\in\mathcal{T}$ be optimal for (x_1,y) and $\theta\in\mathcal{T}$ be a generic stopping time. Then

$$v(x_1,y)-v(x_2,y)\geq \mathbb{E}\bigg\{\int_0^{\tau^*} e^{-rs}\pi_c(X^{0,x_1}(s),y)ds+e^{-r\tau^*} \\ -\int_0^{\theta}\pi_c(X^{0,x_2}(s),y)ds-e^{-r\theta}\bigg\},$$

for any $\theta\in\mathcal{T}$. Take now $\theta\equiv\tau^*$ to obtain

$$v(x_1,y)-v(x_2,y)\geq \mathbb{E}\bigg\{\int_0^{\tau^*} e^{-rs}[\pi_c(X^{0,x_1}(s),y)-\pi_c(X^{0,x_2}(s),y)]ds\bigg\}\geq 0,$$

since $x\mapsto X^x(t)$ is a.s. increasing for any $t\geq 0$ due to the Yamada-Watanabe comparison theorem (see, e.g., [31], Propositions 5.2.13 and 5.2.18) thanks to our conditions (5) and (6). □

COROLLARY 3.8. *Let Assumptions 2.1 and 3.6 hold. Then, the free-boundary $b(\cdot)$ between the continuation region and the stopping region is nondecreasing for any $x\in\mathcal{I}$.*

PROOF. Use the result of Proposition 3.7 and arguments similar to those in [26], proof of Proposition 2.2. □

The next Theorem gives us a new representation for the base capacity l^* in our setting.

THEOREM 3.9. *Let l^* be the unique optional solution of (16) and $b(\cdot)$ the free-boundary defined in (28). Under Assumptions 2.1, 3.3 and 3.6 one has*

$$l^*(t) = b(X^x(t)). \tag{29}$$

PROOF. First of all notice that the right-hand side of (29) is an optional process as well as l^*, being $b(\cdot)$ a Borel-measurable function (since monotone) and X optional. To prove (29) recall that $l^*(t) = -\frac{1}{\xi^*(t)}$ (cf. proof of Proposition 3.4) and that the process ξ^* admits the representation (cf. [5], formula (23) on page 1049)

$$\xi^*(t) = \sup\left\{ l < 0 : \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}\left\{ \int_t^\tau e^{-rs}\pi_c(X^x(s), -\frac{1}{l})\, ds + e^{-r\tau} \middle| \mathcal{F}_t \right\} = e^{-rt} \right\}.$$

To take care of the previous conditional expectation we adapt the arguments of [12], proof of Theorem 4.1. Let $(\overline{\Omega}, \overline{\mathbb{P}})$ be the canonical probability space where $\overline{\mathbb{P}}$ is the Wiener measure on $\overline{\Omega} := \mathcal{C}_0([0,\infty);\mathbb{R}^2)$, the space of all continuous functions from $[0,\infty)$ to $\mathbb{R}^2$ which are zero at $t=0$. We denote by $W(t,\overline{\omega}) = \overline{\omega}(t)$ the coordinate mapping on $\mathcal{C}_0([0,\infty);\mathbb{R}^2)$, with $\overline{\omega}(t) := (\overline{\omega}_1, \overline{\omega}_2)(t)$, $\overline{\omega}_1 := \{W(u), 0 \leq u \leq t\}$ and $\overline{\omega}_2 := \{W(u) - W(t), u \geq t\} = \{W'(u), u \geq 0\}$. Independence of Brownian increments induces a product measure on $\mathcal{C}_0([0,\infty);\mathbb{R}^2) = C_0([0,t];\mathbb{R}) \times C_0([t,\infty);\mathbb{R})$. Then $\tau(\overline{\omega}_1, \overline{\omega}_2) = t + \tau'_{\overline{\omega}_1}(\overline{\omega}_2)$ (where for each $\overline{\omega}_1$, $\tau'_{\overline{\omega}_1}(\cdot)$ is a stopping time with respect to $\{\mathcal{F}^{W'}_u\}_{u\geq 0}$) and we may write

$$\begin{aligned}
&\mathbb{E}\left\{ \int_t^\tau e^{-rs}\pi_c(X^x(s), -\frac{1}{l}) ds + e^{-r\tau} \middle| \mathcal{F}_t \right\} \\
&= e^{-rt}\mathbb{E}\left\{ \int_0^{\tau'_{\overline{\omega}_1}} e^{-ru}\pi_c(X^{t,X^x(t)}(u+t), -\frac{1}{l})\, du + e^{-r\tau'_{\overline{\omega}_1}} \middle| \mathcal{F}_t \right\} \\
&= e^{-rt}\mathbb{E}_{\overline{\omega}_2}\left\{ \int_0^{\tau'_{\overline{\omega}_1}} e^{-ru}\pi_c(X^{t,X^x(t)}(u+t), -\frac{1}{l})\, du + e^{-r\tau'_{\overline{\omega}_1}} \right\} \\
&= e^{-rt}\Psi(X^x(t); \tau'_{\overline{\omega}_1}),
\end{aligned}$$

for any $\overline{\omega}_1$ fixed, for some Ψ and where $\mathbb{E}_{\overline{\omega}_2}\{\cdot\}$ denotes the expectation over $\overline{\omega}_2$ or W'. But now X is a time-homogeneous diffusion, hence

$$\Psi(z; \tau'_{\overline{\omega}_1}) = \mathbb{E}\left\{ \int_0^{\tau'_{\overline{\omega}_1}} e^{-ru}\pi_c(X^{0,z}(u), -\frac{1}{l})\, du + e^{-r\tau'_{\overline{\omega}_1}} \right\},$$

for any $\overline{\omega}_1$ given and fixed, and thus

$$\xi^*(t) = \sup\Big\{ l<0 \,:\, \operatorname*{ess\,inf}_{\tau\geq t} \mathbb{E}\Big\{ \int_t^\tau e^{-rs}\pi_c(X^x(s), -\frac{1}{l})\,ds + e^{-r\tau}\Big|\mathcal{F}_t\Big\} = e^{-rt}\Big\}$$
$$= \sup\Big\{ l<0 \,:\, v(X^x(t), -\frac{1}{l}) = 1\Big\},$$

with v as in (25).

Finally, since $l^*(t) = -\frac{1}{\xi^*(t)}$ (cf. proof of Proposition 3.4), we may write for $y>0$

$$\begin{aligned} l^*(t) &= -\frac{1}{\sup\Big\{ l<0 \,:\, v(X^x(t), -\frac{1}{l}) = 1\Big\}} \\ &= \frac{1}{-\sup\Big\{ -\frac{1}{y}<0 \,:\, v(X^x(t), y) = 1\Big\}} \\ &= \frac{1}{\inf\Big\{ \frac{1}{y}>0 \,:\, v(X^x(t), y) = 1\Big\}} \\ &= \sup\Big\{ y>0 \,:\, v(X^x(t), y) = 1\Big\}, \end{aligned}$$

and then the thesis follows by (28). □

Remark 3.10. *The result of Theorem 3.9 still holds if one introduces depreciation in the production capacity dynamics as in [37]; that is, if*

$$C^{y,\nu}(t) = -\rho C^{y,\nu}(t)dt + d\nu(t), \qquad C^{y,\nu}(0) = y \geq 0,$$

for some $\rho>0$. Moreover, in this case one has (cf. also [37], Theorem 3.2)

$$\nu^*(t) = \int_{[0,t)} e^{-\rho s} d\overline{\nu}^*(s), \qquad \textit{with} \qquad \overline{\nu}^*(t) = \sup_{0\leq s<t}\Big(\frac{b(X^x(s)) - ye^{-\rho s}}{e^{-\rho s}}\Big) \vee 0$$

and $\overline{\nu}^*(0) = 0$.

Theorem 3.9 clarifies why in the literature (cf. [3], [15] or [37], among others) one usually refers to l^* as a 'desirable value of capacity' that the controller aims to maintain in a 'minimal way'. Indeed, as in the classical monotone follower problems (see, e.g., [23] and [30]), the optimal investment policy ν^* (cf. Proposition 3.5) is the solution of a Skorohod problem being

the least effort needed to reflect the production capacity at the moving (random) boundary $l^*(t) = b(X^x(t))$; that is,

$$\nu^*(t) = \sup_{0\le s<t} (b(X^x(s)) - y) \vee 0, \quad t > 0, \qquad \nu^*(0) = 0.$$

The result of Theorem 3.9 resembles those of [4] and [8] in which the connection between the solution of a Bank-El Karoui representation problem and a suitable exercising boundary for parameter-dependent optimal stopping problems has been pointed out. In particular in [8] the authors consider the optimal stopping problem $\sup_{\tau\ge 0} \mathbb{E}\{e^{-r\tau}(u(X^x(\tau)) - k)\}$ where X is a regular, one-dimensional diffusion and k a real parameter which affects linearly the gain function. Under some additional uniform integrability conditions on X, they can show that the solution K of an associated representation problem is given by $K(t) = \gamma(X(t))$, with $\gamma(\cdot)$ the free-boundary on the (x, k)-plane (see also [24], Sections 4 and 5). Moreover, $\gamma(\cdot)$ is characterized in terms of the infimum of an auxiliary function of one variable that can be determined from the Laplace transforms of level passage times for X.

When our marginal profit π_c is multiplicatively separable (i.e. $\pi_c(x, c) = f(x)g(c)$, as in the Cobb-Douglas case) it is not hard to see that our optimal stopping problem (25) may be reduced to that studied in [8] (set $u(x) := \mathbb{E}\{\int_0^\infty e^{-rs} f(X^x(s))ds\}$ and $k := 1/g(y)$ to obtain by the strong Markov property $v(x, y) = g(y)[u(x) - \sup_{\tau\ge 0} \mathbb{E}\{e^{-r\tau}(u(X^x(\tau)) - k)\}]$). However, we shall start from the identification (29) to find, by (16) and by purely probabilistic arguments, an integral equation for the free-boundary (cf. Theorem 3.11 below) which holds for a very general class of concave profit functions not necessarily multiplicatively separable. That is, for example, the case of a CES (constant elasticity of substitution) profit that we will discuss in Section 4.

THEOREM 3.11. *Let Assumptions 2.1, 3.3 and 3.6 hold. Denote by $\mathcal{G}$ the infinitesimal generator associated to X^x, and by $\psi_r(x)$ the increasing solution to the equation $\mathcal{G}u = ru$. Moreover, let $m(dx)$ and $s(dx)$ be the speed measure and the scale function measure, respectively, associated to the diffusion X^x. Then, the free-boundary $b(\cdot)$ between the continuation region and the stopping region is the unique positive nondecreasing solution to the integral equation*

$$\psi_r(x) \int_x^{\overline{x}} \left(\int_{\underline{x}}^z \pi_c(y, b(z))\psi_r(y)m(dy) \right) \frac{s(dz)}{\psi_r^2(z)} = 1. \tag{30}$$

PROOF. Since l^* uniquely solves (16) and $l^*(t) = b(X^x(t))$ (cf. Theorem 3.9), then $b(\cdot)$ satisfies

$$\begin{aligned} r &= \mathbb{E}\Big\{ \int_\tau^\infty re^{-r(s-\tau)}\pi_c(X^x(s), \sup_{\tau\le u<s} b(X^x(u)))ds \Big| \mathcal{F}_\tau \Big\} \\ &= \mathbb{E}\Big\{ \int_0^\infty re^{-rt}\pi_c(X^x(t+\tau), b(\sup_{0\le u<t} X^x(u+\tau)))dt \Big| \mathcal{F}_\tau \Big\}, \end{aligned} \tag{31}$$

for any $\tau \in \mathcal{T}$, where in the second equality we have used the fact that $b(\cdot)$ is nondecreasing by Corollary 3.8. Now, by the strong Markov property, (31) amounts to find $b(\cdot)$ such that

$$\mathbb{E}_x\Big\{ \int_0^\infty re^{-rt}\pi_c(X(t), b(\sup_{0\le u<t} X(u)))dt \Big\} = r;$$

that is, such that

$$\mathbb{E}_x\Big\{ \pi_c(X(\tau_r), b(M(\tau_r))) \Big\} = r,$$

where $M(t) := \sup_{0\le s\le t} X(s)$ and τ_r denotes an independent exponentially distributed random time with parameter r. Integral equation (30) now follows since for a one-dimensional regular diffusion X (cf. [17], page 185) one has

$$\mathbb{P}_x(X(\tau_r) \in dy, M(\tau_r) \in dz) = r\frac{\psi_r(x)\psi_r(y)}{\psi_r^2(z)} m(dy)s(dz), \qquad y \le z, \;\; x \le z.$$

Finally, uniqueness of a positive, nondecreasing $b(\cdot)$ satisfying (30) can be proved arguing by contradiction as follows. Assume there exist two positive, nondecreasing solutions $b_1(\cdot)$ and $b_2(\cdot)$ of (30) such that $b_1(x_o) \neq b_2(x_o)$ for some $x_o \in \mathcal{I}$. Then, proceeding backward from (30) one finds two positive, optional processes $l_1^*(t) := b_1(X^x(t))$ and $l_2^*(t) := b_2(X^x(t))$ both solving (16). By Proposition 3.4 we should have l_1^* and l_2^* indistinguishable. But now X is regular and thus the set $\{\omega \in \Omega : \tau_{x_o}(\omega) < +\infty\}$, with $\tau_{x_o} := \inf\{t \ge 0 : X^x(t) = x_o\}$, has positive probability for any x in the interior of $\mathcal{I}$. It follows that l_1^* and l_2^* are not indistinguishable and such a contradiction concludes the proof. □

Notice that if one deals with an optimal stopping problem of type (25), the common approach consists in writing down the associated free-boundary problem for the value function v and the boundary b and try to solve it on a case by case basis. Alternatively, one could rely on an integral representation for the value function and the free-boundary which follow from the local

time-space calculus for semimartingales on continuous surfaces of Peskir [35]. The latter, indeed, may be seen as the probabilistic counterpart of the free-boundary problem. However, for both these two approaches one needs regularity of v, smooth-fit property or a priori continuity of b.

Our integral equation (30), instead, follows immediately from the backward equation (16) for $l^*(t) = b(X^x(t))$, thanks to (29) and the strong Markov property of X. Therefore, it does not require any regularity of the value function, smooth-fit property or a priori continuity of $b(\cdot)$ itself to be applied. It thus represents an extremely useful tool to determine the free-boundary of the whole class of infinite time horizon, singular stochastic irreversible investment problems of type (9). As we shall see in the next Section, equation (30) may be analytically solved even in some non-trivial cases.

4. Explicit Results. In this Section we aim to explicitly solve the integral equation (30) when the economic shock X^x is a Geometric Brownian motion, a three-dimensional Bessel process and a CEV (constant elasticity of volatility) process. We shall find the free-boundary $b(\cdot)$ of the optimal stopping problem (25) for Cobb-Douglas and CES (constant elasticity of substitution) operating profit functions; that is, for $\pi(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ with $\alpha, \beta \in (0,1)$, and $\pi(x,c) = (x^{\frac{1}{n}} + c^{\frac{1}{n}})^n$, $n \geq 2$, respectively.

To the best of our knowledge, this is the first time that the free-boundary of a singular stochastic control problem of type (9) with a CES profit function is explicitly determined for underlying given by a three-dimensional Bessel process or by a CEV process.

4.1. *The Case of a Cobb-Douglas Operating Profit.* Throughout this Section assume that the operating profit function is of Cobb-Douglas type; that is, $\pi(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ for $\alpha, \beta \in (0,1)$. According to Assumption 3.3 we take $r > 0$.

4.1.1. *Geometric Brownian Motion.* Let $X^x(t) = xe^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$, $x > 0$, with $\sigma^2 > 0$ and $\mu \in \mathbb{R}$. If we denote by $\delta := \frac{\mu}{\sigma^2} - \frac{1}{2}$, then it is well known (cf., e.g., [11]) that

$$m(dx) = \frac{2}{\sigma^2} x^{2\delta - 1} dx$$

and

$$s(dx) := \begin{cases} x^{-2\delta-1}dx, & \delta \neq 0, \\ \\ \frac{1}{x}dx, & \delta = 0. \end{cases}$$

Finally, the ordinary differential equation $\mathcal{G}u = ru$, i.e. $\frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) = ru$, admits the increasing solution

$$\psi_r(x) = x^{\gamma_1},$$

where γ_1 is the positive root of the equation $\frac{1}{2}\sigma^2\gamma(\gamma-1) + \mu\gamma = r$.

PROPOSITION 4.1. *For any $\delta \in \mathbb{R}$ and $x > 0$, one has*

$$b(x) = K_\delta x^{\frac{\alpha}{1-\beta}}, \tag{32}$$

with $K_\delta := \left[\sigma^2\gamma_1(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right)\right]^{-\frac{1}{1-\beta}}$.

PROOF. Let us start with the case $\delta \neq 0$. For any $x > 0$ by (30) we have

$$\int_x^\infty \left(\int_0^z y^{\alpha+\gamma_1+2\delta-1}dy\right) b^{\beta-1}(z) z^{-2\delta-1-2\gamma_1}dz = x^{-\gamma_1}\left(\frac{\alpha+\beta}{2\beta}\right)\sigma^2;$$

that is,

$$\int_x^\infty b^{\beta-1}(z) z^{\alpha-\gamma-1}dz = \sigma^2(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right)x^{-\gamma_1}.$$

Take now $b(z) = (A_\delta z)^{\frac{\alpha}{1-\beta}}$, for some constant A_δ, to obtain

$$A_\delta^{-\alpha}\int_x^\infty z^{-\gamma_1-1}dz = \frac{A_\delta^{-\alpha}}{\gamma_1}x^{-\gamma_1} = \sigma^2(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right)x^{-\gamma_1},$$

which is satisfied by $A_\delta := \left[\sigma^2\gamma_1(\alpha+\gamma_1+2\delta)\left(\frac{\alpha+\beta}{2\beta}\right)\right]^{-\frac{1}{\alpha}}$. Hence $b(x) = K_\delta x^{\frac{\alpha}{1-\beta}}$ with $K_\delta := A_\delta^{\frac{\alpha}{1-\beta}}$. Similar calculations also apply to the case $\delta = 0$ to have $b(x) = K_0 x^{\frac{\alpha}{1-\beta}}$. □

4.1.2. *Three-Dimensional Bessel Process.* Let now X^x be a three-dimensional Bessel process; that is, the strong solution of

$$dX^x(t) = \frac{1}{X^x(t)}dt + dW(t), \qquad X^x(0) = x > 0.$$

It is a diffusion with state space $(0,\infty)$, generator $\mathcal{G} := \frac{1}{2}\frac{d}{dx^2} + \frac{1}{x}\frac{d}{dx}$ and scale and speed measures given by $s(dx) = x^{-2}dx$ and $m(dx) = 2x^2dx$, respectively (cf. [28], Chapter VI). Further, since $X^x(t)$ may be characterized as a killed Brownian motion at zero, conditioned never to hit zero, the three-dimensional Bessel process may be viewed as an excessive transform of a killed Brownian motion with excessive function $h(x) = x$; that is, the scale function of the Brownian motion. Therefore $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$ (cf. [28], Chapter VI or [17], Section 3.2, among others).

REMARK 4.2. *Notice that the first of (5) is not satisfied in this case. However, that condition is not necessary to obtain our Theorem 3.11. In fact, we only need that X is a diffusion process for which a comparison result holds true. One can see that this fact is satisfied by a three-dimensional Bessel process since it is the squared root of a squared Bessel process for which a comparison result (cf. the Yamada-Watanabe Theorem, e.g., [31], Propositions 5.2.13 and 5.2.18) holds.*

The following result holds.

PROPOSITION 4.3. *For any $x > 0$ one has*

$$b(x) = \left[\left(\frac{\alpha+\beta}{2\beta}\right) x^2 \frac{\psi_r'(x)}{g(x)}\right]^{-\frac{1}{1-\beta}}, \tag{33}$$

where $\psi_r'(x)$ denotes the first derivative of the increasing function $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$, and $g(x) := \int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy$.

PROOF. From integral equation (30) we may write

$$\begin{aligned}\left(\frac{\alpha+\beta}{2\beta}\right)\frac{x}{\sinh(\sqrt{2r}x)} &= \int_x^\infty \left(\int_0^z y^{\alpha+1}\sinh(\sqrt{2r}y)dy\right)\frac{b^{\beta-1}(z)}{\sinh^2(\sqrt{2r}z)}dz \\ &= \int_x^\infty g(z)\frac{b^{\beta-1}(z)}{\sinh^2(\sqrt{2r}z)}dz,\end{aligned}$$

with $g(x) := \int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy$. By differentiating, one obtains

$$\begin{aligned}b^{\beta-1}(x) &= \left(\frac{\alpha+\beta}{2\beta}\right)\frac{[x\sqrt{2r}\cosh(\sqrt{2r}x) - \sinh(\sqrt{2r}x)]}{g(x)} \\ &= \left(\frac{\alpha+\beta}{2\beta}\right) x^2\frac{\psi_r'(x)}{g(x)},\end{aligned} \tag{34}$$

i.e.

$$b(x) = \left[\left(\frac{\alpha+\beta}{2\beta}\right) x^2 \frac{\psi_r'(x)}{g(x)}\right]^{-\frac{1}{1-\beta}}.$$

Notice that $b(\cdot)$ is positive since $\psi_r(\cdot)$ is increasing and $g(\cdot)$ is positive.

To complete the proof it now suffices to check that the mapping $x \mapsto b(x)$ is actually nondecreasing as suggested by Proposition 3.8; that is, $x \mapsto b^{\beta-1}(x)$ is nonincreasing. From (34) we have

$$\begin{aligned}\frac{d}{dx}b^{\beta-1}(x) &= \left(\frac{\alpha+\beta}{2\beta g^2(x)}\right)\Big[g(x)(2x\psi_r'(x) + x^2\psi_r''(x)) - g'(x)x^2\psi_r'(x)\Big] \\ &= \left(\frac{x^2(\alpha+\beta)}{2\beta g^2(x)}\right)\Big[2rg(x)\psi_r(x) - g'(x)\psi_r'(x)\Big],\end{aligned} \tag{35}$$

since $\psi_r(x)$ solves $\frac{1}{2}\psi_r''(x) + \frac{1}{x}\psi_r'(x) = r\psi_r(x)$. Recall now that $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$, $g'(x) = x^{\alpha+1}\sinh(\sqrt{2r}x)$ and notice that, by an integration by parts,

$$g(x) = \int_0^x y^{\alpha+1}\sinh(\sqrt{2r}y)dy = \frac{1}{\sqrt{2r}}x^{\alpha+1}\cosh(\sqrt{2r}x) - \frac{\alpha+1}{\sqrt{2r}}I(x),$$

with $I(x) := \int_0^x y^\alpha \cosh(\sqrt{2r}y)dy$. Therefore from (35) we may write

$$\begin{aligned}\frac{d}{dx}b^{\beta-1}(x) &= \left(\frac{x^2(\alpha+\beta)}{2\beta g^2(x)}\right)\frac{\sinh(\sqrt{2r}x)}{x}\Big[-(\alpha+1)\sqrt{2r}I(x) + \sinh(\sqrt{2r}x)x^\alpha\Big] \\ &=: \left(\frac{x^2(\alpha+\beta)}{2\beta g^2(x)}\right)\frac{\sinh(\sqrt{2r}x)}{x}T(x).\end{aligned}$$

Since $T(0) = 0$ and $T'(x) = \alpha x^{\alpha-1}[\sinh(\sqrt{2r}x) - x\sqrt{2r}\cosh(\sqrt{2r}x)] = -\alpha x^{\alpha+1}\psi_r'(x) < 0$, being $x \mapsto \psi_r(x)$ increasing, it follows that $x \mapsto T(x)$ is negative for any $x > 0$. The decreasing property of $x \mapsto b^{\beta-1}(x)$ is therefore proved. □

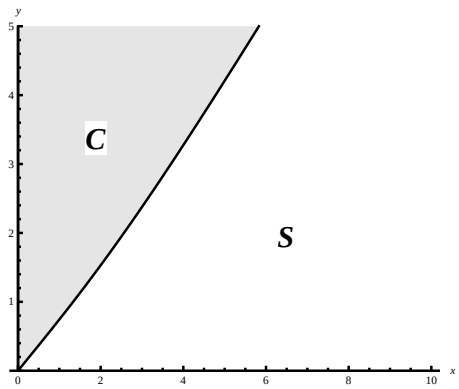

FIGURE 1. *A computer drawing of the free-boundary (33) when $r = \frac{1}{2}$ in the case of a three-dimensional Bessel process and a Cobb-Douglas profit (with $\alpha = \beta = \frac{1}{2}$). The grey area in the figure denotes the continuation (no-action) region, whereas the white one denotes the stopping (action) region.*

4.1.3. *CEV Process.* Let now the diffusion X^x be of CEV (Constant Elasticity of Variance) type; that is,

$$dX^x(t) = rX^x(t)dt + \sigma(X^x)^{1-\gamma}(t)dW(t), \qquad X^x(0) = x > 0, \tag{36}$$

for some $r > 0$, $\sigma > 0$ and $\gamma \in (0, 1/2]$. CEV process was introduced in the financial literature by John Cox in 1975 [16] in order to capture the stylized fact of a negative link between equity volatility and equity price (the so called 'leverage effect'). In this case we have

$$m(dx) = \frac{2}{\sigma^2 x^{2(1-\gamma)}} e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}} dx, \qquad s(dx) = e^{-\frac{r}{\gamma\sigma^2}x^{2\gamma}} dx,$$

and $\psi_r(x) = x$ (cf., e.g., [18], Section 6.2).

PROPOSITION 4.4. *For any $x > 0$ one has*

$$b(x) = \left[\frac{2\beta}{\sigma^2(\alpha+\beta)} g(x) e^{-\frac{r}{\gamma\sigma^2}x^{2\gamma}}\right]^{\frac{1}{1-\beta}}, \tag{37}$$

with $g(x) := \int_0^x y^{2\gamma+\alpha-1} e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}} dy$.

PROOF. From (30) one has

$$\int_x^\infty \left(\int_0^z y^{2\gamma+\alpha-1} e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}} dy\right) \frac{b^{\beta-1}(z)}{z^2} e^{-\frac{r}{\gamma\sigma^2}z^{2\gamma}} dz = \frac{\sigma^2}{x}\left(\frac{\alpha+\beta}{2\beta}\right),$$

i.e.

$$\int_x^\infty g(z) \frac{b^{\beta-1}(z)}{z^2} e^{-\frac{r}{\gamma\sigma^2}z^{2\gamma}} dz = \frac{\sigma^2}{x}\left(\frac{\alpha+\beta}{2\beta}\right),$$

with $g(x) := \int_0^x y^{2\gamma+\alpha-1} e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}} dy$. Take now

$$b^{\beta-1}(x) = \frac{\sigma^2}{g(x)}\left(\frac{\alpha+\beta}{2\beta}\right) e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}}$$

to obtain the sought result.

To conclude the proof we shall now show that $b(x)$ as in (37) is non-decreasing, or, equivalently, that $x \mapsto b^{\beta-1}(x)$ is nonincreasing. Indeed we have

$$\begin{aligned}\frac{d}{dx} b^{\beta-1}(x) &= \frac{\sigma^2}{g^2(x)}\left(\frac{\alpha+\beta}{2\beta}\right) x^{2\gamma-1} e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}} \left[\frac{2r}{\sigma^2} g(x) - x^\alpha e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}}\right] \\ &= -\frac{\alpha\sigma^2}{g^2(x)}\left(\frac{\alpha+\beta}{2\beta}\right) x^{2\gamma-1} e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}} \int_0^x y^{\alpha-1} e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}} dy < 0,\end{aligned}$$

being $g(x) = \frac{\sigma^2}{2r}[e^{\frac{r}{\gamma\sigma^2}x^{2\gamma}} x^\alpha - \alpha \int_0^x y^{\alpha-1} e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}} dy]$, thanks to an integration by parts. □

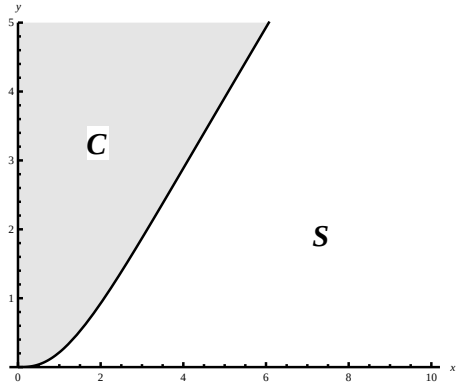

FIGURE 2. *A computer drawing of the free-boundary (37) in the case of a CEV process (with $\gamma = r = \frac{1}{2}$ and $\sigma = 1$) and a Cobb-Douglas profit (with $\alpha = \beta = \frac{1}{2}$). The grey area in the figure denotes the continuation (no-action) region, whereas the white one denotes the stopping (action) region.*

4.2. *The Case of a CES Operating Profit.* In this Section we consider a non-separable operating profit of CES type; that is, $\pi(x,c) = (x^{\frac{1}{n}} + c^{\frac{1}{n}})^n$, $(x,c) \in (0,\infty) \times (0,\infty)$ and $n \geq 2$. Moreover, as in the previous Section, we take X given by a geometric Brownian motion, a three-dimensional Bessel process and a CEV process, respectively.

Notice that CES operating profit does satisfy the first part of Assumption 2.1 with $\kappa = 1$ since $\lim_{c\to\infty} \pi_c(x,c) = \lim_{c\to\infty}[1 + (\frac{x}{c})^{\frac{1}{n}}]^{n-1} = 1$. Then, according to Assumption 3.3 here we take $r > 1$.

Due to our identification $l^*(t) = b(X^x(t))$ (cf. (29)) we expect that Assumption 3.3 might play a role also in the optimal stopping problem (25). Having $r > 1$ guarantees in fact that optimal stopping problem (25) has a non empty continuation region. Indeed, if the economic shock X is a positive diffusion (as we will consider in the examples below) one has

$$\begin{aligned}
1 &\geq \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}\pi_c(X^x(s),y)ds + e^{-r\tau} \Big\} \\
&= \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}\Big[\Big(\frac{X^x(s)}{y}\Big)^{1/n} + 1\Big]^{n-1} ds + e^{-r\tau} \Big\} \\
&\geq \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}\Big[(n-1)\Big(\frac{X^x(s)}{y}\Big)^{1/n} + 1\Big] ds + e^{-r\tau} \Big\} \\
&= \frac{1}{r} + \inf_{\tau\geq 0} \mathbb{E}\Big\{ \int_0^\tau e^{-rs}(n-1)\Big(\frac{X^x(s)}{y}\Big)^{1/n} ds + \Big(\frac{r-1}{r}\Big)e^{-r\tau} \Big\},
\end{aligned}$$

where we have used the generalized Bernoulli inequality for the third step.

If now $r \leq 1$ the two terms in the last expected value above are increasing functions of τ, and therefore it is always optimal to stop immediately; i.e. $\tau^* = 0$ for any $(x, y) \in (0, \infty) \times (0, \infty)$, and thus $\mathcal{C} = \emptyset$.

Before starting with our examples we also need a preliminary Lemma that will be useful in the following.

LEMMA 4.5. *Take $n \geq 2$ and positive continuously differentiable functions $\{\alpha_{k,n}\}_{1 \leq k \leq n-1}$ and h on $(0, \infty)$. Then, for any $x > 0$, the polynomial equation of order $n-1$ for the unknown $f_n(x)$*

$$\sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}(x) f_n^k(x) - h(x) = 0, \tag{38}$$

admits a unique positive solution. Moreover, $f_n(\cdot)$ is continuously differentiable on $(0, \infty)$.

PROOF. The existence of a unique positive solution to (38) follows from a straightforward application of Descartes' rule of signs. To show that such a solution $f_n(\cdot)$, $n \geq 2$, is continuously differentiable on $(0, \infty)$, define the function $\Phi_n : (0, \infty) \times (0, \infty) \mapsto \mathbb{R}$ by

$$\Phi_n(x, y) := \sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}(x) y^k - h(x), \qquad n \geq 2. \tag{39}$$

By the first part of this proof we already know that for any arbitrary but fixed $x_o > 0$ there exists a unique positive $f_n(x_o)$ such that $\Phi_n(x_o, f_n(x_o)) = 0$. Moreover, $\frac{\partial \Phi_n}{\partial y}(x_o, f_n(x_o)) > 0$ because $\alpha_{k,n}$ and f_n are positive. Then f_n is continuously differentiable in a suitable neighborhood of x_o, by the implicit function Theorem. Since $x_o > 0$ was arbitrary, it follows that f_n is continuously differentiable on $(0, \infty)$. □

4.2.1. *Geometric Brownian Motion.* As in Section 4.1.1, let X^x be a geometric Brownian motion with drift $\mu \in \mathbb{R}$ and volatilty $\sigma > 0$.

PROPOSITION 4.6. *Define $\delta := \frac{\mu}{\sigma^2} - \frac{1}{2}$, γ_1 as the positive root of the equation $\frac{1}{2}\sigma^2 \gamma(\gamma - 1) + \mu\gamma = r$ and $\theta := \gamma_1 + 2\delta$. Then, for any $x > 0$ and $n \geq 2$ one has*

$$b_n(x) = \left(\frac{1}{C_n}\right)^n x, \tag{40}$$

where C_n is the unique positive constant solving

$$F_{2,1}(-(n-1), n\theta; n\theta + 1; -C_n) = r, \tag{41}$$

with $F_{2,1}$ the ordinary hypergeometric function (see, e.g., [1], Chapter 15, for details).

PROOF. From (30) one has

$$\begin{aligned}\frac{\sigma^2}{2}x^{-\gamma_1} &= \int_x^\infty \left[\int_0^z (y^{\frac{1}{n}} + b_n^{\frac{1}{n}}(z))^{n-1} y^{\theta-1} dy\right] b_n^{\frac{1}{n}-1}(z) z^{-\theta-\gamma_1-1} dz \\ &= \int_x^\infty \left[\int_0^{z/b_n(z)} (1+t^{\frac{1}{n}})^{n-1} t^{\theta-1} dt\right] b_n^{\theta}(z) z^{-\theta-\gamma_1-1} dz \\ &= \int_x^\infty \left[\int_0^{g_n(z)} (1+t^{\frac{1}{n}})^{n-1} t^{\theta-1} dt\right] g_n^{-\theta}(z) z^{-\gamma_1-1} dz,\end{aligned}$$

where we have performed the change of variable $t := y/b_n(z)$ and we have defined $g_n(z) := z/b_n(z)$ and $\theta := \gamma_1 + 2\delta$. But

$$\int_0^{g_n(z)} (1+t^{\frac{1}{n}})^{n-1} t^{\theta-1} dt = \frac{1}{\theta} g_n^{\theta}(z) F_{2,1}(-(n-1), n\theta; n\theta+1, -g_n^{\frac{1}{n}}(z)),$$

where $F_{2,1}$ is the ordinary hypergeometric function (cf. [1], Chapter 15), and therefore

$$\int_x^\infty F_{2,1}(-(n-1), n\theta; n\theta+1, -g_n^{\frac{1}{n}}(z)) \gamma_1 z^{-\gamma_1-1} dz = \frac{\gamma_1 \theta \sigma^2}{2} x^{-\gamma_1}. \tag{42}$$

Take now $g_n^{\frac{1}{n}}(z) = C_n$ to be constant and notice that $\frac{\gamma_1\theta\sigma^2}{2} = r$ to obtain that (42) is satisfied for C_n solving

$$F_{2,1}(-(n-1), n\theta; n\theta+1, -C_n) = r. \tag{43}$$

According to [1], Chapter 15, equation 15.4.1 at page 561, it is easy to verify that (43) is equivalent to the polynomial equation of order $n-1$

$$\sum_{k=1}^{n-1} \frac{(1-n)_k (n\theta)_k}{(1+n\theta)_k} (-1)^k \frac{C_n^k}{k!} - (r-1) = 0, \tag{44}$$

where $(\cdot)_k$ denotes the Pochhammer symbol. Notice that all the coefficients of the polynomial in (44) are positive except for that of order zero which is instead negative since $r > 1$ (cf. Assumption 3.3); then, (44) admits a unique positive solution by Descartes' rule of signs and (40) is finally obtained recalling that $g_n(z) = z/b_n(z)$. □

4.2.2. *Three-Dimensional Bessel Process.* As in Section 4.1.2 let now X^x be a three-dimensional Bessel process.

PROPOSITION 4.7. *For any $0 \le k \le n-1$ and $n \ge 2$ define the functions $\alpha_{k,n} : (0,\infty) \mapsto (0,\infty)$ by*

$$\alpha_{k,n}(x) := \int_0^x y^{1+\frac{k}{n}} \sinh(\sqrt{2r}y)dy. \tag{45}$$

Then, for any $x > 0$ and $n \ge 2$ the free-boundary $b_n(\cdot)$ is given by

$$b_n(x) = \Big(\frac{1}{f_n(x)}\Big)^n, \tag{46}$$

where $f_n(x)$ is the unique positive solution of the polynomial equation of order $n-1$

$$\sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}(x) f_n^k(x) = (r-1)\alpha_{0,n}(x), \quad x > 0. \tag{47}$$

Moreover, the mapping $x \mapsto b_n(x)$ is nondecreasing.

PROOF. The integral equation (30) takes the form

$$\begin{aligned}
\frac{1}{2}\frac{x}{\sinh(\sqrt{2r}x)} &= \int_x^\infty \Big[\int_0^z \Big(1 + \Big(\frac{y}{b_n(z)}\Big)^{\frac{1}{n}}\Big)^{n-1} y \sinh(\sqrt{2r}y)dy\Big] \frac{dz}{\sinh^2(\sqrt{2r}z)} \\
&= \int_x^\infty \sum_{k=0}^{n-1} \binom{n-1}{k} \Big[\int_0^z \Big(\frac{y}{b_n(z)}\Big)^{\frac{k}{n}} y \sinh(\sqrt{2r}y)dy\Big] \frac{dz}{\sinh^2(\sqrt{2r}z)} \\
&= \int_x^\infty \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha_{k,n}(z) f_n^k(z) \frac{dz}{\sinh^2(\sqrt{2r}z)}, \qquad (48)
\end{aligned}$$

where we have used the binomial expansion and the definitions (45) and (46). Since $\frac{1}{2}\frac{x}{\sinh(\sqrt{2r}x)} = r\int_x^\infty \frac{\alpha_{0,n}(z)dz}{\sinh^2(\sqrt{2r}z)}$, one easily has from (48)

$$\int_x^\infty \sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}(z) f_n^k(z) \frac{dz}{\sinh^2(\sqrt{2r}z)} = (r-1)\int_x^\infty \frac{\alpha_{0,n}(z)dz}{\sinh^2(\sqrt{2r}z)},$$

for any $x > 0$, and thus by differentiating

$$\sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}(x) f_n^k(x) = (r-1)\alpha_{0,n}(x), \tag{49}$$

for a.e. $x > 0$. It now remains to show that (49) actually admits at most one positive solution and that (49) holds for every $x > 0$. Existence of a unique positive solution is guaranteed by Lemma 4.5 with $h(x) := (r-1)\alpha_{0,n}(x)$, which is positive due to Assumption 3.3. Moreover, Lemma 4.5 also ensures that $f_n(\cdot)$ is continuously differentiable on $(0,\infty)$ and therefore (49) actually holds for every $x > 0$.

As for the nondecreasing property of $x \mapsto b_n(x)$, $n \geq 2$, because of (46) it suffices to prove that $x \mapsto f_n(x)$, $n \geq 2$, is nonincreasing. First of all it is not hard to see that $x \mapsto f_2(x)$ is nonincreasing by direct calculations. To prove that any f_n, $n > 2$, is nonincreasing as well we can proceed as follows. Thanks to Lemma 4.5 we can differentiate (49) to obtain

$$\frac{f_n'(x)}{f_n(x)} \sum_{k=1}^{n-1} \binom{n-1}{k} k\alpha_{k,n}(x) f_n^k(x) \;=\; (r-1)\alpha_{0,n}^{'}(x) - \sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}^{'}(x) f_n^k(x),$$

from which

$$\frac{f_n'(x)}{f_n(x)} \sum_{k=1}^{n-1} \binom{n-1}{k} k\alpha_{k,n}(x) f_n^k(x) \leq (r-1)\alpha_{0,n}^{'}(x) - \alpha_{1,2}^{'}(x) f_2(x), \tag{50}$$

because the coefficients $\alpha_{k,n}$ are nondecreasing and f_n is positive. Noticing that $f_2(x) = (r-1)\alpha_{0,n}(x)/\alpha_{1,2}(x)$ and plugging it into (50) we find

$$\frac{f_n'(x)}{f_n(x)} \sum_{k=1}^{n-1} \binom{n-1}{k} k\alpha_{k,n}(x) f_n^k(x) \leq (r-1)\Big[\alpha_{0,n}^{'}(x) - \frac{\alpha_{0,n}(x)\alpha_{1,2}^{'}(x)}{\alpha_{1,2}(x)}\Big].$$

Since now $\alpha_{1,2}(x) \leq \sqrt{x}\alpha_{0,n}(x)$ and $\sqrt{x}\alpha_{0,n}^{'}(x) - \alpha_{1,2}^{'}(x) = 0$, by definition, then

$$\frac{f_n'(x)}{f_n(x)} \sum_{k=1}^{n-1} \binom{n-1}{k} k\alpha_{k,n}(x) f_n^k(x) \leq \frac{(r-1)}{\sqrt{x}}[\sqrt{x}\alpha_{0,n}^{'}(x) - \alpha_{1,2}^{'}(x)] = 0,$$

and the claimed nonincreasing property of $f_n(\cdot)$, $n \geq 2$, follows. □

Notice that finding the free-boundary of a quite intricate non-separable singular control problem has been reduced to determine the positive root of a polynomial equation. Clearly, that can be done analytically up to the second order (i.e. $n = 3$). Then, for higher orders, mathematical softwares can help in solving such a simple computational problem.

4.2.3. *CEV Process.* As in Section 4.1.3, let X^x be a CEV (Constant Elasticity of Variance) process of parameter $\gamma \in (0, \frac{1}{2}]$ (see (36)). Exploiting arguments completely similar to those used in the proof of Proposition 4.7 we can show that

PROPOSITION 4.8. *For any $0 \le k \le n-1$ and $n \ge 2$ define the functions $\alpha_{k,n} : (0,\infty) \mapsto (0,\infty)$ by*

$$\alpha_{k,n}(x) := \int_0^x y^{2\gamma+\frac{k}{n}-1} e^{\frac{r}{\gamma\sigma^2}y^{2\gamma}}\, dy. \tag{51}$$

Then, for any $x > 0$ and $n \ge 2$ the free-boundary $b_n(\cdot)$ is given by

$$b_n(x) = \Big(\frac{1}{f_n(x)}\Big)^n, \tag{52}$$

where $f_n(x)$ is the unique positive solution of the polynomial equation of order $n-1$

$$\sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_{k,n}(x) f_n^k(x) = \frac{\sigma^2}{2} + (r-1)\alpha_{0,n}(x), \quad x > 0. \tag{53}$$

Moreover, the mapping $x \mapsto b_n(x)$ is nondecreasing.

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