ON THE STOCHASTIC BEHAVIOUR OF OPTIONAL PROCESSES UP TO RANDOM TIMES

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Abstract. In this paper, a study of random times on filtered probability spaces is undertaken. The main message is that, as long as distributional properties of optional processes up to the random time are involved, there is no loss of generality in assuming that the random time is actually a randomised stopping time. This perspective has advantages in both the theoretical and practical study of optional processes up to random times. Applications are given to financial mathematics, as well as to the study of the stochastic behaviour of Brownian motion with drift up to its time of overall maximum as well as up to last-passage times over finite intervals. Furthermore, a novel proof of the Jeulin-Yor decomposition formula via Girsanov's theorem is provided.

Introduction

Discussion. Consider a filtered measurable space $(\Omega,\, \mathbf{F})$, where $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a right-continuous filtration, as well as an underlying sigma-algebra $\mathcal{F}$ over Ω such that $\mathcal{F} \supseteq \mathcal{F}_\infty := \bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$, where the last set-inclusion may be strict. A *random time* is a $[0,\infty]$-valued, $\mathcal{F}$-measurable random variable. The interplay between random times and the filtration $\mathbf{F}$ goes a long way back, with the pioneering work of [1], [3], [37]—see also the volume [16]. Interest in random times has been significant, especially in connection with applications in financial mathematics, such as reduced-form credit risk modelling—see, for example, [7], [27] and [15].

A common approach to *constructing* random times is via the use of randomised stopping times (also called *Cox's method*—see [28]). Let $\mathbb{Q}$ be a probability on $(\Omega, \mathcal{F})$, and suppose that there exists an $\mathcal{F}$-measurable random variable U that is stochastically independent of $\mathcal{F}_\infty$ and has the standard uniform law under $\mathbb{Q}$. For a given $\mathbf{F}$-adapted, right-continuous and nondecreasing process $K = (K_t)_{t\in\mathbb{R}_+}$ such that $0 \leq K \leq 1$, define the random time $\psi := \inf\{t \in \mathbb{R}_+ \mid K_t \geq U\}$, where by convention we set $\psi = \infty$ if the last set is empty. For such a duple $(\psi,\, \mathbb{Q})$, we say that ψ is a *randomised stopping time* on $(\Omega,\, \mathcal{F},\, \mathbf{F},\, \mathbb{Q})$. Randomised stopping times have several noteworthy properties:

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- The independence of U and $\mathcal{F}_\infty$ under $\mathbb{Q}$ implies that $\mathbb{Q}[\psi > t \mid \mathcal{F}_t] = 1 - K_t$, for all $t \in \mathbb{R}_+$. Therefore, $1 - K$ represents the conditional survival process associated to ψ under *any* probability $\mathbb{Q}$ which makes U and $\mathcal{F}_\infty$ independent. The latter fact is useful in modelling—for example, since $\mathbb{Q}[\psi \leq t] = \mathbb{E}_{\mathbb{Q}}[K_t]$ holds for $t \in \mathbb{R}_+$, $\mathbb{Q}$ can be chosen in order to control the unconditional distribution of ψ, while keeping the conditional survival probabilities fixed.
- Although ψ is not a stopping time on $(\Omega, \mathbf{F})$, it is in some sense very close to being one. Indeed, ψ is a stopping time of an initially enlarged filtration, defined as the right-continuous augmentation of $(\mathcal{F}_t \vee \sigma(U))_{t\in\mathbb{R}_+}$. Importantly, due to the independence of U and $\mathcal{F}_\infty$ under $\mathbb{Q}$, each martingale on $(\Omega, \mathbf{F}, \mathbb{Q})$ is also a martingale on the space with the enlarged filtration—in other words, the immersion property ([36], also called hypothesis $(\mathcal{H})$ in [3]) holds. This opens the door to major theoretical analysis of such random times using tools of martingale theory.
- From a more practical viewpoint, it is straightforward to simulate processes up to time ψ under $\mathbb{Q}$. One first simulates a uniform random variable U; then, in an independent fashion, one continues with simulating the process K until the point in time that it exceeds U, along with other processes of interest.

In view of the usefulness of randomised stopping times, it is natural to explore their generality. Of course, it is not possible that an arbitrary random time is a randomised stopping time, since for the latter there is a need for the extra "randomisation" coming from the uniform random variable. There is a further, more fundamental reason that an arbitrary random time cannot be realised as a randomised stopping time. Typically, for a random time ρ on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, the nonnegative process $\mathbb{R}_+ \ni t \mapsto \mathbb{P}[\rho > t \mid \mathcal{F}_t]$ fails to be nonincreasing, which would have to be the case if ρ was a randomised stopping time on $(\Omega, \mathbf{F}, \mathbb{P})$. Nevertheless, the main message of the paper is the following:

> With a given a pair $(\rho, \mathbb{P})$ of a random time ρ and a probability $\mathbb{P}$ on $(\Omega, \mathcal{F}, \mathbf{F})$, one can *essentially* associate a pair $(\psi, \mathbb{Q})$, where $\mathbb{Q}$ is a probability on $(\Omega, \mathcal{F})$ and ψ is a randomised stopping time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$, such that for *any* $\mathbf{F}$-optional process Y, the law of $(Y_{\rho\wedge t})_{t\in\mathbb{R}_+}$ under $\mathbb{P}$ is identical to the law of $(Y_{\psi\wedge t})_{t\in\mathbb{R}_+}$ under $\mathbb{Q}$.

Therefore, as long as distributional properties of optional processes on $(\Omega, \mathbf{F})$ under $\mathbb{P}$ *up* to the random time ρ are concerned, there is absolutely no loss of information in passing from $(\rho, \mathbb{P})$ to the more workable pair $(\psi, \mathbb{Q})$.

There is a reason for the qualifying "essentially" in the claim that the above association can be carried out. To begin with, $\mathcal{F}$ should be large enough to support a random variable U that will be independent of $\mathcal{F}_\infty$ under $\mathbb{Q}$. This is hardly a concern; if the original filtered space $(\Omega, \mathcal{F}, \mathbf{F})$ is not rich enough, one can always enlarge it in a minimal way, without affecting the structure of $\mathbf{F}$, in order to make the previous happen. However, there is another, more technical obstacle. As will be

argued in Section 1 of the text, what is guaranteed is the existence of a nonnegative local martingale L on $(\Omega, \mathbf{F}, \mathbb{P})$ with $L_0 = 1$ that is a candidate for a local (through a specific localising sequence of stopping times) density process of $\mathbb{Q}$ with respect to $\mathbb{P}$. Then, an argument ensuring that a consistent family of probabilities constructed in ever-increasing sigma-algebras has a countably additive extension to the limiting sigma-algebra is needed. Such an issue has appeared in different contexts in stochastic analysis—see [10], [29], [4]. Under appropriate topological assumptions on the underlying filtrations—for example, working on canonical path-spaces as discussed in [31]—one can successfully construct a probability $\mathbb{Q}$ out of L.

The aforementioned purely technical issue notwithstanding, the usefulness of the above philosophy is evident. In fact, as will be made clear in the text, even if the probability $\mathbb{Q}$ cannot be constructed, the representation pair consisting of the process K in the definition of ψ and the local martingale L on $(\Omega, \mathbf{F}, \mathbb{P})$ encodes significant information regarding the structure of random times.

In order to carry out the above-described program in practice, given a random time ρ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ one needs to identify the pair (K, L) associated with ρ. There are indeed formulas in the paper that provide (K, L) in terms of the process $\mathbb{R}_+ \ni t \mapsto \mathbb{P}[\rho > t \mid \mathcal{F}_t]$ of conditional survival probabilities of ρ, as well as the optional compensator on $(\Omega, \mathbf{F}, \mathbb{P})$ of the nondecreasing process $\mathbb{R}_+ \ni t \mapsto \mathbb{I}_{\{\rho \leq t\}}$. Closed-form expressions for the previous quantities are sometimes available—this is, for example, the case when times of maximum and last-passage times for certain nonnegative local martingales are considered. In order to illustrate the theoretical results, applications are presented in the context of financial mathematics, and discussion is provided regarding times of maximum and last-passage times for finite time-horizon Brownian motion with drift.

The dominant approach towards the study random times in the literature is that of *progressive enlargement of filtrations*. This theory has produced remarkable results, one of the most important due to Jeulin and Yor [17], providing the canonical representation of semimartingales up to random times under progressive enlargement of filtrations. This result is revisited in the text, where a novel proof of the Jeulin-Yor decomposition formula via the use of Girsanov's theorem—a certainly more familiar result—facilitates understanding by shedding an extra intuitive light.

Structure of the paper. This introductory part ends with general remarks that will be used throughout the text. In Section 1, the canonical pair of processes associated with a random time is introduced, and certain of its properties are explored in Section 1. Section 2 deals with a rigorous statement of the main message of the paper, regarding the law of optional processes up to random times. Section 3 contains some first examples. Section 4 presents applications of the theory in financial settings. Section 5 contains a discussion on the stochastic behaviour of Brownian motion with drift over finite time-intervals until its time of maximum and until last-passage times. Finally, in Section 6 the statement and a new proof of the Jeulin-Yor decomposition formula is provided.

General probabilistic remarks. The underlying filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is assumed to be right-continuous, but it will *not* be assumed that each $\mathcal{F}_t$, $t \in \mathbb{R}_+$, is completed with $\mathbb{P}$-null sets of $\mathcal{F}$. Although this relaxation calls for some technicalities, it is essential in the development; indeed, the need for defining a probability on $(\Omega, \mathcal{F})$ that is not absolutely continuous with respect to $\mathbb{P}$ (not even locally, on each $\mathcal{F}_t$, $t \in \mathbb{R}_+$) will arise. An extensive part of the general theory of stochastic processes can be developed without the completeness assumption on filtrations, as long as properties that hold "everywhere" are asked to hold in an "almost everywhere" sense. (Of course, there are exceptions to the previous rule; for example, the so-called debut theorem fails if the filtration is not completed—see the discussion in [34, II.75].) The interested reader can refer to [14, Chapter I and Chapter II] for results in this slightly non-conventional framework that shall be used throughout the paper. Versions of the section theorem from [12, IV§1], where again the filtration is not assumed to be completed, will also be useful.

For a càdlàg process X, define the process $X_- = (X_{t-})_{t\in\mathbb{R}_+}$, where X_{t-} is the left-limit of X at $t \in (0, \infty)$; by convention, $X_{0-} = 0$. Also, $\Delta X := X - X_-$. Every predictable process H is supposed to satisfy $H_0 = 0$. For any $[0, \infty]$-valued, $\mathcal{F}$-measurable random variable ρ and any process X, $X^\rho = X_{\rho\wedge\cdot}$ is defined as usual to be the process X stopped at ρ. For any càdlàg process X, we set $X^\uparrow := \sup_{t\in[0,\cdot]} X_t$, as well as $X^* = \sup_{t\in[0,\cdot]} |X_t| = (|X|)^\uparrow$.

Whenever H and X are processes such that X is a semimartingale to be used as an integrator and H can be used as integrand with respect to X, we use $\int_{[0,\cdot]} H_t \mathrm{d}X_t$ to denote the integral process. For a detailed account of stochastic integration, see [14].

If not stated otherwise, a property of a stochastic process (such as nonnegativity, path right-continuity, etc.) is assumed to hold *everywhere*; we make explicit note if these properties hold almost surely with respect to some probability on $(\Omega, \mathcal{F})$. When processes that are (local) martingales, supermartingales, etc., are considered, it is tacitly assumed that their paths are almost surely càdlàg with respect to the probability under consideration; for example local martingales on $(\Omega, \mathbf{F}, \mathbb{P})$ have $\mathbb{P}$-a.s. càdlàg paths.

In this paper, we *always* work under the following:

Standing Assumption 0.1. All random times ρ are assumed to satisfy $\mathbb{P}[\rho < \infty] = 1$.

The reason for above assumption is purely conventional; under its force, $t = \infty$ does not appear explicitly in the time-indices involved, something that would be unusual and create unnecessary confusion. We stress, however, that Assumption 0.1 in practice does not entail any loss of generality whatsoever. Indeed, a simple deterministic time-change of $[0, \infty]$ to $[0, 1]$ on the time-indices of filtrations, processes, etc., makes any $[0, \infty]$-valued random time actually bounded; then, all the results of the paper apply.

1. A Canonical Pair Associated with a Random Time

We keep all notation and remarks that appeared in the introductory section. In particular, Assumption 0.1 will always be tacitly in force.

1.1. Construction of the canonical pair.

The following result is the point of our departure.

Theorem 1.1. *Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. Then, there exists a pair of processes (K, L) with the following properties:*

(1) *K is $\mathbf{F}$-adapted, right-continuous, nondecreasing, with $0 \leq K \leq 1$.*
(2) *L is a nonnegative process with $L_0 = 1$ that is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$.*
(3) *For any nonnegative optional processes V on $(\Omega, \mathbf{F})$, it holds that*

$$\mathbb{E}_{\mathbb{P}}[V_\rho] = \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}_+} V_t\, L_t \mathrm{d}K_t\right].$$

(4) *$\int_{\mathbb{R}_+} \mathbb{I}_{\{K_{t-}=1\}}\mathrm{d}L_t = 0$ and $\int_{\mathbb{R}_+} \mathbb{I}_{\{L_t=0\}}\mathrm{d}K_t = 0$ hold $\mathbb{P}$-a.s.*

Furthermore, a pair (L, K) that satisfies the above requirements is essentially unique, in the following sense: if (K', L') is another pair that satisfies the above requirements, then K is $\mathbb{P}$-indistinguishable from K', while $\mathbb{P}[L_t = L'_t, \forall t \in \mathbb{R}_+ \mid K_\infty > 0] = 1$.

Definition 1.2. For a random time ρ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, the pair (K, L) that satisfies requirements (1), (2), (3) and (4) of Theorem 1.1 will be called *the canonical pair associated with ρ*.

Remark 1.3. Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated pair (K, L). Then, ρ is a stopping time on $(\Omega, \mathbf{F})$ if and only if $K = \mathbb{I}_{[\![\rho,\infty[\![}$ (and, in this case, $L \equiv 1$ will hold). Indeed, if ρ is a stopping time, $K' := \mathbb{I}_{[\![\rho,\infty[\![}$ is $\mathbf{F}$-adapted, nonnegative and nondecreasing, and $0 \leq K' \leq 1$ holds. Furthermore, $\mathbb{E}_{\mathbb{P}}[V_\rho] = \mathbb{E}_{\mathbb{P}}[\int_{\mathbb{R}_+} V_t \mathrm{d}K'_t]$ holds for all nonnegative and optional V on $(\Omega, \mathbf{F})$. By the essential uniqueness under $\mathbb{P}$ of the canonical pair associated with ρ, we obtain $K = \mathbb{I}_{[\![\rho,\infty[\![}$ (and $L = 1$). Conversely, assume that $K = \mathbb{I}_{[\![\rho,\infty[\![}$; as K is $\mathbf{F}$-adapted, ρ is a stopping time.

Given a random time ρ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, it will now be explained how the associated canonical pair (K, L) is constructed. We follow the proof of [23, Theorem 2.1], which contains Theorem 1.1 as a special case. Only details which will be essential in the present development are provided. We also introduce some further notation to be used throughout.

Let Z be the nonnegative càdlàg supermartingale on $(\Omega, \mathbf{F}, \mathbb{P})$ that satisfies $Z_t = \mathbb{P}[\rho > t \mid \mathcal{F}_t]$ for all $t \in \mathbb{R}_+$. (The fact that such a $\mathbb{P}$-a.s. càdlàg version Z exists follows from the right-continuity of the filtration $\mathbf{F}$ and the right-continuity of the function $\mathbb{R}_+ \ni t \mapsto \mathbb{P}[\rho > t] \in [0, 1]$ by an application of [12, Theorem II.2.44].) In view of Assumption 0.1, $Z_\infty := \lim_{t\to\infty} Z_t$ is $\mathbb{P}$-a.s. equal to zero. Note that Z is the conditional survival process associated to a random time by Azéma—see [16] and the references therein. Also, let A be the dual optional projection of $\mathbb{I}_{[\![\rho,\infty[\![}$ on $(\Omega, \mathbf{F}, \mathbb{P})$; in other words, A is the unique (up to $\mathbb{P}$-evanescence) $\mathbf{F}$-adapted, càdlàg, nonnegative and nondecreasing

process such that $\mathbb{E}_{\mathbb{P}}[V_\rho] = \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}_+} V_t \mathrm{d}A_t\right]$ holds for all nonnegative optional process V on $(\Omega, \mathbf{F})$. Then, $N := Z + A$ is a nonnegative martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ with $N_t = \mathbb{E}_{\mathbb{P}}[A_\infty \mid \mathcal{F}_t]$, for all $t \in \mathbb{R}_+$.

Remark 1.4. Since we do not assume that the $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, the properties of A being càdlàg, nondecreasing and nonnegative only are valid for $\mathbb{P}$-a.s. every path. However, one can alter A to have them holding identically. Indeed, with $\mathbb{D}$ denoting a countable and dense subset of $\mathbb{R}_+$, define $A' := \lim_{\mathbb{D}\ni t\downarrow} \left(\sup_{s\in[0,t]\cap\mathbb{D}} (\max\{A_s, 0\})\right)$. It is easily seen that this new process A' is $\mathbf{F}$-adapted (the right-continuity of $\mathbf{F}$ is essential here), càdlàg, nondecreasing and nonnegative, and that $A = A'$ up to $\mathbb{P}$-evanescence. It is possible that A can explode to ∞ in finite time, but this happens on a set of zero (outer) $\mathbb{P}$-measure and will not affect the results that follow in any way. Therefore, we might, and shall, assume in the sequel that A is càdlàg, nondecreasing and nonnegative everywhere.

Remark 1.5. The expected total mass of A over $\mathbb{R}_+$ under $\mathbb{P}$ is $\mathbb{E}_{\mathbb{P}}[A_\infty] = 1$. If $\mathbb{P}[A_\infty > 1] = 0$, in which case $\mathbb{P}[A_\infty = 1] = 1$, defining $K := A$ (more precisely, $K := \min\{A, 1\}$) and $L := 1$ would suffice for the purposes of Theorem 1.1. However, in all other cases of random times we have $\mathbb{P}[A_\infty > 1] > 0$, and the pair (K, L) is constructed from (A, Z) as will be shown below.

We continue with providing the definition of the pair (K, L). Consider the stopping time $\zeta_0 := \inf\{t \in \mathbb{R}_+ \mid Z_{t-} = 0 \text{ or } Z_t = 0\}$; in fact, ζ_0 actually is the terminal time of movement for both Z and A. The process K is defined via

(1.1)
$$K = 1 - \mathbb{P}[\rho > 0] \exp\left(-\int_{(0,\zeta_0\wedge\cdot]} \frac{\mathrm{d}A_t}{Z_t + \Delta A_t}\right) \prod_{t\in(0,\zeta_0\wedge\cdot]} \left(\left(1 - \frac{\Delta A_t}{Z_t + \Delta A_t}\right) \exp\left(\frac{\Delta A_t}{Z_t + \Delta A_t}\right)\right),$$

where by convention the product of an empty set of numbers is equal to one. It is clear that K is $\mathbf{F}$-adapted, càdlàg, nondecreasing and $[0, 1]$-valued on $[\![0, \zeta_0[\![$. A little care has to be exercised in the value of K at ζ_0. On $\{\Delta A_{\zeta_0} = 0\}$, it simply holds that $K_{\zeta_0} = K_{\zeta_0-}$. On $\{\Delta A_{\zeta_0} > 0\}$ it holds that $K_{\zeta_0} = 1$ because the product term on the right-hand-side of equation (1.1) is zero. The process K remains constant after ζ_0. In order to get some intuition on the definition of K, note that the differential equation that the process K defined in (1.1) satisfies is

$$\frac{\mathrm{d}K_t}{1 - K_{t-}} = \frac{\mathrm{d}A_t}{Z_t + \Delta A_t}, \qquad \text{for } t \in [0, \zeta_0). \tag{1.2}$$

For fixed $t \in [0, \zeta_0)$, $Z_t + \Delta A_t = \mathbb{P}[\rho \geq t \mid \mathcal{F}_t]$ represents the expected total remaining "life" of ρ on $[t, \infty]$, conditioned on $\mathcal{F}_t$; then, formally, $\mathrm{d}A_t/(Z_t + \Delta A_t)$ is the "fraction of remaining life of ρ spent at t." The equivalent "fraction of remaining life spent at t" for K would be $\mathrm{d}K_t/(1 - K_{t-})$. (The previous quantity is based on the understanding that $\mathbb{P}[K_\infty = 1] = 1$, although this is not always the case as will be shown later in Remark 3.5.) To get a feeling of how L should be defined, observe that $(Z + \Delta A)\Delta K = (1 - K_-)\,\Delta A$ implies that $(Z + \Delta A)(1 - K) = (1 - K_-)Z$. Therefore, from (1.2) we obtain that $\mathrm{d}K_t/(1 - K_t) = \mathrm{d}A_t/Z_t$ holds for $t \in [0, \zeta_0)$, which implies

that $Z_t \mathrm{d}K_t = (1-K_t)\mathrm{d}A_t$ holds for $t \in \mathbb{R}_+$. Since $\mathrm{d}A_t = L_t \mathrm{d}K_t$ has to hold for $t \in \mathbb{R}_+$ in view of property (3) in Theorem 1.1, we obtain $L(1-K) = Z$. Using the previous equality and Itô's formula we obtain the dynamics

$$\frac{\mathrm{d}L_t}{L_{t-}} = \frac{\mathrm{d}N_t}{Z_{t-}}, \quad t \in [0, \zeta_0], \tag{1.3}$$

where recall that $N = Z + A$. Equation (1.3) can actually be used as the definition of L, which becomes equal to the stochastic logarithm of the local martingale $\int_0^{\zeta_0 \wedge \cdot} (1/Z_{t-}) \mathrm{d}N_t$. (One has to be quite careful here: the latter process might not be defined at time ζ_0 and onwards due to explosion, which will imply that, $\mathbb{P}$-a.s., $L_t = 0$ for all $t \geq \zeta_0$. The treatment in [23, §2.3] makes sure that all such issues are dealt with.) Then, the relationship $Z = L(1-K)$ can be shown to hold true. One can check [23, §2.3] for all the remaining technical details of the proof.

Remark 1.6. When ΔK is $\mathbb{P}$-evanescent (which happens exactly when ΔA is $\mathbb{P}$-evanescent), the formula $Z = L(1-K)$ implies that L coincides with the local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ that appears in the multiplicative Doob-Meyer decomposition of the nonnegative $(\Omega, \mathbf{F}, \mathbb{P})$-supermartingale Z. This fact provides the most efficient way to calculate the canonical pair associated with a random time that avoids all stopping times. (For the definition and characterisation of random times avoiding all stopping times, see §1.4.)

1.2. **A consistent family of probabilities associated with a random time.** Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated canonical pair (K, L). Define

$$\eta_u := \inf\{t \in \mathbb{R}_+ \mid K_t \geq u\}, \quad \text{for } u \in [0,1), \tag{1.4}$$

with the convention $\eta_u = \infty$ if the last set is empty. The nondecreasing family $(\eta_u)_{u \in [0,1)}$ of stopping times on $(\Omega, \mathbf{F})$ will play a major role in the development. We start with a "localisation" result.

Lemma 1.7. *Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with canonical pair (K, L). For $u \in [0,1)$, $\mathbb{P}\left[L^*_{\eta_u} \leq 2/(1-u)\right] = 1$ holds. If $\mathbb{P}[\eta_u < \infty, \Delta L_{\eta_u} > 0] = 0$, then $\mathbb{P}\left[L^*_{\eta_u} \leq 1/(1-u)\right] = 1$.*

Proof. Fix $u \in [0,1)$. Since $K_{t-} \leq u$ holds for $t \in [0, \eta_u]$ and $Z_- \leq 1$ holds up to $\mathbb{P}$-evanescence, it follows that

$$L_- = \frac{Z_-}{1-K_-} \leq \frac{1}{1-u} \text{ holds } \mathbb{P}\text{-a.s. on } [\![0, \eta_u]\!],$$

which implies that $\mathbb{P}\left[L^*_{\eta_u -} \leq 1/(1-u)\right] = 1$. It remains to check what happens at η_u. In case $\mathbb{P}[\eta_u < \infty, \Delta L_{\eta_u} > 0] = 0$, $\mathbb{P}\left[L^*_{\eta_u} \leq 1/(1-u)\right] = 1$ is immediate. Now, remove the assumption $\mathbb{P}[\eta_u < \infty, \Delta L_{\eta_u} > 0] = 0$. We shall use that $\Delta A \leq 1$ up to $\mathbb{P}$-evanescence. (Indeed, the equality $\Delta A_\tau = \mathbb{P}[\rho = \tau \mid \mathcal{F}_\tau]$ holds $\mathbb{P}$-a.s. on $\{\tau < \infty\}$ for any stopping time τ, since A is the dual optional projection of $\mathbb{I}_{[\![\rho, \infty[\![}$ on $(\Omega, \mathbf{F}, \mathbb{P})$. It follows that $\mathbb{P}[\Delta A_\tau \leq 1] = 1$ for any stopping time τ and,

therefore, that $\Delta A \leq 1$ up to $\mathbb{P}$-evanescence by [12, Theorem 4.10].) Using (1.3), we obtain, $\mathbb{P}$-a.s.,

$$L_{\eta_u} = L_{\eta_u -} + \frac{\Delta N_{\eta_u}}{1 - K_{\eta_u -}} = \frac{Z_{\eta_u -} + \Delta N_{\eta_u}}{1 - K_{\eta_u -}} = \frac{Z_{\eta_u} + \Delta A_{\eta_u}}{1 - K_{\eta_u -}} \leq \frac{2}{1-u},$$

which completes the proof. □

In view of Lemma 1.7, for any $u \in [0,1)$ one can construct a probability measure $\mathbb{Q}_u$ on $(\Omega, \mathcal{F})$ via the recipe $\mathrm{d}\mathbb{Q}_u = L_{\eta_u}\mathrm{d}\mathbb{P}$. The collection $(\mathbb{Q}_u)_{u\in[0,1)}$ has the following consistency property: $\mathbb{Q}_u = \mathbb{Q}_v$ on $(\Omega, \mathcal{F}_{\eta_u})$ holds whenever $0 \leq u \leq v < 1$. It would be very convenient (but not *a priori* clear and certainly not true in general, as is demonstrated in Example 3.8) if one could find a probability $\mathbb{Q} \equiv \mathbb{Q}_1$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}|_{\mathcal{F}_{\eta_u}} = \mathbb{Q}_u|_{\mathcal{F}_{\eta_u}}$ holds for all $u \in [0,1)$. This is indeed the case in a number of examples, as will be discussed later. The consequences of the existence of such probability are analysed in Section 2. For the time being, we mention an auxiliary result.

Lemma 1.8. *For all $u \in [0,1)$, it holds that $\mathbb{Q}_u[L_{\eta_u} > 0] = 1$ and $\mathbb{Q}_u[\eta_u < \infty] = 1$.*

Proof. Fix $u \in [0,1)$. Then, $\mathbb{Q}_u[L_{\eta_u} > 0] = \mathbb{E}_{\mathbb{P}}[L_{\eta_u}\mathbb{I}_{\{L_{\eta_u}>0\}}] = \mathbb{E}_{\mathbb{P}}[L_{\eta_u}] = 1$. In order to show the equality $\mathbb{Q}_u[\eta_u < \infty] = 1$, first observe that since $0 = Z_\infty = L_\infty(1 - K_\infty)$ holds $\mathbb{P}$-a.s., we have $\mathbb{P}[K_\infty < 1, L_\infty > 0] = 0$. Coupled with the fact that $\{\eta_u = \infty\} \subseteq \{K_\infty < 1\}$, we obtain $\mathbb{P}\left[L_{\eta_u}\mathbb{I}_{\{\eta_u<\infty\}} = L_{\eta_u}\right] = 1$. Therefore, $\mathbb{Q}_u[\eta_u < \infty] = \mathbb{E}_{\mathbb{P}}\left[L_{\eta_u}\mathbb{I}_{\{\eta_u<\infty\}}\right] = \mathbb{E}_{\mathbb{P}}[L_{\eta_u}] = 1$. □

1.3. **Time changes.** For a nonnegative $(\Omega, \mathbf{F})$-optional process V, the change-of-variables formula gives $\int_{\mathbb{R}_+} V_t \mathrm{d}K_t = \int_{[0,1)} V_{\eta_u}\mathbb{I}_{\{\eta_u<\infty\}}\mathrm{d}K_{\eta_u}$. For $a \in [0,1)$, on the event $\{K_{\eta_a -} < K_{\eta_a}\}$ it holds that

$$V_{\eta_a}\Delta K_{\eta_a} = V_{\eta_a}(K_{\eta_a} - K_{\eta_a -}) = \int_{K_{\eta_a -}}^{K_{\eta_a}} V_{\eta_a}\mathrm{d}u = \int_{K_{\eta_a -}}^{K_{\eta_a}} V_{\eta_u}\mathrm{d}u.$$

Therefore, $\int_{\mathbb{R}_+} V_t \mathrm{d}K_t = \int_{[0,1)} V_{\eta_u}\mathbb{I}_{\{\eta_u<\infty\}}\mathrm{d}u$ follows. The last fact helps to establish the following result.

Proposition 1.9. *Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. Then, for any nonnegative $(\Omega, \mathbf{F})$-optional process V, it holds that*

$$\mathbb{E}_{\mathbb{P}}[V_\rho] = \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}[V_{\eta_u}]\,\mathrm{d}u = \lim_{a\uparrow 1} \mathbb{E}_{\mathbb{Q}_a}\left[\int_{[0,a]} V_{\eta_u}\mathrm{d}u\right]. \tag{1.5}$$

Proof. As discussed above, for any V that is nonnegative and $(\Omega, \mathbf{F})$-optional, we have

$$\int_{\mathbb{R}_+} V_t L_t \mathrm{d}K_t = \int_{[0,1)} V_{\eta_u} L_{\eta_u}\mathbb{I}_{\{\eta_u<\infty\}}\mathrm{d}u.$$

Therefore, the first equality in (1.5) is immediate from Theorem 1.1, Fubini's theorem, the definition of the probabilities $(\mathbb{Q}_u)_{u\in[0,1)}$ and Lemma 1.8. The second equality in (1.5) follows from the monotone convergence theorem and the consistency of the family $(\mathbb{Q}_u)_{u\in[0,1)}$. □

Proposition 1.9 has a simple corollary, which states that the law of $K_{\rho-}$ under $\mathbb{P}$ is stochastically dominated (in first order) by the standard uniform law, and that the latter standard uniform law is stochastically dominated by the law of K_ρ under $\mathbb{P}$.

Proposition 1.10. *Let ρ be any random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated pair (K, L). Then, for all nondecreasing functions $f : [0,1) \mapsto \mathbb{R}$, it holds that*

$$\mathbb{E}_{\mathbb{P}}[f(K_{\rho-})] \leq \int_{[0,1)} f(u)\mathrm{d}u \leq \mathbb{E}_{\mathbb{P}}[f(K_\rho)]. \tag{1.6}$$

Proof. Pick any nondecreasing function $f : [0,1) \mapsto \mathbb{R}$. For establishing the inequalities (1.6), it is clearly sufficient to deal with the case where $f(u) \in \mathbb{R}_+$ for $u \in [0,1)$. Since $K_{\eta_u-} \leq u$ and f is nondecreasing, (1.5) gives

$$\mathbb{E}_{\mathbb{P}}[f(K_{\rho-})] = \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[f(K_{\eta_u-})\right] \mathrm{d}u \leq \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[f(u)\right] \mathrm{d}u = \int_{[0,1)} f(u)\mathrm{d}u.$$

The other inequality in (1.6) is proved similarly, using the fact that $\mathbb{Q}_u\left[K_{\eta_u} \geq u\right] = 1$ for $u \in [0,1)$, as follows from Lemma 1.8. □

1.4. **Random times that avoid all stopping times.** A random time ρ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is said to *avoid all stopping times on* $(\Omega, \mathbf{F}, \mathbb{P})$ if $\mathbb{P}[\rho = \tau] = 0$ holds whenever τ is a stopping time on $(\Omega, \mathbf{F})$. The next result states equivalent conditions to ρ avoiding all stopping times.

Proposition 1.11. *Let ρ be any random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated canonical pair (K, L). Then, the following statements are equivalent:*

1. *ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$.*
2. *ΔK is $\mathbb{P}$-evanescent.*
3. *$\mathbb{P}[\Delta K_\rho = 0] = 1$.*
4. *K_ρ has the standard uniform distribution under $\mathbb{P}$.*

Proof. In the course of the proof, we shall be using A, Z, and N for the processes that were introduced in §1.1, associated to the random time ρ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.

For implication (1) ⇒ (2), the fact that $\mathbb{E}_{\mathbb{P}}[\Delta A_\tau] = \mathbb{P}[\rho = \tau] = 0$ implies that $\mathbb{P}[\Delta A_\tau = 0] = 1$ holds for all stopping times τ on $(\Omega, \mathbf{F})$. Then, in view of (1.2), $\mathbb{P}[\Delta K_\tau = 0] = 1$ holds for all stopping times τ on $(\Omega, \mathbf{F})$ as well. An application of [12, Theorem 4.10] shows that ΔK is $\mathbb{P}$-evanescent. Implication (2) ⇒ (3) is trivial. Now, assume (3); from the inequalities (1.6) we get $\mathbb{E}[f(K_\rho)] = \int_{[0,1)} f(u)\mathrm{d}u$ for any nondecreasing Borel function $f : [0,1) \mapsto \mathbb{R}_+$, which implies that K_ρ has a standard uniform distribution under $\mathbb{P}$. In the next three paragraphs, we shall show (4) ⇒ (3) ⇒ (2) ⇒ (1).

We show (4) ⇒ (3). By (1.5), we have

$$\mathbb{E}_{\mathbb{P}}\left[K_\rho + K_{\rho-}\right] = \lim_{a \uparrow 1} \mathbb{E}_{\mathbb{Q}_a}\left[\int_{[0,a]} \left(K_{\eta_u} + K_{\eta_u-}\right) \mathrm{d}u\right].$$

For $a \in [0,1)$, on the event $\{K_{\eta_a} \geq a\}$ it holds that

$$a^2 = \int_{[0,a]} 2u\mathrm{d}u \leq \int_{[0,a]} (K_{\eta_u} + K_{\eta_u -})\,\mathrm{d}u \leq 1.$$

With the help of Lemma 1.8, we obtain $\mathbb{E}_{\mathbb{P}}[K_\rho + K_{\rho-}] = 1$. Since $\mathbb{E}_{\mathbb{P}}[K_\rho] = 1/2$ holds in view of the fact that K_ρ has the standard uniform distribution under $\mathbb{P}$, we obtain $\mathbb{E}[K_{\rho-}] = 1/2$. As K is nondecreasing and $\mathbb{E}_{\mathbb{P}}[\Delta K_\rho] = 0$, we obtain $\mathbb{P}[\Delta K_\rho = 0] = 1$, i.e., statement (3).

For (3) $\Rightarrow$ (2), start with the following observation: for any stopping time τ, on $\{\tau < \infty\}$ it holds that

$$L_\tau = L_{\tau-} + \Delta L_\tau = L_{\tau-} + \frac{\Delta N_\tau}{1 - K_{\tau-}} = \frac{L_{\tau-}(1 - K_{\tau-}) + Z_\tau - Z_{\tau-} + \Delta A_\tau}{1 - K_{\tau-}} = \frac{Z_\tau + \Delta A_\tau}{1 - K_{\tau-}}.$$

Since $\{\Delta K_\tau > 0\} \subseteq \{\Delta A_\tau > 0\}$ holds on $\{\tau < \infty\}$, it follows that $\{\Delta K_\tau > 0\} \subseteq \{L_\tau > 0\}$ modulo $\mathbb{P}$ holds on $\{\tau < \infty\}$ for all stopping times τ. Continuing, note that

$$0 = \mathbb{E}_{\mathbb{P}}[\Delta K_\rho] = \mathbb{E}_{\mathbb{P}}\left[\int_{\mathbb{R}_+} (K_t - K_{t-})\, L_t \mathrm{d}K_t\right] = \mathbb{E}_{\mathbb{P}}\left[\sum_{t \in \mathbb{R}_+} L_t (\Delta K_t)^2\right].$$

Consider a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with disjoint graphs that exhausts the jumps of K; then, $\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}} L_{\tau_n}(\Delta K_{\tau_n})^2\right] = 0$. This means that $\sum_{n\in\mathbb{N}} L_{\tau_n}(\Delta K_{\tau_n})^2 = 0$, $\mathbb{P}$-a.s.; since $\{\Delta K_{\tau_n} > 0\} \subseteq \{L_{\tau_n} > 0\}$ modulo $\mathbb{P}$ holds on $\{\tau_n < \infty\}$ for all $n \in \mathbb{N}$, we obtain $\mathbb{P}[\Delta K_{\tau_n} = 0] = 1$ for all $n \in \mathbb{N}$. The last implies that $\mathbb{P}[\Delta K_\tau = 0] = 1$ for all stopping times τ. In view of [12, Theorem 4.10], this is exactly statement (2).

Finally, we establish (2) $\Rightarrow$ (1). Since

$$\{\Delta A_\tau > 0\} = \{L_\tau \Delta K_\tau > 0\} = \{L_\tau > 0\} \cap \{\Delta K_\tau > 0\} = \{\Delta K_\tau > 0\}$$

modulo $\mathbb{P}$ holds for all stopping times τ, we have $\mathbb{P}[\rho = \tau] = \mathbb{E}_{\mathbb{P}}[\Delta A_\tau] = 0$, the latter being valid because $\mathbb{P}[\Delta A_\tau > 0] = \mathbb{P}[\Delta K_\tau > 0] = 0$. Therefore, ρ avoids all stopping times under $\mathbb{P}$. □

1.5. **An optimality property of L amongst all nonnegative local $\mathbb{P}$-martingales.** Let $\mathcal{S}$ be the set of all nonnegative supermartingales S on $(\Omega, \mathbf{F}, \mathbb{P})$ with $\mathbb{P}[S_0 = 1] = 1$. The set $\mathcal{S}$ contains in particular all nonnegative local martingales M on $(\Omega, \mathbf{F}, \mathbb{P})$ with $\mathbb{P}[M_0 = 1] = 1$. For a random time ρ with associated canonical pair (K, L), it is reasonable to expect that the local martingale L has some optimality property within the class $\mathcal{S}$ when sampled at ρ. Indeed, the next result shows that, in the jargon of [23], L_ρ is the numéraire under $\mathbb{P}$ in the convex set $\{S_\rho \mid S \in \mathcal{S}\}$.

Proposition 1.12. *Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated canonical pair (K, L). Then, $\mathbb{P}[L_\rho > 0] = 1$ and $\mathbb{E}_{\mathbb{P}}[S_\rho / L_\rho] \leq 1$ holds for all $S \in \mathcal{S}$. If, furthermore, ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$, then the stronger inequality $\mathbb{E}_{\mathbb{P}}[S_\rho / L_\rho \mid K_\rho] \leq 1$ holds for all $S \in \mathcal{S}$.*

Proof. By Lemma 1.8, $\mathbb{Q}_u\left[L_{\eta_u} > 0\right] = 1$ holds for all $u \in [0,1)$. Then, by Proposition 1.9,

$$\mathbb{P}\left[L_\rho > 0\right] = \int_{[0,1)} \mathbb{Q}_u\left[L_{\eta_u} > 0\right] \mathrm{d}u = 1.$$

Fix $S \in \mathcal{S}$. Observe that $\mathbb{E}_{\mathbb{Q}_u}[S_{\eta_u}/L_{\eta_u}] = \mathbb{E}_{\mathbb{P}}[S_{\eta_u}\mathbb{I}_{\{L_{\eta_u}>0\}}] \leq 1$ holds for all $u \in [0,1)$. Then,

$$\mathbb{E}_{\mathbb{P}}[S_\rho/L_\rho] = \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[S_{\eta_u}/L_{\eta_u}\right] \mathrm{d}u \leq 1.$$

Assume now that ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$. By a straightforward extension of Lemma 1.8, $\mathbb{Q}_u\left[K_{\eta_u} = u\right] = 1$ holds for all $u \in [0,1)$. Therefore, for all functions $f : [0,1) \mapsto \mathbb{R}_+$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[(S_\rho/L_\rho)\, f(K_\rho)\right] &= \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[(S_{\eta_u}/L_{\eta_u})\, f(K_{\eta_u})\right] \mathrm{d}u \\
&= \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[(S_{\eta_u}/L_{\eta_u})\, f(u)\right] \mathrm{d}u \\
&\leq \int_{[0,1)} f(u)\mathrm{d}u \;=\; \mathbb{E}_{\mathbb{P}}\left[f(K_\rho)\right],
\end{aligned}$$

the last equality following from Proposition 1.11. Since the function $f : [0,1) \mapsto \mathbb{R}_+$ is arbitrary, we obtain $\mathbb{E}_{\mathbb{P}}\left[S_\rho/L_\rho \mid K_\rho\right] \leq 1$. □

2. Random Times and Randomised Stopping Times

2.1. **The one probability $\mathbb{Q}$.** Recall the consistent family of probabilities $(\mathbb{Q}_u)_{u\in[0,1)}$ from §1.2. For the purposes of Section 2, we shall be working under the following assumption.

Assumption 2.1. There exists a probability measure $\mathbb{Q} \equiv \mathbb{Q}_1$ on $(\Omega, \mathcal{F})$, as well as a random variable $U : \Omega \mapsto [0,1)$, such that:

(1) $\mathbb{Q}|_{\mathcal{F}_{\eta_u}} = \mathbb{Q}_u|_{\mathcal{F}_{\eta_u}}$ holds for all $u \in [0,1)$.
(2) Under $\mathbb{Q}$, U is independent of $\mathcal{F}_\infty$ and has the standard uniform law.

Remark 2.2. Given that there exists a probability measure $\mathbb{Q} \equiv \mathbb{Q}_1$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}|_{\mathcal{F}_{\eta_u}} = \mathbb{Q}_u|_{\mathcal{F}_{\eta_u}}$ holds for all $u \in [0,1)$, asking that there also exists a random variable $U : \Omega \mapsto [0,1)$ such that U is independent of $\mathcal{F}_\infty$ and has the standard uniform law under $\mathbb{Q}$ entails no loss of generality whatsoever. Indeed, if such random variable does not exist, the underlying probability space can always be enlarged in order to support one. More precisely, define $\overline{\Omega} := \Omega \times [0,1)$, a filtration $\overline{\mathbf{F}} = (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$ via $\overline{\mathcal{F}}_t = \mathcal{F}_t \otimes \{\emptyset, [0,1)\}$ for $t \in \mathbb{R}_+$, as well as $\overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}([0,1))$, where $\mathcal{B}([0,1))$ is the Borel sigma-algebra on $[0,1)$. It is immediate that $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $(\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$ are in one-to-one correspondence. (However, $\mathcal{F}$ and $\overline{\mathcal{F}}$ are not isomorphic.) On $(\overline{\Omega}, \overline{\mathcal{F}})$, define $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathsf{Leb}$, $\overline{\mathbb{Q}} := \mathbb{Q} \otimes \mathsf{Leb}$, as well as $\overline{\mathbb{Q}}_u := \mathbb{Q}_u \otimes \mathsf{Leb}$ for $u \in [0,1)$, where "Leb" denotes Lebesgue measure on $\mathcal{B}([0,1))$. Any process X on the original stochastic basis is identified on the new stochastic basis with the process $\overline{X}$ defined via $\overline{X}(\omega, u) = X(\omega)$ for all $(\omega, u) \in \overline{\Omega}$—this way, properties like adaptedness and optionality of processes are in one-to-one correspondence. The random variable

$U : \overline{\Omega} \mapsto [0,1)$ defined via $U(\omega, u) = u$ for all $(\omega, u) \in \overline{\Omega}$ has the standard uniform distribution, and is independent of $\overline{\mathcal{F}}_\infty$, the previous holding under both $\overline{\mathbb{P}}$ and $\overline{\mathbb{Q}}$. Note that the pair associated with ρ on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{F}}, \overline{\mathbb{P}})$ is $(\overline{K}, \overline{L})$ in the previously-introduced notation, which is identified with (K, L). Furthermore, $\overline{\mathbb{Q}}|_{\overline{\mathcal{F}}_{\eta_u}} = \overline{\mathbb{Q}}_u|_{\overline{\mathcal{F}}_{\eta_u}}$ holds for all $u \in [0,1)$.

Remark 2.3. Even though item (2) of Assumption 2.1 is not really an assumption in view of Remark 2.2 above, item (1) *is*, as Example 3.8 will reveal. In fact, Example 3.8 will make an additional point: even if $\mathbb{Q}$ exists, it is in general possible that neither of the conditions $\mathbb{Q} \ll_{\mathcal{F}_t} \mathbb{P}$ nor $\mathbb{P} \ll_{\mathcal{F}_t} \mathbb{Q}$ holds, for any choice of $t \in (0, \infty)$. This clarifies the absolute need to refrain from completing $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ with $\mathbb{P}$-null sets, even if the null sets come from $\bigcup_{t \in \mathbb{R}_+} \mathcal{F}_t$ and not from the larger, in general, sigma-field $\mathcal{F}_\infty = \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t$.

2.2. **The stochastic behavior of optional processes up to random times.** We now turn to the topic discussed in the introductory section: as long as distributional properties of optional processes on $(\Omega, \mathbf{F})$ up to a random time are concerned, one can pass from the original random time ρ and probability $\mathbb{P}$ to a randomised stopping time ψ on $(\Omega, \mathbf{F}, \mathbb{Q})$, where $\mathbb{Q}$ is the probability of Assumption 2.1.

Theorem 2.4. *Let ρ be a random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated canonical pair (K, L). Under the validity of Assumption 2.1, let $\mathbb{Q}$ the probability that appears there. Define*

$$\psi := \inf\{t \in \mathbb{R}_+ \mid K_t \geq U\} = \eta_U.$$

Then, ψ is a randomised stopping time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{Q})$ with associated canonical pair $(K, 1)$. Furthermore, for any optional process Y on $(\Omega, \mathbf{F})$, the finite-dimensional distributions of $Y^\rho = (Y_{\rho \wedge t})_{t \in \mathbb{R}_+}$ under $\mathbb{P}$ coincide with the finite-dimensional distributions of $Y^\psi = (Y_{\psi \wedge t})_{t \in \mathbb{R}_+}$ under $\mathbb{Q}$.

Proof. Observe that $\{\psi > t\} = \{U > K_t\}$ holds for $t \in \mathbb{R}_+$. Therefore,

$$\mathbb{Q}[\psi > t \mid \mathcal{F}_t] = \mathbb{Q}[U > K_t \mid \mathcal{F}_t] = 1 - K_t, \quad \text{for } t \in \mathbb{R}_+.$$

It follows that the pair associated with ψ on $(\Omega, \mathbf{F}, \mathbb{Q})$ is $(K, 1)$.

Pick any nonnegative optional process V on $(\Omega, \mathbf{F})$. Then,

$$\mathbb{E}_{\mathbb{P}}[V_\rho] = \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}[V_{\eta_u}]\,\mathrm{d}u = \int_{[0,1)} \mathbb{E}_{\mathbb{Q}}[V_{\eta_u}]\,\mathrm{d}u = \mathbb{E}_{\mathbb{Q}}\left[\int_{[0,1)} V_{\eta_u}\mathrm{d}u\right] = \mathbb{E}_{\mathbb{Q}}[V_{\eta_U}] = \mathbb{E}_{\mathbb{Q}}[V_\psi]. \tag{2.1}$$

Continuing, fix an optional process Y on $(\Omega, \mathbf{F})$ and times $\{t_1, \ldots, t_n\} \subseteq \mathbb{R}_+$. For any nonnegative Borel-measurable function $f : \mathbb{R}^n \mapsto \mathbb{R}_+$, the process $V = f(Y^{t_1}, \ldots, Y^{t_n})$ is optional on $(\Omega, \mathbf{F})$. Since $V_\rho = f(Y^\rho_{t_1}, \ldots, Y^\rho_{t_n})$ and $V_\psi = f(Y^\psi_{t_1}, \ldots, Y^\psi_{t_n})$, (2.1) gives

$$\mathbb{E}_{\mathbb{P}}\left[f(Y^\rho_{t_1}, \ldots, Y^\rho_{t_n})\right] = \mathbb{E}_{\mathbb{Q}}\left[f(Y^\psi_{t_1}, \ldots, Y^\psi_{t_n})\right].$$

As the collection $\{t_1, \dots, t_n\} \subseteq \mathbb{R}_+$ and the nonnegative Borel-measurable function f are arbitrary, the finite-dimensional distributions of Y^ρ under $\mathbb{P}$ coincide with the finite-dimensional distributions of Y^ψ under $\mathbb{Q}$. □

Remark 2.5. In the setting of Theorem 2.4, assume that τ is a stopping time on $(\Omega, \mathbf{F})$ and that E is an $\mathcal{F}_\tau$-measurable set. Then, since the process $\mathbb{I}_E \mathbb{I}_{]\!]\tau,\infty[\![}$ is optional, we obtain

$$\mathbb{P}\left[E, \rho > \tau\right] = \mathbb{Q}\left[E, \eta_U > \tau\right] = \mathbb{Q}\left[E, K_\tau < U\right] = \int_{[0,1)} \mathbb{Q}\left[E, K_\tau < u\right] \mathrm{d}u = \mathbb{E}_{\mathbb{Q}}\left[(1 - K_\tau)\mathbb{I}_E\right].$$

3. First Examples

3.1. **Finite-horizon discrete-time models.** Models where the time-set is discrete can be naturally embedded in a continuous-time framework. *Only for the purposes of §3.1*, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$, where $\mathbb{T} = \{0, \dots, T\}$ for $T \in \mathbb{N}$. We assume that $\mathcal{F} = \mathcal{F}_T \vee \sigma(U)$, where U is a random variable with uniform distribution under $\mathbb{P}$, independent of $\mathcal{F}_T$. A random time ρ in this setting is a $\mathbb{T}$-valued random variable.

It is straightforward to check that $A = \sum_{t \leq \cdot} \mathbb{P}[\rho = t \mid \mathcal{F}_t]$ is the dual optional projection on $(\Omega, \mathbf{F}, \mathbb{P})$ of $\mathbb{I}_{[\![\rho,T]\!]}$. Recall from §1.1 the stopping time $\zeta_0 := \min\{t \in \mathbb{T} \mid Z_t = 0\}$. The discrete-time versions of (1.2) and (1.3) on $\{t \leq \zeta_0\}$ read

$$K_t = K_{t-1} + (1 - K_{t-1})\left(\frac{A_t - A_{t-1}}{Z_t + A_t - A_{t-1}}\right) = K_{t-1} + (1 - K_{t-1})\frac{\mathbb{P}[\rho = t \mid \mathcal{F}_t]}{\mathbb{P}[\rho \geq t \mid \mathcal{F}_t]}$$

and

$$L_t = L_{t-1}\left(1 + \frac{N_t - N_{t-1}}{Z_{t-1}}\right) = L_{t-1}\frac{Z_t + A_t - A_{t-1}}{Z_{t-1}} = L_{t-1}\frac{\mathbb{P}[\rho \geq t \mid \mathcal{F}_t]}{\mathbb{P}[\rho \geq t \mid \mathcal{F}_{t-1}]}.$$

On $\{t > \zeta_0\}$, $K_t = K_{\zeta_0}$ and $L_t = L_{\zeta_0}$ holds.

In finite-horizon discrete-time settings like the one considered here, nonnegative local martingales are actually martingales—see [13]. Therefore, one may define a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ that has density L_T with respect to $\mathbb{P}$; then, $\mathbb{Q}|_{\mathcal{F}_{\eta_u}} = \mathbb{Q}_u|_{\mathcal{F}_{\eta_u}}$ holds for all $u \in [0, 1)$. The probability $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. (Observe also that Assumption 2.1 is always valid in this setting. Indeed, L_T is $\mathcal{F}_T$-measurable and, therefore, independent of U under $\mathbb{P}$, which implies that U is independent of $\mathcal{F}_T$ under $\mathbb{Q}$.) The next result shows that the stochastic behaviour of ρ under $\mathbb{P}$ and $\mathbb{Q}$ might be radically different.

Proposition 3.1. *Let ρ be a random time on $(\Omega, \mathbf{F}, \mathbb{P})$. If $\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}]$ is $\mathbb{P}$-a.s. $\{0, 1\}$-valued, then $\mathbb{Q}[\rho = \zeta_0] = 1$.*

Proof. On $\{\zeta_0 > 0\}$ it holds that $L_{\zeta_0} = L_{\zeta_0 - 1}\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}]/\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0 - 1}]$, which implies that $\{L_{\zeta_0} > 0\} = \{\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}] > 0\}$. Since $\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}]$ is $\mathbb{P}$-a.s. $\{0, 1\}$-valued, it follows that $\{L_{\zeta_0} > 0\} = \{\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}] = 1\}$ holds modulo $\mathbb{P}$ on $\{\zeta_0 > 0\}$. On $\{\zeta_0 = 0\}$ both $L_{\zeta_0} = 1$ and $\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}] = 1$ hold modulo $\mathbb{P}$. Therefore,

$$\mathbb{Q}[\rho = \zeta_0] = \mathbb{E}_{\mathbb{P}}[L_{\zeta_0}\mathbb{I}_{\{\rho = \zeta_0\}}] = \mathbb{E}_{\mathbb{P}}[L_{\zeta_0}\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}]] = \mathbb{E}_{\mathbb{P}}[L_{\zeta_0}] = 1,$$

which completes the proof. □

Random times that satisfy the condition in the statement of Proposition 3.1 are $\mathbb{Q}$-a.s. equal to a stopping time. The next example shows that familiar random times that are far from being stopping times under $\mathbb{P}$ become $\mathbb{Q}$-a.s. equal to a constant.

Example 3.2. Let X be an adapted process on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ such that $\mathbb{P}[X_t \geq X_{t-1} \mid \mathcal{F}_{t-1}] > 0$ holds $\mathbb{P}$-a.s. for all $t \in \mathbb{T} \setminus \{0\}$. Define $\rho := \max\left\{t \in \mathbb{T} \mid X_t = X_T^{\uparrow}\right\}$ to be the last time of maximum of X. On the event $\{\zeta_0 < T\}$, and in view of $\mathbb{P}[X_{\zeta_0+1} \geq X_{\zeta_0} \mid \mathcal{F}_{\zeta_0}] > 0$ holding $\mathbb{P}$-a.s., we have $\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}] = 0$ holding $\mathbb{P}$-a.s. On the other hand, on the event $\{\zeta_0 = T\}$ we have $\mathbb{P}[\rho = \zeta_0 \mid \mathcal{F}_{\zeta_0}] = \mathbb{I}_{\{\rho=T\}}$, which is $\mathbb{P}$-a.s. $\{0,1\}$-valued. From statement (2) of Proposition 3.1, it follows that $\mathbb{Q}[\rho = \zeta_0] = 1$. Since $\mathbb{P}[\rho = \zeta_0 < T] = 0$ and $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, we obtain $\mathbb{Q}[\rho = T] = 1$.

A continuous-time version of Example 3.2 involving Brownian motion with drift over finite time-intervals will be given in §5.2, where it will be demonstrated in particular that the corresponding probabilities $\mathbb{P}$ and $\mathbb{Q}$ in that setting are singular.

3.2. **Time of maximum of nonnegative local martingales with zero terminal value, continuous running supremum and no jumps while at their running supremum.** For special cases of random times, the calculation of the canonical pair becomes relatively easy. More information and extensive discussion on the material of §3.2 can be found in [25], where exact connections with so-called *honest times* are presented.

Let us introduce some notation: $\mathcal{L}_0$ denotes the class of all nonnegative local martingales M such that $\mathbb{P}[M_0 = 1, M_\infty = 0] = 1$ (where $M_\infty := \lim_{t\to\infty} M_t$, noting that the limit in the definition of L_∞ exists in the $\mathbb{P}$-a.s. sense, in view of the nonnegative supermartingale convergence theorem), the running supremum process $M^* = M^{\uparrow}$ is continuous and $\left\{M_- = M_-^*\right\} \subseteq \{\Delta M = 0\}$ holds up to a $\mathbb{P}$-evanescent set. For $M \in \mathcal{L}_0$, define

$$\rho_M := \sup\left\{t \in \mathbb{R}_+ \mid M_{t-} = M_{t-}^*\right\}. \tag{3.1}$$

(The convention $M_{0-} = 0 = M_{0-}^*$ implies that the random set $\left\{t \in \mathbb{R}_+ \mid M_{t-} = M_{t-}^*\right\}$ is non-empty.) Since $\mathbb{P}[M_\infty = 0] = 1$ holds for $M \in \mathcal{L}_0$, it follows that $\mathbb{P}[\rho_M < \infty] = 1$. Whenever $M \in \mathcal{L}_0$, it $\mathbb{P}$-a.s. holds that $M_{\rho_M -} = M_{\rho_M} = M_{\rho_M}^*$; in fact, as [25, Theorem 1.2] implies, the previous random variables are also equal to M_∞^*, which makes ρ_M a time of *overall* maximum of $M \in \mathcal{L}_0$.

Proposition 3.3. *Let $M \in \mathcal{L}_0$, and let ρ be any time of maximum of M, in the sense that $\mathbb{P}[M_\rho = M_\infty^*] = 1$. Then, the following are true:*

- *The canonical pair associated with ρ is $(K, L) = (1 - 1/M^*, M)$.*
- *ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$.*

- $\mathbb{P}[\rho = \rho_M] = 1$.

Proof. Only a sketch of the proof is provided; as already mentioned, more information can be found in [25]. Note that $\mathbb{P}[\rho \leq \rho_M] = 1$ holds by definition on ρ_M; in particular, $\mathbb{P}[\rho < \infty] = 1$. The fact that ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$ follows from Doob's maximal identity, as presented in [30]; more precisely, $\mathbb{P}[\rho = \tau \mid \mathcal{F}_\tau] = 0$ holds on $\{\tau < \infty, M_\tau < M^*_\tau\}$, while on $\{\tau < \infty, M_\tau = M^*_\tau\}$ it follows that

$$\mathbb{P}[\rho = \tau \mid \mathcal{F}_\tau] = \mathbb{P}\left[\sup_{t\in[\tau,\infty)} M_t > M_\tau \;\middle|\; \mathcal{F}_\tau\right] = 1 - \frac{M_\tau}{M^*_\tau} = 0.$$

Doob's maximal identity applied again implies that $Z = M/M^*$ (see [30]); then, since ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$, one can use Remark 1.6 to conclude that the canonical pair associated with ρ is $(1 - 1/M^*, M)$.

Since ρ_M is a special instance of a random time that achieves the maximum of M, it follows that the pair associated with ρ_M is also $(1 - 1/M^*, M)$. Since the canonical pair associated to a random time completely determines its distribution, the laws of ρ and ρ_M are the same under $\mathbb{P}$. Combined with $\mathbb{P}[\rho \leq \rho_M] = 1$, we obtain $\mathbb{P}[\rho = \rho_M] = 1$. □

Remark 3.4. Proposition 3.3 implies in particular that there exists an almost surely unique time of maximum of processes in $\mathcal{L}_0$.

Remark 3.5. It was already hinted out in the discussion at §1.1 that the canonical pair (K, L) associated with a random time may be such that $\mathbb{P}[K_\infty < 1] > 0$ holds; additionally, L may fail to be a true martingale. Indeed, in the context of Proposition 3.3, $M = L$ can be freely chosen to be a strict local martingale in the terminology of [8]; furthermore, $\mathbb{P}[K_\infty < 1] = \mathbb{P}[L^*_\infty < \infty] = 1$.

Remark 3.6. Recall the set $\mathcal{S}$ from §1.5. Specialising to the setting of Proposition 3.3, let ρ be the time of maximum of $M \in \mathcal{L}_0$. In this case, and since $K_\rho = 1 - 1/M_\rho$, we obtain from Proposition 1.12 that $\mathbb{E}_\mathbb{P}[S_\rho \mid M_\rho] \leq M_\rho$ for all $S \in \mathcal{S}$. This result is quite interesting—it states that *no matter* what the level of M at its maximum, no other nonnegative supermartingale with unit initial value is expected to lie above that.

Since $\mathcal{S}$ is convex, the condition $\mathbb{E}_\mathbb{P}[S_\rho \mid M_\rho] \leq M_\rho$ for all $S \in \mathcal{S}$ is actually equivalent to the fact that M_ρ stochastically dominates all random variables in $\{S_\rho \mid S \in \mathcal{S}\}$ in second order, meaning that $\mathbb{E}_\mathbb{P}[U(S_\rho)] \leq \mathbb{E}_\mathbb{P}[U(M_\rho)]$ holds for all nondecreasing concave functions $U : \mathbb{R}_+ \mapsto \mathbb{R}$. In fact, a stronger statement is true. Since S is a nonnegative supermartingale on $(\Omega, \mathbf{F}, \mathbb{P})$ with $\mathbb{P}[S_0 = 1] = 1$ for all $S \in \mathcal{S}$, Doob's maximal inequality implies that $\mathbb{P}[S_\rho > x] \leq 1 \wedge (1/x)$ holds for all $x \in (0, \infty)$. On the other hand, since $M \in \mathcal{L}_0$, it follows from Doob's maximal identity [30] that $\mathbb{P}[M_\rho > x] = 1 \wedge (1/x)$ holds for all $x \in (0, \infty)$. Therefore, $\sup_{S\in\mathcal{S}} \mathbb{P}[S_\rho > x] = \mathbb{P}[M_\rho > x]$ holds for all $x \in (0, \infty)$, which implies that M_ρ stochastically dominates all random variables in $\{S_\rho \mid S \in \mathcal{S}\}$, even in first order.

Example 3.7. Let Ω be the canonical space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. Take X to be the coordinate process and $\mathbf{F}$ be the right-continuous augmentation of the natural filtration of X. For the time being, $\mathcal{F}$ is taken to be equal to $\mathcal{F}_\infty$. Let $\mathbb{P}$ be the unique probability on $(\Omega, \mathcal{F})$ under which X is a Brownian motion with (strictly negative) drift $\mu < 0$ and unit diffusion coefficient. Since $\mathbb{P}\left[\lim_{t\to\infty} X_t = -\infty\right] = 1$, consider a random time ρ that is a time of overall maximum of X. Note that ρ is also a time of maximum of the process $M := \exp(-2\mu X)$, which satisfies all the conditions of Proposition 3.3. We obtain that the canonical representation pair (K, L) of ρ on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is such that $K = 1 - \exp(2\mu X^\uparrow)$ and $L = \exp(-2\mu X)$. An application of Proposition 1.11 gives that $\sup_{t\in\mathbb{R}_+} X_t = (1/2\mu)\log(1 - K_\rho)$ has the exponential distribution with rate -2μ under $\mathbb{P}$—of course, this fact is well known.

Note that the process $L = \exp(-2\mu X)$ is a martingale on $(\Omega, \mathbf{F}, \mathbb{P})$. Since we are working on the canonical space, a joint application of the extension theorem of Daniell-Kolmogorov [21, §2.2A] and Girsanov's theorem [21, Section 3.5] imply there exists a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}, \mathbf{F})$ such that $\mathrm{d}\mathbb{Q} = L_t \mathrm{d}\mathbb{P}$ holds on each $\mathcal{F}_t$ for $t \in \mathbb{R}_+$, and under which X is a Brownian motion with drift $-\mu > 0$ and unit diffusion coefficient. In order to be in par with Assumption 2.1, we carry out the enlargement of the probability space as discussed in Remark 2.2. Then, it comes as a consequence of Theorem 2.4 that a path of X^ρ under $\mathbb{P}$ can be stochastically realised as follows:

(1) With U being a standard uniform random variable, set $X^\uparrow_\infty = X_\rho = (1/2\mu)\log(U)$.
(2) Given $x = X_\rho$, generate X^{τ_x} under $\mathbb{Q}$, where $\tau_x := \inf\{t \in \mathbb{R}_+ \mid X_t = x\}$.

The next example will settle a couple of claims that were previously made in Remark 2.3.

Example 3.8. Consider the interval $(0, \infty)$, with an extra "cemetery" state $\triangle$ appended in a way so that $\triangle$ is a topologically isolated point of $(0, \infty) \cup \{\triangle\}$. For a right-continuous path $\omega : \mathbb{R}_+ \mapsto (0, \infty) \cup \{\triangle\}$, define $\zeta(\omega) := \inf\{t \in \mathbb{R}_+ \mid \omega(t) = \triangle\}$. With the previous understanding, define Ω to be the space of all right-continuous paths $\omega : \mathbb{R}_+ \mapsto (0, \infty) \cup \{\triangle\}$ such that $\omega(0) \in (0, \infty)$, that are actually continuous on the interval $[0, \zeta(\omega))$ and $\omega(t) = \triangle$ holds for all $t \in [\zeta(\omega), \infty)$. Let X denote the coordinate process on Ω and $\mathbf{F}$ be the right-continuous augmentation of the natural filtration of X; then ζ becomes a stopping time on $(\Omega, \mathbf{F})$. Defining Ω as above is essential for ensuring that Assumption 2.1 is valid—see the discussion on standard systems, and, more particularly, [29, Example (6.3)].

Set $\beta(x) = 1 \vee x^2$ for $x \in (0, \infty)$. From the treatment of [21, Section 5.5], there exists a probability $\mathbb{P}$ on $\mathcal{F}$ such that the coordinate process X satisfies $\mathbb{P}[X_0 = 1] = 1$ and has dynamics $\mathrm{d}X_t = \beta(X_t)\mathrm{d}W^{\mathbb{P}}_t$, for $t \in [0, \zeta)$, where $W^{\mathbb{P}}$ is a standard Brownian motion under $\mathbb{P}$. (In general, $W^{\mathbb{P}}$ is defined only up to time ζ.) In fact, X is a strict local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ in the terminology of [8], as follows from results in [5]. Using Feller's test for explosions and the local martingale property, it is straightforward to check that $\mathbb{P}\left[\zeta \le t, X_{\zeta-} = 0\right] = \mathbb{P}\left[\zeta \le t\right] > 0$ holds for all $t \in (0, \infty)$. Let ρ denote a time of overall maximum of X. By Proposition 3.3, it follows that

$L = X\mathbb{I}_{[\![0,\zeta[\![}$. In order to characterise the probability $\mathbb{Q}$ that L induces as in Assumption 2.1, note that, *if* L was actually the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$, Girsanov's theorem would imply that the dynamics of X under $\mathbb{Q}$ are $\mathrm{d}X_t = \left(\beta^2(X_t)/X_t\right)\mathrm{d}t + \beta(X_t)\mathrm{d}W^{\mathbb{Q}}_t$ for $t \in [0,\zeta)$, with $W^{\mathbb{Q}}$ being a standard Brownian motion on $(\Omega, \mathbf{F}, \mathbb{Q})$. Even though L is not a martingale on $(\Omega, \mathbf{F}, \mathbb{P})$, the treatment of [21, Section 5.5] implies that there exists a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that the coordinate process X indeed satisfies $\mathbb{Q}[X_0 = 1] = 1$ and $\mathrm{d}X_t = \left(\beta^2(X_t)/X_t\right)\mathrm{d}t + \beta(X_t)\mathrm{d}W^{\mathbb{Q}}_t$ for $t \in [0,\zeta)$, where $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$, in general defined until time ζ. It is also clear that $\mathbb{Q}$ is exactly the probability that appears in Assumption 2.1. Writing the formal dynamics under $\mathbb{Q}$ of $1/X$ on the stochastic interval $[\![0,\zeta[\![$, it is straightforward to conclude that the law of $(1/X_t)_{t\in[0,\zeta)}$ under $\mathbb{Q}$ is the same as the law of $(X_t)_{t\in[0,\zeta)]}$ under $\mathbb{P}$. It follows that $\mathbb{P}\left[\zeta \le t,\, X_{\zeta-} = \infty\right] = \mathbb{P}\left[\zeta \le t\right] > 0$ holds for all $t \in (0,\infty)$. Coupled with the fact that $\mathbb{P}\left[\zeta \le t,\, X_{\zeta-} = 0\right] = \mathbb{P}\left[\zeta \le t\right] > 0$ holds for all $t \in (0,\infty)$ that was established above, we conclude that neither $\mathbb{Q} \ll_{\mathcal{F}_t} \mathbb{P}$ nor $\mathbb{P} \ll_{\mathcal{F}_t} \mathbb{Q}$ holds, for any $t \in (0,\infty)$.

The above example also illustrates that the filtration $\mathbf{F}$ should not be completed in any way by $\mathbb{P}$, if $\mathbb{Q}$ is to be defined. In fact, let $\mathbf{F}^{\mathbb{P}} = (\mathcal{F}^{\mathbb{P}}_t)_{t\in\mathbb{R}_+}$ be *any* right-continuous filtration such that:

- $\mathbf{F} \subseteq \mathbf{F}^{\mathbb{P}}$, and
- if $B \subseteq \bigcup_{n\in\mathbb{N}} B_n$, where $B_n \in \bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t$ and $\mathbb{P}\left[B_n\right] = 0$ holds for all $n \in \mathbb{N}$, then $B \in \mathcal{F}^{\mathbb{P}}_0$.

(Note that we are *not* asking that each $\mathcal{F}^{\mathbb{P}}_t$, $t \in \mathbb{R}_+$, contains all $\mathbb{P}$-null sets of $\mathcal{F}_\infty$, but a weaker condition that is tailored to avoid problems with singularities of probabilities at infinity—see [2] for the concept of such *natural*, as opposed to *usual*, augmentations.) For any $n \in \mathbb{N}$, $B_n := \{\zeta \le n,\, X_{\zeta-} = \infty\} \in \mathcal{F}_n$ and $\mathbb{P}\left[B_n\right] = 0$. In view of the assumptions on $\mathbf{F}^{\mathbb{P}}$, $\{\zeta < \infty,\, X_{\zeta-} = \infty\} \in \mathcal{F}^{\mathbb{P}}_0$. If $\mathbb{Q}$ could be defined, $\mathbb{Q}|_{\mathcal{F}^{\mathbb{P}}_{\eta_u}} \ll \mathbb{P}|_{\mathcal{F}^{\mathbb{P}}_{\eta_u}}$ would hold for $u \in [0,1)$; in particular, $\mathbb{Q}^{\mathbb{P}}|_{\mathcal{F}^{\mathbb{P}}_0} \ll \mathbb{P}|_{\mathcal{F}^{\mathbb{P}}_0}$. This is impossible: indeed, we should have $\mathbb{Q}[\zeta < \infty,\, X_{\zeta-} = \infty] = 1$, while it is true that $\mathbb{P}[\zeta < \infty,\, X_{\zeta-} = \infty] = 0$. Of course, since the filtration is *not* enlarged in order to include $\mathbb{P}$-null sets, one can define $\mathbb{Q}$ without problems.

3.3. **Last-passage times of nonnegative continuous-path local martingales vanishing at infinity.** Let M be a nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ with $M_0 = 1$, M having continuous paths and $\lim_{t\to\infty} M_t = 0$, all holding $\mathbb{P}$-a.s. In particular, and in the notation of §3.2, $M \in \mathcal{L}_0$. We fix $y \in \mathbb{R}_+$ and define $\rho := \sup\{t \in \mathbb{R}_+ \mid M_t = y\}$, setting $\rho = 0$ when the last set is empty. In words, ρ is the last passage time of M at level y. In this case, it is straightforward that

$$Z_t = \mathbb{P}[\rho > t \mid \mathcal{F}_t] = \frac{M_t}{y} \wedge 1, \quad \text{for all } t \in \mathbb{R}_+.$$

(The set-inclusion $\{M > y\} \subseteq \{Z = 1\}$ certainly holds modulo $\mathbb{P}$; the fact that $Z = M/y$ holds on $\{M \le y\}$ follows from Doob's maximal identity [30] because M has $\mathbb{P}$-a.s. continuous paths.)

Recall from §1.1 that $Z = N - A$ holds for an appropriate local martingale N on $(\Omega, \mathbf{F}, \mathbb{P})$. In order to compute N and A in the decomposition of Z, information on the jumps of A is required. Since A is the dual optional projection of $\mathbb{I}_{[\![\rho,\infty[\![}$ on $(\Omega, \mathbf{F}, \mathbb{P})$, $\Delta A_\tau = \mathbb{P}\left[\rho = \tau \mid \mathcal{F}_\tau\right]$

holds for any finite stopping time τ. Note that $A_0 = \mathbb{P}[\rho = 0] = 1 - Z_0 = 0 \vee (1 - 1/y)$. Furthermore, on $\{\tau > 0, M_\tau \neq y\}$, it is clear that $\mathbb{P}[\rho = \tau \mid \mathcal{F}_\tau] = 0$ holds for any finite stopping time τ. Furthermore, $\mathbb{P}[\rho \geq \tau \mid \mathcal{F}_\tau] = 1$ holds on $\{M_\tau = y\} \subseteq \{Z_\tau = 1\}$, which implies that on $\{\tau > 0, M_\tau = y\}$ it holds that $\mathbb{P}[\rho = \tau \mid \mathcal{F}_\tau] = 1 - \mathbb{P}[\rho > \tau \mid \mathcal{F}_\tau] = 1 - Z_\tau = 0$. We conclude that $\Delta A_\tau = 0$ on $\{\tau > 0\}$, which implies that A is a continuous-path process. It follows that $Z = N - A$ coincides with the Doob-Mayer decomposition of Z, where N is (necessarily) a continuous-path martingale with $N_0 = 1$. By the Meyer-Itô-Tanaka formula [33, Theorem IV.70], it holds that $\mathrm{d}N_t = (1/y)\mathbb{I}_{\{M_t \leq y\}}\mathrm{d}M_t$ and $\mathrm{d}A_t = (1/2y)\mathrm{d}\Lambda^M_t(y)$ for $t \in (0, \infty)$, where $(\Lambda^M_t(y))_{t\in\mathbb{R}_+}$ denotes the semimartingale local time of M at level y—see [33, page 216]. A bit of algebra on (1.1) gives

$$K = 1 - \left(1 \wedge \frac{1}{y}\right) \exp\left(-\frac{1}{2y}\Lambda^M(y)\right). \tag{3.2}$$

Furthermore, since $\{M \leq y\} \subseteq \{yZ = M\}$, the dynamics $\mathrm{d}N_t = (1/y)\mathbb{I}_{\{M_t \leq y\}}\mathrm{d}M_t$ for $t \in \mathbb{R}_+$ and (1.3) give

$$\frac{\mathrm{d}L_t}{L_t} = \mathbb{I}_{\{M_t \leq y\}} \frac{\mathrm{d}M_t}{M_t}, \text{ for } t \in [0, \zeta_0). \tag{3.3}$$

Remark 3.9. If Assumption 2.1 is valid, the dynamics in (3.3) suggest that the stochastic behaviour of processes under $\mathbb{Q}$ is like the one under $\mathbb{P}$ when $M > y$; furthermore, when $M \leq y$, the stochastic behaviour of processes under $\mathbb{Q}$ is like the one under the corresponding probability $\mathbb{Q}$ when the random time is the time of maximum of M, studied in §3.2. The reader should also check Example 4.8 in §4.2 for dynamics under $\mathbb{Q}$ in a one-dimensional diffusion setting.

Remark 3.10. Suppose that $y \in (0, 1]$. In this case, $K = 1 - \exp\big(-(1/2y)\Lambda^M(y)\big)$, so that $\Delta K = 0$ up to a $\mathbb{P}$-evanescent set. By Proposition 1.11, $K_\rho = K_\infty$ has the standard uniform distribution under $\mathbb{P}$. It follows that $\Lambda^M_\infty(y) = \Lambda^M_\rho(y)$ has the exponential distribution with rate parameter $2y$ under $\mathbb{P}$. Also, note that in this case that the last exit time ρ is actually the time of maximum of L, which becomes apparent once one writes

$$L = \frac{Z}{1 - K} = \left(\frac{M}{y} \wedge 1\right) \exp\left(\frac{1}{2y}\Lambda^M(y)\right)$$

and use the facts that $\mathbb{P}[M_\rho = y] = 1$ and $\mathbb{P}\big[\Lambda^M_\rho(y) = \Lambda^M_\infty(y)\big] = 1$.

Example 3.11. Recall the Brownian setting of Example 3.7. Suppose that $x \in \mathbb{R}$. Define $\rho := \sup\{t \in \mathbb{R}_+ \mid X_t = x\}$, where we set $\rho = 0$ when the last set is empty. Recalling that $M = \exp(-2\mu X)$, it holds that $\rho := \sup\{t \in \mathbb{R}_+ \mid M_t = y\}$, where $y = \exp(-2\mu x)$. Furthermore, straightforward computations using a combination of the two occupation-times formulas for Λ^X and Λ^M imply that we can choose the local times in a way so that $(1/y)\Lambda^M(y) = -2\mu\Lambda^X(x)$. Therefore, equation (3.2) in this case reads $K = 1 - (1 \wedge \exp(2\mu x)) \exp\big(\mu\Lambda^X(x)\big)$. By Proposition 1.11 it follows that $\Lambda^X_\infty(x) = \Lambda^X_\rho(x)$ is such that $\mathbb{P}\left[\Lambda^X_\infty(x) = 0\right] = 1 - \exp(2\mu x)$ when $x \in (0, \infty)$ and

$\mathbb{P}\left[\Lambda^X_\infty(x) = 0\right] = 0$ when $x \in (-\infty, 0]$; furthermore, given $\Lambda^X_\infty(x) > 0$, $\Lambda^X_\infty(x)$ has the exponential distribution with rate parameter $-\mu$ under $\mathbb{P}$.

Using Novikov's condition [21, Section 3.5.D], it is straightforward to check that the local martingale L in (3.3) is an actual martingale. The extension theorem of Daniell-Kolmogorov [21, §2.2A] implies that Assumption 2.1 is valid in this case (modulo the enlargement of the probability space in order to accommodate a uniform random variable). It is straightforward to check that, under $\mathbb{Q}$, the process X has dynamics $\mathrm{d}X_t = \mu\mathsf{sign}(X_t - x)\mathrm{d}t + \mathrm{d}W^{\mathbb{Q}}_t$ for $t \in \mathbb{R}_+$, where $\mathsf{sign} = \mathbb{I}_{(0,\infty)} - \mathbb{I}_{(-\infty,0]}$ and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. Dynamics like the ones of X under $\mathbb{Q}$ have been the object of study in previous literature; see, for example, [35] and [9, §5.2, page 96].

4. Applications to Financial Mathematics

4.1. **Market behavior up to the time of overall minimum of the numéraire portfolio.** For the purposes of §4.1 we shall not be needing Assumption 2.1; $(\Omega, \mathbf{F}, \mathbb{P})$ is taken to be a filtered probability space, where $\mathbf{F}$ actually satisfies the usual conditions of right-continuity and augmentation by $\mathbb{P}$-null sets of $\mathcal{F}$. On $(\Omega, \mathbf{F}, \mathbb{P})$, let $S = (S^i)_{i=1,\dots,d}$ be a sigma-bounded d-dimensional semimartingale. (The condition of sigma-boundedness is weaker than local boundedness of S—in fact, it is equivalent to the existence of strictly positive and nonincreasing predictable processes ϑ^i such that $\int_0^\cdot \vartheta^i_t \mathrm{d}S^i_t$ is a uniformly bounded process for each $i \in \{1, \dots, d\}$. For the concepts of sigma-localisation and sigma-martingales, the reader can refer to [19]. The concept of sigma-boundedness has also appeared in [26].) For each $i \in \{1, \dots, d\}$, S^i represents the discounted, with respect to some baseline security, price of a liquid asset in the market. This baseline security should be thought as a locally riskless asset. Starting with normalised unit capital, and investing according to some d-dimensional, $\mathbf{F}$-predictable and S-integrable strategy ϑ (modelling the number of liquid assets held in the portfolio), an economic agent's discounted wealth is given by $X^\vartheta = 1 + \int_0^\cdot \vartheta_t \cdot \mathrm{d}S_t$. (Stochastic integrals with respect to S are to be understood in the sense of vector stochastic integration—see [14].) Define $\mathcal{X}$ as the set of all processes X^ϑ in the previous notation that remain nonnegative at all times.

Assumption 4.1. In the above set-up, assume the following:

(1) There exists $\widehat{X} \in \mathcal{X}$ with the following properties:
   (a) $X/\widehat{X}$ is a supermartingale for all $X \in \mathcal{X}$.
   (b) $\Delta\widehat{X} \geq 0$ up to $\mathbb{P}$-evanescence. Furthermore, with $\widehat{I} := \inf_{t\in[0,\cdot]} \widehat{X}$, the set-inclusion $\{\widehat{X}_- = \widehat{I}_-\} \subseteq \{\Delta\widehat{X} = 0\}$ holds up to $\mathbb{P}$-evanescence.
(2) There exists $X \in \mathcal{X}$ such that $\mathbb{P}\left[\lim_{t\to\infty} X_t = \infty\right] = 1$.

Remark 4.2. Condition (1) in Assumption 4.1 is connected to market viability, and in particular to *absence of arbitrage of the first kind*, i.e., condition NA_1. (The market allows for arbitrage of the first kind if there exists $T \in \mathbb{R}_+$ and an $\mathcal{F}_T$-measurable random variable ξ with the properties

$\mathbb{P}[\xi \geq 0] = 1$ and $\mathbb{P}[\xi > 0] > 0$, and such that for all $x > 0$ there exists $X \in x\mathcal{X}$, which may depend on x, satisfying $\mathbb{P}[X_T \geq \xi] = 1$.) Condition NA_1 is actually equivalent to the requirement that $\lim_{m\to\infty} \sup_{X\in\mathcal{X}} \mathbb{P}[X_T > m] = 0$ holds for all $T \in \mathbb{R}_+$—see [22, Proposition 1]. It then comes as a consequence of results in [20] that absence of arbitrage of the first kind is equivalent to existence of $\widehat{X} \in \mathcal{X}$ such that $X/\widehat{X}$ is a supermartingale for all $X \in \mathcal{X}$, which is exactly condition (1a). Condition (1b) in Assumption 4.1 additionally forces certain requirements which will enable use of results from §3.2 and are crucial for the development below.

Condition (1) of Assumption 4.1 implies in particular that $1/\widehat{X}$ is a supermartingale on $(\Omega, \mathbf{F}, \mathbb{P})$. The next result refines this observation.

Lemma 4.3. *Under condition (1) of Assumption 4.1, $1/\widehat{X}$ is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$.*

Proof. Since both $\widehat{X}_- > 0$ and $\widehat{X} > 0$ hold, we have $\widehat{X} = 1 + \int_0^\cdot \widehat{X}_{t-} (\varphi_t \cdot \mathrm{d}S_t)$ for some d-dimensional predictable and S-integrable process φ. A straightforward application of [20, Lemma 3.4] shows that $L := 1/\widehat{X} = 1 - \int_0^\cdot L_{t-} \left(\varphi_t \cdot \mathrm{d}\overline{S}_t\right)$, where

$$\overline{S} := S - \left[{}^{\mathrm{c}}S, \int_0^\cdot (\varphi_t \cdot \mathrm{d}{}^{\mathrm{c}}S_t)\right] - \sum_{t\leq\cdot} \frac{\Delta\widehat{X}_t}{\widehat{X}_t} \Delta S_t,$$

with ${}^{\mathrm{c}}S$ denoting the uniquely defined continuous local martingale part of S (see, for example, [14]) and $[\cdot,\cdot]$ denotes the operator returning the quadratic covariation of semimartingales. Since $L_- > 0$ and $L > 0$, L is a local martingale if and only if $\int_0^\cdot \left(\varphi_t \cdot \mathrm{d}\overline{S}_t\right)$ is a local martingale. The supermartingale property of L already gives that $\int_0^\cdot \left(\varphi_t \cdot \mathrm{d}\overline{S}_t\right)$ is a local submartingale. We shall show that $\int_0^\cdot \left(\varphi_t, \mathrm{d}\overline{S}_t\right)$ is also a local supermartingale. Since $2\varphi \cdot \Delta S = 2(\Delta\widehat{X}/\widehat{X}_-) \geq 0$, the process X' defined implicitly via $X' = 1 + \int_0^\cdot X'_{t-} (2\varphi_t \cdot \mathrm{d}S_t)$ is an element of $\mathcal{X}$ with $X' > 0$ and $X'_- > 0$. Therefore, $X'/\widehat{X}$ is a nonnegative supermartingale. Again, [20, Lemma 3.4] shows that $X'/\widehat{X} = 1 + \int_0^\cdot (X'_{t-}/\widehat{X}_{t-}) \left(\varphi_t \cdot \mathrm{d}\overline{S}_t\right)$. The supermartingale property of $X'/\widehat{X}$ implies that $\int_0^\cdot \left(\varphi_t \cdot \mathrm{d}\overline{S}_t\right)$ is a local supermartingale, which completes the argument. □

Remark 4.4. Lemma 4.3 above follows part of the proof of [23, Theorem 2.15]. While the latter result really requires the full force of condition (1) in Assumption 4.1 in order to be valid, the set-inclusion $\{\widehat{X}_- = \widehat{I}_-\} \subseteq \{\Delta\widehat{X} = 0\}$ was erroneously neglected in [23, Theorem 2.15].

Given condition (1a) in Assumption 4.1, the nonnegative supermartingale convergence theorem implies that condition (2) in Assumption 4.1 is actually equivalent to $\mathbb{P}\left[\lim_{t\to\infty} \widehat{X}_t = \infty\right] = 1$. Let $L := 1/\widehat{X}$. Since $L_0 = 1$ and Assumption 4.1 implies that L^* is continuous and $\mathbb{P}[L_\infty = 0] = 1$, Lemma 4.3 and condition (1) of Assumption 4.1 imply that $L \in \mathcal{L}_0$, in the notation of §3.2. By Proposition 3.3, it follows that there exists a $\mathbb{P}$-a.s. unique time ρ of minimum of $\widehat{X}$, and that $(1 - 1/L^*, L)$ is the canonical representation pair associated with ρ. Let $\mathbf{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be the smallest right-continuous filtration that contains $\mathbf{F}$ and makes the random variable $\widehat{I}_\infty = \inf_{t\in\mathbb{R}_+} \widehat{X}_t$

be $\mathcal{G}_0$-measurable. In this case, ρ is $\mathbb{P}$-a.s. equal to the first time that $\widehat{X}$ equals $\widehat{I}_\infty$, which is a stopping time on $(\Omega, \mathbf{G})$; since $\mathbf{F}$ satisfies the usual conditions, we conclude that ρ is a stopping time on $(\Omega, \mathbf{G})$.

When S consists of continuous-path semimartingales, a version of the next result appears in [24, Theorem 1.4]. The strengthened result that is presented here has a short proof due to the previously-built theory.

Theorem 4.5. *Under Assumption 4.1 and the above notation, the d-dimensional process* $S^\rho = (S_{\rho\wedge t})_{t\in\mathbb{R}_+}$ *is a sigma-martingale on* $(\Omega, \mathbf{G}, \mathbb{P})$.

Proof. Let $X \in \mathcal{X}$. In the notation of §1.5, since $(X/\widehat{X}) \in \mathcal{S}$ and ρ is a time of maximum of $L := 1/\widehat{X}$, which in particular avoids all stopping times in view of Proposition 3.3, it follows that $\mathbb{E}_\mathbb{P}\big[X_\rho/\widehat{X}_\rho \mid K_\rho\big] \leq 1/\widehat{X}_\rho$. Since $K_\rho = 1 - 1/\widehat{X}_\rho$, the last equality translates to $\mathbb{E}_\mathbb{P}\left[X_\rho \mid K_\rho\right] \leq 1$; in other words, $\mathbb{E}_\mathbb{P}\left[X_\rho f(K_\rho)\right] \leq \mathbb{E}_\mathbb{P}[f(K_\rho)]$ is valid for all $X \in \mathcal{X}$ and Borel-measurable $f : [0,1) \mapsto \mathbb{R}_+$. Now, fix $t_1 \in \mathbb{R}_+$, $t_2 \in (t_1, \infty)$, $A \in \mathcal{F}_{t_1}$ and $X \in \mathcal{X}$ with $X \geq 1/2$. Let ϑ be so that $X = 1 + \int_0^\cdot \vartheta_t \cdot \mathrm{d}S_t$, and define $\vartheta' := (1/X_{t_1})\mathbb{I}_A\mathbb{I}_{]\!]t_1,t_2]\!]}\vartheta$ and $X' := 1 + \int_0^\cdot \vartheta'_t \cdot \mathrm{d}S_t$. It is straightforward to check that $X' \in \mathcal{X}$ and that $X'_\rho = \mathbb{I}_{\Omega\setminus A} + \big(X^\rho_{t_2}/X^\rho_{t_1}\big)\,\mathbb{I}_A$. Therefore, the inequality $\mathbb{E}_\mathbb{P}\left[X'_\rho f(K_\rho)\right] \leq \mathbb{E}_\mathbb{P}[f(K_\rho)]$ gives $\mathbb{E}_\mathbb{P}\left[\big(X^\rho_{t_2}/X^\rho_{t_1}\big)\, f(K_\rho)\mathbb{I}_A\right] \leq \mathbb{E}_\mathbb{P}[f(K_\rho)\mathbb{I}_A]$. Defining $\mathcal{G}^0_t = \mathcal{F}_t \vee \sigma(K_\rho)$ for all $t \in \mathbb{R}_+$ and ranging $A \in \mathcal{F}_{t_1}$, we obtain that $\mathbb{E}_\mathbb{P}\left[X^\rho_{t_2} \mid \mathcal{G}^0_{t_1}\right] \leq X^\rho_{t_1}$ holds for all $t_1 \in \mathbb{R}_+$, $t_2 \in (t_1, \infty)$ and $X \in \mathcal{X}$ with $X \geq 1/2$. By definition of the filtration $\mathbf{G}$, $\mathcal{G}_{t_1} = \bigcap_{t>t_1} \mathcal{G}^0_t$ holds; then, the conditional version of Fatou's lemma gives that $\mathbb{E}_\mathbb{P}\left[X^\rho_{t_2} \mid \mathcal{G}_{t_1}\right] \leq X^\rho_{t_1}$ holds for all $t_1 \in \mathbb{R}_+$, $t_2 \in (t_1, \infty)$ and $X \in \mathcal{X}$ with $X \geq 1/2$. Ranging $t_1 \in \mathbb{R}_+$ and $t_2 \in (t_1, \infty)$, we obtain that X^ρ is a supermartingale on $(\Omega, \mathbf{G}, \mathbb{P})$ for all $X \in \mathcal{X}$ with $X \geq 1/2$.

For each $i \in \{1, \ldots, d\}$ pick a strictly positive and nonincreasing predictable process process ϑ^i such that $|\int_0^\cdot \vartheta^i_t \mathrm{d}S^i_t| \leq 1/2$ identically holds. In this case, both processes $1 + \int_0^\cdot \vartheta^i_t \mathrm{d}S^i_t$ and $1 - \int_0^\cdot \vartheta^i_t \mathrm{d}S^i_t$ are elements of $\mathcal{X}$ and bounded below by $1/2$. It follows that $\int_0^{\rho\wedge\cdot} \vartheta^i_t \mathrm{d}S^i_t$ is both a supermartingale and a submartingale on $(\Omega, \mathbf{G}, \mathbb{P})$, which means that it is a martingale on $(\Omega, \mathbf{G}, \mathbb{P})$. Since ϑ^i is strictly positive, this implies that $(S^i_{\rho\wedge t})_{t\in\mathbb{R}_+}$ is a sigma-martingale on $(\Omega, \mathbf{G}, \mathbb{P})$ for all $i \in \{1, \ldots, d\}$. □

The importance of Theorem 4.5 lies in the following observation: with the "insider information" flow $\mathbf{G}$, investing in the risky assets before time ρ gives the same instantaneous return as the (locally) riskless asset, but entails (locally) higher risk; therefore, before ρ an insider would not be willing to take any position on the risky assets. In a sense, Theorem 4.5 endows $\widehat{X}$ the quality of an index of market status. Extensive discussion on this and further remarks can be found in [24].

4.2. **Valuation of exchange options and last-passage times.** In recent literature there has been considerable interest in representations of the value of plain vanilla options in terms of last passage times—in fact, the monograph [32] contains much of this development. Last-passage times

for continuous local martingales that vanish at infinity were considered in §3.3; that discussion will be used here to provide a further representation for the value of exchange options.

On $(\Omega, \mathbf{F}, \mathbb{P})$, let S^0 and S^1 be two nonnegative continuous-path semimartingales. The process S^0 satisfies $S^0_0 = 1$ and $\mathbb{P}\left[\inf_{t\in[0,T]} S^0_t > 0\right] = 1$ for all $T \in \mathbb{R}_+$, and should be considered as a baseline security. Set $R := S^1/S^0$ to denote the "exchange rate," i.e., the price process S^1 denominated in units of the baseline asset with price process S^0.

In the above market, consider an option to exchange at time $T \in \mathbb{R}_+$ a unit of a security with price process S^1 for κ units of the baseline security S^0. The option will be valid at time T only if the event $\{\sigma \leq T\}$ has occurred, where σ is a stopping time on $(\Omega, \mathbf{F})$. For example, one could take $\sigma = \inf\{t \in \mathbb{R}_+ \mid R_t > \lambda\}$ for some $\lambda > \kappa$, in which case the security is really an "up-and-in" exchange option. For a plain vanilla exchange option, one may set $\sigma = 0$.

Given that $\mathbb{P}$ is the valuation measure and that discounting is done using the baseline security, as is typically the case, the value of a European exchange option of the aforementioned type, to be exercised at time $T \in \mathbb{R}_+$, is $\mathsf{EE}_T = \mathbb{E}_{\mathbb{P}}\left[(\kappa - R_T)_+ \mathbb{I}_{\{\sigma\leq T\}}\right]$. Note that $\mathbb{P}$ is an equivalent local martingale measure for R, which means that R is a nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$.

Remark 4.6. In fact, the valuation formula for the European option is valid also for the value of the corresponding American option. In order to see this, let $\mathcal{T}_{[0,T]}$ be the class of all stopping times τ on $(\Omega, \mathbf{F})$ satisfying $0 \leq \tau \leq T$. Using $\mathbb{P}$ as valuation measure, an American option of the previous type has value $\mathsf{AE}_T := \sup_{\tau\in\mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{P}}\left[(\kappa - R_\tau)_+ \mathbb{I}_{\{\sigma\leq\tau\}}\right]$. Given that R is a nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$, thus a supermartingale on $(\Omega, \mathbf{F}, \mathbb{P})$, it is straightforward that the process $((\kappa - R_t)_+)_{t\in\mathbb{R}_+}$ is a submartingale on $(\Omega, \mathbf{F}, \mathbb{P})$. Then, for any $\tau \in \mathcal{T}_{[0,T]}$ it holds that

$$\mathbb{E}_{\mathbb{P}}\left[(\kappa - R_T)_+ \mathbb{I}_{\{\sigma\leq T\}} \mid \mathcal{F}_\tau\right] \geq \mathbb{E}_{\mathbb{P}}\left[(\kappa - R_T)_+ \mathbb{I}_{\{\sigma\leq\tau\}} \mid \mathcal{F}_\tau\right] \geq (\kappa - R_\tau)_+ \mathbb{I}_{\{\sigma\leq\tau\}},$$

which readily gives

$$\mathsf{AE}_T = \sup_{\tau\in\mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{P}}\left[(\kappa - R_\tau)_+ \mathbb{I}_{\{\sigma\leq\tau\}}\right] = \mathbb{E}_{\mathbb{P}}\left[(\kappa - R_T)_+ \mathbb{I}_{\{\sigma\leq T\}}\right] = \mathsf{EE}_T.$$

For $\kappa \in \mathbb{R}_+$, define the random time $\rho := \sup\{t \in \mathbb{R}_+ \mid R_t = \kappa\}$, where we set $\rho = 0$ if the last set is empty. Under the force of Assumption 2.1, denote by $\mathbb{Q}$ the probability corresponding to ρ.

Proposition 4.7. *In the above set-up, suppose that* $\mathbb{P}\left[\lim_{t\to\infty} R_t = 0\right] = 1$ *and that the validity of Assumption 2.1 is in force for the random time* ρ*. Then, it holds that*

$$\mathsf{EE}_T = \kappa\mathbb{P}\left[\rho \wedge \sigma \leq T\right] = \kappa\mathbb{P}\left[\sigma \leq T\right] - \kappa(1\wedge\kappa)\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\frac{\kappa}{2}\Lambda^R_T(\kappa)\right)\mathbb{I}_{\{\sigma\leq T\}}\right]. \tag{4.1}$$

Proof. Under the validity of $\mathbb{P}\left[\lim_{t\to\infty} R_t = 0\right] = 1$, the equality $(\kappa - R_T)_+ = \kappa\mathbb{P}\left[\rho \leq T \mid \mathcal{F}_T\right]$ holds in view of [32, Theorem 2.5]; then, the first equality in (4.1) follows from the fact that $\{\sigma \leq T\} \in \mathcal{F}_T$. For the second equality in (4.1), note that, in view of (3.2), the process K in the canonical representation pair of ρ on $(\Omega, \mathbf{F}, \mathbb{P})$ is such that $1 - K = (1\wedge\kappa)\exp\left(-(\kappa/2)\Lambda^R(\kappa)\right)$.

By Remark 2.5, and since $\{\sigma \leq T\} \in \mathcal{F}_T$,

$$\mathbb{P}[\rho \wedge \sigma \leq T] = \mathbb{P}[\sigma \leq T] - \mathbb{P}[\sigma \leq T, \rho > T] = \mathbb{P}[\sigma \leq T] - \mathbb{E}_{\mathbb{Q}}\left[(1 - K_T)\,\mathbb{I}_{\{\sigma \leq T\}}\right],$$

which concludes the proof. □

Example 4.8. We present here an example where the "exchange rate" process R behaves as a one-dimensional diffusion under $\mathbb{P}$. Exact modelling of S^0 and S^1 is not necessary.

The filtered measurable space will be the exact one considered in Example 3.8, where the reader is referred to for all the details. Recall that X denotes the coordinate process and $\mathbf{F}$ be the right-continuous augmentation of the natural filtration of X. The sigma-algebra $\mathcal{F}$ is taken to be equal to $\mathcal{F}_\infty$. Note that this set-up is essential for ensuring that Assumption 2.1 is valid (modulo the enlargement discussed in Remark 2.2 in order to accommodate for an independent uniform random variable).

Fix a function $\beta : (0, \infty) \mapsto (0, \infty)$ such that $1/\beta^2$ is locally integrable on $(0, \infty)$. From the treatment of [21, Section 5.5], for any $x_0 \in \mathbb{R}_+$ there exists a probability $\mathbb{P}$ on $\mathcal{F}$ (which coincides with the Borel sigma-algebra on Ω) such that $\mathbb{P}[X_0 = x_0] = 1$, and X has dynamics

$$\frac{\mathrm{d}X_t}{X_t} = \beta(X_t)\mathrm{d}W_t^{\mathbb{P}}, \quad \text{for } t \in [0, \zeta),$$

where recall that $\zeta := \inf\{t \in \mathbb{R}_+ \mid X_t = \triangle\}$, and $W^{\mathbb{P}}$ is a standard Brownian motion (defined only up to time ζ) under $\mathbb{P}$. Due to the nonnegative local martingale convergence theorem and the fact that $\beta : (0, \infty) \mapsto (0, \infty)$ is such that $1/\beta^2$ is locally integrable on $(0, \infty)$, it follows in straightforward way that $\mathbb{P}[X_{\zeta-} = 0] = 1$. Letting $R := X\mathbb{I}_{[\![0,\zeta[\![}$, note that the assumptions of Proposition 4.7 are satisfied.

Regarding the probability $\mathbb{Q}$, (3.3) implies that the local martingale L on $(\Omega, \mathbf{F}, \mathbb{P})$ in the canonical representation pair of ρ is such that $\mathrm{d}L_t/L_t = \mathbb{I}_{\{X_t \leq \kappa\}}(\mathrm{d}X_t/X_t) = \mathbb{I}_{\{X_t \leq \kappa\}}\beta(X_t)\mathrm{d}W_t^{\mathbb{P}}$, for $t \in [0, \zeta)$. Using Girsanov's theorem, it is straightforward to then check that

$$\frac{\mathrm{d}X_t}{X_t} = \beta^2(X_t)\mathbb{I}_{\{X_t \leq \kappa\}}\mathrm{d}t + \beta(X_t)\mathrm{d}W_t^{\mathbb{Q}}, \quad \text{for } t \in [0, \zeta), \tag{4.2}$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. (Even though L may fail to be a true martingale on $(\Omega, \mathbf{F}, \mathbb{P})$, one infers the existence of the probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that the dynamics of X are given by (4.2) using knowledge of weak solutions of stochastic differential equations with possible explosions from the treatment of [21, §5.5].) By employing Feller's test for explosions, it can be easily seen that X under $\mathbb{Q}$ does not explode, i.e., does not exit $(0, \infty)$ in finite time; i.e, $R = X$ under $\mathbb{Q}$. In fact, by calculating the scale function of X, one may conclude that $R = X$ becomes a recurrent Markov process under $\mathbb{Q}$.

5. Time of Maximum and Last-Passage Times of Brownian Motion with Drift over Finite Time-Intervals

5.1. **Set-up.** For the purposes of Section 5, $T \in \mathbb{R}_+$ will be fixed. Define Ω as the canonical path-space of continuous functions from $[0,T)$ to $\mathbb{R}$. Call X the coordinate process, let $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T)}$ be the right-continuous augmentation of the natural filtration of X, and set $\mathcal{F} = \bigvee_{t\in[0,T)} \mathcal{F}_t$.

Remark 5.1. It is important to note that the canonical space of processes with time-index $[0,T)$, as opposed to $[0,T]$, is considered here. As will become clear, it is in this setting that we can ensure later the validity of Assumption 2.1 (modulo the enlargement of the space in order to accommodate a random variable with the uniform law and independent of $\mathcal{F}_\infty$, as discussed in Remark 2.2).

Fix $\mu \in \mathbb{R}$. On $(\Omega, \mathcal{F})$, let $\mathbb{P}$ be the probability under which X is a Brownian motion with drift μ and unit diffusion coefficient. In the rest of Section 5, and using the previously-developed theory, we discuss the behaviour of X up to the time of maximum and last-passage times of X. We shall calculate the canonical associated pair (K, L) in each case, and via L we shall describe the dynamics of X under $\mathbb{Q}$ (generated by L). In view of Section 2, this gives a complete characterisation of the stochastic behaviour of optional processes up to the random times that are considered.

5.2. **Time of maximum.** Define $\rho := \sup\left\{t \in [0,T) \mid X_t = \sup_{s\in[0,T)} X_s\right\}$, where by convention one sets $\rho = T$ if the previous set is empty.

In the sequel, we shall make use of the following functions, related to the standard normal law:

$$\overline{\Phi}(x) = \int_x^\infty \phi(y)\mathrm{d}y, \quad \text{where } \phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad \text{for } x \in \mathbb{R}.$$

Define the function $F_\mu : (0,\infty) \times \mathbb{R}_+ \mapsto [0,1]$ via

$$F_\mu(\tau, z) := \exp(2\mu z)\,\overline{\Phi}\left(\frac{z+\mu\tau}{\sqrt{\tau}}\right) + \overline{\Phi}\left(\frac{z-\mu\tau}{\sqrt{\tau}}\right) = \int_0^\tau \left(\frac{z}{\sqrt{2\pi s^3}} \exp\left(-\frac{(z-\mu s)^2}{2s}\right)\right) \mathrm{d}s, \tag{5.1}$$

for $(\tau, z) \in (0,\infty) \times \mathbb{R}_+$. The second equality follows upon differentiation of the defining quantity giving F_μ with respect to the temporal variable. The fact that F_μ is $[0,1]$-valued follows from the second representation, since the quantity inside the integral is the density of the first hitting time of the level z for Brownian motion with drift μ—see [21, page 197, equation (5.12)]. By this last fact and the Markovian property of Brownian motion, it is straightforward that

$$Z_t = \mathbb{P}[\rho > t \mid \mathcal{F}_t] = F_\mu\left(T - t, X_t^\uparrow - X_t\right), \quad \text{for } t \in [0,T),$$

where recall that $X^\uparrow = \sup_{t\in[0,\cdot]} X$. In preparation for the formulas below, note that

$$\frac{\partial F_\mu}{\partial z}(\tau, z) = 2\mu \exp(2\mu z)\,\overline{\Phi}\left(\frac{z+\mu\tau}{\sqrt{\tau}}\right) - \frac{2}{\sqrt{\tau}}\phi\left(\frac{z-\mu\tau}{\sqrt{\tau}}\right), \quad \text{for } (\tau, z) \in (0,\infty) \times \mathbb{R}_+, \tag{5.2}$$

where the fact that $\exp(2\mu z)\,\phi\left(z/\sqrt{\tau}+\mu\sqrt{\tau}\right) = \phi\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)$ for $(\tau, z) \in (0,\infty)\times\mathbb{R}_+$ holds was used in the above calculation. Define also the function $f_\mu : (0,\infty) \mapsto \mathbb{R}$ via

$$f_\mu(\tau) := -\frac{\partial F_\mu}{\partial z}(\tau, 0) = \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{\mu^2\tau}{2}\right) - 2\mu\overline{\Phi}(\mu\sqrt{\tau}), \quad \text{for } \tau \in (0,\infty).$$

Upon simple differentiation it is easy to check that the function f_μ is decreasing in $\tau \in (0,\infty)$. As $\lim_{\tau\to\infty} f_\mu(\tau) = \max\{0, -2\mu\} \in \mathbb{R}_+$, f_μ is nonnegative.

Since Z has continuous paths and all martingales on $(\Omega, \mathbf{F}, \mathbb{P})$ have continuous paths as well, it follows that A is the continuous nondecreasing process appearing in the additive Doob-Meyer decomposition of $-Z$. In view of Proposition 1.11, ρ avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$. A simple use of Itô's formula gives, after some term cancellations, that

$$\mathrm{d}Z_t = -\frac{\partial F_\mu}{\partial z}\left(T-t, X_t^{\uparrow} - X_t\right)\mathrm{d}\left(X_t - \mu t\right) - f_\mu(T-t)\mathrm{d}X_t^{\uparrow}, \quad \text{for } t \in [0,T). \tag{5.3}$$

In particular, it holds that $A = \int_0^{\cdot} f_\mu(T-t)\mathrm{d}X_t^{\uparrow}$. From (1.1), it then follows that

$$K_t = 1 - \exp\left(-\int_0^t f_\mu(T-s)\mathrm{d}X_s^{\uparrow}\right), \quad \text{for } t \in [0,T). \tag{5.4}$$

Using the equality $L = Z/(1-K)$, it follows that

$$L_t = F_\mu\left(T-t, X_t^{\uparrow} - X_t\right)\exp\left(\int_0^t f_\mu(T-s)\mathrm{d}X_s^{\uparrow}\right), \quad \text{for } t \in [0,T). \tag{5.5}$$

The next result ensures that Assumption 2.1 will be valid in this setting.

Lemma 5.2. *For all $t \in [0,T)$, it holds that $\mathbb{E}_{\mathbb{P}}[L_t] = 1$.*

Proof. Since $(L_t)_{t\in[0,T)}$ is a nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ with $L_0 = 1$, $\mathbb{E}_{\mathbb{P}}[L_t] = 1$ for all $t \in [0,T)$ will follow if $\mathbb{E}_{\mathbb{P}}[L_t^*] < \infty$ for all $t \in [0,T)$ is established. Given that the function F_μ is a $[0,1]$-valued and that the function f_μ is decreasing, (5.5) implies that $L_t^* \le \exp\left(f_\mu(T-t)X_t^{\uparrow}\right)$ holds for all $t \in [0,T)$. Therefore, $\mathbb{E}_{\mathbb{P}}[L_t^*] < \infty$ for all $t \in [0,T)$ will follow if it is established that $\mathbb{E}_{\mathbb{P}}\left[\exp(aX_t^{\uparrow})\right] < \infty$ holds for all $a \in \mathbb{R}$ and $t \in \mathbb{R}_+$. To see this, note first that in view of Girsanov's theorem and Hölder's inequality, one may assume that $\mu = 0$. Then, the claim follows because, for $\mu = 0$, the law of $X_t^{\uparrow}$ under $\mathbb{P}$ is the same as the law of $|X_t|$ under $\mathbb{P}$, and all exponential moments of the latter law are finite. □

By Lemma 5.2 and the extension theorem of Daniell-Kolmogorov [21, §2.2A], there exists a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that L_t is the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$ for all $t \in [0,T)$. (It is exactly here that the point of Remark 5.1 becomes relevant.) It follows either from (5.3) of from (5.5) that the dynamics of L are

$$\frac{\mathrm{d}L_t}{L_t} = -\frac{(\partial F_\mu/\partial z)\left(T-t, X_t^{\uparrow} - X_t\right)}{F_\mu\left(T-t, X_t^{\uparrow} - X_t\right)}\mathrm{d}\left(X_t - \mu t\right), \quad \text{for } t \in [0,T).$$

A straightforward application of Girsanov's theorem imply that, under $\mathbb{Q}$, the dynamics of X are

$$\mathrm{d}X_t = G_\mu\left(T-t, X_t^{\uparrow} - X_t\right)\mathrm{d}t + \mathrm{d}W_t^{\mathbb{Q}}, \quad \text{for } t \in [0,T), \tag{5.6}$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion on $(\Omega, \mathbf{F}, \mathbb{Q})$ and $G_\mu : (0,\infty)\times\mathbb{R}_+ \mapsto \mathbb{R}$ is a function satisfying $G_\mu(\tau,z) = \mu - (\partial F_\mu/\partial z)(\tau,z)/F_\mu(\tau,z)$ for $(\tau,z)\in(0,\infty)\times\mathbb{R}_+$. A use of (5.2) gives

$$G_\mu(\tau,z) = \mu + \frac{(2/\sqrt{\tau})\,\phi\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right) - 2\mu\exp\left(2\mu z\right)\overline{\Phi}\left(z/\sqrt{\tau}+\mu\sqrt{\tau}\right)}{\overline{\Phi}\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)+\exp\left(2\mu z\right)\overline{\Phi}\left(z/\sqrt{\tau}+\mu\sqrt{\tau}\right)}, \quad \text{for } (\tau,z)\in(0,\infty)\times\mathbb{R}_+. \tag{5.7}$$

Remark 5.3. When $\mu\in(-\infty,0)$, it is straightforward to calculate $\lim_{\tau\to\infty}F_\mu(\tau,z) = \exp(2\mu z)$ and $\lim_{\tau\to\infty}G_\mu(\tau,z) = -\mu$ for all $z\in\mathbb{R}_+$, as well as $\lim_{\tau\to\infty}f_\mu(\tau,z) = -2\mu$. Formally plugging these long-run limits in (5.4), (5.5) and (5.6), the set-up and results of Example 3.7 are recovered.

Remark 5.4. When $\mu = 0$, previous formulas simplify significantly. In this case, $F_0(\tau,z) = 2\overline{\Phi}\left(z/\sqrt{\tau}\right)$ for $(\tau,z)\in(0,\infty)\times\mathbb{R}_+$, $f_0(\tau) = 1/\sqrt{2\pi\tau}$ for $\tau\in(0,\infty)$, and the function G_0 appearing in the dynamics (5.6) is given by $G_0(\tau,z) = (1/\sqrt{\tau})\left(\phi\left(z/\sqrt{\tau}\right)/\overline{\Phi}\left(z/\sqrt{\tau}\right)\right)$, for $(\tau,z)\in(0,\infty)\times\mathbb{R}_+$. Upon differentiation, it is straightforward to check that $(0,\infty)\times\mathbb{R}_+\ni(\tau,z)\mapsto G_0(\tau,z)$ is decreasing in τ and increasing in z. This is a very plausible behaviour: recalling the dynamics (5.6) under $\mathbb{Q}$, one would expect the drift to increase both when X is moving away from its maximum and when the "time to maturity" $\tau = T-t$ is getting shorter.

It is conjectured that the function $(0,\infty)\times(0,\infty)\ni(\tau,z)\mapsto G_\mu(\tau,z)$ is decreasing in τ and increasing in z for all $\mu\in\mathbb{R}$—this was discussed for the case $\mu=0$ in Remark 5.4. However, the calculations towards proving such a statement for all $\mu\in\mathbb{R}$ seem quite tedious. Proposition 5.5 that follows gives important information on G_μ for arbitrary $\mu\in\mathbb{R}$.

Proposition 5.5. *The function G_μ is $\mathbb{R}_+$-valued and such that $\liminf_{\tau\downarrow 0}\left(\inf_{z\in[w,\infty)}\left(\tau G_\mu(\tau,z)\right)\right)\geq w$ holds for all $w\in(0,\infty)$. In particular, it follows that X is a local submartingale on $(\Omega,\mathbf{F},\mathbb{Q})$ and that $\mathbb{Q}\big[\liminf_{t\to T}(X_t^{\uparrow}-X_t)=0\big]=1$.*

Proof. Let $c\in\mathbb{R}$ and $d\in\mathbb{R}$. A simple change of variables implies that

$$\begin{aligned}\exp(2cd)\overline{\Phi}(c+d) &= \int_{c+d}^{\infty}\exp\left(2cd-\frac{x^2}{2}\right)\frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{d-c}^{\infty}\exp\left(2cd-\frac{(x+2c)^2}{2}\right)\frac{\mathrm{d}x}{\sqrt{2\pi}}\\ &= \int_{d-c}^{\infty}\exp\left(2c(d-c-x)\right)\exp\left(-\frac{x^2}{2}\right)\frac{\mathrm{d}x}{\sqrt{2\pi}}.\end{aligned}$$

When $x\geq d-c$, it holds that $c\exp\left(2c(d-c-x)\right)\leq c$, for any $c\in\mathbb{R}$. Therefore, from the equalities above we obtain $c\exp(2cd)\overline{\Phi}(c+d)\leq c\overline{\Phi}(d-c)$. Applying the previous inequality above with $c=\mu\sqrt{\tau}$ and $d=z/\sqrt{\tau}$, it follows that $\mu\overline{\Phi}\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)-\mu\exp\left(2\mu z\right)\overline{\Phi}\left(z/\sqrt{\tau}+\mu\sqrt{\tau}\right)\geq 0$ for all $(\tau,z)\in(0,\infty)\times\mathbb{R}_+$. By (5.7), we obtain

$$G_\mu(\tau,z)\geq\frac{(2/\sqrt{\tau})\,\phi\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)}{\overline{\Phi}\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)+\exp\left(2\mu z\right)\overline{\Phi}\left(z/\sqrt{\tau}+\mu\sqrt{\tau}\right)}, \quad \text{for all } (\tau,z)\in(0,\infty)\times\mathbb{R}_+, \tag{5.8}$$

from which it immediately follows that G_μ is a nonnegative function. The fact that X is a local submartingale in $(\Omega, \mathbf{F}, \mathbb{Q})$ then follows from the dynamics (5.6).

Continuing, fix $w \in (0,\infty)$. Using the uniform estimates $1 - 1/x^2 \leq x\overline{\Phi}(x)/\phi(x) \leq 1$, valid for $x \in (0,\infty)$ (see, for example, [6, Theorem 1.2.3, page 11]), and the fact that the equality $\exp(2\mu z)\,\phi(z/\sqrt{\tau} + \mu\sqrt{\tau}) = \phi(z/\sqrt{\tau} - \mu\sqrt{\tau})$ holds for all $(\tau, z) \in (0,\infty) \times \mathbb{R}_+$, we obtain that

$$\lim_{\tau\downarrow 0}\left(\inf_{z\geq w}\left(\frac{2\sqrt{\tau}\phi\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)}{\overline{\Phi}\left(z/\sqrt{\tau}-\mu\sqrt{\tau}\right)+\exp\left(2\mu z\right)\Phi\left(z/\sqrt{\tau}+\mu\sqrt{\tau}\right)}\right)\right) = w.$$

Therefore, (5.8) gives $\liminf_{\tau\downarrow 0}(\inf_{z\geq w}(\tau G(\tau,z))) \geq w$ for all $w \in (0,\infty)$. According to this fact and the dynamics given in (5.6), on the event $\left\{\liminf_{t\to T}(X_t^{\uparrow} - X_t) > 0\right\}$ one would obtain $\lim_{t\to T} X_t = \infty$ under $\mathbb{Q}$—indeed, the drift term in the dynamics (5.6) would dominate (up to a strictly positive random variable) the quantity $1/(T-t)$ when t approaches T, implying that the behaviour of X itself near T would be explosive. However, in that case $\lim_{t\to T}(X_t^{\uparrow} - X_t) = 0$ would hold on $\left\{\liminf_{t\to T}(X_t^{\uparrow} - X_t) > 0\right\}$ under $\mathbb{Q}$, since $X_t < \infty$ holds for all $t \in [0,T)$. We conclude that $\mathbb{Q}\left[\liminf_{t\to T}(X_t^{\uparrow} - X_t) = 0\right] = 1$. □

Remark 5.6. The fact that $\mathbb{Q}\left[\liminf_{t\to T}(X_t^{\uparrow} - X_t) = 0\right] = 1$ is the equivalent of $\mathbb{Q}[\rho = T] = 1$ that was obtained in the finite-horizon discrete-time analogue discussed in Example 3.2. However, in contrast to Example 3.2, the fact that $\mathbb{P}\left[\lim_{t\to T}(X_t^{\uparrow} - X_t) > 0\right] = 1$ implies that in the present setting $\mathbb{P}$ and $\mathbb{Q}$ are singular probabilities on $\mathcal{F}$. (Note also that $\mathbb{P}\left[\liminf_{t\to T}(X_t^{\uparrow} - X_t) > 0\right] = 1$ implies $\mathbb{P}\left[\lim_{t\to T} L_t = 0\right] = 1$, which directly shows the singularity of $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{F}$.)

5.3. **Last-passage times.** Fix $x \in \mathbb{R}$ and define $\rho := \sup\{t \in [0,T) \mid X_t = x\}$, where one sets $\rho = 0$ if the previous set is empty. Recalling the definition of the function F_μ from (5.1), it is straightforward to compute

(5.9)
$$Z_t = \mathbb{P}[\rho > t \mid \mathcal{F}_t] = F_\mu\left(T-t, x - X_t\right)\mathbb{I}_{\{X_t \leq x\}} + F_{-\mu}\left(T-t, X_t - x\right)\mathbb{I}_{\{X_t > x\}}, \quad \text{for } t \in [0,T).$$

In particular, $Z_0 = \mathbb{P}[\rho > 0] = 1 - F_{\mathsf{sign}(x)\mu}(T, |x|)$. Define also the function $h_\mu : (0,\infty) \mapsto \mathbb{R}_+$ via

$$h_\mu(\tau) := -\frac{1}{2}\left(\frac{\partial F_\mu}{\partial z} + \frac{\partial F_{-\mu}}{\partial z}\right)(\tau, 0) = \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{\mu^2\tau}{2}\right) - \mu\left(1 - 2\overline{\Phi}(\mu\sqrt{\tau})\right), \quad \text{for } \tau \in (0,\infty).$$

Upon differentiation, one checks that the nonnegative function h_μ is decreasing in $\tau \in (0,\infty)$.

By a straightforward generalisation of the Itô-Tanaka formula, one can write $Z = N - A$, where N is a local martingale (with necessarily continuous paths) and $A = \int_0^{\cdot} h_\mu(T-t)\mathrm{d}\Lambda_t^X(x)$. Recalling that $\mathbb{P}[\rho > 0] = 1 - F_{\mathsf{sign}(x)\mu}(T, |x|)$, it follows from (1.1) that

$$K_t = 1 - \left(1 - F_{\mathsf{sign}(x)\mu}(T, |x|)\right)\exp\left(-\int_0^t h_\mu(T-s)\mathrm{d}\Lambda_s^X(x)\right), \quad \text{for } t \in [0,T). \tag{5.10}$$

Since $L = Z/(1-K)$, (5.9) and (5.10) give a closed-form expression for L.

Lemma 5.7. *For all $t \in [0,T)$, it holds that $\mathbb{E}_{\mathbb{P}}[L_t] = 1$.*

Proof. As in the proof of Lemma 5.2, it will be shown that $\mathbb{E}_{\mathbb{P}}[L_t^*] < \infty$ holds for all $t \in [0,T)$. Since $L \leq 1/(1-K)$ and h_μ is a decreasing function, for all $t \in [0,T)$ we obtain the inequality $L_t^* \leq \left(1 - F_{\mathsf{sign}(x)\mu}(T,|x|)\right)^{-1} \exp\left(h_\mu(T-t)\Lambda_t^X(x)\right)$. Therefore, it suffices to show that $\mathbb{E}_{\mathbb{P}}\left[\exp(a\Lambda_t^X(x))\right] < \infty$ holds for all $a \in \mathbb{R}$ and $t \in \mathbb{R}_+$. For this, and in view of Girsanov's theorem and Hölder's inequality, one may assume that $\mu = 0$. Then, the properties of standard Brownian motion imply that, for $\mu = 0$, the law of $\Lambda_t^X(x)$ under $\mathbb{P}$ is stochastically dominated in the first order by the law of $\Lambda_t^X(0)$ under $\mathbb{P}$. Furthermore, Lévy's equivalence theorem on Brownian local time and maximum of Brownian motion [21, Theorem 3.6.17] implies that the law of $\Lambda_t^X(0)$ under $\mathbb{P}$ is the same as the law of $X_t^\uparrow$ under $\mathbb{P}$; the latter is also the same as the law of $|X_t|$ under $\mathbb{P}$, for which all exponential moments are finite. □

By Lemma 5.7 and the Daniell-Kolmogorov extension theorem, there exists a probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $L_t = (\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_t}$ holds for all $t \in [0,T)$. (Remark 5.1 becomes again relevant at this point.) Since $L = Z/(1-K)$, using (5.9) and (5.10) we obtain the dynamics of L as

$$\frac{\mathrm{d}L_t}{L_t} = \left(-\frac{(\partial F_\mu/\partial z)(T-t, x-X_t)}{F_\mu(T-t, x-X_t)}\mathbb{I}_{\{X_t \leq x\}} + \frac{(\partial F_{-\mu}/\partial z)(T-t, X_t-x)}{F_{-\mu}(T-t, X_t-x)}\mathbb{I}_{\{X_t > x\}}\right)\mathrm{d}\left(X_t - \mu t\right),$$

for $t \in [0,T)$. Then, a straightforward application of Girsanov's theorem and (5.2) imply that, under $\mathbb{Q}$, the dynamics of X are given by

$$\mathrm{d}X_t = \left(G_\mu(T-t, x-X_t)\,\mathbb{I}_{\{X_t \leq x\}} - G_{-\mu}(T-t, X_t-x)\,\mathbb{I}_{\{X_t > x\}}\right)\mathrm{d}t + \mathrm{d}W_t^{\mathbb{Q}}, \quad \text{for } t \in [0,T),$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion on $(\Omega, \mathbf{F}, \mathbb{Q})$ and the function G_μ is defined in (5.7).

Remark 5.8. As was the case in §5.2, when the Brownian motion has zero drift the formulas simplify. In particular, when $\mu = 0$,

$$K_t = 1 - \left(1 - 2\overline{\Phi}\left(\frac{|x|}{\sqrt{T}}\right)\right)\exp\left(-\frac{1}{\sqrt{2\pi}}\int_0^t \frac{1}{\sqrt{T-s}}\mathrm{d}\Lambda_s^X(x)\right), \quad \text{for } t \in [0,T)$$

and, under $\mathbb{Q}$, the dynamics of X are given by

$$\mathrm{d}X_t = -\mathsf{sign}(X_t - x)\left(\frac{1}{\sqrt{T-t}}\frac{\phi\left(|X_t - x|/\sqrt{T-t}\right)}{\overline{\Phi}\left(|X_t - x|/\sqrt{T-t}\right)}\right)\mathrm{d}t + \mathrm{d}W_t^{\mathbb{Q}}, \quad \text{for } t \in [0,T).$$

6. The Decomposition Result of Jeulin and Yor

Let ρ be a $\mathcal{F}_\infty$-measurable random time on $(\Omega, \mathcal{F}, \mathbf{F})$. Let $\mathbf{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ be defined via

$$\mathcal{G}_t = \{B \in \mathcal{F}_\infty \mid B \cap \{\rho > t\} = B_t \cap \{\rho > t\} \text{ for some } B_t \in \mathcal{F}_t\}, \quad t \in \mathbb{R}_+.$$

It is straightforward to check that $\mathbf{G}$ is a right-continuous filtration that contains $\mathbf{F}$, as well as that ρ is a stopping time on $(\Omega, \mathbf{G})$.

Whenever X is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$, the Jeulin-Yor decomposition theorem identifies the Doob-Meyer decomposition of X^ρ on $(\Omega, \mathbf{G}, \mathbb{P})$. Here, we provide the statement (Theorem 6.2) and a novel proof of the result of Jeulin and Yor that uses the tools developed in this paper and

does not rely on elements of the theory of progressive filtration enlargements. The following result, which is basically a consequence of Proposition 1.9, provides a main ingredient of our approach. It is useful to recall the collection $(\eta_u)_{u\in[0,1)}$ from (1.4).

Lemma 6.1. *Let ρ be a $\mathcal{F}_\infty$-measurable random time, and Y be a process such that $\mathbb{E}_{\mathbb{P}}\left[Y^*_\rho\right] < \infty$ and Y^{η_u} is local martingale on $(\Omega, \mathbf{F}, \mathbb{Q}_u)$ for all $u \in [0,1)$. Then Y^ρ is a martingale on $(\Omega, \mathbf{G}, \mathbb{P})$.*

Proof. Using (1.5), observe that $\int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[Y^*_{\eta_u}\right] \mathrm{d}u = \mathbb{E}_{\mathbb{P}}\left[Y^*_\rho\right] < \infty$. Furthermore, the mapping $[0,1) \ni u \mapsto \mathbb{E}_{\mathbb{Q}_u}\left[Y^*_{\eta_u}\right]$ is nondecreasing, as follows from consistency of the family $(\mathbb{Q}_u)_{u\in[0,1)}$. Therefore, $\mathbb{E}_{\mathbb{Q}_u}\left[Y^*_{\eta_u}\right] < \infty$ for all $u \in [0,1)$. This implies that, actually, Y^{η_u} is a uniformly integrable martingale on $(\Omega, \mathbf{F}, \mathbb{Q}_u)$ for all $u \in [0,1)$.

Fix $s \in \mathbb{R}_+$ and $t \in (s,\infty)$. Pick $B \in \mathcal{G}_s$ and $B_s \in \mathcal{F}_s$ such that $B \cap \{\rho > s\} = B_s \cap \{\rho > s\}$. Note that the process $Y^t \mathbb{I}_{B_s \cap]\!]s,\infty[\![}$ is optional on $(\Omega, \mathbf{F})$ and $Y^t_\rho \mathbb{I}_{B_s\cap\{s<\rho\}} = Y^\rho_t \mathbb{I}_{B_s}\mathbb{I}_{\{\rho>s\}}$. In view of Proposition 1.9 (with the usual trick of splitting into positive and negative parts) and the martingale property of Y^{η_u} on $(\Omega, \mathbf{F}, \mathbb{Q}_u)$ for all $u \in [0,1)$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[Y^\rho_t \mathbb{I}_{B_s}\mathbb{I}_{\{\rho>s\}}\right] &= \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[Y^{\eta_u}_t \mathbb{I}_{B_s}\mathbb{I}_{\{\eta_u>s\}}\right] \mathrm{d}u \\ &= \int_{[0,1)} \mathbb{E}_{\mathbb{Q}_u}\left[Y^{\eta_u}_s \mathbb{I}_{B_s}\mathbb{I}_{\{\eta_u>s\}}\right] \mathrm{d}u \;=\; \mathbb{E}_{\mathbb{P}}\left[Y^\rho_s \mathbb{I}_{B_s}\mathbb{I}_{\{\rho>s\}}\right].\end{aligned}$$

The last equation and the fact that $Y^\rho_t \mathbb{I}_B = Y^\rho_s \mathbb{I}_B \mathbb{I}_{\{\rho\le s\}} + Y^\rho_t \mathbb{I}_{B_s}\mathbb{I}_{\{\rho>s\}}$ imply that $\mathbb{E}_{\mathbb{P}}\left[Y^\rho_t \mathbb{I}_B\right] = \mathbb{E}_{\mathbb{P}}\left[Y^\rho_s \mathbb{I}_B\right]$. Since $B \in \mathcal{G}_s$ is arbitrary, we obtain $\mathbb{E}_{\mathbb{P}}\left[Y^\rho_t \mid \mathcal{G}_s\right] = Y^\rho_s$, which establishes the claim. □

What follows is the decomposition theorem of Jeulin and Yor (see [17], as well as [11] for further development), which in particular implies that for any semimartingale X on $(\Omega, \mathbf{F}, \mathbb{P})$, X^ρ is a semimartingale on $(\Omega, \mathbf{G}, \mathbb{P})$.

Theorem 6.2. *Let ρ be a $\mathcal{F}_\infty$-measurable random time on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with associated canonical pair (K, L). Recall the processes Z and N from §1.1. Furthermore, let X be a process such that X^{η_u} is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $u \in [0,1)$. Then:*

1. *The set-inclusion $[\![0,\rho]\!] \subseteq \Gamma := \bigcup_{u\in[0,1)} [\![0,\eta_u]\!]$ holds modulo $\mathbb{P}$-evanescence.*
2. *The processes $\langle L, X\rangle$ and $\langle N, X\rangle$, each being the predictable compensator under $\mathbb{P}$ of $[L,X]$ and $[N,X]$ respectively, are well defined on Γ.*
3. *$\mathbb{P}\left[\inf_{t\in\mathbb{R}_+} L^\rho_{t-} > 0\right] = 1$ and $\mathbb{P}\left[\inf_{t\in\mathbb{R}_+} Z^\rho_{t-} > 0\right] = 1$; therefore, $\mathbb{P}$-a.s.,*
$$\int_0^\rho \frac{1}{L_{t-}} \mathrm{d}\,\mathsf{Var}\left(\langle L, X\rangle\right)_t = \int_0^\rho \frac{1}{Z_{t-}} \mathrm{d}\,\mathsf{Var}\left(\langle N, X\rangle\right)_t < \infty,$$
where "Var" is the operator returning the first variation of a process.
4. *The process*
$$Y^\rho := X^\rho - \int_0^{\rho\wedge\cdot} \frac{1}{L_{t-}} \mathrm{d}\langle L, X\rangle_t = X^\rho - \int_0^{\rho\wedge\cdot} \frac{1}{Z_{t-}} \mathrm{d}\langle N, X\rangle_t \tag{6.1}$$
is a local martingale on $(\Omega, \mathbf{G}, \mathbb{P})$.

Remark 6.3. Technicalities aside, intuition on the important statement (4) of Theorem 6.2 follows from Lemma 6.1 coupled with an application of Girsanov's theorem. Indeed, if X^{η_u} is a martingale on $(\Omega, \mathbf{F}, \mathbb{P})$, Y^{η_u} (in obvious notation) has (some kind of) the martingale property on $(\Omega, \mathbf{F}, \mathbb{Q}_u)$ in view of Girsanov's theorem and the fact that $L_{\eta_u} = (\mathrm{d}\mathbb{Q}_u/\mathrm{d}\mathbb{P})|_{\mathcal{F}_{\eta_u}}$ for all $u \in [0,1)$. Then, Y^ρ should have (some kind of) the martingale property on $(\Omega, \mathbf{G}, \mathbb{P})$, as follows from Lemma 6.1.

The idea of proving the Jeulin-Yor decomposition theorem via Girsanov's theorem has also been used by Yoeurp [18, Chapitre III, page 172]. However, Girsanov's theorem there is applied on the product space $\Omega \times \mathbb{R}_+$ equipped with the predictable sigma-algebra. The approach here is more transparent, as we are dealing with probabilities on $(\Omega, \mathcal{F}, \mathbf{F})$.

Proof of Theorem 6.2. Since $\mathbb{P}[\rho \leq \eta_a] = \int_{[0,1)} \mathbb{Q}_u[\eta_u \leq \eta_a]\,\mathrm{d}u \geq a$ holds for all $a \in [0,1)$ by Proposition 1.9, it follows that $\lim_{a\uparrow 1} \mathbb{P}[\rho \leq \eta_a] = 1$. Therefore, statement (1) is established.

Fix $u \in [0,1)$. As L^{η_u} is locally bounded (see Lemma 1.7) and X^{η_u} is locally integrable (being a local martingale) on $(\Omega, \mathbf{F}, \mathbb{P})$, it follows that $\mathsf{Var}([L,X])^{\eta_u}$ is locally integrable on $(\Omega, \mathbf{F}, \mathbb{P})$. By (1.3) and $Z = L(1-K)$, $\mathsf{Var}([N,X])^{\eta_u} = (1/(1-K_-)) \cdot \mathsf{Var}([L,X])^{\eta_u} \leq (1-u)^{-1}\mathsf{Var}([L,X])^{\eta_u}$ implies that $\mathsf{Var}([N,X])^{\eta_u}$ is also locally integrable on $(\Omega, \mathbf{F}, \mathbb{P})$. Since this holds for all $u \in [0,1)$, $\langle L, X\rangle$ and $\langle N, X\rangle$ are well defined on Γ, which establishes statement (2).

By Proposition 1.12 $\mathbb{P}[L_\rho > 0] = 1$; since L is a nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$, we obtain $\mathbb{P}\left[\inf_{t\in\mathbb{R}_+} L^\rho_{t-} > 0\right] = 1$. Then, $\mathbb{P}\left[\inf_{t\in\mathbb{R}_+} Z^\rho_{t-} > 0\right] = 1$ follows from $\mathbb{P}\left[\inf_{t\in\mathbb{R}_+} L^\rho_{t-} > 0\right] = 1$, coupled with $\mathbb{P}\left[\sup_{t\in\mathbb{R}_+} K^\rho_{t-} < 1\right] = \mathbb{P}[K_{\rho-} < 1] = 1$ (see Proposition 1.10) and the relationship $Z = L(1-K)$, holding up to $\mathbb{P}$-evanescence. This shows the validity of statement (3).

We proceed to the proof of statement (4). Since $[\![0,\rho]\!] \subseteq \Gamma$ holds modulo $\mathbb{P}$-evanescence, standard localisation arguments imply the existence of a nondecreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times on $(\Omega, \mathbf{F})$ and a $(0,\infty)$-valued nondecreasing sequence $(C_n)_{n\in\mathbb{N}}$ such that all the following conditions are met: $\tau_n \leq \eta_{1-1/n}$ for all $n \in \mathbb{N}$; $\lim_{n\to\infty} \mathbb{P}[\rho \leq \tau_n] = 1$; $\lim_{n\to\infty} C_n = \infty$; $\mathbb{P}\big[\inf_{t\in\mathbb{R}_+} L^{\tau_n}_{t-} \geq C_n^{-1/2}\big] = 1$ for all $n \in \mathbb{N}$; $\mathbb{P}\big[[L,L]_{\tau_n} \leq C_n\big] = 1$ for all $n \in \mathbb{N}$; $\mathbb{E}_{\mathbb{P}}\left[X^*_{\tau_n}\right] < \infty$ for all $n \in \mathbb{N}$. (In particular, the last condition implies that X^{τ_n} is a martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $n \in \mathbb{N}$.)

Suppose we can show that $Y^{\rho\wedge\tau_n}$ is a local martingale on $(\Omega, \mathbf{G}, \mathbb{P})$ for all $n \in \mathbb{N}$. Then, setting $\tau'_n := \tau_n \mathbb{I}_{\{\rho > \tau_n\}} + \infty \mathbb{I}_{\{\rho \leq \tau_n\}}$, we have that $(\tau'_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of stopping times on $(\Omega, \mathbf{G})$ such that $\mathbb{P}[\lim_{n\to\infty} \tau'_n = \infty] = 1$ and $Y^{\rho\wedge\tau'_n} = Y^{\rho\wedge\tau_n}$ is a local martingale on $(\Omega, \mathbf{G}, \mathbb{P})$ for all $n \in \mathbb{N}$; it will then follow that Y^ρ is a local martingale on $(\Omega, \mathbf{G}, \mathbb{P})$. Therefore, it suffices to show that $Y^{\rho\wedge\tau_n}$ is a local martingale on $(\Omega, \mathbf{G}, \mathbb{P})$ for all $n \in \mathbb{N}$.

We estimate $\mathsf{Var}([L,X])_{\tau_n} \leq [L,L]^{1/2}_{\tau_n}[X,X]^{1/2}_{\tau_n} \leq C_n^{1/2}[X,X]^{1/2}_{\tau_n}$. Using (6.1) and the fact that $\inf_{t\in\mathbb{R}_+} L^{\tau_n}_{t-} \geq C_n^{1/2}$, we obtain $Y^*_{\rho\wedge\tau_n} \leq X^*_{\tau_n} + C_n[X,X]^{1/2}_{\tau_n}$. In view of the Davis inequality, $\mathbb{E}_{\mathbb{P}}\left[X^*_{\tau_n}\right] < \infty$ implies $\mathbb{E}_{\mathbb{P}}\big[[X,X]^{1/2}_{\tau_n}\big] < \infty$; therefore, $\mathbb{E}_{\mathbb{P}}\left[Y^*_{\rho\wedge\tau_n}\right] < \infty$. Furthermore, $Y^{\tau_n\wedge\eta_u}$ is a local martingale on $(\Omega, \mathbf{F}, \mathbb{Q}_u)$ for all $u \in [0,1)$. Indeed, given that, $\mathbb{Q}$-a.s., $\int_0^{\tau_n\wedge\eta_u}(1/L_{t-})\mathrm{d}\mathsf{Var}(\langle L,X\rangle)_t < \infty$, this follows in a straightforward way from Girsanov's theorem. Then, $Y^{\rho\wedge\tau_n}$ is a martingale on $(\Omega, \mathbf{G}, \mathbb{P})$, as follows from Lemma 6.1. $\square$

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